STUDIA MATHEMATICA 202 (2) (2011)

The canonical injection of the Hardy–Orlicz space H^Ψ into the Bergman–Orlicz space $\mathfrak{B}^\Psi$

by

Pascal Lefèvre (Lens), Daniel Li (Lens),
Hervé Queffélec (Lille) and Luis Rodríguez-Piazza (Sevilla)

Abstract. We study the canonical injection from the Hardy–Orlicz space H^Ψ into the Bergman–Orlicz space $\mathfrak{B}^\Psi$.

1. Introduction and notation

1.1. Introduction. There are two natural Orlicz spaces of analytic functions on the unit disk $\mathbb{D}$ of the complex plane: the Hardy space H^p and the Bergman space $\mathfrak{B}^p$. It is well-known that $H^p \subseteq \mathfrak{B}^p$ and the canonical injection J_p from H^p to $\mathfrak{B}^p$ is bounded, and even compact. Recently, we introduced natural generalizations of these two spaces, the Hardy–Orlicz space H^Ψ and the Bergman–Orlicz space $\mathfrak{B}^\Psi$, associated to an Orlicz function Ψ, and studied composition operators C_φ acting on either of those spaces ([7], [9]). It turns out that, in most cases, the compactness of $C_\varphi\colon H^\Psi \to H^\Psi$ implies the compactness of $C_\varphi\colon \mathfrak{B}^\Psi \to \mathfrak{B}^\Psi$. Therefore, it seems natural to study directly the link between H^Ψ and $\mathfrak{B}^\Psi$.

In fact, for any Orlicz function Ψ, one has $H^\Psi \subseteq \mathfrak{B}^\Psi$ and the canonical injection $J_\Psi\colon H^\Psi \to \mathfrak{B}^\Psi$ is bounded. In this paper, we investigate the compactness and weak compactness of this injection, as well as other properties, like being Dunford–Pettis, absolutely summing, order bounded. We show that the compactness of J_Ψ requires that Ψ does not grow too fast. In Section 2 we actually characterize the compactness: J_Ψ is compact if and only if $\lim_{x\to\infty} \Psi(Ax)/[\Psi(x)]^2 = 0$ for every $A > 1$, and the weak compactness: J_Ψ is weakly compact if and only if $\limsup_{x\to\infty} \Psi(Ax)/[\Psi(x)]^2 < \infty$ for every $A > 1$. We show that even though these two properties are "often" equivalent (this happens for example if $\Psi(2x)/x$ is non-decreasing for x

2010 *Mathematics Subject Classification*: Primary 46E15; Secondary 30H10, 30H20, 32A35, 32A36, 42B30, 46E30.
Key words and phrases: absolutely summing operator, Bergman–Orlicz space, compactness, Dunford–Pettis operator, Hardy–Orlicz space, weak compactness.

DOI: 10.4064/sm202-2-2

large enough), it is not always the case. We actually show a stronger result in Section 4: there is an Orlicz function Ψ such that J_Ψ is weakly compact and Dunford–Pettis, but not compact. We also prove in Section 3.2 that J_Ψ is compact if it is p-summing with $p < 2$ (Theorem 3.3). Finally, we show that J_Ψ is order bounded into the weak Orlicz space $L^{\Psi,\infty}(\mathbb{D}, m_2)$ (Proposition 3.7).

1.2. Notation. An *Orlicz function* is a non-decreasing convex function $\Psi\colon [0,\infty) \to [0,\infty)$ such that $\Psi(0) = 0$, $\Psi(x) > 0$ for $x > 0$, and $\Psi(\infty) = \infty$. One says that the Orlicz function Ψ has *property* Δ_2 ($\Psi \in \Delta_2$) if $\Psi(2x) \le C\Psi(x)$ for some constant $C > 0$ and x large enough. This is equivalent to saying that, for every $\beta > 1$, $\Psi(\beta x) \le C_\beta \Psi(x)$. It is known that if $\Psi \in \Delta_2$, then $\Psi(x) = O(x^p)$ for some $1 \le p < \infty$. One says (see [6], [7]) that Ψ satisfies the condition Δ^0 if, for some $\beta > 1$, one has $\lim_{x\to\infty} \Psi(\beta x)/\Psi(x) = \infty$.

If $\Psi \in \Delta^0$, then $\Psi(x)/x^p \to \infty$ as $x \to \infty$ for every $1 \le p < \infty$. Indeed, let $1 \le p < \infty$. For every $\beta > 1$ one can find $x_0 > 0$ such that $\Psi(\beta x)/\Psi(x) \ge \beta^p$ for $x \ge x_0$; then $\Psi(\beta^n x_0) \ge \beta^{np}\Psi(x_0)$ for every $n \ge 1$. That implies that $\Psi(x) \ge C_p\, x^p$ for every $x > 0$ large enough. Since $p \ge 1$ is arbitrary, we get $x^p = o\,[\Psi(x)]$.

We say that $\Psi \in \nabla_0(1)$ if, for every $A > 1$, $\Psi(Ax)/\Psi(x)$ is non-decreasing for x large enough. This is equivalent to saying (see [7, Proposition 4.7]) that $\log \Psi(e^x)$ is convex. When $\Psi \in \nabla_0(1)$, one has either $\Psi \in \Delta_2$, or $\Psi \in \Delta^0$.

If $(S, \mathcal{S}, \mu)$ is a finite measure space, one defines the *Orlicz space* $L^\Psi(\mu)$ as the set of all (classes of) measurable functions $f\colon S \to \mathbb{C}$ for which there is a $C > 0$ such that $\int_S \Psi(|f|/C)\, d\mu$ is finite. The norm $\|f\|_\Psi$ is the infimum of all $C > 0$ for which the above integral is ≤ 1. The *Morse–Transue space* $M^\Psi(\mu)$ is the subspace of $f \in L^\Psi(\mu)$ for which $\int_S \Psi(|f|/C)\, d\mu$ is finite for all $C > 0$; it is the closure of $L^\infty(\mu)$ in $L^\Psi(\mu)$. One has $M^\Psi(\mu) = L^\Psi(\mu)$ if and only if $\Psi \in \Delta_2$.

If $\Psi(x)/x \to \infty$ as $x \to \infty$, the *conjugate function* Φ of Ψ is defined by $\Phi(y) = \sup_{x>0}(xy - \Psi(x))$. It is an Orlicz function and $[M^\Psi(\mu)]^* = L^\Phi(\mu)$ isomorphically.

Note that if $\Psi(x)/x \not\to \infty$ as $x \to \infty$, we must have $\Psi(x) \le ax$ for some $a \ge 1$ and x large enough. Then $L^\Psi(\mu) = L^1(\mu)$ isomorphically and so $\Phi(y) = \infty$ for $y > a$ (giving $L^\Phi(\mu) = L^\infty(\mu)$ isomorphically).

We denote by $\mathbb{D}$ the open unit disk of $\mathbb{C}$ and by $\mathbb{T} = \partial\mathbb{D}$ the unit circle. The normalized area-measure on $\mathbb{D}$ is denoted by m_2 and the normalized Lebesgue measure on $\mathbb{T}$ is denoted by m.

The *Hardy–Orlicz space* H^Ψ is defined as $\{f \in H^1;\ f^* \in L^\Psi(m)\}$, where f^* is the boundary value function of f, and $HM^\Psi = H^\Psi \cap M^\Psi(m)$ is the closure of H^∞ in H^Ψ. The *Bergman–Orlicz space* $\mathfrak{B}^\Psi$ is the subspace of analytic $f \in L^\Psi(m_2)$, and $\mathfrak{B}M^\Psi = \mathfrak{B}^\Psi \cap M^\Psi(m_2)$ is the closure of H^∞ in $\mathfrak{B}^\Psi$.

Since, for $f \in H^\Psi$, $\|f\|_{H^\Psi} = \sup_{0<r<1} \|f_r\|_{H^\Psi}$ (see [7, Proposition 3.1]), where $f_r(z) = f(rz)$, one has

$$\int_0^{2\pi} \Psi\biggl(\frac{|f(re^{it})|}{\|f\|_{H^\Psi}}\biggr)\frac{dt}{2\pi} \le \int_0^{2\pi} \Psi\biggl(\frac{|f(re^{it})|}{\|f_r\|_{H^\Psi}}\biggr)\frac{dt}{2\pi} \le 1;$$

hence

$$\int_{\mathbb{D}} \Psi\biggl(\frac{|f(re^{it})|}{\|f\|_{H^\Psi}}\biggr)\,dm_2 = \int_0^1 \biggl[\int_0^{2\pi} \Psi\biggl(\frac{|f(re^{it})|}{\|f\|_{H^\Psi}}\biggr)\frac{dt}{2\pi}\biggr] 2r\,dr \le 1,$$

so $f \in \mathfrak{B}^\Psi$ and $\|f\|_{\mathfrak{B}^\Psi} \le \|f\|_{H^\Psi}$. It follows that $H^\Psi \subseteq \mathfrak{B}^\Psi$ and the canonical injection $J_\Psi\colon H^\Psi \to \mathfrak{B}^\Psi$ is bounded, and has norm 1. Let us point out that the boundedness also follows from [7, Theorem 4.10, 2]), since J_Ψ is a Carleson embedding $J_\Psi\colon H^\Psi \to \mathfrak{B}^\Psi \subseteq L^\Psi(m_2)$.

This injection is not onto, since there are functions $f \in \mathfrak{B}^\Psi$ with no radial limit on a subset of $\mathbb{T}$ of positive measure (the proof is the same as in $\mathfrak{B}^p$: see [4, §3.2, Lemma 2, p. 81]). Note that J_Ψ is not an into-isomorphism (i.e. is not an isomorphism between H^Ψ and $J_\Psi(H^\Psi)$): take $f_n(z) = z^n$ for every $n \in \mathbb{N}$; it is easy to see that $\{f_n\}_n$ tends to 0 in $\mathfrak{B}^\Psi$, but not in H^Ψ.

2. Compactness and weak compactness. In order to characterize the compactness and weak compactness of J_Ψ, we introduce the quantity

$$Q_A = \limsup_{x\to\infty} \frac{\Psi(Ax)}{[\Psi(x)]^2}, \qquad A > 1, \tag{2.1}$$

which will turn out to be essential.

We start with compactness.

Theorem 2.1. *The canonical injection $J_\Psi\colon H^\Psi \to \mathfrak{B}^\Psi$ is compact if and only if*

$$\lim_{x\to\infty} \frac{\Psi(Ax)}{[\Psi(x)]^2} = 0 \quad \textit{for every } A > 1. \tag{2.2}$$

Remarks. 1) Condition (2.2) means that $Q_A = 0$ for every $A > 1$. This is equivalent to saying that

$$\sup_{A>1} Q_A < \infty. \tag{2.3}$$

Indeed, assume that $M := \sup_{A>1} Q_A < \infty$. Let $0 < \varepsilon \le 1$ and $A > 1$; we can find $x_A = x_A(\varepsilon) > 0$ such that $\Psi(Ax/\varepsilon)/[\Psi(x)]^2 \le 2M$ for $x \ge x_A$. By convexity, $\Psi(Ax) \le \varepsilon\Psi(Ax/\varepsilon)$, and hence $\Psi(Ax)/[\Psi(x)]^2 \le 2\varepsilon M$ for $x \ge x_A$. We get $Q_A = 0$.

2) It is clear that condition (2.2) is satisfied whenever $\Psi \in \Delta_2$, but $\Psi(x) = e^{[\log(x+1)]^2} - 1$ satisfies (2.2) without being in Δ_2. However, condition (2.2) implies that Ψ cannot grow too fast. More precisely, we must have

$$\Psi(x) = o(e^{x^\alpha}) \qquad \text{for every } \alpha > 0.$$

Indeed, $\Psi(At) \leq [\Psi(t)]^2$ for $t \geq t_A$, and, by iteration, $\Psi(A^n t_A) \leq [\Psi(t_A)]^{2^n}$ for every $n \geq 1$. For every $x > 0$ large enough, taking $n \geq 1$ such that $A^n t_A \leq x < A^{n+1} t_A$, we get $\Psi(x) \leq C_1\, e^{C_2 x^\alpha}$ with $\alpha = \log 2/\log A$. Since $A > 1$ is arbitrary, α may be any positive number. The little-oh condition follows from the fact that the inequality is true for all $\alpha > 0$.

Proof of Theorem 2.1. By definition, $\mathfrak{B}^\Psi$ is a subspace of $L^\Psi(\mathbb{D}, m_2)$; hence we can view J_Ψ as a Carleson embedding $J_\Psi\colon H^\Psi \to L^\Psi(\mathbb{D}, m_2)$. If $S(\xi, h) = \{z \in \mathbb{D};\ |z - \xi| < h\}$, the compactness of J_Ψ implies, by [7, Theorem 4.11], that, for every $A > 1$, every $\varepsilon > 0$, and $h > 0$ small enough,

$$h^2 \leq 4m_2[S(\xi, h)] \leq \frac{4\varepsilon}{\Psi[A\Psi^{-1}(1/h)]},$$

that is, setting $x = \Psi^{-1}(1/h)$, we have $\Psi(Ax) \leq 4\varepsilon[\Psi(x)]^2$, and (2.2) is satisfied.

Conversely,

$$\sup_{0<t\leq h}\ \sup_{|\xi|=1} \frac{m_2[S(\xi,t)]}{t} \leq \sup_{0<t\leq h} \frac{t^2}{t} = h,$$

which is $o((1/h)/\Psi[A\Psi^{-1}(1/h)])$ for every $A > 1$ if (2.2) holds; hence, by [7, Theorem 4.11] again, J_Ψ is compact. ∎

We now turn to weak compactness.

Theorem 2.2. *The following assertions are equivalent:*

(a) $J_\Psi\colon H^\Psi \to \mathfrak{B}^\Psi$ *is weakly compact;*
(b) J_Ψ *fixes no copy of* c_0*;*
(c) J_Ψ *fixes no copy of* ℓ_∞*;*
(d) $Q_A < \infty$ *for every* $A > 1$*;*
(e) $H^\Psi \subseteq \mathfrak{B}M^\Psi$*;*
(f) J_Ψ *is strictly singular.*

Recall that an operator $T\colon X \to Y$ between two Banach spaces is said to be *strictly singular* if there is no infinite-dimensional subspace X_0 of X on which T is an into-isomorphism.

The proof will be somewhat long, and before beginning it, we remark that if $\Psi \in \Delta^0$, then the condition

$$Q_A < \infty \quad \text{for every } A > 1 \tag{2.4}$$

implies (2.2). Indeed, if $\lim_{x\to\infty} \Psi(\beta x)/\Psi(x) = \infty$, we get, for every $A > 1$,

$$\limsup_{x\to\infty} \frac{\Psi(Ax)}{[\Psi(x)]^2} = \limsup_{x\to\infty} \frac{\Psi(Ax)}{\Psi(\beta Ax)}\,\frac{\Psi(\beta Ax)}{[\Psi(x)]^2} \leq \limsup_{x\to\infty} \frac{\Psi(Ax)}{\Psi(\beta Ax)} Q_{\beta A} = 0.$$

This remark yields:

Proposition 2.3. *If, for some $A > 1$, $\Psi(Ax)/\Psi(x)$ is non-decreasing for x large enough, then the weak compactness of J_Ψ is equivalent to its compactness.*

Proof. If, for some $A > 1$, $\Psi(Ax)/\Psi(x)$ is non-decreasing for x large enough (in particular if $\Psi \in \nabla_0(1)$), one has the dichotomy: either $\Psi \in \Delta_2$, and then J_Ψ is compact; or $\Psi \in \Delta^0$, and hence the weak compactness of J_Ψ implies its compactness, by the above two theorems. ∎

However, it is easy to construct an Orlicz function Ψ which satisfies condition (2.4), but not (2.2). We do not give an example here because we have a stronger result in Section 4.

In order to prove Theorem 2.2, we shall need several lemmas.

Lemma 2.4. *Let Ψ be any Orlicz function and define $\Psi_1(t) = [\Psi(t)]^2$, $t \geq 0$. Then Ψ_1 is an Orlicz function for which $H^\Psi \subseteq \mathfrak{B}^{\Psi_1}$ and the canonical injection of H^Ψ into $\mathfrak{B}^{\Psi_1}$ is continuous.*

Proof. It is enough to see that H^Ψ continuously embeds into $L^{\Psi_1}(m_2)$, and for this we can use Theorem 4.10 of [7] for the measure $\mu = m_2$. Recall that

$$\rho_\mu(h) = \sup_{|\xi|=1} \mu[W(\xi,h)] \quad \text{and} \quad K_\mu(h) = \sup_{0<t\leq h} \frac{\rho_\mu(t)}{t},$$

where $W(\xi,h) = \{z \in \mathbb{D};\ |z| \geq 1-h \text{ and } \arg(z\overline{\xi}) \leq h\}$ is the Carleson window of size h centered at ξ.

It is easy to see that, as $h \to 0^+$, $\rho_{m_2}(h) \approx h^2$ and $K_{m_2}(h) \approx h$. Observe that, for $t > 1$, we have $\Psi_1[\Psi^{-1}(t)] = t^2$, and so, for $h \in (0,1)$,

$$\frac{1/h}{\Psi_1[\Psi^{-1}(1/h)]} = \frac{1/h}{1/h^2} = h \succeq K_{m_2}(h).$$

Using part 2) of Theorem 4.10 in [7], the lemma follows. ∎

Lemma 2.5. *Let $M > \delta > 0$ and $\{f_n\}_n$ be a sequence in $H^\Psi \cap \mathfrak{B}M^\Psi$ such that:*

(a) *$\{f_n\}_n$ tends to 0 uniformly on compact subsets of $\mathbb{D}$;*
(b) *$\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$ for every $n \geq 1$;*
(c) *$\|f_n\|_{H^\Psi} \leq M$ for every $n \geq 1$.*

Then there exists a subsequence $\{f_{n_k}\}_k$ such that $\sum_k |f_{n_k}(z)| < \infty$ for every $z \in \mathbb{D}$, and for every $\alpha = (\alpha_k)_k \in \ell_\infty$ one has, writing $T\alpha(z) = \sum_{k=1}^\infty \alpha_k f_{n_k}(z)$,

$$T\alpha \in \mathfrak{B}^\Psi \quad \textit{and} \quad (\delta/2)\|\alpha\|_\infty \leq \|T\alpha\|_{\mathfrak{B}^\Psi} \leq 2M\|\alpha\|_\infty. \tag{2.5}$$

Remark. It is clear that, by (2.5), we are defining an operator T from ℓ_∞ into $\mathfrak{B}^\Psi$ which is an isomorphism between ℓ_∞ and its image. In particular, the subsequence $\{f_{n_k}\}_k$ is equivalent, in $\mathfrak{B}^\Psi$, to the canonical basis of c_0.

Proof. First we are going to construct, inductively, a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}$, and an increasing sequence $\{r_k\}_k$ in $(0,1)$, such that $\lim_{k\to\infty} r_k = 1$ and, setting

$$D_k = \{z \in \mathbb{D};\ |z| \le r_k\} \quad \text{for } k \ge 1$$

and

$$C_1 = D_1, \quad C_k = D_k \setminus D_{k-1} = \{z \in \mathbb{D};\ r_{k-1} < |z| \le r_k\}, \quad k \ge 2,$$

we have

$$|f_{n_k}(z)| \le 2^{-k} \quad \text{for every } z \in D_{k-1} \text{ and every } k \ge 2, \tag{2.6}$$

and

$$\|f_{n_k} \mathbb{1}_{\mathbb{D}\setminus C_k}\|_{L^\Psi} < \delta 2^{-k-2} \quad \text{for every } k \ge 1. \tag{2.7}$$

Start the construction by taking $n_1 = 1$. It is a known fact (see [12, Theorem III.14], for example) that, for every function f in the Morse–Transue space $M^\Psi(m_2)$,

$$\lim_{m_2(A)\to 0} \|f \mathbb{1}_A\|_{L^\Psi} = 0. \tag{2.8}$$

Now, using (2.8) with $f = f_{n_1}$ and $A = \{z \in \mathbb{D};\ r < |z| < 1\}$, we get $r_1 \in (0,1)$ such that, for $C_1 = D_1 = \{z \in \mathbb{D};\ |z| \le r_1\}$,

$$\|f_1 \mathbb{1}_{\mathbb{D}\setminus C_1}\|_{L^\Psi} < \delta 2^{-3}.$$

By the uniform convergence of $\{f_n\}_n$ to 0 on D_1, we can find $n_2 > n_1$ such that

$$|f_{n_2}(z)| \le 1/4 \quad \text{for every } z \in D_1 \quad \text{and} \quad \|f_{n_2} \mathbb{1}_{D_1}\|_{L^\Psi} < \delta 2^{-5}.$$

Using this last inequality and (2.8) again (for $f = f_{n_2}$), we get $r_2 \in (r_1, 1)$, $r_2 > 1 - 1/2$, such that, setting $C_2 = \{z \in \mathbb{D};\ r_1 < |z| \le r_2\}$, we have

$$\|f_{n_2} \mathbb{1}_{\mathbb{D}\setminus C_2}\|_{L^\Psi} < \delta 2^{-4}.$$

Now that we have (2.6) and (2.7) for $k = 1$ and $k = 2$, it is clear how we must iterate the inductive construction. When choosing $r_k \in (r_{k-1}, 1)$, we also impose the condition $r_k > 1 - 1/k$ in order to get $\lim_{k\to\infty} r_k = 1$.

Once the construction is finished, let us see why the subsequence $\{f_{n_k}\}_k$ works. The condition (2.6) and the fact that $\lim_{k\to\infty} r_k = 1$ imply that, for every compact set K in $\mathbb{D}$ and $z \in \mathbb{D}$, there exists $l_K \in \mathbb{N}$ such that

$$|f_{n_k}(z)| \le 2^{-k} \quad \text{for every } z \in K \text{ and every } k \ge l_K.$$

This yields two facts. First, $\sum_k |f_{n_k}(z)| < \infty$ for every $z \in \mathbb{D}$, and secondly, for every bounded complex sequence $\alpha = \{\alpha_k\}_k \in \ell_\infty$, the series $\sum_k \alpha_k f_{n_k}$ converges uniformly on compact subsets of $\mathbb{D}$, and its sum, the function $T\alpha$, is analytic on $\mathbb{D}$.

It remains to prove the estimates in (2.5) for the norm of $T\alpha$ in $L^\Psi(m_2)$. By homogeneity, we may assume that $\|\alpha\|_\infty = 1$. Let us write $g_k = f_{n_k} \mathbb{1}_{C_k}$

and $h_k = f_{n_k} \mathbb{1}_{\mathbb{D}\setminus C_k}$, for every $k \geq 1$, and

$$g = \sum_{k=1}^{\infty} \alpha_k g_k \quad \text{and} \quad h = \sum_{k=1}^{\infty} \alpha_k h_k.$$

We have $T\alpha = g + h$. By (2.7) and the fact that $|\alpha_k| \leq 1$, we deduce that $h \in L^\Psi(m_2)$ and $\|h\|_{L^\Psi} \leq \delta/4$.

By the condition (c) in the statement, and the definition of the norm in H^Ψ, we have, for every n and every $r \in (0,1)$,

$$\frac{1}{2\pi} \int_0^{2\pi} \Psi(|f_n(re^{it})|/M)\, dt \leq 1. \tag{2.9}$$

The function g_k is 0 outside of C_k, and the sequence $\{C_k\}_k$ is a partition of $\mathbb{D}$. Therefore

$$\begin{aligned}\int_{\mathbb{D}} \Psi(|g|/M)\, dm_2 &= \sum_{k=1}^{\infty} \int_{C_k} \Psi(|g|/M)\, dm_2 = \sum_{k=1}^{\infty} \int_{C_k} \Psi(|\alpha_k|\,|f_{n_k}|/M)\, dm_2 \\ &\leq \sum_{k=1}^{\infty} \int_{C_k} \Psi(|f_{n_k}|/M)\, dm_2.\end{aligned}$$

Integrating in polar coordinates, setting $r_0 = 0$, and using (2.9), we get

$$\begin{aligned}\int_{\mathbb{D}} \Psi(|g|/M)\, dm_2 &\leq \sum_{k=1}^{\infty} \int_{r_{k-1}}^{r_k} 2r\, \frac{1}{2\pi} \int_0^{2\pi} \Psi(|f_{n_k}(re^{it})|/M)\, dt\, dr \\ &\leq \sum_{k=1}^{\infty} \int_{r_{k-1}}^{r_k} 2r\, dr = 1,\end{aligned}$$

and therefore $\|g\|_{L^\Psi} \leq M$, and $\|T\alpha\|_{L^\Psi} \leq \delta/4 + M \leq 2M$.

On the other hand, for every k, we have

$$\|g\|_{L^\Psi} \geq \|g\mathbb{1}_{C_k}\|_{L^\Psi} = |\alpha_k|\, \|f_{n_k} - h_k\|_{L^\Psi} \geq |\alpha_k|\, (\delta - \delta/2^{2+k}) \geq \frac{3\delta}{4} |\alpha_k|.$$

Taking the supremum over k, we get $\|g\|_{L^\Psi} \geq (3\delta/4)\|\alpha\|_\infty = 3\delta/4$. Consequently,

$$\|T\alpha\|_{L^\Psi} \geq \|g\|_{L^\Psi} - \|h\|_{L^\Psi} \geq 3\delta/4 - \delta/4 = \delta/2,$$

and Lemma 2.5 is fully proved. ∎

In the following lemma we isolate the proof of the implication (c)⇒(d) in Theorem 2.2.

Lemma 2.6. *Assume that the Orlicz function Ψ is such that, for some $A > 1$,*

$$\limsup_{x \to \infty} \frac{\Psi(Ax)}{[\Psi(x)]^2} = \infty. \tag{2.10}$$

Then the injection $J_\Psi \colon H^\Psi \to \mathfrak{B}^\Psi$ fixes a copy of ℓ_∞.

Proof. Let us take a sequence $\{d_n\}_n$ of positive numbers and a sequence $\{\xi_n\}_n$ in $\mathbb{T}$ such that the disks $\{D(\xi_n, d_n)\}_n$ are pairwise disjoint in $\mathbb{D}$. In particular, we should have $\lim_{n\to\infty} d_n = 0$.

The convexity of Ψ implies the existence of some $c > 0$ such that $\Psi(x) \geq cx$ for every $x \geq 1$. Given a sequence $\{\beta_n\}_n$ in $(4, \infty)$ to be fixed later, we can find, thanks to (2.10), an increasing sequence $\{x_n\}$ satisfying

$$x_n > 1, \quad \Psi(x_n) > 1, \quad \Psi(Ax_n) > \beta_n[\Psi(x_n)]^2, \quad \text{for every } n \in \mathbb{N}. \tag{2.11}$$

Define y_n as the point in the interval (x_n, Ax_n) such that

$$[\Psi(y_n)]^2 = \Psi(Ax_n). \tag{2.12}$$

Put now $h_n = 1/\Psi(y_n)$ and $r_n = 1 - h_n$. By (2.11) and (2.12), we have $[\Psi(y_n)]^2 > \beta_n > 4$, and therefore $h_n \in (0, 1/2)$. Define

$$u_n(z) = \left(\frac{h_n}{1 - r_n\bar{\xi}_n z}\right)^2 \quad \text{and} \quad f_n(z) = y_n\, u_n(z).$$

It is easy to see that $\|u_n\|_\infty = 1$ and $\|u_n\|_{H^1} \leq h_n$.

We first impose on β_n the condition $\beta_n > 16/d_n^2$. That gives $[\Psi(y_n)]^2 > 16/d_n^2$ and $h_n < d_n/4$. Let us write D_n for the disk $D(\xi_n, d_n)$. Observe that, for $z \in \overline{\mathbb{D}} \setminus D_n$, we have

$$|1-r_n\bar{\xi}_n z| = |1-r_n+r_n\,\xi_n\bar{\xi}_n-r_n\bar{\xi}_n z| \geq r_n|\xi_n-z|-h_n \geq (1/2)d_n-h_n \geq d_n/4,$$

and therefore, since $[\Psi(x_n)]^2 \geq \Psi(x_n) \geq cx_n$,

$$|f_n(z)| \leq y_n\left(\frac{4h_n}{d_n}\right)^2 = \frac{16y_n}{d_n^2[\Psi(y_n)]^2} \leq \frac{16Ax_n}{d_n^2\beta_n[\Psi(x_n)]^2} \leq \frac{16A}{cd_n^2\beta_n}.$$

We also impose the condition $\beta_n > 16An^2/(cd_n^2)$, and so

$$|f_n(z)| \leq 1/n^2 \quad \text{for } z \in \overline{\mathbb{D}} \setminus D_n. \tag{2.13}$$

From (2.13) we deduce that $\{f_n\}_n$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. Moreover (2.13) implies that, for every bounded sequence $\{\alpha_n\}_n$ of complex numbers, the series $\sum_{n\geq1} \alpha_n f_n$ is uniformly convergent on compact subsets of $\mathbb{D}$. Let us write f_n^* for the boundary value (on $\mathbb{T} = \partial\mathbb{D}$) of the function f_n. We claim that

$$S = \sum_{n=1}^{\infty} |f_n^*| \in L^\Psi(\mathbb{T}, m). \tag{2.14}$$

From this, it is not difficult to deduce that, for every bounded sequence $\{\alpha_n\}_n$ of complex numbers, the function $\sum_{n=1}^\infty \alpha_n f_n$ is in H^Ψ and, for $M = \|S\|_{L^\Psi(\mathbb{T})}$,

$$\Big\|\sum_{n=1}^{\infty} \alpha_n f_n\Big\|_{H^\Psi} \leq M\|\{\alpha_n\}_n\|_\infty. \tag{2.15}$$

On the other hand, taking $A_n = \{z \in \mathbb{D};\, |z-\xi_n| \le h_n\}$, there exists a constant $\gamma \in (0,1)$ such that $m_2(A_n) \ge \gamma h_n^2$, and, for every $z \in A_n$,

$$|1 - r_n\bar{\xi}_n z| \le |1-r_n| + |r_n\xi_n\bar{\xi}_n - r_n\bar{\xi}_n z| = h_n + r_n|z-\xi_n| \le 2h_n,$$

and consequently $|u_n(z)| \ge 1/4$. If $\delta = \gamma/(4A)$, we have, for every n,

$$\begin{aligned}\int_{\mathbb{D}} \Psi\left(\frac{|f_n|}{\delta}\right) dm_2 \ge \int_{A_n} \Psi\left(\frac{y_n}{4\delta}\right) dm_2 &\ge \gamma h_n^2 \Psi\left(\frac{1}{\gamma} A y_n\right)\\ &\ge h_n^2\Psi(Ay_n) > h_n^2\Psi(Ax_n) = 1.\end{aligned}$$

Thus $\|f_n\|_{\mathfrak{B}^\Psi} \ge \delta$ for every $n \in \mathbb{N}$. Using Lemma 2.5 and (2.15), we get a subsequence $\{f_{n_k}\}_k$ such that, for every $\alpha = (\alpha_k)_k \in \ell_\infty$,

$$(\delta/2)\,\|\{\alpha_k\}_k\|_\infty \le \Big\|\sum_{k=1}^\infty \alpha_k f_{n_k}\Big\|_{\mathfrak{B}^\Psi} \le \Big\|\sum_{k=1}^\infty \alpha_k f_{n_k}\Big\|_{H^\Psi} \le M\|\{\alpha_k\}_k\|_\infty.$$

This clearly says that J_Ψ fixes a copy of ℓ_∞.

It remains to prove (2.14). To do this we impose the last condition on the sequence $\{\beta_n\}_n$:

$$\sum_{n=1}^\infty 1/\sqrt{\beta_n} \le 1. \tag{2.16}$$

Let us set $g_n = |f_n^*|\mathbb{1}_{D_n}$. Thanks to (2.13), $S - \sum_{n=1}^\infty g_n$ is a bounded function. Thus we just need to prove that $G = \sum_{n=1}^\infty g_n$ is in $L^\Psi(\mathbb{T})$. We have $\|G\|_{L^\Psi(\mathbb{T})} \le A$. Indeed, recalling that the D_n's are pairwise disjoint, and that each g_n is 0 outside D_n, we have

$$\begin{aligned}\int_{\mathbb{T}} \Psi\left(\frac{G}{A}\right) dm &= \sum_{n=1}^\infty \int_{D_n\cap\mathbb{T}} \Psi\left(\frac{G}{A}\right) dm = \sum_{n=1}^\infty \int_{D_n\cap\mathbb{T}} \Psi\left(\frac{|f_n^*|}{A}\right) dm\\ &\le \sum_{n=1}^\infty \int_{\mathbb{T}} \Psi\left(\frac{y_n|u_n^*|}{A}\right) dm,\end{aligned}$$

and by the convexity of Ψ and the fact that $|u_n| \le 1$,

$$\begin{aligned}&\le \sum_{n=1}^\infty \int_{\mathbb{T}} |u_n^*|\Psi\left(\frac{y_n}{A}\right) dm = \sum_{n=1}^\infty \|u_n\|_{H_1}\Psi\left(\frac{y_n}{A}\right)\\ &\le \sum_{n=1}^\infty \frac{\Psi(y_n/A)}{\Psi(y_n)} \le \sum_{n=1}^\infty \frac{\Psi(x_n)}{\Psi(y_n)} = \sum_{n=1}^\infty \frac{\Psi(x_n)}{\sqrt{\Psi(Ax_n)}}\\ &\le \sum_{n=1}^\infty \frac{1}{\sqrt{\beta_n}} \le 1,\end{aligned}$$

by the required condition (2.16); this ends the proof of Lemma 2.6. ■

We are now in a position to prove Theorem 2.2.

Proof of Theorem 2.2. We shall prove that

$$\text{(a)}\Rightarrow\text{(b)}\Rightarrow\text{(c)}\Rightarrow\text{(d)}\Rightarrow\text{(e)}\Rightarrow\text{(a)},$$

and that (b)⇔(f).

The implications (a)⇒(b)⇒(c) and (f)⇒(b) are trivial, and we have seen in Lemma 2.6 that (c)⇒(d).

(d)⇒(e). By Lemma 2.4, there exists a constant $C > 0$ such that, for every f in the unit ball of H^{Ψ},

$$\int_{\mathbb{D}} [\Psi(|f|/C)]^2 \, dm_2 \leq 1. \tag{2.17}$$

For every $A > 0$, there exists x_A such that $\Psi(Ax) \leq (Q_A+1)[\Psi(x)]^2$ for every $x \geq x_A$. Thus for every $x \geq 0$ we have $\Psi(Ax) \leq (Q_A+1)[\Psi(x)]^2 + \Psi(Ax_A)$. Then, by (2.17),

$$\int_{\mathbb{D}} \Psi(A|f|/C)\, dm_2 < \infty \quad \text{for every } A > 0.$$

Therefore $f \in \mathfrak{B}M^{\Psi}$ for every f in the unit ball of H^{Ψ}, and thus for every f in H^{Ψ}.

(e)⇒(a). Let $\{f_n\}_n$ be in the unit ball of H^{Ψ}. We have to prove that $\{f_n\}_n$ has a subsequence which converges in the weak topology of $\mathfrak{B}^{\Psi}$. By Montel's Theorem, $\{f_n\}_n$ has a subsequence converging uniformly on compact subsets of $\mathbb{D}$ to a function g, which, by Fatou's lemma, also belongs to the unit ball of H^{Ψ}. If this subsequence converges to g in the norm of $\mathfrak{B}^{\Psi}$, we are done. If not, after perhaps a new extraction of a subsequence, there exist $\delta > 0$ and a subsequence $\{f_{n_k}\}_k$ such that

$$\|f_{n_k} - g\|_{\mathfrak{B}^{\Psi}} \geq \delta \quad \text{and} \quad \|f_{n_k} - g\|_{H^{\Psi}} \leq 2.$$

Since moreover $\{f_{n_k} - g\}_k$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ and, by condition (e), $f_{n_k} - g \in \mathfrak{B}M^{\Psi}$, we may apply Lemma 2.5 to conclude that $\{f_{n_k} - g\}_k$ has a subsequence equivalent to the canonical basis of c_0 in $\mathfrak{B}^{\Psi}$, and is therefore weakly null. This shows that $\{f_n\}_n$ has a subsequence converging to g in the weak topology of $\mathfrak{B}^{\Psi}$.

(b)⇒(f). Suppose there exists an infinite-dimensional subspace X of H^{Ψ} on which the norms $\|\cdot\|_{\mathfrak{B}^{\Psi}}$ and $\|\cdot\|_{H^{\Psi}}$ are equivalent. We shall have finished if we prove that X contains a subspace isomorphic to c_0, because then J_{Ψ} will fix a copy of c_0.

We can assume that X is contained in $\mathfrak{B}M^{\Psi}$ because we already know that (b) implies (e). X being infinite-dimensional, there exists, for every $n \in \mathbb{N}$, $f_n \in X$ such that $\|f_n\|_{H^{\Psi}} = 1$ and $\widehat{f_n}(k) = 0$ for $k = 0, 1, \ldots, n$. By the equivalence of the norms in X, there exists $\delta > 0$ such that $\|f_n\|_{\mathfrak{B}^{\Psi}} \geq \delta$ for every n. The unit ball of H^{Ψ} is compact in the topology of $\mathcal{H}(\mathbb{D})$. Since

$$\lim_{n\to\infty} \widehat{f_n}(k) = 0 \quad \text{for every } k \geq 0,$$

the only possible limit of a subsequence of $\{f_n\}_n$ is the function 0. So $\{f_n\}_n$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. As $f_n \in X \subseteq \mathfrak{B}M^{\Psi}$ for every n, we can apply Lemma 2.5 to conclude that $\{f_n\}_n$ has a subsequence generating a space Y isomorphic to c_0 in $\mathfrak{B}^{\Psi}$. The space Y is contained in X, where the norms are equivalent, so Y is also isomorphic to c_0 for the norm of H^{Ψ}. ∎

3. Other properties

3.1. Dunford–Pettis. Recall that an operator $T\colon X \to Y$ between two Banach spaces X and Y is said to be *Dunford–Pettis* if $\{Tx_n\}_n$ converges in norm whenever $\{x_n\}_n$ converges weakly. Every compact operator is Dunford–Pettis. The next proposition shows that, in "most" cases, these two properties are equivalent for J_{Ψ}.

PROPOSITION 3.1. *Assume that the conjugate function of Ψ satisfies the condition Δ_2. Then $J_{\Psi}\colon H^{\Psi} \to \mathfrak{B}^{\Psi}$ is a Dunford–Pettis operator if and only if it is compact.*

We shall see in Section 4 that without condition Δ_2 for the conjugate function, J_{ψ} may be Dunford–Pettis without being compact.

Proof. We remark first that when we speak of the conjugate function of Ψ, we implicitly assume that $\Psi(x)/x$ tends to ∞ as x goes to ∞.

Assume that J_{Ψ} is not compact. By Theorem 2.1, there are some $A > 1$ and a sequence $\{x_j\}_j$ going to ∞ such that $\Psi(Ax_j) \geq [\Psi(x_j)]^2$. Setting $r_j = 1 - 1/\Psi(x_j)$, this is equivalent to saying that $A\Psi^{-1}(1/(1 - r_j)) \geq \Psi^{-1}(1/(1 - r_j)^2)$. Define

$$f_j(z) = x_j \left(\frac{1 - r_j}{1 - r_j z}\right)^2.$$

One has $f_j \in HM^{\Psi}$ and $\|f_j\|_{H^{\Psi}} \leq 1$ (see [7, Corollary 3.10]). Since $\{f_j\}_j$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, $\{f_j\}_j$ converges to 0 in the weak-star topology of H^{Ψ} ([7, Proposition 3.7]). But, since the conjugate function of Ψ satisfies condition Δ_2, H^{Ψ} is the bidual of HM^{Ψ} ([7, Corollary 3.3]); hence $\{f_j\}_j$ converges weakly to 0 in HM^{Ψ}.

On the other hand, if $S_j = D(1, 1 - r_j) \cap \mathbb{D}$, then $|1 - r_j z| \leq 2(1 - r_j)$ for $z \in S_j$; hence, writing $K = \|f_j\|_{\mathfrak{B}^{\Psi}}$, one has

$$1 = \int_{\mathbb{D}} \Psi(|f_j|/K)\, dm_2 \geq \int_{S_j} \Psi(|f_j|/K)\, dm_2 \geq m_2(S_j)\Psi(x_j/(4K)).$$

Since $m_2(S_j) \geq \alpha(1 - r_j)^2$ with $0 < \alpha < 1$, we get (as $\Psi(\alpha x_j/(4K)) \leq \alpha\Psi(x_j/(4K))$, by convexity)

$$\|f_j\|_{\mathfrak{B}^{\Psi}} \geq (\alpha/4)\,\frac{x_j}{\Psi^{-1}(1/(1 - r_j)^2)} = (\alpha/4)\,\frac{\Psi^{-1}(1/(1 - r_j))}{\Psi^{-1}(1/(1 - r_j)^2)} \geq \frac{\alpha}{4A}.$$

Therefore J_{Ψ} is not Dunford–Pettis. ∎

On the other hand, one has:

Proposition 3.2. *If J_Ψ is Dunford–Pettis, then it is weakly compact.*

Proof. By Theorem 2.2, if J_Ψ is not weakly compact, there is a subspace X_0 of H^Ψ isomorphic to c_0 on which J_Ψ is an into-isomorphism; hence J_Ψ cannot be Dunford–Pettis. ■

We shall see in the next section that J_Ψ may be weakly compact without being Dunford–Pettis.

3.2. Absolutely summing. Every p-summing operator is weakly compact and Dunford–Pettis; so it might be expected that J_Ψ is p-summing for some $p < \infty$. The next results show that this is never the case as soon as Ψ grows faster than all the power functions.

Recall that an operator $T\colon X \to Y$ between two Banach spaces X and Y is called (p,q)-*summing* if there is a constant $C > 0$ such that

$$\Big(\sum_{k=1}^{n} \|Tx_k\|^p\Big)^{1/p} \le C \sup_{\|x^*\|_{X^*}\le 1} \Big(\sum_{k=1}^{n} |x^*(x_k)|^q\Big)^{1/q}$$

for every finite sequence $(x_1,\dots,x_n)$ in X. If $q = p$, then T is said to be p-*summing*. Every p-summing operator is (p,q)-summing for all $q \le p$.

Theorem 3.3. *If $J_\Psi\colon H^\Psi \to \mathfrak{B}^\Psi$ is p-summing, then, for every $q > p$, $\Psi(x) = O(x^q)$ for x large enough. Moreover, if $p < 2$, then J_Ψ is compact.*

In order to prove this, we need two lemmas.

Lemma 3.4. *If the canonical injection $I_\Psi\colon A \to \mathfrak{B}^\Psi$ is $(p,1)$-summing, where $A = A(\mathbb{D})$ is the disk algebra, then $\Psi(x) = O(x^{2p})$ for x large enough.*

In particular, $J_r\colon H^r \to \mathfrak{B}^r$ is $(p,1)$-summing for no $p < r/2$, and if $\Psi \in \Delta^0$, then J_Ψ is $(p,1)$-summing for no $p < \infty$.

Recall that the *disk algebra* is the space of continuous functions on $\overline{\mathbb{D}}$ which are analytic in $\mathbb{D}$.

We refer to [10] for a detailed study of r-summing Carleson embeddings $H^r \to L^r(\mu)$. In particular, it follows from these results that $J_r\colon H^r \to \mathfrak{B}^r$ is 1-summing for $1 \le r < 2$. On the other hand, it is easy to see that $J_2\colon H^2 \to \mathfrak{B}^2$ is not Hilbert–Schmidt (i.e. not 2-summing): for the canonical orthonormal bases $\{z^n\}_n$ and $\{\sqrt{n+1}\, z^n\}_n$ of H^2 and $\mathfrak{B}^2$, J_2 is the diagonal operator of multiplication by $\{1/\sqrt{n+1}\}_n$. It also follows from [10] that, for $r \ge 2$, J_r is p-summing for no finite p.

Proof of Lemma 3.4. Assume that we do not have $\Psi(x) = O(x^{2p})$ for x large enough. Then $\limsup_{x\to\infty} \Psi(x)/x^{2p} = \infty$. Given any $K > 0$, take $y > 0$ such that $\Psi(y)/y^{2p} \ge K$ and $h = 1/\sqrt{\Psi(y)} \le 1/2$. Let N be the integer part of $1/h+1$. Writing $\xi_j = e^{2\pi ij/N}$, we set

$$u_j(z) = \frac{h^2}{[1-(1-h)\overline{\xi}_j z]^2}.$$

Then $u_j \in A(\mathbb{D})$. By [7, Lemma 5.6], one has, since $h \geq 1/N$,

$$\sum_{j=0}^{N-1} |u_j(e^{it})| \leq Nh^2 \frac{1-(1-h)^{2N}}{[1-(1-h)^2][1-(1-h)^N]^2} \leq \frac{e^2}{(1-e)^2} =: C.$$

In fact, $\frac{1}{1-(1-h)^N} \leq \frac{1}{1-(1-1/N)^N} \leq \frac{1}{1-1/e}$ and $\frac{Nh^2}{1-(1-h)^2} = \frac{Nh}{2-h} \leq 1$ since we have assumed that $h \leq 1/2$. Hence

$$\sup_{\|x^*\|_{A^*}\leq 1} \sum_{j=0}^{N-1} |x^*(u_j)| \leq C.$$

On the other hand, it is easy to see that $|u_j(z)| \geq 1/9$ if $|z-(1-h)\xi_j| < h$; hence, for $S_j = \{z \in \mathbb{D};\ |z-(1-h)\xi_j| < h\}$, one has, since $m_2(S_j) = h^2$,

$$1 = \int_{\mathbb{D}} \Psi\bigg(\frac{|u_j(z)|}{\|u_j\|_{\mathfrak{B}^\Psi}}\bigg)\, dm_2(z) \geq \int_{S_j} \Psi\bigg(\frac{1/9}{\|u_j\|_{\mathfrak{B}^\Psi}}\bigg)\, dm_2 \geq h^2 \Psi\bigg(\frac{1/9}{\|u_j\|_{\mathfrak{B}^\Psi}}\bigg),$$

so $\|u_j\|_{\mathfrak{B}^\Psi} \geq 1/[9\Psi^{-1}(1/h^2)]$. Since $y = \Psi^{-1}(1/h^2)$, one gets

$$\sum_{j=0}^{N-1} \|u_j\|_{\mathfrak{B}^\Psi}^p \geq (1/9)^p \frac{N}{y^p} \geq (1/9)^p \bigg[\frac{\Psi(y)}{y^{2p}}\bigg]^{1/2} \geq \frac{K^{1/2}}{9^p}.$$

This shows that the $(p,1)$-summing norm of I_Ψ should be greater than $K^{1/2p}/(9C)$, and, as K is arbitrary, I_Ψ is not $(p,1)$-summing. ∎

Remark. When $I_\Psi\colon A \hookrightarrow \mathfrak{B}^\Psi$ is p-summing, we have this shorter argument. By Pietsch's factorization theorem, I_Ψ factors through H^p. It follows from [7, Theorem 4.10] that $\alpha h^2 \leq \rho_{m_2}(h) \leq 1/\Psi^{-1}(A/h^{1/p})$ for some constants $0 < \alpha < 1$ and $A > 0$, and h small enough. This means that $\Psi(x) \leq Cx^{2p}$ for x large enough.

Lemma 3.5. *If the canonical injection $I_\Psi\colon A \to \mathfrak{B}^\Psi$ is 1-summing, then J_Ψ is compact.*

Proof. The canonical injection $J_1\colon H^1 \to \mathfrak{B}^1$ (as well as J_Ψ whenever $\Psi \in \Delta_2$) is compact. Hence we may assume that H^Ψ is not H^1 and hence that $\Psi(x)/x$ tends to ∞ as x tends to ∞.

Assume that J_Ψ is not compact. Then, as in the proof of Proposition 3.1, there are some $A > 1$ and a sequence $\{x_k\}_k$ going to ∞ such that $\Psi(Ax_k) \geq [\Psi(x_k)]^2$. Setting $h_k = 1/\Psi(x_k)$, we define, as in the proof of

Lemma 3.4,

$$u_{k,j}(z) = \frac{h_k^2}{[1-(1-h_k)\overline{\xi}_{k,j}z]^2},$$

where $\xi_{k,j} = e^{2\pi ij/N_k}$, with N_k the integer part of $1/h_k + 1$. Then $u_{k,j} \in A$ and (see the proofs of the two cited propositions)

$$\sum_{j=0}^{N_k-1} |u_{k,j}(e^{it})| \le C \quad \text{and} \quad \|u_{k,j}\|_{\mathfrak{B}^\Psi} \ge \frac{\delta\alpha}{A}\,\frac{1}{\Psi^{-1}(1/h_k)}.$$

It follows that

$$\sum_{j=0}^{N_k-1} \|u_{k,j}\|_{\mathfrak{B}^\Psi} \ge \frac{\delta\alpha}{A}\,\frac{N_k}{\Psi^{-1}(1/h_k)} \ge \frac{\delta\alpha}{A}\,\frac{1/h_k}{\Psi^{-1}(1/h_k)} = \frac{\delta\alpha}{A}\,\frac{\Psi(x_k)}{x_k} \xrightarrow[k\to\infty]{} \infty.$$

Hence I_Ψ is not 1-summing. ∎

Proof of Theorem 3.3. Since $J_\Psi\colon H^\Psi \to \mathfrak{B}^\Psi$ is p-summing and the canonical injection $I_\Psi\colon A \to \mathfrak{B}^\Psi$ factors as $I_\Psi\colon A \to H^\Psi \to \mathfrak{B}^\Psi$, this injection is p-summing. By Lemma 3.4, $\Psi(x) = O(x^{2p})$ for x large enough. Hence we have the factorization $A \to H^{2p} \to H^\Psi \to \mathfrak{B}^\Psi$. Since the first injection is $2p$-summing and the last one is p-summing, the composition is $\max(1, p_1)$-summing, with $1/p_1 = 1/(2p) + 1/p$ (see [2, Theorem 2.22]), i.e. $p_1 = \frac{2}{3}p$. If $p_1 > 1$, we can use again Lemma 3.4 with p_1 instead of $2p$; we find that $\Psi(x) = O(x^{2p_1})$ for x large enough, and the factorization $I_\Psi\colon A \to H^{2p_1} \to H^\Psi \to \mathfrak{B}^\Psi$ is $\max(1, p_2)$-summing with $1/p_2 = 1/(2p_1) + 1/p$. In the same way, we get a decreasing sequence $\{p_n\}_n$ such that the canonical injection $A \to \mathfrak{B}^\Psi$ is $\max(1, p_n)$-summing and $1/p_{n+1} = 1/(2p_n) + 1/p$. Writing $p_n = \alpha_n p$, we get $\alpha_{n+1} = 2\alpha_n/(2\alpha_n + 1)$; hence $p_n \to p/2$ as $n \to \infty$. In particular, $\Psi(x) = O(x^q)$ for every $q > p$.

If $p < 2$, one has $\max(1, p_n) = 1$ for n large enough, and Lemma 3.4 implies that J_Ψ is compact. ∎

Remarks. 1) It is not clear whether J_Ψ p-summing, with $p \ge 2$, implies that J_Ψ is compact. However, when $r \ge 2$, $J_r\colon H^r \to \mathfrak{B}^r$ is p-summing for no $p < \infty$ (see [10]).

2) An operator $T\colon X \to Y$ between two Banach spaces is said to be *finitely strictly singular* (or *superstrictly singular*) if for every $\varepsilon > 0$, there is an integer $N_\varepsilon \ge 1$ such that, for every subspace X_0 of X of dimension $\ge N_\varepsilon$, there is an $x \in X_0$ such that $\|Tx\| \le \varepsilon\|x\|$. Every finitely strictly singular operator is strictly singular. It is not difficult to see that every compact operator is finitely strictly singular, and it is shown in [11] (see also [5, Corollary 2.3]) that every p-summing operator is finitely strictly singular. We do not know when J_Ψ is finitely strictly singular.

3.3. Order boundedness. Recall that an operator $T\colon X \to Y$ from a Banach space X into a Banach lattice Y is said to be *order bounded* if there is $y \in Y_+$ (Y_+ denoting the cone of non-negative elements of Y) such that $|Tx| \le y$ for every x in the unit ball of X. Since the Bergman–Orlicz space $\mathfrak{B}^\Psi$ is a subspace of the Banach lattice $L^\Psi(\mathbb{D}, m_2)$, we may study the order boundedness of J_Ψ. Actually, we are going to see that the natural space for the order boundedness of J_Ψ is not $L^\Psi(\mathbb{D}, m_2)$, but the *weak Orlicz space* $L^{\Psi,\infty}(\mathbb{D}, m_2)$, the definition of which we recall below (see [7, Definition 3.16]).

Definition 3.6. Let $(S, \mathcal{S}, \mu)$ be a measure space; the *weak-L^Ψ space* $L^{\Psi,\infty}$ is the set of all (classes of) measurable functions $f\colon S \to \mathbb{C}$ such that, for some constant $c > 0$,

$$\mu(|f| > t) \le \frac{1}{\Psi(ct)} \quad \text{for every } t > 0.$$

One has $L^\Psi \subseteq L^{\Psi,\infty}$ and ([7, Proposition 3.18]) the equality $L^\Psi = L^{\Psi,\infty}$ implies that $\Psi \in \Delta^0$. On the other hand, this equality holds when Ψ grows sufficiently quickly, for example, if Ψ satisfies the condition Δ^1: $x\Psi(x) \le \Psi(\alpha x)$ for some constant $\alpha > 1$ and x large enough.

Proposition 3.7. *$J_\Psi\colon H^\Psi \to \mathfrak{B}^\Psi$ is order bounded into $L^{\Psi,\infty}(\mathbb{D}, m_2)$.*

Proof. One has (see [7, Lemma 3.11])

$$\frac{1}{4}\Psi^{-1}\Big(\frac{1}{1-|z|}\Big) \le \sup_{\|f\|_{H^\Psi} \le 1} |f(z)| \le 4\Psi^{-1}\Big(\frac{1}{1-|z|}\Big). \tag{3.1}$$

Hence, denoting

$$F(z) = \sup_{\|f\|_{H^\Psi} \le 1} |f(z)|, \tag{3.2}$$

one has, for t large enough,

$$m_2(|F| > t) \le m_2(\{z \in \mathbb{D};\ |z| > 1 - 1/\Psi(t/4)\}) \le \frac{2}{\Psi(t/4)} \le \frac{1}{\Psi(t/8)},$$

and the result follows. ∎

Since we also have, for t large enough,

$$m_2(|F| > t) \ge m_2(\{z \in \mathbb{D};\ |z| > 1 - 1/\Psi(4t)\}) \ge \frac{1}{\Psi(4t)},$$

we get:

Corollary 3.8. *J_Ψ is order bounded into $L^\Psi(\mathbb{D}, m_2)$ if and only if $L^\Psi = L^{\Psi,\infty}$. This is the case if $\Psi \in \Delta^1$.*

Remark. Unlike compactness, or weak compactness, which requires that Ψ does not grow too fast, the order boundedness of J_Ψ into $L^\Psi(\mathbb{D}, m_2)$ holds when Ψ grows fast enough. Nevertheless, for the Orlicz function $\Psi(x) = \exp[(\log(x+1))^2] - 1$, J_Ψ is compact and order bounded into $L^\Psi(\mathbb{D}, m_2)$.

When J_Ψ is weakly compact, J_Ψ maps H^Ψ into $\mathfrak{B}M^\Psi$ (Theorem 2.2); hence, we may ask whether J_Ψ may be order bounded into $M^\Psi(\mathbb{D}, m_2)$; however, we have:

PROPOSITION 3.9. *J_Ψ is never order bounded into $M^\Psi(\mathbb{D}, m_2)$.*

Proof. If it were, we should have $F \in M^\Psi(\mathbb{D}, m_2)$ (where F is defined in (3.2)), and hence

$$\int_{\mathbb{D}} \Psi\left[4 \cdot \frac{1}{4}\Psi^{-1}\left(\frac{1}{1-|z|}\right)\right] dm_2(z) < \infty,$$

which is false. ∎

4. An example

THEOREM 4.1. *There exists an Orlicz function Ψ such that J_Ψ is weakly compact and Dunford–Pettis, but not compact.*

Note that such an Orlicz function is very irregular: $\Psi \notin \Delta_2$, $\Psi \notin \Delta^0$, so, for every $A > 1$, $\Psi(Ax)/\Psi(x)$ is not non-decreasing for x large enough, and the conjugate function of Ψ does not satisfy condition Δ_2.

The following lemma is undoubtedly well-known, but we have found no reference, so we shall give a proof. Recall that a sublattice X of $L^0(\mu)$ is *solid* if $|f| \le |g|$ and $g \in X$ implies $f \in X$ and $\|f\| \le \|g\|$.

LEMMA 4.2. *Let $(S, \mathcal{S}, \mu)$ be a measure space, and let X be a solid Banach sublattice of $L^0(\mu)$, the space of all measurable functions. Then, for every weakly null sequence $\{f_n\}_n$ in X and every sequence $\{A_n\}_n$ of disjoint measurables sets, the sequence $\{f_n \mathbb{1}_{A_n}\}_n$ converges weakly to 0 in X.*

Proof. If the conclusion does not hold, there are a continuous linear functional $\sigma\colon X \to \mathbb{C}$ and some $\delta > 0$ such that, up to taking a subsequence, $|\sigma(f_n \mathbb{1}_{A_n})| \ge \delta$. Set, for every $A \in \mathcal{S}$,

$$\mu_n(A) = \sigma(f_n \mathbb{1}_A).$$

Then μ_n is a finitely additive measure with bounded variation. By Rosenthal's lemma (see [3, Lemma I.4.1, p. 18] or [1, Chapter VII, p. 82]), there is an increasing sequence $\{n_k\}_k$ of integers such that

$$\Big|\mu_{n_k}\Big(\bigcup_{l\ne k} A_{n_l}\Big)\Big| \le |\mu_{n_k}|\Big(\bigcup_{l\ne k} A_{n_l}\Big) \le \delta/2.$$

Now, if $A = \bigcup_{l\ge 1} A_{n_l}$, then $\{f_{n_k}\mathbb{1}_A\}_k$ is weakly null, but

$$|\sigma(f_{n_k}\mathbb{1}_A)| \ge |\sigma(f_{n_k}\mathbb{1}_{A_{n_k}})| - |\mu_{n_k}|\Big(\bigcup_{l\ne k} A_{n_l}\Big) \ge \delta - \delta/2 = \delta/2,$$

so we get a contradiction. ∎

Proof of Theorem 4.1. We begin by defining a sequence $\{x_n\}_n$ of positive numbers in the following way: set $x_1 = 4$ and, for every $n \geq 1$, $x_{n+1} > 2x_n$ is the abscissa of the second intersection point of the parabola $y = x^2$ with the straight line containing (x_n, x_n^2) and $(2x_n, x_n^4)$; we have $x_{n+1} = x_n^3 - 2x_n$ (for example, $x_2 = 56$). Define $\Psi\colon [0,\infty) \to [0,\infty)$ by $\Psi(x) = 4x$ for $0 \leq x \leq 4$, and, for $n \geq 1$,

$$\Psi(x_n) = x_n^2, \quad \Psi(2x_n) = x_n^4, \quad \Psi \text{ affine between } x_n \text{ and } x_{n+1}. \tag{4.1}$$

Then Ψ is an Orlicz function and

$$x^2 \leq \Psi(x) \leq x^4 \quad \text{for } x \geq 4. \tag{4.2}$$

For this Orlicz function Ψ, J_Ψ is not compact, since $\Psi(2x)/[\Psi(x)]^2$ does not tend to 0. However, J_Ψ is weakly compact, because one has the factorization $H^\Psi \hookrightarrow H^2 \hookrightarrow \mathfrak{B}^4 \hookrightarrow \mathfrak{B}^\Psi$ (by (4.2) and Lemma 2.4).

Assume that J_Ψ is not Dunford–Pettis: there exists a weakly null sequence $\{f_n\}_n$ in the unit ball of H^Ψ which does not converge in the norm of $\mathfrak{B}^\Psi$. Then $\{f_n\}_n$ converges uniformly to 0 on compact subsets of $\mathbb{D}$ (since it is weakly null) and we may assume that $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$ for some $\delta > 0$. We may also assume that $\|f_n\|_\infty \to \infty$ as $n \to \infty$, because if $\{f_n\}_n$ were uniformly bounded, we should have $\|f_n\|_{\mathfrak{B}^\Psi} \to 0$, by dominated convergence.

We are going to show that there exist a subsequence $\{f_{n_k}\}_k$ and pairwise disjoint measurable sets $A_k \subseteq \mathbb{T}$ such that the sequence $\{f_{n_k}\mathbb{1}_{A_k}\}_k \subseteq L^\Psi(\mathbb{T}, m)$ is equivalent to the canonical basis of ℓ_1, which contradicts Lemma 4.2.

Let us note that the Poisson integral $\mathcal{P}$ maps boundedly $L^2(\mathbb{T})$ into $L^4(\mathbb{D})$. Indeed, $L^2(\mathbb{T}) = H^2 \oplus \overline{H_0^2}$ and the canonical injection is bounded from H^2 into $\mathfrak{B}^4$, by Lemma 2.4. If $\|\mathcal{P}\|$ stands for the norm of $\mathcal{P}\colon L^2(\mathbb{T}) \to L^4(\mathbb{D})$, it follows that $\mathcal{P}\colon L^\Psi(\mathbb{T}) \to L^\Psi(\mathbb{D})$ is bounded and its norm is $\leq \|\mathcal{P}\|$, thanks to the factorization $L^\Psi(\mathbb{T}) \hookrightarrow L^2(\mathbb{T}) \xrightarrow{\mathcal{P}} L^4(\mathbb{D}) \hookrightarrow L^\Psi(\mathbb{D})$.

We have seen in the proof of Lemma 2.5 that there exist a subsequence $\{f_{n_k}\}_k$ and disjoint measurable annuli $C_1 = \{z \in \mathbb{D};\, |z| \leq r_1\}$ and $C_k = \{z \in \mathbb{D};\, r_{k-1} < |z| \leq r_k\}$, $k \geq 2$, with $0 < r_1 < r_2 < \cdots < 1$, such that $\|f_{n_k}\mathbb{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \geq \delta/2$. The assumptions of that lemma are satisfied here: $\|f_n\|_{H^\Psi} \leq 1$, $\|f_n\|_{\mathfrak{B}^\Psi} \geq \delta$, $\{f_n\}_n$ converges uniformly to 0 on compact subsets of $\mathbb{D}$, and $f_n \in \mathfrak{B}M^\Psi$ because $H^\Psi \subseteq \mathfrak{B}M^\Psi$, since J_Ψ is weakly compact. Then we have:

FACT 1. *There exist two sequences $\{\alpha_k\}_k$ and $\{\beta_k\}_k$, with $\beta_n > \alpha_n \to \infty$ as $n \to \infty$, such that if $g_k = f_{n_k}^* \mathbb{1}_{\{\alpha_k \leq |f_{n_k}^*| \leq \beta_k\}}$, then*

$$\|\mathcal{P}(g_k)\|_{L^\Psi(\mathbb{D})} \geq \delta/3,$$

where $f_{n_k}^*$ is the boundary value of f_{n_k} on $\mathbb{T}$.

Proof. 1) Let $\alpha_k = \frac{\delta}{12}\Psi^{-1}(1/m_2(C_k))$ and $v_k = \mathcal{P}(f_{n_k}^* \mathbb{1}_{\{|f_{n_k}^*|<\alpha_k\}})\mathbb{1}_{C_k}$. One has

$$\int_{\mathbb{D}} \Psi(|v_k|/(\delta/12))\, dm_2 = \int_{C_k} \Psi(|v_k|/(\delta/12))\, dm_2 \le \Psi(\alpha_k/(\delta/12))\, m_2(C_k) = 1,$$

so $\|v_k\|_{L^\Psi(\mathbb{D})} \le \delta/12$. Since $\mathcal{P}(f_{n_k}^*) = f_{n_k}$, we have $\|\mathcal{P}(f_{n_k}^*)\mathbb{1}_{C_k}\|_{L^\Psi(\mathbb{D})} = \|f_{n_k}\mathbb{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \ge \delta/2$, and we get

$$\|\mathcal{P}(f_{n_k}^* \mathbb{1}_{\{|f_{n_k}^*|\ge\alpha_k\}})\mathbb{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \ge \|f_{n_k}\mathbb{1}_{C_k}\|_{L^\Psi(\mathbb{D})} - \|v_k\|_{L^\Psi(\mathbb{D})} \ge \frac{\delta}{2} - \frac{\delta}{12} = \frac{5\delta}{12}.$$

2) Let $w_k = f_{n_k}^* \mathbb{1}_{\{|f_{n_k}^*|\ge\alpha_k\}}$. Since $\mathcal{P}(w_k \mathbb{1}_{\{|w_k|>\beta\}}) \to 0$ uniformly on C_k as $\beta \to \infty$, Lebesgue's dominated convergence theorem gives

$$\|\mathcal{P}(w_k \mathbb{1}_{\{|w_k|>\beta\}})\mathbb{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \le \|\mathcal{P}(w_k \mathbb{1}_{\{|w_k|>\beta\}})\mathbb{1}_{C_k}\|_{L^4(\mathbb{D})} \xrightarrow[\beta\to\infty]{} 0,$$

so there is some $\beta_k > \alpha_k$ such that $\|\mathcal{P}(w_k \mathbb{1}_{\{|w_k|>\beta\}})\mathbb{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \le \delta/12$.

We then have, with $g_k = f_{n_k}^* \mathbb{1}_{\{\alpha_k \le |f_{n_k}^*| \le \beta_k\}}$,

$$\|\mathcal{P}(g_k)\|_{L^\Psi(\mathbb{D})} \ge \|\mathcal{P}(g_k)\mathbb{1}_{C_k}\|_{L^\Psi(\mathbb{D})} \ge \frac{5\delta}{12} - \frac{\delta}{12} = \frac{\delta}{3},$$

and that ends the proof of Fact 1. ∎

FACT 2. *There are a further subsequence, still denoted by* $\{f_{n_k}\}_k$, *and pairwise disjoint measurable subsets* $E_k \subseteq \{\alpha_k \le |f_{n_k}^*| \le \beta_k\}$ *such that if* $h_k = f_{n_k}^* \mathbb{1}_{E_k}$, *then*

$$\|\mathcal{P}(h_k)\|_{L^\Psi(\mathbb{D})} \ge \delta/4.$$

Proof. First, since $g_k \in L^\infty(\mathbb{T}) \subseteq M^\Psi(\mathbb{T})$, there exists $\varepsilon_k > 0$ such that $m(A) \le \varepsilon_k$ implies $\|g_k \mathbb{1}_A\|_{L^\Psi(\mathbb{T})} \le \delta/(12\|\mathcal{P}\|)$. Hence $\|\mathcal{P}(g_k \mathbb{1}_A)\|_{L^\Psi(\mathbb{D})} \le \delta/12$ for $m(A) \le \varepsilon_k$.

Let $B_k = \{\alpha_k \le |f_{n_k}^*| \le \beta_k\}$. Up to taking a subsequence, we may assume that $\sum_{l>k} m(B_l) \le \varepsilon_k$. Let

$$E_k = B_k \setminus \bigcup_{l>k} B_l.$$

The sets E_k, $k \ge 1$, are pairwise disjoint, and

$$\|\mathcal{P}(g_k \mathbb{1}_{E_k})\|_{L^\Psi(\mathbb{D})} \ge \|\mathcal{P}(g_k \mathbb{1}_{B_k})\|_{L^\Psi(\mathbb{D})} - \|\mathcal{P}(g_k \mathbb{1}_{\bigcup_{l>k} B_l})\|_{L^\Psi(\mathbb{D})} \ge \frac{\delta}{3} - \frac{\delta}{12} = \frac{\delta}{4};$$

so we get the conclusion with $h_k = g_k \mathbb{1}_{E_k} = f_{n_k}^* \mathbb{1}_{E_k}$. ∎

Set

$$F_k = \{z \in E_k;\ \Psi(|f_{n_k}^*(z)|) \le M|f_{n_k}^*(z)|^2\}.$$

For $z \in E_k \setminus F_k$, one has

$$\int\limits_{E_k\setminus F_k} |f^*_{n_k}|^2\, dm \le \frac{1}{M}\int\limits_{\mathbb{T}} \Psi(|f^*_{n_k}|)\, dm \le \frac{1}{M},$$

so $\|f^*_{n_k}\mathbb{1}_{E_k\setminus F_k}\|_{L^2(\mathbb{T})} \le 1/\sqrt{M}$ and

$$\begin{aligned}\|\mathcal{P}(f^*_{n_k}\mathbb{1}_{E_k\setminus F_k})\|_{L^\Psi(\mathbb{D})} &\le \|\mathcal{P}(f^*_{n_k}\mathbb{1}_{E_k\setminus F_k})\|_{L^4(\mathbb{D})}\\ &\le \|\mathcal{P}\|\,\|f^*_{n_k}\mathbb{1}_{E_k\setminus F_k}\|_{L^2(\mathbb{T})} \le \frac{\|\mathcal{P}\|}{\sqrt{M}} \le \frac{\delta}{8}\end{aligned}$$

for M large enough. It follows that, for M large enough, $\|\mathcal{P}(f^*_{n_k}\mathbb{1}_{F_k})\|_{L^\Psi(\mathbb{D})}$ $\ge \delta/8$ and

$$\|f^*_{n_k}\mathbb{1}_{F_k}\|_{L^\Psi(\mathbb{D})} \ge \delta/(8\|\mathcal{P}\|). \tag{4.3}$$

Now, we may assume that, for some $\alpha > 0$,

$$\int\limits_{\mathbb{T}} |f^*_{n_k}|^2\mathbb{1}_{F_k}\, dm \ge \alpha,$$

because, if not, there would be a subsequence $\{f^*_{n_{k_j}}\mathbb{1}_{F_{k_j}}\}_j$ converging to 0 in $L^2(\mathbb{T})$; but then $\{\mathcal{P}(f_{n_{k_j}}\mathbb{1}_{F_{k_j}})\}_j$ would converge to 0 in $\mathfrak{B}^4$, and hence in $\mathfrak{B}^\Psi$, contrary to (4.3). It follows, using (4.2), that

$$\int\limits_{F_k} \Psi(|f^*_{n_k}|)\, dm \ge \alpha. \tag{4.4}$$

The following lemma is now the key of the proof.

Lemma 4.3. *Let $\delta_n = 2x_{n-1}/x_n = 2/(x^2_{n-1} - 2)$. If $\Psi(x) \le Mx^2$ and $x \ge x_n$, then, for n large enough ($n \ge N$), one has $\Psi(\varepsilon x) \ge C_M\, \varepsilon\Psi(x)$ for $\delta_n \le \varepsilon \le 1$.*

Proof. We may assume that $x_n \le x < x_{n+1}$, because if $x_k \le x < x_{k+1}$ with $k \ge n$, then $\varepsilon \ge \delta_n$ implies $\varepsilon \ge \delta_k$.

We shall first show that

$$\frac{\Psi(y)}{\Psi(x)} \le 4\,\frac{y}{x} \quad \text{for } 2x_n \le x \le y \le x_{n+1}. \tag{4.5}$$

Indeed, on the one hand,

$$\frac{\Psi(y) - \Psi(x_n)}{\Psi(x) - \Psi(x_n)} = \frac{y - x_n}{x - x_n} \le \frac{y}{x/2} = 2\frac{y}{x};$$

and, on the other hand, $\Psi(y) - \Psi(x_n) \ge \Psi(y) - \Psi(y/2) \ge \Psi(y) - \frac{1}{2}\Psi(y) = \frac{1}{2}\,\Psi(y)$, so

$$\frac{\Psi(y)}{\Psi(x)} \le \frac{\Psi(y)}{\Psi(x) - \Psi(x_n)} \le 2\,\frac{\Psi(y) - \Psi(x_n)}{\Psi(x) - \Psi(x_n)} \le 4\,\frac{y}{x}.$$

We shall separate three cases:

1) $\varepsilon x \le x_n \le x \le 2x_n$. Then $\varepsilon x \ge \varepsilon x_n$ and hence $\Psi(\varepsilon x) \ge \Psi(\varepsilon x_n)$. But $2x_{n-1} \le \varepsilon x_n \le x_n$, since $\varepsilon \ge \delta_n$; hence (4.5) implies that $\Psi(\varepsilon x) \ge (\varepsilon/4)\Psi(x_n) = (\varepsilon/4)x_n^2$. On the other hand, by hypothesis, $\Psi(x) \le Mx^2 \le M(2x_n)^2$, so we get $\Psi(\varepsilon x) \ge (\varepsilon/(16M))\Psi(x)$.

2) $x_n \le \varepsilon x \le x \le 2x_n$. Then, since $1 \le 1/\varepsilon$,

$$\frac{\Psi(x)}{\Psi(\varepsilon x)} \le \frac{Mx^2}{\Psi(x_n)} \le \frac{M(2x_n)^2}{x_n^2} = 4M \le \frac{4M}{\varepsilon}.$$

3) For $x \ge 2x_n$, the conditions $\Psi(x) \le Mx^2$ and $x \ge 2x_n$ imply that $x \ge x_n^2/\sqrt{M}$. Indeed, if $x \ge 2x_n$, then $\Psi(x) \ge \Psi(2x_n) = x_n^4$, and the condition $\Psi(x) \le Mx^2$ implies $x_n^4 \le Mx^2$, i.e. $x \ge x_n^2/\sqrt{M}$.

In this case, one has $\varepsilon x \ge \varepsilon x_n^2/\sqrt{M} \ge \delta_n x_n^2/\sqrt{M} = 2(x_{n-1}/x_n)x_n^2/\sqrt{M} = 2x_{n-1}x_n/\sqrt{M} \ge 2x_n$ if $x_{n-1} \ge \sqrt{M}$. Hence (4.5) gives, for $2x_n \le x < x_{n+1}$ (since then $2x_n \le \varepsilon x \le x < x_{n+1}$),

$$\frac{\Psi(x)}{\Psi(\varepsilon x)} \le 4\,\frac{x}{\varepsilon x} = \frac{4}{\varepsilon}.$$

That ends the proof of Lemma 4.3. ∎

Extract now a further subsequence of $\{f_{n_k}\}$, still denoted by $\{f_{n_k}\}$, in order that (see Fact 1) $\alpha_k \ge x_{N+k}$. The assumption of Lemma 4.3 holds for $x = \Psi(|f_{n_k}^*(z)|)$, $z \in F_k$, for every $k \ge 1$; one has (since, by definition, $\Psi(|f_{n_k}|) \le M\,|f_{n_k}|^2$ on F_k)

$$\int_{F_k} \Psi(\varepsilon\,|f_{n_k}^*|)\,dm \ge \varepsilon C/\alpha := c\varepsilon \quad \text{ for } \delta_{N+k} \le \varepsilon \le 1.$$

We have now reached the final part of the proof of Theorem 4.1: put $u_k = f_{n_k}^* \mathbb{1}_{F_k}$, and take an arbitrary sequence $\{\lambda_k\}_k$ of complex numbers such that $\sum_{k\ge1} |\lambda_k| = 1$. Let $\delta_0 = \sum_{k\ge N} \delta_k$. Then $\delta_0 < 1$, because we may assume that N had been taken large enough. One gets

$$\begin{aligned}\int_{\mathbb{T}} \Psi\Big(\Big|\sum_{k\ge1} \lambda_k u_k\Big|\Big)\,dm &= \sum_{k\ge1}\int_{F_k} \Psi(|\lambda_k f_{n_k}|)\,dm \\ &\ge \sum_{|\lambda_k|\ge\delta_{N+k}} c\,|\lambda_k| + \sum_{|\lambda_k|<\delta_{N+k}} \int_{F_k} \Psi(|\lambda_k f_{n_k}|)\,dm \\ &\ge \sum_{|\lambda_k|\ge\delta_{N+k}} c\,|\lambda_k| = c\Big(1 - \sum_{|\lambda_k|<\delta_{N+k}} |\lambda_k|\Big) \\ &\ge c\Big(1 - \sum_{k\ge N} \delta_k\Big) = c\,(1-\delta_0) =: c_0.\end{aligned}$$

Since $c_0 < 1$, this implies, by convexity, that

$$\Big\| \sum_{k\geq 1} \lambda_k u_k \Big\|_{L^\Psi(\mathbb{T})} \geq c_0.$$

Hence $\{u_k\}_k$ is equivalent to the canonical basis of ℓ_1, and this concludes the proof of Theorem 4.1. ∎

REMARKS. 1) It follows from Theorem 3.3 that, for this Ψ, J_Ψ is not p-summing for $p < 4$. By modifying the definition of Ψ (taking $\Psi(x_n) = x_n^{r/2}$ and $\Psi(2x_n) = x_n^r$), we get, for every $4 \leq r < \infty$, an Orlicz function Ψ such that J_Ψ is Dunford–Pettis and weakly compact, without being p-summing for $p < r$, and without being compact. We do not know whether it is possible to have J_Ψ p-summing for no finite p.

2) Let us point out that the fact that J_Ψ is Dunford–Pettis does not trivially follow from its weak compactness: H^Ψ does not have the Dunford–Pettis property. In fact, if it had, the weakly compact injection $H^\Psi \hookrightarrow H^2$ would be Dunford–Pettis, and hence so would be $H^4 \hookrightarrow H^2$ (since $H^4 \hookrightarrow H^\Psi \hookrightarrow H^2$). But the latter is not the case: the sequence $\{z^n\}_n$ converges weakly to 0 in H^4, whereas it does not converge in norm to 0 in H^2.

PROPOSITION 4.4. *There is an Orlicz function Ψ for which J_Ψ is weakly compact, but not Dunford–Pettis.*

Proof. Let Ψ_0 be the Orlicz function constructed in Theorem 4.1, and set $\Psi(x) = \Psi_0(x^2)$. Then, with $\beta = 2$, $\Psi(\beta x) = \Psi_0(4x^2) \geq 4\Psi_0(x^2) = (2\beta)\Psi(x)$; that means that the conjugate function of Ψ satisfies Δ_2.

J_Ψ is weakly compact (since it factors as $H^\Psi \hookrightarrow H^4 \hookrightarrow \mathfrak{B}^8 \hookrightarrow \mathfrak{B}^\Psi$), but not compact, since $[\Psi(\sqrt{x_n})]^2 = \Psi(\sqrt{2}\sqrt{x_n})$. Since the conjugate function satisfies Δ_2, J_Ψ is not Dunford–Pettis, by Proposition 3.1. ∎

Acknowledgements. This work is partially supported by a Spanish research project MTM 2009-08934. Part of this paper was written during an invitation of the second-named author by the *Departamento de Análisis Matemático* of the *Universidad de Sevilla*. It is a pleasure to thank the members of this department for their warm hospitality.

We thank the referee for a quick and very careful reading of this paper.

References

[1] J. Diestel, *Sequences and Series in Banach Spaces*, Grad. Texts in Math. 92, Springer, 1984.

[2] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely Summing Operators*, Cambridge Stud. Adv. Math. 43, Cambridge Univ. Press, 1995.

[3] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., 1977.

[4] P. Duren and A. Schuster, *Bergman Spaces*, Math. Surveys Monogr. 100, Amer. Math. Soc., 2004.

[5] P. Lefèvre, *When strict singularity of operators coincides with weak compactness*, J. Operator Theory, to appear.

[6] P. Lefèvre, D. Li, H. Queffélec, and L. Rodríguez-Piazza, *A criterion of weak compactness for operators on subspaces of Orlicz spaces*, J. Funct. Spaces Appl. 6 (2008), 277–292.

[7] —, —, —, —, *Composition operators on Hardy–Orlicz spaces*, Mem. Amer. Math. Soc. 207 (2010), no. 974.

[8] —, —, —, —, *Nevanlinna counting function and Carleson function of analytic maps*, Math. Ann., to appear; doi: 10.1007/s00208-010-0596-1.

[9] —, —, —, —, *Compact composition operators on Bergman–Orlicz spaces*, arXiv: 0910.5368, hal-00426831.

[10] P. Lefèvre and L. Rodríguez-Piazza, *Absolutely summing Carleson embeddings*, in preparation.

[11] A. Plichko, *Superstrictly singular and superstrictly cosingular operators*, in: Functional Analysis and its Applications, North-Holland Math. Stud. 197, Elsevier, 2004, 239–255.

[12] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Pure Appl. Math. 146, Dekker, 1991.

Pascal Lefèvre, Daniel Li
Université Lille Nord de France
F-59 000 Lille, France
U-Artois
Laboratoire de Mathématiques de Lens EA 2462
Fédération CNRS Nord-Pas-de-Calais FR 2956
Faculté des Sciences Jean Perrin
Rue Jean Souvraz, S.P. 18
F-62 300 Lens, France
E-mail: pascal.lefevre@euler.univ-artois.fr
daniel.li@euler.univ-artois.fr

Hervé Queffélec
Université Lille Nord de France
F-59 000 Lille, France
USTL, Laboratoire Paul Painlevé
U.M.R. CNRS 8524
F-59 655 Villeneuve d'Ascq, France
E-mail: queff@math.univ-lille1.fr

Luis Rodríguez-Piazza
Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad de Sevilla
Apartado de Correos 1160
41 080 Sevilla, Spain
E-mail: piazza@us.es

Received May 10, 2010
Revised version September 10, 2010 (6888)