STUDIA MATHEMATICA 202 (2) (2011)

Quasi *-algebras and generalized inductive limits of C^*-algebras

by

Giorgia Bellomonte and Camillo Trapani (Palermo)

Abstract. A generalized procedure for the construction of the inductive limit of a family of C^*-algebras is proposed. The outcome is no more a C^*-algebra but, under certain assumptions, a locally convex quasi *-algebra, named a C^*-inductive quasi *-algebra. The properties of positive functionals and representations of C^*-inductive quasi *-algebras are investigated, in close connection with the corresponding properties of positive functionals and representations of the C^*-algebras that generate the structure. The typical example of the quasi *-algebra of operators acting on a rigged Hilbert space is analyzed in detail.

1. Introduction. The construction of the inductive limit of a system $\{\mathfrak{B}_\alpha, J_{\beta\alpha} : \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$ where $\mathbb{F}$ is a directed set of indices, $\mathfrak{B}_\alpha$ a C^*-algebra and $J_{\beta\alpha}$ a *-isomorphism of $\mathfrak{B}_\alpha$ into $\mathfrak{B}_\beta$ is a well-known procedure whose outcome is a C^*-algebra $\mathfrak{B}$ (see, e.g., [5, 10]) which contains copies of the C^*-algebras $\{\mathfrak{B}_\alpha : \alpha \in \mathbb{F}\}$ of the given system. The main reason why $\mathfrak{B}$ is a C^*-algebra is that the injective maps $J_{\beta\alpha}$ entering the construction preserve not only the vector space operations, but also the multiplication; this fact, in turn, implies that the norms are also preserved when passing from a C^*-algebra $\mathfrak{B}_\alpha$ to a *larger* C^*-algebra $\mathfrak{B}_\beta$. However, situations where one can easily recognize inside a locally convex space an indexed family of vector subspaces which can be viewed as the image under some vector space isomorphism of C^*-algebras abound. This is, for instance, the case of the space $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ of all continuous linear maps from $\mathcal{D}$ into $\mathcal{D}^\times$, where $\mathcal{D}$ and $\mathcal{D}^\times$ are the extreme spaces of a rigged Hilbert space $(\mathcal{D}[t], \mathcal{H}, \mathcal{D}^\times[t^\times])$, if the topology t of $\mathcal{D}$ is the graph topology defined by a *-algebra $\mathfrak{M}$ of unbounded operators (an O^*-algebra, in the terminology of [12, 1]; precise definitions will be given in Section 2) and $t^\times$ is the corresponding strong dual topology. Similarly, certain spaces of distributions contain natural families of C^*-algebras, typically *-algebras of continuous functions on some (locally) compact set X. Then it is natural to ask whether, by weakening the assump-

2010 *Mathematics Subject Classification*: Primary 47L60; Secondary 47L40.
Key words and phrases: quasi *-algebras, inductive limit of C^*-algebras, partial *-algebras.

DOI: 10.4064/sm202-2-4

tions on the family of maps $\{J_{\beta\alpha} : \alpha, \beta \in \mathbb{F},\ \beta \geq \alpha\}$ it is possible to recover, by a generalization of the procedure of inductive limit, more general spaces and structures. Both the space of operators in a rigged Hilbert space and the space of distributions can be viewed as locally convex *quasi* **-algebras* over appropriate distinguished *-algebras contained in them [1, Ch. 10]. This is exactly the structure we will get as a result of our approach.

Our starting point will be again the system $\{\mathfrak{B}_\alpha, J_{\beta\alpha} : \alpha, \beta \in \mathbb{F},\ \beta \geq \alpha\}$ with the proviso that the maps $J_{\beta\alpha}$ are only *-isomorphisms of vector spaces, i.e., they do not necessarily preserve the multiplication; but we will require a *control* on their behavior on positive elements (namely, we suppose that the $J_{\beta\alpha}$'s are Schwarz maps). In Section 3, we will show how this generalized inductive limit can be constructed.

Other generalizations of the construction of the inductive limit of C^*-algebras have been considered in the literature: one of them consists in supposing that the embedding maps $J_{\beta\alpha}$ act as *-homomorphisms at least asymptotically, and assuming a boundedness condition on the $J_{\beta\alpha}$'s (see the review paper by Blackadar and Kirchberg [4] and references therein). The result of the construction is then also a C^*-algebra.

Our approach goes one step further: what we get at the end of our construction is an involutive locally convex space $\mathfrak{A}$ with an underlying C^*-structure: we will call it a C^**-inductive locally convex space*, for short. In the same section we introduce an order, reflecting that of the C^*-algebras which generate the structure, and show that positive elements behave similarly to positive elements of a C^*-algebra. Then we consider positive linear functionals on $\mathfrak{A}$ and give conditions for the existence of a sort of GNS *-representation of $\mathfrak{A}$.

Finally we go back to the main question and investigate the possibility of giving $\mathfrak{A}$ the structure of a partial *-algebra or quasi *-algebra in close connection with the family of C^*-algebras $\{\mathfrak{B}_\alpha : \alpha \in \mathbb{F}\}$. This is indeed possible, but it depends on a family $\{w_\alpha : \alpha \in \mathbb{F}\}$ which *weighs* the multiplication. This ambiguous behavior is not surprising since the same ambiguity arises for multiplication of operators in rigged Hilbert spaces and for multiplication of distributions.

Section 4 is devoted to examining, in the light of the results of the preceding section, the problem of existence of a GNS construction for a general quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$, starting from a linear functional which is positive on $\mathfrak{A}_0$. The representations constructed in this section take values in the quasi *-algebra of operators acting on a rigged Hilbert space. In particular, the role of an *admissibility* condition (called (Q3)) for these functionals is discussed. Finally, in Section 5, we describe in full detail some examples: the main one is that of the quasi *-algebra of operators in a rigged Hilbert space, which has been, in a sense, the starting point of this paper. The last examples show that C^*-algebras of functions may give rise, by inductive limit, either to a locally convex *-algebra of functions or to a locally convex quasi *-algebra of distributions.

2. Notation and preliminaries. For general aspects of the theory of partial *-algebras and of their representations, we refer to the monograph [1]. For the convenience of the reader, however, we repeat here the essential definitions.

A *partial* *-*algebra* $\mathfrak{A}$ is a complex vector space with conjugate linear involution * and a distributive partial multiplication $\cdot$, defined on a subset $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$, with the property that $(x, y) \in \Gamma$ if, and only if, $(y^*, x^*) \in \Gamma$ and $(x \cdot y)^* = y^* \cdot x^*$. From now on, we will write simply xy instead of $x \cdot y$ whenever $(x, y) \in \Gamma$. For every $y \in \mathfrak{A}$, the set of left (resp. right) multipliers of y is denoted by $L(y)$ (resp. $R(y)$), i.e., $L(y) = \{x \in \mathfrak{A} : (x, y) \in \Gamma\}$ (resp. $R(y) = \{x \in \mathfrak{A} : (y, x) \in \Gamma\}$). We denote by $L\mathfrak{A}$ (resp. $R\mathfrak{A}$) the space of universal left (resp. right) multipliers of $\mathfrak{A}$. In general, a partial *-algebra is not associative.

The *unit* of a partial *-algebra $\mathfrak{A}$, if any, is an element $e \in \mathfrak{A}$ such that $e = e^*$, $e \in R\mathfrak{A} \cap L\mathfrak{A}$ and $xe = ex = x$ for every $x \in \mathfrak{A}$.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all (closable) linear operators X such that $D(X) = \mathcal{D}$ and $D(X^*) \supseteq \mathcal{D}$. The map $X \mapsto X^\dagger = X^* \upharpoonright \mathcal{D}$ defines an involution on $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, which can be made into a partial *-algebra with respect to *weak* multiplication [1]; however, this fact will not be used in this paper.

Let $\mathcal{L}^\dagger(\mathcal{D})$ be the subspace of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ consisting of all its elements which, together with their adjoints, leave the domain $\mathcal{D}$ invariant. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a *-algebra with respect to the usual operations. A *-subalgebra $\mathfrak{M}$ of $\mathcal{L}^\dagger(\mathcal{D})$ is called an O^*-*algebra*.

Let $\mathfrak{M}$ be an O^*-algebra. The *graph topology* $t_{\mathfrak{M}}$ on $\mathcal{D}$ is the locally convex topology defined by the family $\{\|\cdot\|, \|\cdot\|_X : X \in \mathfrak{M}\}$ of seminorms, where $\|\xi\|_X = \|X\xi\|$, $\xi \in \mathcal{D}$. The topology $t_{\mathfrak{M}}$ is finer than the norm topology, unless $\mathfrak{M}$ consists of bounded operators only. If the locally convex space $\mathcal{D}[t_{\mathfrak{M}}]$ is complete, then $\mathfrak{M}$ is said to be *closed*. More generally, we denote by $\widetilde{\mathcal{D}}(\mathfrak{M})$ the completion of the locally convex space $\mathcal{D}[t_{\mathfrak{M}}]$ and put

$$\widetilde{X} := \overline{X} \upharpoonright \widetilde{\mathcal{D}}(\mathfrak{M}) \quad \text{and} \quad \widetilde{\mathfrak{M}} := \{\widetilde{X} : X \in \mathfrak{M}\}.$$

Then $\widetilde{\mathfrak{M}}$ is a closed O^*-algebra on $\widetilde{\mathcal{D}}(\mathfrak{M})$ which is called the *closure* of $\mathfrak{M}$, since it is the smallest closed extension of $\mathfrak{M}$. By $\mathfrak{L}^\dagger(\mathcal{D})$ we denote the set of $t_{\mathfrak{M}}$-continuous elements of $\mathcal{L}^\dagger(\mathcal{D})$.

If $\mathfrak{A}_0$ is a *-algebra, a *-homomorphism $\pi : \mathfrak{A}_0 \to \mathcal{L}^\dagger(\mathcal{D}_\pi)$, where $\mathcal{D}_\pi$ is a dense domain in Hilbert space $\mathcal{H}_\pi$, is called a *-*representation* of $\mathfrak{A}_0$. A *-representation is called *closed* if the O^*-algebra $\pi(\mathfrak{A}_0)$ is closed. The graph topology $t_{\pi(\mathfrak{A}_0)}$ will be briefly denoted by t_π.

Let $\mathfrak{A}$ be a complex vector space and $\mathfrak{A}_0$ a *-algebra contained in $\mathfrak{A}$. We say that $(\mathfrak{A}, \mathfrak{A}_0)$ is a *quasi *-algebra* if

(i) the left multiplication ax and the right multiplication xa of an element a of $\mathfrak{A}$ and an element x of $\mathfrak{A}_0$ which extend the multiplication of $\mathfrak{A}_0$ are always defined and bilinear;
(ii) $x_1(x_2a) = (x_1x_2)a$ and $x_1(ax_2) = (x_1a)x_2$, for each $x_1, x_2 \in \mathfrak{A}_0$ and $a \in \mathfrak{A}$;
(iii) an involution $*$ which extends the involution of $\mathfrak{A}_0$ is defined in $\mathfrak{A}$ with the property $(ax)^* = x^*a^*$ and $(xa)^* = a^*x^*$ for each $x \in \mathfrak{A}_0$ and $a \in \mathfrak{A}$.

Of course, every quasi *-algebra is a partial *-algebra.

Let $\mathcal{D}$ be a dense linear subspace of a Hilbert space $\mathcal{H}$ and t a locally convex topology on $\mathcal{D}$, finer than the topology induced by the Hilbert norm. Then the space $\mathcal{D}^\times$ of all continuous conjugate linear functionals on $\mathcal{D}[t]$, i.e., the conjugate dual of $\mathcal{D}[t]$, is a vector space and *contains* $\mathcal{H}$, in the sense that $\mathcal{H}$ can be identified with a subspace of $\mathcal{D}^\times$ (to avoid confusion, we denote by $B(\cdot,\cdot)$ the bilinear form that puts $\mathcal{D}$ and $\mathcal{D}^\times$ in duality; the identifications made imply that $B(h,\xi) = \langle h \,|\, \xi\rangle$ for $h \in \mathcal{H}$ and $\xi \in \mathcal{D}$). The space $\mathcal{D}^\times$ will always be considered as endowed with the *strong dual topology* $t^\times = \beta(\mathcal{D}^\times, \mathcal{D})$. The Hilbert space $\mathcal{H}$ is dense in $\mathcal{D}^\times[t^\times]$.

We get in this way a *Gel'fand triplet* or *rigged Hilbert space* (RHS)

$$\mathcal{D}[t] \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times[t^\times],$$

where $\hookrightarrow$ denotes a continuous embedding with dense range.

Let $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ denote the vector space of all continuous linear maps from $\mathcal{D}[t]$ into $\mathcal{D}^\times[t^\times]$. In $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ an involution $X \mapsto X^\dagger$ can be introduced by the equality

$$B(X\xi, \eta) = \overline{B(X^\dagger\eta, \xi)}, \qquad \forall \xi, \eta \in \mathcal{D}.$$

Hence $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ is a *-invariant vector space.

To every $X \in \mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ there corresponds a separately continuous sesquilinear form θ_X on $\mathcal{D} \times \mathcal{D}$ defined by

$$\theta_X(\xi, \eta) = B(X\xi, \eta), \qquad \xi, \eta \in \mathcal{D}.$$

The space of all *jointly* continuous sesquilinear forms on $\mathcal{D} \times \mathcal{D}$ will be denoted by $\mathsf{B}(\mathcal{D}, \mathcal{D})$. We denote by $\mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times)$ the subspace of all $X \in \mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ such that $\theta_X \in \mathsf{B}(\mathcal{D}, \mathcal{D})$. By [1, Prop. 10.2.4], $(\mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times), \mathfrak{L}^\dagger(\mathcal{D}))$ is a quasi *-algebra.

In what follows we will use extensively the notion of *joint topological limit* (a generalized inductive limit) of a directed *contractive* family of Hilbert spaces. We give the definitions below, referring to [3] for more details.

Let $\{\mathcal{H}_\alpha : \alpha \in \mathbb{F}\}$ be a family of Hilbert spaces indexed by a set $\mathbb{F}$ upward directed by $\leq$ (we denote by $\langle \cdot \mid \cdot \rangle_\alpha$ and $\|\cdot\|_\alpha$, respectively, the inner product and the norm of $\mathcal{H}_\alpha$). Suppose that, for every $\alpha, \beta \in \mathbb{F}$ with $\beta \geq \alpha$, there exists a linear map $U_{\beta\alpha} : \mathcal{H}_\alpha \to \mathcal{H}_\beta$ with the properties

(i) $U_{\beta\alpha}$ is injective;
(ii) $\|U_{\beta\alpha}\xi_\alpha\|_\beta \leq \|\xi_\alpha\|_\alpha$ for all $\xi_\alpha \in \mathcal{H}_\alpha$;
(iii) $U_{\alpha\alpha} = I_\alpha$, the identity of $\mathcal{H}_\alpha$;
(iv) $U_{\gamma\alpha} = U_{\gamma\beta}U_{\beta\alpha}$ if $\alpha \leq \beta \leq \gamma$.

The family $\{\mathcal{H}_\alpha, U_{\beta\alpha} : \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$ is called a *directed contractive system of Hilbert spaces.*

If $\{\mathcal{H}_\alpha, U_{\beta\alpha} : \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$ is a directed contractive system of Hilbert spaces, the following statements hold:

(d_1) There exists a conjugate dual pair $(\mathcal{D}^\times, \mathcal{D})$ and, for every $\alpha \in \mathbb{F}$, a pair of injective linear maps $(\Pi_\alpha, \Theta_\alpha)$, where $\Pi_\alpha : \mathcal{D} \to \mathcal{H}_\alpha$ and $\Theta_\alpha : \mathcal{H}_\alpha \to \mathcal{D}^\times$, both with dense range, such that

(I_1) $\Pi_\alpha = V_{\alpha\beta}\Pi_\beta$ if $\alpha \leq \beta$ (where $V_{\alpha\beta} = U^*_{\beta\alpha}$);
(I_2) $\Theta_\alpha = \Theta_\beta U_{\beta\alpha}$ if $\alpha \leq \beta$;
(I_3) $\mathcal{D}^\times = \bigcup_{\alpha\in\mathbb{F}} \Theta_\alpha(\mathcal{H}_\alpha)$;
(I_4) if $\xi \in \mathcal{D}$ and $\eta \in \mathcal{D}^\times$ with $\eta = \Theta_\alpha\eta_\alpha$ for some $\alpha \in \mathbb{F}$ and $\eta_\alpha \in \mathcal{H}_\alpha$, then

$$B(\eta, \xi) = B(\Theta_\alpha\eta_\alpha, \xi) = \overline{\langle \Pi_\alpha\xi \mid \eta_\alpha\rangle_\alpha},$$

independently of α such that $\eta \in \Theta_\alpha(\mathcal{H}_\alpha)$.

(d_2) The pair $(\mathcal{D}^\times, \mathcal{D})$ occurring in (d_1) is uniquely determined by the conditions given in (d_1), in the following sense: if $(\mathcal{D}_1^\times, \mathcal{D}_1)$ is another conjugate dual pair for which there exists, for every $\alpha \in \mathbb{F}$, a pair $(\Delta_\alpha, \Gamma_\alpha)$, with $\Delta_\alpha : \mathcal{D}_1 \to \mathcal{H}_\alpha$ and $\Gamma_\alpha : \mathcal{H}_\alpha \to \mathcal{D}_1^\times$, such that the statements corresponding to (I_1)–(I_4) are satisfied, then there exists an injective linear map $T : \mathcal{D}^\times \to \mathcal{D}_1^\times$ such that $\Gamma_\alpha = T\Theta_\alpha$ and $\Delta_\alpha = \Pi_\alpha T^\times$ for every $\alpha \in \mathbb{F}$, where $T^\times : \mathcal{D}_1 \to \mathcal{D}$ denotes the adjoint map of T.

The conjugate dual pair $(\mathcal{D}^\times, \mathcal{D})$ described above is called the *joint topological limit* of the directed contractive system $\{\mathcal{H}_\alpha, U_{\beta\alpha} : \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$ of Hilbert spaces. The spaces $\mathcal{D}^\times$ and $\mathcal{D}$ are, respectively, the inductive limit and the projective limit of the family $\{\mathcal{H}_\alpha : \alpha \in \mathbb{F}\}$.

Let $(\mathcal{D}, \mathcal{D}^\times)$ be the joint topological limit of a directed contractive family $\{\mathcal{H}_\alpha, U_{\beta\alpha} : \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$ of Hilbert spaces. We denote [1] by $\mathsf{L}_\mathsf{B}(\mathcal{D}, \mathcal{D}^\times)$

[1] We notice that $\mathsf{L}_\mathsf{B}(\mathcal{D}, \mathcal{D}^\times) = \mathfrak{L}_\mathsf{B}(\mathcal{D}, \mathcal{D}^\times)$ when $\mathcal{D}$ and $\mathcal{D}^\times$ are the extreme spaces of a RHS.

the space of all linear maps $X : \mathcal{D} \to \mathcal{D}^\times$ for which there exist $\gamma \in \mathbb{F}$ and $C > 0$ such that

$$|B(X\eta, \xi)| \leq C\|\xi_\gamma\|_\gamma \|\eta_\gamma\|_\gamma, \quad \forall \xi = (\xi_\alpha),\, \eta = (\eta_\alpha) \in \mathcal{D}. \tag{2.1}$$

Assume that, for each $\alpha \in \mathbb{F}$, an operator $X_\alpha \in \mathfrak{B}(\mathcal{H}_\alpha)$ (the C^*-algebra of bounded operators in $\mathcal{H}_\alpha$) is given and that there exists $\overline{\alpha} \in \mathbb{F}$ for which $X_\beta = U_{\beta\alpha} X_\alpha V_{\alpha\beta}$ whenever $\overline{\alpha} \leq \alpha \leq \beta$. Then [3] there exists a unique linear map $X \in \mathsf{L}_\mathsf{B}(\mathcal{D}, \mathcal{D}^\times)$ such that $X(\xi_\gamma) = \Theta_\beta X_\beta \Pi_\beta(\xi_\gamma)$ whenever $\beta \geq \overline{\alpha}$. The map X is called the *inductive limit of the operators* X_α and denoted by $X = \varinjlim X_\alpha$.

3. Vector spaces with underlying C^*-inductive structure. We will consider here a class of locally convex vector spaces which can be obtained, in a certain sense, as the inductive limit of a family of C^*-algebras.

3.1. Definitions and basic facts. Let $\mathfrak{A}$ be a vector space over $\mathbb{C}$. Let $\mathbb{F}$ be an upward directed set of indices and assume that, for every $\alpha \in \mathbb{F}$, there is a Banach space $\mathfrak{A}_\alpha \subset \mathfrak{A}$ such that:

(I.1) $\mathfrak{A}_\alpha \subseteq \mathfrak{A}_\beta$ if $\alpha \leq \beta$;
(I.2) $\mathfrak{A} = \bigcup_{\alpha\in\mathbb{F}} \mathfrak{A}_\alpha$;
(I.3) for each $\alpha \in \mathbb{F}$, there exists a C^*-algebra $\mathfrak{B}_\alpha$ (with unit e_α and norm $\|\cdot\|_\alpha$) and a norm-preserving isomorphism of vector spaces $\Phi_\alpha : \mathfrak{B}_\alpha \to \mathfrak{A}_\alpha$;
(I.4) $x_\alpha \in \mathfrak{B}_\alpha^+ \Rightarrow x_\beta = (\Phi_\beta^{-1}\Phi_\alpha)(x_\alpha) \in \mathfrak{B}_\beta^+$, for all $\alpha, \beta \in \mathbb{F}$ with $\beta \geq \alpha$.

We put $J_{\beta\alpha} = \Phi_\beta^{-1}\Phi_\alpha$ if $\alpha, \beta \in \mathbb{F}$, $\beta \geq \alpha$.

If $x \in \mathfrak{A}$, there exist $\alpha \in \mathbb{F}$ such that $x \in \mathfrak{A}_\alpha$ and (a unique) $x_\beta \in \mathfrak{B}_\beta$ such that $x = \Phi_\beta(x_\beta)$ for all $\beta \geq \alpha$. Then we put

$$J_{\beta\alpha}(x_\alpha) := x_\beta \quad \text{if } \alpha \leq \beta.$$

Remark 3.1. By (I.4), $J_{\beta\alpha}$ preserves positivity, i.e., $J_{\beta\alpha}(x_\alpha) \geq 0$ if $x_\alpha \in \mathfrak{B}_\alpha^+$. From this, it follows easily that $J_{\beta\alpha}$ also preserves involution, i.e., $J_{\beta\alpha}(x_\alpha^*) = (J_{\beta\alpha}(x_\alpha))^*$.

The family $\{\mathfrak{B}_\alpha, J_{\beta\alpha} : \beta \geq \alpha\}$ is a *directed system of C^*-algebras*, in the sense that:

(J.1) for every $\alpha, \beta \in \mathbb{F}$ with $\beta \geq \alpha$, $J_{\beta\alpha} : \mathfrak{B}_\alpha \to \mathfrak{B}_\beta$ is a linear and injective map; $J_{\alpha\alpha}$ is the identity of $\mathfrak{B}_\alpha$;
(J.2) for every $\alpha, \beta \in \mathbb{F}$ with $\alpha \leq \beta$, $\Phi_\alpha = \Phi_\beta J_{\beta\alpha}$;
(J.3) $J_{\gamma\beta} J_{\beta\alpha} = J_{\gamma\alpha}$ if $\alpha \leq \beta \leq \gamma$.

We assume that, in addition, the $J_{\beta\alpha}$'s are Schwarz maps (see, e.g., [9]), i.e.,

(sch) $J_{\beta\alpha}(x_\alpha)^* J_{\beta\alpha}(x_\alpha) \leq J_{\beta\alpha}(x_\alpha^* x_\alpha)$ for all $x_\alpha \in \mathfrak{B}_\alpha$, $\alpha \leq \beta$.

For every $\alpha, \beta \in \mathbb{F}$ with $\alpha \leq \beta$, $J_{\beta\alpha}$ is continuous [9], and moreover

$$\|J_{\beta\alpha}(x_\alpha)\|_\beta \leq \|x_\alpha\|_\alpha, \quad \forall x_\alpha \in \mathfrak{B}_\alpha.$$

REMARK 3.2. We notice that $J_{\beta\alpha}$ is not, in general, a *-homomorphism of C^*-algebras, since it might not preserve multiplication.

The fact that the $J_{\beta\alpha}$'s preserve the involution allows us to define an involution in $\mathfrak{A}$. Let $x \in \mathfrak{A}$. Then $x \in \mathfrak{A}_\alpha$ for some $\alpha \in \mathbb{F}$, i.e., $x = \Phi_\alpha(x_\alpha)$ for a unique $x_\alpha \in \mathfrak{B}_\alpha$. Put $x^* := \Phi_\alpha(x_\alpha^*)$. Then if $\beta \geq \alpha$, we have

$$\Phi_\beta^{-1}(x^*) = \Phi_\beta^{-1}(\Phi_\alpha(x_\alpha^*)) = J_{\beta\alpha}(x_\alpha^*) = (J_{\beta\alpha}(x_\alpha))^* = x_\beta^*.$$

It is easily seen that the map $x \mapsto x^*$ is an involution in $\mathfrak{A}$. Moreover, by the definition itself, it follows that every map Φ_α *preserves involution*, i.e., $\Phi_\alpha(x_\alpha^*) = (\Phi_\alpha(x_\alpha))^*$ for all $x_\alpha \in \mathfrak{B}_\alpha$, $\alpha \in \mathbb{F}$.

DEFINITION 3.3. A locally convex vector space $\mathfrak{A}$ with involution * is called a *C^*-inductive locally convex space* if

(i) there exists a family $\{\{\mathfrak{B}_\alpha, \Phi_\alpha\} : \alpha \in \mathbb{F}\}$, where $\mathbb{F}$ is a directed set and, for every $\alpha \in \mathbb{F}$, $\mathfrak{B}_\alpha$ is a C^*-algebra and Φ_α is a linear injective map of $\mathfrak{B}_\alpha$ into $\mathfrak{A}$, satisfying the above conditions (I.1)–(I.4) and (sch), with $\mathfrak{A}_\alpha = \Phi_\alpha(\mathfrak{B}_\alpha)$, $\alpha \in \mathbb{F}$;
(ii) $\mathfrak{A}$ is endowed with the locally convex inductive topology τ_{ind} generated by the family $\{\{\mathfrak{B}_\alpha, \Phi_\alpha\} : \alpha \in \mathbb{F}\}$.

For brevity the family $\{\{\mathfrak{B}_\alpha, \Phi_\alpha\} : \alpha \in \mathbb{F}\}$ will be called the *defining system* of $\mathfrak{A}$. We notice that the involution is automatically continuous in $\mathfrak{A}[\tau_{\text{ind}}]$.

In the following subsections we will study some properties of the structure introduced above. Even if not mentioned explicitly, throughout Section 3 we will always denote by $\mathfrak{A}$ a C^*-inductive locally convex space.

3.2. Positive elements

DEFINITION 3.4. An element $x \in \mathfrak{A}$ is called *positive* if there exists $\gamma \in \mathbb{F}$ such that $\Phi_\alpha^{-1}(x) \in \mathfrak{B}_\alpha^+$ for all $\alpha \geq \gamma$. We denote by $\mathfrak{A}^+$ the set of all positive elements of $\mathfrak{A}$.

LEMMA 3.5. *The following statements hold.*

(i) *Every positive element $x \in \mathfrak{A}$ is hermitian, i.e., $x \in \mathfrak{A}_h := \{y \in \mathfrak{A} : y^* = y\}$.*
(ii) *$\mathfrak{A}^+$ is a nonempty convex pointed cone.*
(iii) *If $\alpha \in \mathbb{F}$ and $x_\alpha \in \mathfrak{B}_\alpha^+$, then $\Phi_\alpha(x_\alpha)$ is positive.*

PROPOSITION 3.6. *Every hermitian element $x = x^*$ is the difference of two positive elements, i.e. there exist $x^+, x^- \in \mathfrak{A}^+$ such that $x = x^+ - x^-$.*

Proof. Let $x \in \mathfrak{A}_h$. Then there exists $\alpha \in \mathbb{F}$ such that $x \in \mathfrak{A}_\alpha$. Hence $\Phi_\alpha^{-1}(x) \in \mathfrak{B}_\alpha$ and $\Phi_\alpha^{-1}(x) = \Phi_\alpha^{-1}(x)^*$. Being a hermitian element of a C^*-algebra, $\Phi_\alpha^{-1}(x)$ can be decomposed into the difference of two positive elements. Thus, there exist $y_\alpha^+, y_\alpha^- \in \mathfrak{B}_\alpha^+$, with $y_\alpha^+ y_\alpha^- = y_\alpha^- y_\alpha^+ = 0$, such that $\Phi_\alpha^{-1}(x) = y_\alpha^+ - y_\alpha^-$, from which it follows that $x = \Phi_\alpha(y_\alpha^+) - \Phi_\alpha(y_\alpha^-)$.

Define $x^+ = \Phi_\beta(y_\beta^+)$, $x^- = \Phi_\beta(y_\beta^-)$ for $\beta \geq \alpha$, and $x^+ = x^- = 0$, $\beta \not\geq \alpha$. Then, by Lemma 3.5(iii), x^+ and x^- are positive. ■

REMARK 3.7. In every C^*-algebra $\mathfrak{C}$, the decomposition of $z \in \mathfrak{C}_h$ as $z = z^+ - z^-$, with $z^+, z^- \in \mathfrak{C}^+$ and $z^+ z^- = z^- z^+ = 0$ (*orthogonal* decomposition), is unique and has the property

$$\|z\| = \max\{\|z^+\|, \|z^-\|\}. \tag{3.1}$$

In our case, the same statement is true for every *representative* of a hermitian element x in the sense that, for every $\alpha \in \mathbb{F}$ such that $x \in \mathfrak{A}_\alpha$, $\Phi_\alpha^{-1}(x)$ decomposes as $\Phi_\alpha^{-1}(x) = x_\alpha^+ - x_\alpha^-$ with $x_\alpha^+, x_\alpha^- \in \mathfrak{B}_\alpha^+$, $x_\alpha^+ x_\alpha^- = x_\alpha^- x_\alpha^+ = 0$, and hence obeying (3.1). Thus, $x = \Phi_\alpha(x_\alpha^+) - \Phi_\alpha(x_\alpha^-)$, but $\Phi_\alpha(x_\alpha^+), \Phi_\alpha(x_\alpha^-)$ are positive but not necessarily orthogonal; so the uniqueness of decomposition fails.

3.3. Linear functionals. Let ω be a linear functional on $\mathfrak{A}$. Then, for every $\alpha \in \mathbb{F}$, $\omega \circ \Phi_\alpha$ is a linear functional on $\mathfrak{B}_\alpha$.

DEFINITION 3.8. A linear functional $\omega : \mathfrak{A} \to \mathbb{C}$ is called *positive* ($\omega \geq 0$) if $\omega(x) \geq 0$ for every $x \in \mathfrak{A}^+$.

PROPOSITION 3.9. *Let ω be a linear functional on $\mathfrak{A}$. The following statements are equivalent.*

(i) *ω is positive on $\mathfrak{A}$.*
(ii) *$\omega \circ \Phi_\alpha \geq 0$ on $\mathfrak{B}_\alpha$ for every $\alpha \in \mathbb{F}$.*
(iii) *ω is continuous on $\mathfrak{A}[\tau_{\text{ind}}]$ and $\|\omega \circ \Phi_\alpha\| = (\omega \circ \Phi_\alpha)(e_\alpha)$ for every $\alpha \in \mathbb{F}$.*

Proof. (i)⇒(ii): Assume that $\omega \geq 0$ and let $x_\alpha \in \mathfrak{B}_\alpha^+$. Put $x = \Phi_\alpha(x_\alpha)$; then $x = \Phi_\beta(J_{\beta\alpha}(x_\alpha))$ for all $\beta \geq \alpha$ and $J_{\beta\alpha}(x_\alpha) = x_\beta \in \mathfrak{B}_\beta^+$, since $J_{\beta\alpha}$ preserves positivity. This implies that $x \in \mathfrak{A}^+$; indeed, $\Phi_\gamma^{-1}(\Phi_\beta(x_\beta)) = x_\gamma \in \mathfrak{B}_\gamma^+$ for all $\gamma \geq \beta$. In conclusion, $(\omega \circ \Phi_\alpha)(x_\alpha) = \omega(x) \geq 0$ for all $\alpha \in \mathbb{F}$.

(ii)⇒(i): Assume $\omega \circ \Phi_\alpha \geq 0$ for all $\alpha \in \mathbb{F}$, and let $x \in \mathfrak{A}^+$. Then there exists $\gamma \in \mathbb{F}$ such that $\Phi_\alpha^{-1}(x) \in \mathfrak{B}_\alpha^+$ for all $\alpha \geq \gamma$. Thus, $\omega(x) = (\omega \circ \Phi_\alpha)(\Phi_\alpha^{-1}(x)) \geq 0$ for all $\alpha \geq \gamma$, i.e. $\omega \geq 0$.

(iii)⇔(ii): This equivalence follows from well-known properties of inductive limits and from elementary properties of C^*-algebras. ■

For each $\alpha \in \mathbb{F}$, let ω_α be a positive linear functional on $\mathfrak{B}_\alpha$. Assume that

$$\omega_\beta(J_{\beta\alpha}(x_\alpha)) = \omega_\alpha(x_\alpha), \quad \forall x_\alpha \in \mathfrak{B}_\alpha,\ \beta \geq \alpha. \tag{3.2}$$

If $x \in \mathfrak{A}_\alpha$, then $x = \Phi_\alpha(x_\alpha)$, $x_\alpha \in \mathfrak{B}_\alpha$. We define

$$\widetilde{\omega}_\alpha(x) := \omega_\alpha(\Phi_\alpha^{-1}(x)) = \omega_\alpha(x_\alpha).$$

If $\beta \geq \alpha$, we have

$$\begin{aligned}\widetilde{\omega}_\beta(x) &= \omega_\beta(\Phi_\beta^{-1}(x)) = \omega_\beta(\Phi_\beta^{-1}\Phi_\alpha(x_\alpha)) = \omega_\beta(J_{\beta\alpha}(x_\alpha)) \\ &= \omega_\alpha(x_\alpha) = \omega_\alpha(\Phi_\alpha^{-1}(x)) = \widetilde{\omega}_\alpha(x).\end{aligned}$$

Thus, we can define a linear functional ω on $\mathfrak{A}$ by putting

$$\omega(x) = \widetilde{\omega}_\alpha(x), \qquad x \in \mathfrak{A}_\alpha.$$

The functional ω is called the *inductive limit* of the ω_α's : $\omega = \varinjlim \omega_\alpha$. It is easily seen that ω is a positive linear functional on $\mathfrak{A}$ and $\omega_\alpha = \omega \circ \Phi_\alpha$ for every $\alpha \in \mathbb{F}$.

Proposition 3.10. *A linear functional ω on $\mathfrak{A}$ is positive if, and only if, it is the inductive limit of a family $\{\omega_\alpha\}$, where each ω_α is a positive linear functional on $\mathfrak{B}_\alpha$.*

Proof. Let ω be positive on $\mathfrak{A}$. Then, by Proposition 3.9, $\omega_\alpha := \omega \circ \Phi_\alpha$ is positive on $\mathfrak{B}_\alpha$. We have, for every $x_\alpha \in \mathfrak{B}_\alpha$ and for $\beta \geq \alpha$,

$$\omega_\beta(J_{\beta\alpha}(x_\alpha)) = (\omega \circ \Phi_\beta)(\Phi_\beta^{-1} \circ \Phi_\alpha(x_\alpha)) = (\omega \circ \Phi_\alpha)(x_\alpha) = \omega_\alpha(x_\alpha).$$

Hence (3.2) is satisfied and $\varinjlim \omega_\alpha$ is well defined. Let us denote it by ω'. It remains to prove that $\omega' = \omega$.

Let $x \in \mathfrak{A}$. Then $x = \Phi_\alpha(x_\alpha)$, $x_\alpha \in \mathfrak{B}_\alpha$. By the definition of ω' we have

$$\omega'(x) = \omega_\alpha(x_\alpha) = \omega(x). \ \blacksquare$$

3.4. Inductive limit of representations

Proposition 3.11. *Let $\{\mathcal{H}_\alpha, U_{\beta\alpha} : \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$ be a directed contractive system of Hilbert spaces and denote by $(\mathcal{D}, \mathcal{D}^\times)$ the joint topological limit of this system. Let $\mathfrak{A}$ be the C^*-inductive locally convex space defined by the system $\{\{\mathfrak{B}_\alpha, \Phi_\alpha\} : \alpha \in \mathbb{F}\}$ as in Definition* 3.3. *For each $\alpha \in \mathbb{F}$, let π_α be a *-representation of $\mathfrak{B}_\alpha$ in $\mathcal{H}_\alpha$ and assume that*

$$\pi_\beta(J_{\beta\alpha}(x_\alpha)) = U_{\beta\alpha}\pi_\alpha(x_\alpha)U_{\beta\alpha}^*, \qquad \forall x_\alpha \in \mathfrak{B}_\alpha,\ \beta \geq \alpha. \tag{3.3}$$

Then there exists a unique linear map $\pi : \mathfrak{A} \to \mathsf{L}_B(\mathcal{D}, \mathcal{D}^\times)$, preserving involution, such that

$$\pi(\Phi_\alpha(x_\alpha)) = \Theta_\alpha \pi_\alpha(x_\alpha) \Pi_\alpha, \qquad \forall x_\alpha \in \mathfrak{B}_\alpha,$$

where Θ_α is the embedding of $\mathcal{H}_\alpha$ into $\mathcal{D}^\times$ and Π_α is the embedding of $\mathcal{D}$ into $\mathcal{H}_\alpha$.

Proof. Let $x \in \mathfrak{A}$. Then $\Phi_\alpha^{-1}(x) \in \mathfrak{B}_\alpha$ for some $\alpha \in \mathbb{F}$. The equality (3.3) implies that

$$\pi_\beta(\Phi_\beta^{-1}(x)) = U_{\beta\alpha}\pi_\alpha(\Phi_\alpha^{-1}(x))U_{\beta\alpha}^*, \qquad \beta \geq \alpha.$$

Indeed, we have

$$\begin{aligned}U_{\beta\alpha}\pi_\alpha(\Phi_\alpha^{-1}(x))U_{\beta\alpha}^* &= \pi_\beta(J_{\beta\alpha}\Phi_\alpha^{-1}(x)) = \pi_\beta((\Phi_\beta^{-1}\Phi_\alpha)(\Phi_\alpha^{-1}(x))) \\ &= \pi_\beta(\Phi_\beta^{-1}(x)).\end{aligned}$$

Hence $\varinjlim \pi_\alpha(\Phi_\alpha^{-1}(x))$ is well-defined. Thus, we put

$$\pi(x) := \varinjlim \pi_\alpha(\Phi_\alpha^{-1}(x)), \qquad x \in \mathfrak{A}_\alpha \subset \mathfrak{A}. \tag{3.4}$$

Then π satisfies the requirements. We shorten (3.4) by writing $\pi = \varinjlim \pi_\alpha$. The uniqueness follows from the corresponding uniqueness of the inductive limit of operators. ■

THEOREM 3.12. *Let $\mathfrak{A}$ be a C^*-inductive locally convex space and let $\{\{\mathfrak{B}_\alpha, \Phi_\alpha\} : \alpha \in \mathbb{F}\}$ be the corresponding defining system. Let $\omega = \varinjlim \omega_\alpha$ ($\omega_\alpha = \omega \circ \Phi_\alpha$) be a positive linear functional on $\mathfrak{A}$ such that*

(A) *if $\alpha \in \mathbb{F}$ and $\omega_\beta(J_{\beta\alpha}(x_\alpha^*)J_{\beta\alpha}(x_\alpha)) = 0$ for some $\beta \geq \alpha$ and $x_\alpha \in \mathfrak{B}_\alpha$, then $\omega_\alpha(x_\alpha^* x_\alpha) = 0$.*

Let $\{\pi_\alpha, \mathcal{H}_\alpha, \xi_\alpha\}$ be the GNS construction for $\mathfrak{B}_\alpha$ defined by ω_α. Then:

(i) *for every $\alpha, \beta \in \mathbb{F}$ with $\alpha \leq \beta$, there exists a contractive injective linear map $U_{\beta\alpha}$ such that $\{\mathcal{H}_\alpha, U_{\beta\alpha} : \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$ is a directed contractive system of Hilbert spaces;*

(ii) *if $(\mathcal{D}, \mathcal{D}^\times)$ is the joint topological limit generated by the directed contractive system of Hilbert spaces in* (i)*, there exists a unique linear map $\pi : \mathfrak{A} \to \mathsf{L}_B(\mathcal{D}, \mathcal{D}^\times)$, preserving involution, such that*

$$\pi(\Phi_\alpha(x_\alpha)) = \Theta_\alpha \pi_\alpha(x_\alpha) \Pi_\alpha, \qquad \forall x_\alpha \in \mathfrak{B}_\alpha,$$

where Θ_α is the embedding of $\mathcal{H}_\alpha$ into $\mathcal{D}^\times$ and Π_α is the embedding of $\mathcal{D}$ into $\mathcal{H}_\alpha$, i.e. $\pi = \varinjlim \pi_\alpha$;

(iii) *the inductive limit of the ω_α's is *-representable, i.e., for every $x \in \mathfrak{A}$ there exists $\xi \in \mathcal{D}$ such that*

$$\omega(x) = B(\pi(x)\xi, \xi),$$

where $B(\cdot,\cdot)$ is the form that puts $\mathcal{D}^\times$ and $\mathcal{D}$ in conjugate duality.

Proof. (i): Making use of (sch), we have

$$\begin{aligned}\|\pi_\beta(J_{\beta\alpha}(x_\alpha))\xi_\beta\|_\beta^2 &= \langle \pi_\beta(J_{\beta\alpha}(x_\alpha))\xi_\beta \,|\, \pi_\beta(J_{\beta\alpha}(x_\alpha))\xi_\beta\rangle_\beta \\ &= \langle \pi_\beta(J_{\beta\alpha}(x_\alpha^*)J_{\beta\alpha}(x_\alpha))\xi_\beta \,|\, \xi_\beta\rangle_\beta = \omega_\beta(J_{\beta\alpha}(x_\alpha^*)J_{\beta\alpha}(x_\alpha)) \\ &\leq \omega_\beta(J_{\beta\alpha}(x_\alpha^* x_\alpha)) = \omega_\alpha(x_\alpha^* x_\alpha) = \langle \pi_\alpha(x_\alpha^* x_\alpha)\xi_\alpha \,|\, \xi_\alpha\rangle_\alpha \\ &= \|\pi_\alpha(x_\alpha)\xi_\alpha\|_\alpha^2.\end{aligned}$$

Hence, if we put

$$U_{\beta\alpha}\pi_\alpha(x_\alpha)\xi_\alpha := \pi_\beta(J_{\beta\alpha}(x_\alpha))\xi_\beta, \qquad x_\alpha \in \mathfrak{B}_\alpha,$$

the above inequality implies that $U_{\beta\alpha}$ is a well-defined linear map from the dense subspace $\{\pi_\alpha(x_\alpha)\xi_\alpha : x_\alpha \in \mathfrak{B}_\alpha\}$ of $\mathcal{H}_\alpha$ into $\mathcal{H}_\beta$ satisfying

$$\|U_{\beta\alpha}\pi_\alpha(x_\alpha)\xi_\alpha\|_\beta \leq \|\pi_\alpha(x_\alpha)\xi_\alpha\|_\alpha, \quad \forall x_\alpha \in \mathfrak{B}_\alpha, \tag{3.5}$$

thus it extends to a contraction of $\mathcal{H}_\alpha$ into $\mathcal{H}_\beta$ which we denote by the same symbol. Every map $U_{\beta\alpha}$, $\beta \geq \alpha$, is injective. Indeed, suppose $U_{\beta\alpha}\pi_\alpha(x_\alpha)\xi_\alpha = 0$. Then

$$\omega_\beta(J_{\beta\alpha}(x_\alpha^*)J_{\beta\alpha}(x_\alpha)) = \|U_{\beta\alpha}\pi_\alpha(x_\alpha)\xi_\alpha\|_\beta^2 = 0.$$

Hence, by (A), $\|\pi_\alpha(x_\alpha)\xi_\alpha\|^2 = \omega_\alpha(x_\alpha^* x_\alpha) = 0$. Moreover, $U_{\alpha\alpha} = I_\alpha$ is the identity of $\mathcal{H}_\alpha$ and, if $\alpha \leq \beta \leq \gamma$, by the cyclicity of ξ_α and by

$$\begin{aligned} U_{\gamma\beta}U_{\beta\alpha}\pi_\alpha(x_\alpha)\xi_\alpha &= U_{\gamma\beta}\pi_\beta(J_{\beta\alpha}(x_\alpha))\xi_\beta = \pi_\gamma(J_{\gamma\beta}J_{\beta\alpha}(x_\alpha))\xi_\gamma \\ &= U_{\gamma\alpha}\pi_\alpha(x_\alpha)\xi_\alpha, \end{aligned}$$

the equality holds all over $\mathcal{H}_\alpha$.

(ii): Let $U_{\beta\alpha}^* : \mathcal{H}_\beta \to \mathcal{H}_\alpha$ be the adjoint of $U_{\beta\alpha}$. Then, using the equality $\omega_\alpha(x_\alpha) = \omega_\beta(J_{\beta\alpha}(x_\alpha))$ for every $x_\alpha \in \mathfrak{B}_\alpha$, we have

$$\begin{aligned} \langle \pi_\alpha(x_\alpha)\xi_\alpha \,|\, \xi_\alpha\rangle_\alpha &= \langle \pi_\beta(J_{\beta\alpha}(x_\alpha))\xi_\beta \,|\, \xi_\beta\rangle_\beta = \langle U_{\beta\alpha}\pi_\alpha(x_\alpha)\xi_\alpha \,|\, \xi_\beta\rangle_\beta \\ &= \langle \pi_\alpha(x_\alpha)\xi_\alpha \,|\, U_{\beta\alpha}^*\xi_\beta\rangle_\alpha. \end{aligned}$$

The density of $\{\pi_\alpha(x_\alpha)\xi_\alpha : x_\alpha \in \mathfrak{B}_\alpha\}$ implies that $U_{\beta\alpha}^*\xi_\beta = \xi_\alpha$. Hence (3.3) holds true and we can apply Proposition 3.11 to the representations π_α's, proving the statement.

(iii): If $x \in \mathfrak{A}$, then there exists $\overline{\alpha} \in \mathbb{F}$ such that $x \in \mathfrak{A}_\alpha$ for all $\alpha \geq \overline{\alpha}$; now, if $\beta \geq \alpha \geq \overline{\alpha}$ then

$$\begin{aligned} \omega(x) &= \omega_\alpha(\Phi_\alpha^{-1}(x)) = \langle \pi_\alpha(\Phi_\alpha^{-1}(x))\xi_\alpha \,|\, \xi_\alpha\rangle_\alpha = \langle \pi_\beta(J_{\beta\alpha}(\Phi_\alpha^{-1}(x)))\xi_\beta \,|\, \xi_\beta\rangle_\beta \\ &= \langle U_{\beta\alpha}\pi_\alpha(\Phi_\alpha^{-1}(x))U_{\beta\alpha}^*\xi_\beta \,|\, \xi_\beta\rangle_\beta = \langle \pi_\alpha(\Phi_\alpha^{-1}(x))U_{\beta\alpha}^*\xi_\beta \,|\, U_{\beta\alpha}^*\xi_\beta\rangle_\alpha \\ &= \langle \pi_\alpha(\Phi_\alpha^{-1}(x))U_{\beta\alpha}^*\Pi_\beta\xi \,|\, U_{\beta\alpha}^*\Pi_\beta\xi\rangle_\alpha = \langle \pi_\alpha(\Phi_\alpha^{-1}(x))\Pi_\alpha\xi \,|\, \Pi_\alpha\xi\rangle_\alpha \\ &= B(\Theta_\alpha\pi_\alpha(\Phi_\alpha^{-1}(x))\Pi_\alpha\xi, \xi) = B(\pi(x)\xi, \xi). \end{aligned}$$

The uniqueness of π follows once more from Proposition 3.11. ■

Remark 3.13. In the statement of Theorem 3.12 there is a seeming ambiguity: the GNS representation π_α of $\mathfrak{B}_\alpha$, constructed from ω_α, is in fact determined only up to unitary equivalence. It is not difficult to realize, however, that changing the π_α's to unitarily equivalent representations gives *essentially* the same global representation π.

3.5. Sufficient families of positive linear functionals

Theorem 3.14. *Let $\mathfrak{A}$ be a C^*-inductive locally convex space and let $\{\{\mathfrak{B}_\alpha, \Phi_\alpha\} : \alpha \in \mathbb{F}\}$ be the corresponding defining system. Assume that*

(i) *if $x_\alpha \in \mathfrak{B}_\alpha$ and $J_{\beta\alpha}(x_\alpha) \geq 0$ for all $\beta \geq \alpha$, then $x_\alpha \geq 0$;*
(ii) *$e_\beta \in J_{\beta\alpha}(\mathfrak{B}_\alpha)$ for all $\alpha, \beta \in \mathbb{F}$ with $\beta \geq \alpha$.*

Then, for every $y \in \mathfrak{A}^+$, $y \neq 0$, *there exists a positive linear functional* ω *on* $\mathfrak{A}$ *such that* $\omega(y) > 0$.

Proof. Let $y \in \mathfrak{A}^+$, $y \neq 0$. Then there exists $\alpha \in \mathbb{F}$ such that $y = \Phi_\alpha(x_\alpha)$ for a unique positive element x_α of $\mathfrak{B}_\alpha$. Clearly, $x_\alpha \neq 0$ and $x_\alpha = w_\alpha^* w_\alpha$ for some $w_\alpha \in \mathfrak{B}_\alpha$. Therefore, there exists a positive linear functional ω_α on $\mathfrak{B}_\alpha$, with $\omega_\alpha(e_\alpha) = 1$, such that $\omega_\alpha(x_\alpha) = \|w_\alpha\|_\alpha^2 > 0$.

Let $\beta \geq \alpha$ and $\mathfrak{M}_\beta := J_{\beta\alpha}(\mathfrak{B}_\alpha)$. Then $\mathfrak{M}_\beta$ is a subspace of $\mathfrak{B}_\beta$, stable under the involution and containing the unit e_β. We define, for $y_\beta \in \mathfrak{M}_\beta$, $\omega_\beta(y_\beta) := \omega_\alpha(x_\alpha)$, where x_α is the unique element of $\mathfrak{B}_\alpha$ such that $y_\beta = J_{\beta\alpha}(x_\alpha)$. By (i), ω_β is positive on $\mathfrak{M}_\beta$. Thus, it is bounded: $|\omega_\beta(y_\beta)| \leq \omega_\beta(e_\beta)\|y_\beta\|_\beta$ for every $y_\beta \in \mathfrak{M}_\beta$ [5, Th. 4.3.2]. By the Hahn–Banach theorem, ω_β has an extension, denoted by the same symbol, to $\mathfrak{B}_\beta$, which is continuous and has norm equal to $\omega_\beta(e_\beta)$. Hence, it is positive on $\mathfrak{B}_\beta$. We also define $\omega_\gamma \equiv 0$ if $\gamma < \alpha$. By the definition itself, $\{\omega_\alpha : \alpha \in \mathbb{F}\}$ satisfies (3.2). Hence it defines, by taking the inductive limit, a positive linear functional ω on $\mathfrak{A}$. One has $\omega(y) = \omega_\alpha(x_\alpha) > 0$. ∎

Remark 3.15. Under the assumptions of Theorem 3.14, the previous proof shows also that, if f_α is a positive linear functional on $\mathfrak{B}_\alpha$, then there exists a positive linear functional f on $\mathfrak{A}$ such that $f_\alpha = f \circ \Phi_\alpha$ for every $\alpha \in \mathbb{F}$.

Theorem 3.14 shows that the set of positive linear functionals on a C^*-inductive locally convex space is, at least under certain circumstances, sufficiently large to separate the points of the cone of positive elements. So we expect the existence of a *faithful representation* of $\mathfrak{A}$ in this case.

Let $\mathcal{F}$ be a family of *representable* positive linear functionals on $\mathfrak{A}$, by which we mean that, for each $\omega \in \mathcal{F}$, its *components* $\{\omega_\alpha\}$ satisfy the condition (A) of Theorem 3.12. For every $\omega \in \mathcal{F}$, we denote by π_ω the linear map of $\mathfrak{A}$ into $\mathsf{L_B}(\mathcal{D}_\omega, \mathcal{D}_\omega^\times)$ constructed in Theorem 3.12. Every space $\mathcal{D}_\omega$ is built up from a directed contractive system of Hilbert spaces $\{\mathcal{H}_\alpha^\omega, U_{\beta\alpha}^\omega : \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$. For each fixed $\alpha \in \mathbb{F}$, we can construct the direct sum of the corresponding spaces:

$$\mathcal{H}_\alpha^{\mathcal{F}} = \bigoplus_{\omega \in \mathcal{F}} \mathcal{H}_\alpha^\omega = \Big\{ \oplus \xi_\alpha^\omega : \xi_\alpha^\omega \in \mathcal{H}_\alpha^\omega, \sum_\omega \|\xi_\alpha^\omega\|^2 < \infty \Big\}.$$

If $\beta \geq \alpha$, the map

$$U_{\beta\alpha}^{\mathcal{F}} := \oplus U_{\beta\alpha}^\omega, \quad \text{where} \quad (\oplus U_{\beta\alpha}^\omega)(\oplus \xi_\alpha^\omega) = \oplus (U_{\beta\alpha}^\omega \xi_\alpha^\omega),$$

defines a contraction of $\mathcal{H}_\alpha^{\mathcal{F}}$ into $\mathcal{H}_\beta^{\mathcal{F}}$, and $\{\mathcal{H}_\alpha^{\mathcal{F}}, U_{\beta\alpha}^{\mathcal{F}} : \alpha, \beta \in \mathbb{F}, \beta \geq \alpha\}$ is a directed contractive system of Hilbert spaces. Hence, it defines a joint topological limit denoted by $(\mathcal{D}(\pi_{\mathcal{F}})^\times, \mathcal{D}(\pi_{\mathcal{F}}))$.

For every $\alpha \in \mathbb{F}$ we define $\pi_\alpha^{\mathcal{F}}$ to be the ordinary direct sum of the family of *-representations π_α^ω. It is easily seen that

$$\pi_\beta^{\mathcal{F}}(J_{\beta\alpha}(x_\alpha)) = U_{\beta\alpha}^{\mathcal{F}}\pi_\alpha^{\mathcal{F}}(x_\alpha)(U_{\beta\alpha}^{\mathcal{F}})^*, \quad \forall x_\alpha \in \mathfrak{B}_\alpha,\ \beta \geq \alpha. \tag{3.6}$$

Then by Proposition 3.11, the inductive limit $\pi_{\mathcal{F}} = \varinjlim \pi_\alpha^{\mathcal{F}}$ is well-defined.

PROPOSITION 3.16. *Let $\mathfrak{A}$ be a C^*-inductive locally convex space and $\{\{\mathfrak{B}_\alpha, \Phi_\alpha\} : \alpha \in \mathbb{F}\}$ the corresponding defining system. Assume that the conditions* (i) *and* (ii) *of Theorem* 3.14 *are satisfied and that every positive linear functional on $\mathfrak{A}$ fulfills condition* (A)*. Then there exists a representation π of $\mathfrak{A}$ such that $\pi(y) > 0$ for every $y \in \mathfrak{A}^+ \setminus \{0\}$.*

Proof. Let us consider the *-representation $\pi_{\mathcal{F}}$ constructed above, with $\mathcal{F}$ the family of all positive linear functionals on $\mathfrak{A}$. By Theorem 3.14, for every $y \in \mathfrak{A}^+$, there exists a positive linear functional ω such that $\omega(y) > 0$. Put, as before, $\omega_\alpha = \omega \circ \Phi_\alpha$. Then $\omega_\alpha(\Phi_\alpha^{-1}(y)) > 0$. This in turn implies that $\pi_\alpha^{\mathcal{F}}(y) > 0$ and so $\pi^{\mathcal{F}}(y) > 0$. ∎

REMARK 3.17. It is clear that all the assumptions of Theorem 3.14 and condition (A) too are satisfied if $J_{\beta\alpha}$ is, for $\beta \geq \alpha$, a *-isomorphism or, in particular, the identity of $\mathfrak{B}_\alpha$ into $\mathfrak{B}_\beta$ (of course, this means that $\mathfrak{B}_\alpha$ is a true subspace of $\mathfrak{B}_\beta$). This case is not necessarily trivial, as shown in Example 5.5.

REMARK 3.18. The condition (sch), which has played an important role in our construction, is certainly satisfied if every $J_{\beta\alpha}$ is completely positive and $\|J_{\beta\alpha}(e_\alpha)\|_\beta \leq 1$ [9, Proposition 9.9.4]. Hence, it is natural to ask what changes in the previous construction if these stronger assumptions are satisfied. For instance, one may conjecture that the space which comes out from our set-up has a richer structure, e.g. is a Banach space. Example 5.5 (where $J_{\beta\alpha}$ is the identity map from a certain C^*-algebra $\mathfrak{B}_\alpha$ into another $\mathfrak{B}_\beta$, with $\mathfrak{B}_\alpha \subseteq \mathfrak{B}_\beta$) shows that this is not the case: strengthening in this way the assumptions does not essentially modify the final structure that one obtains. Nevertheless, the hypothesis of complete positivity of the $J_{\beta\alpha}$'s is intermediate between (sch) and the $J_{\beta\alpha}$'s being *-homomorphisms, and thus, certainly, it deserves a deeper analysis, which we hope to undertake in the future.

3.6. An algebraic structure for $\mathfrak{A}$. In some cases, as we shall see, it is possible to define a partial multiplication in $\mathfrak{A}$. This can be introduced by means of a family $w = \{w_\alpha\}$, $w_\alpha \in \mathfrak{B}_\alpha$. The outcome is the structure of a partial *-algebra, depending, clearly, on the chosen family w.

Let $w = \{w_\alpha\}$ be a family of elements such that $w_\alpha \in \mathfrak{B}_\alpha^+$ and $J_{\beta\alpha}(w_\alpha) = w_\beta$ for all $\alpha, \beta \in \mathbb{F}$ with $\beta \geq \alpha$.

Let $x, y \in \mathfrak{A}$. There exists $\alpha \in \mathbb{F}$ such that $x, y \in \mathfrak{A}_\alpha$. This implies that also $x, y \in \mathfrak{A}_\beta$ for all $\beta \geq \alpha$. For every $\beta \geq \alpha$, there exist $x_\beta, y_\beta \in \mathfrak{B}_\beta$ such that $x = \Phi_\beta(x_\beta)$ and $y = \Phi_\beta(y_\beta)$. Put $z_\beta = x_\beta w_\beta y_\beta \in \mathfrak{B}_\beta$. Let $z_{(\beta)} \in \mathfrak{A}_\beta$, for all $\beta \geq \alpha$, be such that $z_{(\beta)} = \Phi_\beta(z_\beta) = \Phi_\beta(\Phi_\beta^{-1}(x)\, w_\beta\, \Phi_\beta^{-1}(y))$. If $\beta' > \beta$, then $z_{(\beta')} = \Phi_{\beta'}(z_{\beta'}) \in \mathfrak{A}_{\beta'}$ but in general $\Phi_{\beta'}(z_{\beta'}) \neq \Phi_\beta(z_\beta)$. Hence, we can multiply two elements $x, y \in \mathfrak{A}$ if there exists $\gamma \in \mathbb{F}$ such that $z_{(\beta)} = z_{(\beta')}$ for all $\beta, \beta' \geq \gamma$.

Definition 3.19. In $\mathfrak{A}$, partial multiplication $x \cdot y$ of $x, y \in \mathfrak{A}$ is defined by the conditions:

$$\exists \gamma \in \mathbb{F}: \quad \Phi_\beta(\Phi_\beta^{-1}(x) w_\beta \Phi_\beta^{-1}(y)) = \Phi_{\beta'}(\Phi_{\beta'}^{-1}(x) w_{\beta'} \Phi_{\beta'}^{-1}(y)), \quad \forall \beta, \beta' \geq \gamma,$$
$$x \cdot y = \Phi_\beta(\Phi_\beta^{-1}(x) w_\beta \Phi_\beta^{-1}(y)), \qquad \beta \geq \gamma.$$

Proposition 3.20. *$\mathfrak{A}$ is a partial *-algebra with respect to the usual operations and the above defined multiplication.*

Proof. Let $x, y \in \mathfrak{A}$; we want to prove that if $x \cdot y$ is well defined, so also is $y^* \cdot x^*$ and $(x \cdot y)^* = y^* \cdot x^*$. From $x \cdot y \in \mathfrak{A}$, using the fact that every Φ_α preserves involution, it follows that $(x \cdot y)^* \in \mathfrak{A}$ and there exists $\gamma \in \mathbb{F}$ such that, for every $\alpha \geq \gamma$,

$$\begin{aligned}(x \cdot y)^* &= (\Phi_\alpha(\Phi_\alpha^{-1}(x) w_\alpha \Phi_\alpha^{-1}(y)))^* = \Phi_\alpha((\Phi_\alpha^{-1}(x) w_\alpha \Phi_\alpha^{-1}(y))^*) \\ &= \Phi_\alpha(\Phi_\alpha^{-1}(y^*) w_\alpha \Phi_\alpha^{-1}(x^*)) = y^* \cdot x^*.\end{aligned}$$

It remains to prove that, if $x \cdot y$ and $x \cdot z$ are well-defined then, for every $\lambda, \mu \in \mathbb{C}$, $x \cdot (\lambda y + \mu z)$ is well-defined too. From the assumptions, there exists $\gamma_1 \in \mathbb{F}$ such that for all $\alpha \geq \gamma_1$, $x \cdot y = \Phi_\alpha(\Phi_\alpha^{-1}(x) w_\alpha \Phi_\alpha^{-1}(y))$. Since $x \cdot z$ is also well-defined, there exists $\gamma_2 \in \mathbb{F}$ such that for all $\beta \geq \gamma_2$, $x \cdot z = \Phi_\beta(\Phi_\beta^{-1}(x) w_\beta \Phi_\beta^{-1}(z))$. If $\gamma \geq \gamma_1, \gamma_2$, then for all $\alpha \geq \gamma$,

$$\begin{aligned}&\lambda \Phi_\alpha(\Phi_\alpha^{-1}(x)\, w_\alpha\, \Phi_\alpha^{-1}(y)) + \mu \Phi_\alpha(\Phi_\alpha^{-1}(x)\, w_\alpha\, \Phi_\alpha^{-1}(z)) \\ &\qquad = \Phi_\alpha(\Phi_\alpha^{-1}(x)\, w_\alpha\, \Phi_\alpha^{-1}(\lambda y)) + \Phi_\alpha(\Phi_\alpha^{-1}(x)\, w_\alpha\, \Phi_\alpha^{-1}(\mu z)) \\ &\qquad = \Phi_\alpha(\Phi_\alpha^{-1}(x)\, w_\alpha\, (\Phi_\alpha^{-1}(\lambda y) + \Phi_\alpha^{-1}(\mu z))) \\ &\qquad = \Phi_\alpha(\Phi_\alpha^{-1}(x)\, w_\alpha\, \Phi_\alpha^{-1}(\lambda y + \mu z)) = x \cdot (\lambda y + \mu z). \ \blacksquare\end{aligned}$$

Proposition 3.21. *The spaces $R\mathfrak{A}^{(w)}$ and $L\mathfrak{A}^{(w)}$ of the universal right, respectively, left multipliers* ([2]) *of $\mathfrak{A}$ are algebras. Hence, $\mathfrak{A}_0^{(w)} := L\mathfrak{A}^{(w)} \cap R\mathfrak{A}^{(w)}$ is a *-algebra.*

Proof. Let $y, z \in R\mathfrak{A}^{(w)}$; then $y \cdot z$ is well-defined. To prove that $y \cdot z \in R\mathfrak{A}^{(w)}$, consider any $x \in \mathfrak{A}$. Since $x \cdot y$ is well-defined, there exists $\gamma_1 \in \mathbb{F}$ such that for all $\delta, \delta' \geq \gamma_1$, $\Phi_\delta(\Phi_\delta^{-1}(x) w_\delta \Phi_\delta^{-1}(y)) = \Phi_{\delta'}(\Phi_{\delta'}^{-1}(x) w_{\delta'} \Phi_{\delta'}^{-1}(y))$

([2]) The symbol $^{(w)}$ is a reminder of the dependence of the spaces of multipliers on $w = \{w_\alpha\}$; we omit a similar reminder for the multiplication itself, to lighten notation.

and $x \cdot y = \Phi_\delta(\Phi_\delta^{-1}(x) w_\delta \Phi_\delta^{-1}(y))$. Now $(x \cdot y) \cdot z$ is well-defined too, so there exists $\gamma_2 \in \mathbb{F}$ such that for all $\sigma, \sigma' \geq \gamma_2$ we have $\Phi_\sigma(\Phi_\sigma^{-1}(x \cdot y) w_\sigma \Phi_\sigma^{-1}(z)) = \Phi_{\sigma'}(\Phi_{\sigma'}^{-1}(x \cdot y) w_{\sigma'} \Phi_{\sigma'}^{-1}(z))$ and $(x \cdot y) \cdot z = \Phi_\sigma(\Phi_\sigma^{-1}(x \cdot y) w_\sigma \Phi_\sigma^{-1}(z))$. If now $\tau \geq \gamma_1, \gamma_2$ then

$$(x \cdot y) \cdot z = \Phi_\tau(\Phi_\tau^{-1}(x \cdot y) w_\tau \Phi_\tau^{-1}(z)) = \Phi_\tau(\Phi_\tau^{-1}(x) w_\tau \Phi_\tau^{-1}(y) w_\tau \Phi_\tau^{-1}(z)).$$

By associativity of $\mathfrak{A}_\tau$, for all $\lambda, \lambda' \geq \tau$ we can write

$$\Phi_\lambda(\Phi_\lambda^{-1}(x) w_\lambda (\Phi_\lambda^{-1}(y) w_\lambda \Phi_\lambda^{-1}(z))) = \Phi_{\lambda'}(\Phi_{\lambda'}^{-1}(x)\, w_{\lambda'}\, (\Phi_{\lambda'}^{-1}(y) w_{\lambda'} \Phi_{\lambda'}^{-1}(z))).$$

Hence $x \cdot (y \cdot z)$ is well-defined, so we conclude that $y \cdot z \in R\mathfrak{A}$. The statement for $L\mathfrak{A}^{(w)}$ follows by observing that $L\mathfrak{A}^{(w)} = (R\mathfrak{A}^{(w)})^*$. ∎

Corollary 3.22. *The following statements hold.*

(i) $(\mathfrak{A}, \mathfrak{A}_0^{(w)})$ *is a quasi* **-algebra.*

(ii) *If* $\mathfrak{A}$ *is endowed with* τ_{ind}*, then the maps* $x \mapsto x^*$, $x \mapsto a \cdot x$, $x \mapsto x \cdot b$, $a, b \in \mathfrak{A}_0^{(w)}$*, are continuous.*

Proof. (i): Because of Proposition 3.20 and the fact $\mathfrak{A}_0^{(w)}$ is a *-algebra, we need only prove the module associativity. But this is done by computations analogous to those in the proof of Lemma 3.21.

(ii): The continuity of the involution follows immediately from the definition. Let now $a \in \mathfrak{A}_0^{(w)}$ and $x \in \mathfrak{A}$. Then, there exists $\alpha \in \mathbb{F}$ such that $a, x \in \mathfrak{A}_\alpha$. Hence, for $\beta \geq \alpha$, $a \cdot x = \Phi_\beta(\Phi_\beta^{-1}(a) w_\beta \Phi_\beta^{-1}(x))$ and

$$\begin{aligned}\|a \cdot x\|_{(\beta)} &= \|\Phi_\beta(\Phi_\beta^{-1}(a) w_\beta \Phi_\beta^{-1}(x))\|_{(\beta)} = \|\Phi_\beta^{-1}(a) w_\beta \Phi_\beta^{-1}(x)\|_\beta \\ &\leq \|\Phi_\beta^{-1}(a)\|_\beta \|w_\beta\|_\beta \|\Phi_\beta^{-1}(x)\|_\beta = \|w_\beta\|_\beta \|a\|_{(\beta)} \|x\|_{(\beta)}.\end{aligned}$$

Thus $x \mapsto a \cdot x$ maps $\mathfrak{A}_\beta$ continuously into itself for sufficiently large $\beta \in \mathbb{F}$. This implies the continuity with respect to the inductive topology [11, II, 6.3]. The proof for the right multiplication is similar. ∎

Proposition 3.23. *The quasi* **-algebra* $(\mathfrak{A}, \mathfrak{A}_0^{(w)})$ *has a unit* e *if, and only if, every element* w_α *of the family* $\{w_\alpha\}$ *defining the multiplication is invertible and*

$$J_{\beta\alpha}(w_\alpha^{-1}) = w_\beta^{-1}, \quad \forall \alpha, \beta \in \mathbb{F}, \beta \geq \alpha. \tag{3.7}$$

In this case, $e = \Phi_\alpha(w_\alpha^{-1})$*, independently of* $\alpha \in \mathbb{F}$*.*

Proof. Assume that (3.7) holds. If we put $e := \Phi_\alpha(w_\alpha^{-1})$, we also have, for $\beta \geq \alpha$, $\Phi_\beta(w_\beta^{-1}) = \Phi_\beta(J_{\beta\alpha}(w_\alpha^{-1})) = \Phi_\alpha(w_\alpha^{-1}) = e$. Thus, for every $x \in \mathfrak{A}$,

$$x \cdot e = \Phi_\beta(\Phi_\beta^{-1}(x) w_\beta \Phi_\beta^{-1}(e)) = \Phi_\beta(\Phi^{-1}(x)) = x, \quad \beta \geq \alpha.$$

Similarly, $e \cdot x = x$. Hence, $e \in \mathfrak{A}_0^{(w)}$ and it is the unit of $(\mathfrak{A}, \mathfrak{A}_0^{(w)})$.

Conversely, assume that $(\mathfrak{A}, \mathfrak{A}_0^{(w)})$ has a unit e. Then $e = \Phi_\alpha(w_\alpha)$ for some $\alpha \in \mathbb{F}$ and $w_\alpha \in \mathfrak{B}_\alpha$. Put $w_\beta = J_{\beta\alpha}(w_\alpha)$ for $\beta \geq \alpha$. If $x \in \mathfrak{A}$, then for sufficiently large β, we have $x = \Phi_\beta(x_\beta)$, $x_\beta \in \mathfrak{B}_\beta$, and

$$x = x \cdot e = \Phi_\beta(\Phi_\beta^{-1}(x) w_\beta \Phi_\beta^{-1}(e)).$$

This implies that

$$x_\beta = \Phi_\beta^{-1}(x) w_\beta \Phi_\beta^{-1}(e) = x_\beta w_\beta \Phi_\beta^{-1}(e).$$

Analogously, we can prove that

$$x_\beta = \Phi_\beta^{-1}(e) w_\beta x_\beta.$$

Since x_β is an arbitrary element of $\mathfrak{B}_\beta$, we conclude that $\Phi_\beta^{-1}(e) = w_\beta^{-1}$. This, in turn, implies that $J_{\gamma\beta}(w_\beta^{-1}) = w_\gamma^{-1}$ for all $\gamma \geq \beta$. ∎

Proposition 3.24. *The following statements hold.*

(i) *If $a \in \mathfrak{A}$ and $a^* \cdot a$ is well-defined, then $a^* \cdot a \in \mathfrak{A}^+$. In particular, $a^* \cdot a \in \mathfrak{A}^+$ for every $a \in \mathfrak{A}_0^{(w)}$.*

(ii) *If $x \in \mathfrak{A}^+$ and $a \in \mathfrak{A}_0^{(w)}$, then $a^* \cdot x \cdot a \in \mathfrak{A}^+$.*

Proof. (i): If the element $a^* \cdot a$ is well-defined, there exists $\gamma \in \mathbb{F}$ such that, for all $\beta, \beta' \geq \gamma$,

$$\Phi_\beta(\Phi_\beta^{-1}(a^*) w_\beta \Phi_\beta^{-1}(a)) = \Phi_{\beta'}(\Phi_{\beta'}^{-1}(a^*) w_{\beta'} \Phi_{\beta'}^{-1}(a)),$$
$$a^* \cdot a = \Phi_\beta(\Phi_\beta^{-1}(a^*) w_\beta \Phi_\beta^{-1}(a)).$$

The positivity of $a^* \cdot a$ follows from

$$\Phi_\beta^{-1}(a^* \cdot a) = \Phi_\beta^{-1}(a^*) w_\beta \Phi_\beta^{-1}(a) = \Phi_\beta^{-1}(a)^* w_\beta \Phi_\beta^{-1}(a) \in \mathfrak{B}_\beta^+, \quad \forall \beta \geq \gamma.$$

(ii): Let $x \in \mathfrak{A}^+$; then there exists $\gamma \in \mathbb{F}$ such that $\Phi_\alpha^{-1}(x) \in \mathfrak{B}_\alpha^+$ for all $\alpha \geq \gamma$. The product $a^* \cdot x \cdot a$ is well-defined; hence, there exists $\gamma' \in \mathbb{F}$ such that, for all $\delta \geq \gamma'$,

$$a^* \cdot x \cdot a = \Phi_\delta(\Phi_\delta^{-1}(a^*) w_\delta \Phi_\delta^{-1}(x) w_\delta \Phi_\delta^{-1}(a)).$$

If $\sigma \geq \gamma, \gamma'$, then (taking into account the associativity of $\mathfrak{B}_\sigma$)

$$\begin{aligned}\Phi_\sigma^{-1}(a^* \cdot x \cdot a) &= \Phi_\sigma^{-1}\Phi_\sigma(\Phi_\sigma^{-1}(a^*) w_\sigma \Phi_\sigma^{-1}(x) w_\sigma \Phi_\sigma^{-1}(a)) \\ &= \Phi_\sigma^{-1}(a^*) w_\sigma \Phi_\sigma^{-1}(x) w_\sigma \Phi_\sigma^{-1}(a) \in \mathfrak{B}_\sigma^+.\end{aligned}$$

It is then clear that $a^* \cdot x \cdot a \in \mathfrak{A}^+$. ∎

By Propositions 3.9 and 3.24, it is easy to prove the following corollary.

Corollary 3.25. *Let $\omega \geq 0$, $a \in \mathfrak{A}_0^{(w)}$ and define $\omega_a : \mathfrak{A} \to \mathbb{C}$ by $\omega_a(x) := \omega(a^* \cdot x \cdot a)$. Then $\omega_a \geq 0$.*

Remark 3.26. The fact that several different multiplications can be defined in a C^*-inductive locally convex space, depending on the choice of the family $w = \{w_\alpha\}$, deserves a comment. The reader may suspect there is something artificial in our construction. Why not choose, for instance, $w_\alpha = e_\alpha$, the unit of $\mathfrak{B}_\alpha$? This is certainly a possible choice. But, in some examples, the corresponding spaces of multipliers become too small to make the partial multiplication of any use. A certain ambiguity in the definition of partial multiplication occurs, on the other hand, in familiar examples, like spaces of distributions or in spaces of operators acting in the rigged Hilbert space $(\mathcal{D}, \mathcal{H}, \mathcal{D}^\times)$ considered in Example 5.1. In the latter case, introducing partial multiplication is really a touchy business, because of the (sometimes dramatic) pathologies pointed out by Kürsten and collaborators [6, 7, 8]. An unambiguous definition of multiplication can only be given by fixing suitably chosen families of *interspaces* [14] between $\mathcal{D}$ and $\mathcal{D}^\times$ (see, also, [1, 2]). In conclusion, the definition of multiplication through the family $w = \{w_\alpha\}$ corresponds on one hand to this known ambiguity, and on the other hand yields a certain flexibility.

4. Quasi *-algebras with C^*-inductive structure. Let now $(\mathfrak{A}, \mathfrak{A}_0)$ be a given quasi *-algebra and assume that $\mathfrak{A}$ is a C^*-inductive locally convex space whose involution coincides with the involution of $(\mathfrak{A}, \mathfrak{A}_0)$.

Definition 4.1. A quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is called a *C^*-inductive quasi *-algebra* if $\mathfrak{A}$ is a C^*-inductive locally convex space with respect to a directed system of C^*-algebras $\{\mathfrak{B}_\alpha, J_{\beta\alpha} : \beta \geq \alpha\}$ and the following conditions hold:

(i) all maps $x \in \mathfrak{A} \mapsto xa \in \mathfrak{A}$, $x \in \mathfrak{A} \mapsto bx \in \mathfrak{A}$, $a, b \in \mathfrak{A}_0$, are continuous with respect to $\tau_{\rm ind}$;
(ii) $a^*a, a^*xa \in \mathfrak{A}^+$ for every $a \in \mathfrak{A}_0$ and $x \in \mathfrak{A}^+$.

Remark 4.2. If there exists $w = \{w_\alpha\}$ such that $\mathfrak{A}_0 \subset \mathfrak{A}_0^{(w)}$ and $ab = a \cdot b$ for every $a, b \in \mathfrak{A}_0$, then, as shown in Section 3.6, conditions (i) and (ii) are automatically satisfied.

Proposition 4.3. *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a C^*-inductive quasi *-algebra and ω be a positive linear functional on $\mathfrak{A}$. Then:*

(i) $\omega(a^*a) \geq 0$ *for every* $a \in \mathfrak{A}_0$*;*
(ii) *for every $a \in \mathfrak{A}_0$, the linear functional ω_a defined by*

$$\omega_a(x) = \omega(a^*xa), \qquad x \in \mathfrak{A},$$

is also positive;
(iii) $\omega(b^*xa) = \overline{\omega(a^*x^*b)}$ *for all $a, b \in \mathfrak{A}_0$ and $x \in \mathfrak{A}$;*
(iv) *if $x \in \mathfrak{A}_\alpha$, there exists $y \in \mathfrak{A}_\alpha^+ := \mathfrak{A}_\alpha \cap \mathfrak{A}^+$, depending only on α, such that*

$$|\omega(b^*xa)| \leq \|x\|_\alpha \omega(a^*ya)^{1/2} \omega(b^*yb)^{1/2}, \qquad \forall a, b \in \mathfrak{A}_0. \tag{4.1}$$

Proof. (i) and (ii) follow immediately from the definition of positivity and from (ii) of Definition 4.1. As for (iii), it is enough to consider the equality

$$\omega(b^*xa) = \frac{1}{4}\sum_{k=0}^{3} i^k \omega((a+i^k b)^* x(a+i^k b)).$$

To prove (iv) we begin by showing that, for every $x \in \mathfrak{A}$, there exists $\beta \in \mathbb{F}$ such that $x \in \mathfrak{A}_\beta$ and

$$|\omega(a^*xa)| \leq \|x\|_{(\beta)}\omega(a^*\Phi_\beta(e_\beta)a), \qquad a \in \mathfrak{A}_0. \tag{4.2}$$

Indeed, if $x \in \mathfrak{A}_\beta$, then $x = \Phi_\beta(x_\beta)$ for some $x_\beta \in \mathfrak{B}_\beta$. Hence, for every $a \in \mathfrak{A}_0$,

$$\begin{aligned}|(\omega_a \circ \Phi_\beta)(x_\beta)| &= |(\omega_a \circ \Phi_\beta)(\Phi_\beta^{-1}(x))| \\ &\leq ((\omega_a \circ \Phi_\beta)(e_\beta))\|\Phi_\beta^{-1}(x)\|_\beta = \omega_a(\Phi_\beta(e_\beta))\|x\|_{(\beta)}.\end{aligned}$$

On the other hand, $(\omega_a \circ \Phi_\beta)(\Phi_\beta^{-1}(x)) = \omega_a(x) = \omega(a^*xa)$. Hence

$$|\omega(a^*xa)| \leq \|x\|_{(\beta)}\omega_a(\Phi_\beta(e_\beta)) = \|x\|_{(\beta)}\omega(a^*\Phi_\beta(e_\beta)a), \qquad \forall a \in \mathfrak{A}_0.$$

Now, let $x \in \mathfrak{A}^+$ and $a, b \in \mathfrak{A}_0$. Define $\Omega_\omega^x(a,b) := \omega(b^*xa)$. Then it is easily checked that Ω_ω^x is a positive sesquilinear form on $\mathfrak{A}_0 \times \mathfrak{A}_0$.

Hence, for fixed $x \in \mathfrak{A}^+$, the Cauchy–Schwarz inequality holds:

$$|\Omega_\omega^x(a,b)| \leq \Omega_\omega^x(a,a)^{1/2}\Omega_\omega^x(b,b)^{1/2}, \qquad \forall a, b \in \mathfrak{A}_0.$$

Then using (4.2), for a suitable $\beta \in \mathbb{F}$, we get

$$\begin{aligned}|\omega(b^*xa)| &\leq \omega(a^*xa)^{1/2}\omega(b^*xb)^{1/2} \\ &\leq \|x\|_{(\beta)}\omega(a^*\Phi_\beta(e_\beta)a)^{1/2}\omega(b^*\Phi_\beta(e_\beta)b)^{1/2}.\end{aligned} \tag{4.3}$$

Now we turn to the general case. If $x \in \mathfrak{A}$, then x can be decomposed as $x = u + iv = u^+ - u^- + iv^+ - iv^-$ with $u^+, u^-, v^+, v^- \in \mathfrak{A}^+$, by Proposition 3.6. If $x \in \mathfrak{A}_\beta$, then also $u^+, u^-, v^+, v^- \in \mathfrak{A}_\beta^+$. Then using (4.3), we get

$$\begin{aligned}|\omega(b^*xa)| &\leq |\omega(b^*u^+a)| + |\omega(b^*u^-a)| + |\omega(b^*v^+a)| + |\omega(b^*v^+a)| \\ &\leq 2\|u\|_{(\beta)}\omega(a^*\Phi_\beta(e_\beta)a)^{1/2}\,\omega(b^*\Phi_\beta(e_\beta)b)^{1/2} \\ &\quad + 2\|v\|_{(\beta)}\omega(a^*\Phi_\beta(e_\beta)a)^{1/2}\,\omega(b^*\Phi_\beta(e_\beta)b)^{1/2} \\ &\leq 4\|x\|_{(\beta)}\omega(a^*\Phi_\beta(e_\beta)a)^{1/2}\omega(b^*\Phi_\beta(e_\beta)b)^{1/2}.\end{aligned}$$

Finally, taking $y = 4\,\Phi_\beta(e_\beta) \in \mathfrak{A}_\beta^+$, we get the desired inequality (4.1). ∎

As we have seen, every positive linear functional over a C^*-inductive quasi *-algebra satisfies the conditions (i)–(iv) of Proposition 4.3. It is then natural to ask whether these conditions are sufficient in order to get a GNS-construction of a general quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ taking its values in some space $\mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times)$.

Before going forth, we need an explicit definition of *-representation of a quasi *-algebra in RHS.

Definition 4.4. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with identity e, and $\mathcal{D}_\pi$ a dense domain in a certain Hilbert space $\mathcal{H}_\pi$ endowed with the graph topology $t_{\mathfrak{M}}$ defined by an O^*-algebra $\mathfrak{M} \subset \mathcal{L}^\dagger(\mathcal{D}_\pi)$. A linear map π from $\mathfrak{A}$ into $\mathfrak{L}_\mathsf{B}(\mathcal{D}_\pi, \mathcal{D}_\pi^\times)$ such that:

(i) $\pi(a^*) = \pi(a)^\dagger$ for all $a \in \mathfrak{A}$,
(ii) if $a \in \mathfrak{A}$, $x \in \mathfrak{A}_0$, then $\pi(ax) = \pi(a)\pi(x)$,
(iii) $\pi(\mathfrak{A}_0) \subset \mathcal{L}^\dagger(\mathcal{D}_\pi)$,

is called a **-representation* of $\mathfrak{A}$ in the RHS $(\mathcal{D}_\pi[t_{\mathfrak{M}}], \mathcal{H}_\pi, \mathcal{D}_\pi^\times[t_{\mathfrak{M}}^\times])$.

The following lemma, which allows one to extend a *-representation defined on a domain $\mathcal{D}$ to its completion, can be easily proved.

Lemma 4.5. *Let $\mathcal{D}$ be endowed with the graph topology $t_{\mathfrak{M}}$ defined by an O^*-algebra $\mathfrak{M}$ on $\mathcal{D}$ and let $X \in \mathfrak{L}_B(\mathcal{D}, \mathcal{D}^\times)$. Let $\widetilde{\mathcal{D}}[t_{\mathfrak{M}}]$ denote the completion of $\mathcal{D}$ with respect to the topology $t_{\mathfrak{M}}$. Then X has a unique extension $\widetilde{X}$ such that $\widetilde{X} \in \mathfrak{L}_B(\widetilde{\mathcal{D}}, \mathcal{D}^\times)$.*

Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with unit e, and ω a linear functional on $\mathfrak{A}$. Assume that ω satisfies the following conditions:

(Q1) $\omega(a^*a) \geq 0$ for every $a \in \mathfrak{A}_0$;
(Q2) $\omega(b^*x^*a) = \overline{\omega(a^*xb)}$ for every $x \in \mathfrak{A}$ and $a, b \in \mathfrak{A}_0$;
(Q3) for all $x \in \mathfrak{A}$, there exist $\gamma_x > 0$ and $c \in \mathfrak{A}_0$ such that

$$|\omega(a^*x^*b)|^2 \leq \gamma_x \omega(a^*c^*ca)\omega(b^*c^*cb), \qquad \forall a, b \in \mathfrak{A}_0.$$

Then, starting from $\omega^0 := \omega \restriction \mathfrak{A}_0$ one can define a closed strongly cyclic *-representation π_ω^0, with strongly cyclic vector ξ_ω, defined on a domain $\mathcal{D}_{\pi_\omega}$. The space $\mathcal{D}_{\pi_\omega}$ can be endowed with several topologies finer than that induced by the Hilbert norm. Each of them can be used to construct a RHS having $\mathcal{D}_{\pi_\omega}$ as the smaller space. In [1] it was proved that if we endow $\mathcal{D}_{\pi_\omega}$ with $t_\dagger$, the graph topology generated by $\mathcal{L}^\dagger(\mathcal{D}_{\pi_\omega})$, and ω is a linear functional on $\mathfrak{A}$ satisfying (Q1)–(Q3), then there exists a *-representation π_ω of $(\mathfrak{A}, \mathfrak{A}_0)$ into the corresponding space $\mathfrak{L}_\mathsf{B}(\mathcal{D}_{\pi_\omega}, \mathcal{D}_{\pi_\omega}^\times)$ which reduces to π_ω^0 on $\mathfrak{A}_0$ (π_ω was called the *-representation *canonically associated* with π_ω^0).

Definition 4.6. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra and π^0 a *-representation of $\mathfrak{A}_0$, with domain $\mathcal{D}_{\pi^0}$. We say that π^0 is *extensible* to $\mathfrak{A}$ if there exists a *-representation π in the RHS $(\mathcal{D}_{\pi^0}[t_{\pi^0}], \mathcal{H}_{\pi^0}, \mathcal{D}_{\pi^0}^\times[t_{\pi^0}^\times])$ such that $\pi \restriction \mathfrak{A}_0 = \pi^0$.

Proposition 4.7. *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with unit e and ω a linear functional on $\mathfrak{A}$ satisfying* (Q1) *and* (Q2). *Let π_ω^0 denote the closed GNS-representation of $\mathfrak{A}_0$. Then π_ω^0 is extensible to $\mathfrak{A}$ if and only if ω satisfies* (Q3).

Proof. Assume that (Q3) is satisfied. The following argument modifies that given in [1] only in some points. For this reason we skip all details.

As is known, the GNS representation of $\mathfrak{A}_0$ acts on the pre-Hilbert space $\mathfrak{A}_0/N_\omega$, with $N_\omega = \{a \in \mathfrak{A}_0 : \omega(a^*a) = 0\}$, as

$$\pi^0_\omega(b)\lambda_\omega(a) = \lambda_\omega(ba), \qquad a, b \in \mathfrak{A}_0,$$

where $\lambda_\omega(a) = a + N_\omega$ for $a \in \mathfrak{A}_0$, and then it is extended to the completion $\mathcal{D}_\omega := \widetilde{\lambda_\omega(\mathfrak{A}_0)}[t_{\pi^0_\omega}]$ (the extension is denoted by the same symbol). The vector $\eta_\omega := \lambda_\omega(e)$ is strongly cyclic for π^0_ω.

If $x \in \mathfrak{A}$, the linear functional x^ω on $\mathcal{D}_\omega$ defined by

$$x^\omega(\lambda_\omega(a)) = \omega(x^*a), \qquad a \in \mathfrak{A}_0,$$

is continuous, since, by (Q3), there exist $\gamma_x > 0$ and $c \in \mathfrak{A}_0$ such that

$$|x^\omega(\lambda_\omega(a))| = |\omega(x^*a)| \le \gamma_x \omega(c^*c)^{1/2}\, \omega(a^*c^*ca)^{1/2} = \gamma'_x \|\pi^0_\omega(c)\lambda_\omega(a)\|.$$

Hence, there exists a unique $\xi_\omega(x) \in \mathcal{D}^\times_\omega$, the conjugate dual of $\mathcal{D}_\omega[t_{\pi^0_\omega}]$, such that

$$x^\omega(\lambda_\omega(a)) = \langle \lambda_\omega(a) \,|\, \xi_\omega(x)\rangle, \qquad \forall a \in \mathfrak{A}_0.$$

Thus for every $x \in \mathfrak{A}$, $\pi_\omega(x)\lambda_\omega(a) := \xi_\omega(xa)$, $a \in \mathfrak{A}_0$, is well-defined and maps $\lambda_\omega(\mathfrak{A}_0)$ into $\mathcal{D}^\times_\omega$. By Lemma 4.5, $\pi_\omega(x)$ extends to $\mathcal{D}_\omega$. The fact that π_ω is a *-representation is easily checked.

Finally, consider the sesquilinear form $\theta_{\pi_\omega(x)}$ associated to $\pi_\omega(x)$, $x \in \mathfrak{A}$:

$$\begin{aligned} |\theta_{\pi_\omega(x)}(\lambda_\omega(a), \lambda_\omega(b))| &= \langle \pi_\omega(x)\lambda_\omega(a) \,|\, \lambda_\omega(b)\rangle = |\omega(b^*xa)| \\ &\le \gamma_x\, \omega(b^*c^*cb)^{1/2}\omega(a^*c^*ca)^{1/2} \\ &= \gamma_x \|\pi^0_\omega(c)\lambda_\omega(b)\|\, \|\pi^0_\omega(c)\lambda_\omega(a)\|, \qquad \forall a, b \in \mathfrak{A}_0. \end{aligned}$$

This means that $\theta_{\pi_\omega(x)}$ is jointly continuous in $\mathcal{D}_\omega[t_{\pi^0_\omega}]$. Hence $\pi_\omega(x) \in \mathfrak{L}_{\mathsf{B}}(\mathcal{D}_\omega, \mathcal{D}^\times_\omega)$.

Vice versa, assume that π^0_ω is extensible to $\mathfrak{A}$; then for every $x \in \mathfrak{A}$, $\pi_\omega(x) \in \mathfrak{L}_{\mathsf{B}}(\mathcal{D}_\omega, \mathcal{D}^\times_\omega)$, where $\mathcal{D}_\omega$ is endowed with $t_{\pi^0_\omega}$. Hence, there exist $\gamma_x > 0$ and $c \in \mathfrak{A}_0$ such that for every $a, b \in \mathfrak{A}_0$,

$$\begin{aligned} |\omega(b^*xa)| = |\theta_{\pi_\omega(x)}(\lambda_\omega(a), \lambda_\omega(b))| &\le \gamma_x \|\pi^0_\omega(c)\lambda_\omega(b)\|\, \|\pi^0_\omega(c)\lambda_\omega(a)\| \\ &= \gamma_x \omega(b^*c^*cb)^{1/2}\omega(a^*c^*ca)^{1/2}, \qquad \forall a, b \in \mathfrak{A}_0. \ \blacksquare \end{aligned}$$

EXAMPLE 4.8. Assume that the topology t on a domain $\mathcal{D}$, in a Hilbert space $\mathcal{H}$, makes $(\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times), \mathcal{L}^\dagger(\mathcal{D}))$ a quasi *-algebra. For every fixed $\xi \in \mathcal{D}$, the linear functional ω_ξ on $\mathfrak{L}(\mathcal{D}, \mathcal{D}^\times)$ defined by

$$\omega_\xi(X) = \langle X\xi \,|\, \xi\rangle, \qquad X \in \mathfrak{L}(\mathcal{D}, \mathcal{D}^\times),$$

satisfies the conditions (i)–(iii) of Proposition 4.3. As for (iv), we get the stronger condition (Q3). Indeed, by the definition itself, for every $X \in$

$\mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times)$, there exist $\gamma_X > 0$ and $A \in \mathcal{L}^\dagger(\mathcal{D})$ such that

$$|\omega_\xi(X)| = |\langle X\xi \,|\, \xi\rangle| \le \gamma_X \|A\xi\|^2,$$

or in other terms

$$|\omega_\xi(X)| \le \gamma_X \omega_\xi(A^\dagger A),$$

which, in turn, implies

$$|\omega_\xi(C^\dagger X B)| \le \gamma_X \omega_\xi(B^\dagger A^\dagger AB)\omega_\xi(C^\dagger A^\dagger AC), \quad \forall B, C \in \mathcal{L}^\dagger(\mathcal{D}).$$

Remark 4.9. The conditions (Q1)–(Q3) given above look very close to the conditions (L1)–(L3) used in [13]. The two groups of assumptions differ essentially in the third condition (i.e., (L3) and (Q3)). The first one implies that the corresponding representation lives in Hilbert space, giving rise to ordinary closable operators, while (Q3) forces the operators to go beyond Hilbert space.

5. Examples. In this section we collect some examples that illustrate the ideas developed so far.

5.1. Sesquilinear forms and operators. We will now show that certain spaces of sesquilinear forms or spaces of operators acting on a RHS (see [1, 12] for details) provide examples of C^*-inductive locally convex spaces or C^*-inductive quasi *-algebras.

Example 5.1. Let $\mathcal{D}$ be a dense domain in a Hilbert space $\mathcal{H}$. The graph topology $t_\dagger$, defined by $\mathcal{L}^\dagger(\mathcal{D})$, is also generated by the system of seminorms $\{\|\cdot\|_A\}_{A\in\mathcal{L}^\dagger(\mathcal{D})}$, where

$$\|\xi\|_A = \sqrt{\|\xi\|^2 + \|A\xi\|^2} = \|(I + A^*\overline{A})^{1/2}\xi\|, \quad \xi \in \mathcal{D}.$$

The completion of $\mathcal{D}[\|\cdot\|_A]$ is a Hilbert space denoted by $\mathcal{H}_A$.

For $A, B \in \mathcal{L}^\dagger(\mathcal{D})$, we write $A \preceq B$ if $\|A\xi\| \le \|B\xi\|$ for every $\xi \in \mathcal{D}$. If $A \preceq B$, we define, as in (I_1) of Section 2, a linear map V_{AB} from $\mathcal{H}_B$ into $\mathcal{H}_A$ in the following way. If $\xi \in \mathcal{H}_B$, there exists a sequence $\{\xi_n\}$ of elements of $\mathcal{D}$ which converges to ξ with respect to $\|\cdot\|_B$. This implies that $\{\xi_n\}$ is Cauchy with respect to $\|\cdot\|_A$, and therefore it converges to $\xi' \in \mathcal{H}_A$. Clearly, $\xi' = \xi$ and so $\mathcal{H}_B \subseteq \mathcal{H}_A$. Thus V_{AB} is nothing but the identity I_{AB} of $\mathcal{H}_B$ into $\mathcal{H}_A$. One has $\|I_{AB}\xi\|_A \le \|\xi\|_B$ for every $\xi \in \mathcal{H}_B$. We denote by $U_{BA} : \mathcal{H}_A \to \mathcal{H}_B$, the adjoint map of I_{AB}, i.e. $U_{BA} = I^*_{AB}$. One also has $\|U_{BA}\eta\|_B \le \|\eta\|_A$ for every $\eta \in \mathcal{H}_A$.

Let, as before, $\mathsf{B}(\mathcal{D}, \mathcal{D})$ denote the vector space of all jointly continuous sesquilinear forms on $\mathcal{D} \times \mathcal{D}$, i.e. $\theta \in \mathsf{B}(\mathcal{D}, \mathcal{D})$ if, and only if, there exists $c > 0$ and $A \in \mathcal{L}^\dagger(\mathcal{D})$ such that

$$|\theta(\xi, \eta)| \le c\|\xi\|_A \|\eta\|_A, \quad \forall \xi, \eta \in \mathcal{D}. \tag{5.1}$$

The involution $\theta \mapsto \theta^*$ in $\mathsf{B}(\mathcal{D},\mathcal{D})$ is defined by

$$\theta^*(\xi,\eta) = \overline{\theta(\eta,\xi)}, \qquad \xi,\eta \in \mathcal{D}.$$

The subspace $\mathsf{B}^A(\mathcal{D},\mathcal{D})$ of $\mathsf{B}(\mathcal{D},\mathcal{D})$ consisting of all $\theta \in \mathsf{B}(\mathcal{D},\mathcal{D})$ such that (5.1) holds, for a fixed $A \in \mathcal{L}^\dagger(\mathcal{D})$, is stable under involution.

Then, if $\theta \in \mathsf{B}^A(\mathcal{D},\mathcal{D})$, θ extends to $\mathcal{H}_A$ (we use the same symbol for this extension) and it is a bounded sesquilinear form on $\mathcal{H}_A$. Hence, there exists a unique operator $X_A^\theta \in \mathfrak{B}(\mathcal{H}_A)$ such that

$$\theta(\xi,\eta) = \langle X_A^\theta \xi \,|\, \eta\rangle_A, \qquad \forall \xi,\eta \in \mathcal{H}_A.$$

Conversely, if $X_A \in \mathfrak{B}(\mathcal{H}_A)$, then the sesquilinear form θ_{X_A} defined by

$$\theta_{X_A}(\xi,\eta) = \langle X_A \xi \,|\, \eta\rangle_A, \qquad \xi,\eta \in \mathcal{D},$$

satisfies (5.1). Thus the map

$$\Phi_A : X_A \in \mathfrak{B}(\mathcal{H}_A) \mapsto \theta_{X_A} \in \mathsf{B}^A(\mathcal{D},\mathcal{D})$$

defines a *-isomorphism of vector spaces with involution.

If $B \succeq A$, then

$$|\theta_{X_A}(\xi,\eta)| = |\langle X_A\xi \,|\, \eta\rangle_A| \le \|X_A\|_{A,A}\|\xi\|_A\|\eta\|_A \le \|X_A\|_{A,A}\|\xi\|_B\|\eta\|_B,$$

where $\|\cdot\|_{A,A}$ denotes the operator norm in $\mathfrak{B}(\mathcal{H}_A)$. Hence, there exists a unique $X_B \in \mathfrak{B}(\mathcal{H}_B)$ such that

$$\langle X_A\xi \,|\, \eta\rangle_A = \langle X_B\xi \,|\, \eta\rangle_B, \qquad \forall \xi,\eta \in \mathcal{D}.$$

So it is natural to define

$$J_{BA}(X_A) = X_B, \qquad \forall X_A \in \mathfrak{B}(\mathcal{H}_A).$$

It is easily seen that $J_{BA} = \Phi_B^{-1}\Phi_A$. A more explicit expression of J_{BA} is obtained as follows. Let $\xi,\eta \in \mathcal{D} \subset \mathcal{H}_B$. Then

$$\langle J_{BA}(X_A)\xi \,|\, \eta\rangle_B = \langle X_A\xi \,|\, \eta\rangle_A = \langle X_A I_{AB}\xi \,|\, I_{AB}\eta\rangle_A = \langle U_{BA}X_A I_{AB}\xi \,|\, \eta\rangle_B.$$

The density of $\mathcal{D}$ in $\mathcal{H}_B$ implies that $J_{BA}(X_A)\xi = U_{BA}X_AI_{AB}\xi$ and that this equality extends to $\mathcal{H}_B$. Hence,

$$J_{BA}(X_A) = U_{BA}X_AI_{AB}, \qquad \forall X_A \in \mathfrak{B}(\mathcal{H}_A),\ B \succeq A. \tag{5.2}$$

It is readily checked that

$$J_{BA}(X_A) = (I + B^*\overline{B})^{-1}(I + A^*\overline{A})X_A, \qquad X_A \in \mathfrak{B}(\mathcal{H}_A).$$

Now we prove that (sch) is satisfied. Let $X_A \in \mathfrak{B}(\mathcal{H}_A)$ and $\xi_B \in \mathcal{H}_B$; put $\xi_A = I_{AB}\xi_B$ for $A \preceq B$. Then, using (5.2),

$$\begin{aligned}
\langle J_{BA}(X_A^*)J_{BA}(X_A)\xi_B \,|\, \xi_B\rangle_B &= \langle J_{BA}(X_A)\xi_B \,|\, J_{BA}(X_A)\xi_B\rangle_B \\
&= \|U_{BA}X_AI_{AB}\xi_B\|_B^2 \\
&\le \|X_AI_{AB}\xi_B\|_A^2 = \langle X_A\xi_A \,|\, X_A\xi_A\rangle_A \\
&= \langle X_A^*X_A\xi_A \,|\, \xi_A\rangle_A = \langle J_{BA}(X_A^*X_A)\xi_B \,|\, \xi_B\rangle_B.
\end{aligned}$$

Hence, $\mathsf{B}(\mathcal{D},\mathcal{D})$ is a C^*-inductive locally convex space.

EXAMPLE 5.2. We denote by $\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)$ the space of all linear maps X from $\mathcal{D}$ into $\mathcal{D}^\times$ such that the sesquilinear form on $\mathcal{D}\times\mathcal{D}$ defined by

$$\theta_X(\xi,\eta)=\langle X\xi\,|\,\eta\rangle,\qquad \xi,\eta\in\mathcal{D}, \tag{5.3}$$

is *jointly* continuous in $\mathcal{D}[t_\dagger]$; i.e., $X\in\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)$ if, and only if, there exist $\gamma_X>0$ and $A\in\mathcal{L}^\dagger(\mathcal{D})$ such that

$$|\theta_X(\xi,\eta)|=|\langle X\xi\,|\,\eta\rangle|\le\gamma_X\|\xi\|_A\|\eta\|_A,\qquad\forall\xi,\eta\in\mathcal{D}. \tag{5.4}$$

Then $\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)\subset\mathfrak{L}(\mathcal{D},\mathcal{D}^\times)$, the space of all continuous linear maps from $\mathcal{D}[t_\dagger]$ into $\mathcal{D}^\times[t_\dagger^\times]$. Clearly, if $X\in\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)$, then $\theta_X\in\mathsf{B}(\mathcal{D},\mathcal{D})$.

Conversely, if $\theta\in\mathsf{B}(\mathcal{D},\mathcal{D})$, there exists a unique $X\in\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)$ such that $\theta=\theta_X$. The operator $X^\dagger\in\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)$ corresponding to θ^* is the adjoint map of X, since

$$\langle X\xi\,|\,\eta\rangle=\overline{\langle X^\dagger\eta\,|\,\xi\rangle},\qquad\forall\xi,\eta\in\mathcal{D}.$$

Thus, the map

$$\mathbb{I}:X\in\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)\mapsto\theta_X\in\mathsf{B}(\mathcal{D},\mathcal{D})$$

is an isomorphism of vector spaces preserving involution.

Let $\mathfrak{L}^A_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)=\mathbb{I}^{-1}\mathsf{B}^A(\mathcal{D},\mathcal{D})$. Clearly $\mathfrak{L}^A_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)$ is a subspace of $\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)$ and a Banach space, with norm

$$\|X\|^A:=\sup_{\|\xi\|_A,\|\eta\|_A\le1}|\theta_X(\xi,\eta)|.$$

If $A\preceq B$, then $\mathfrak{L}^A_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)\subset\mathfrak{L}^B_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)$ and $\|\cdot\|^B\le\|\cdot\|^A$ on $\mathfrak{L}^A_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)$.

The space $\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)$ endowed with the inductive topology $\tau_{\rm ind}$ defined by the family of subspaces $\{\mathfrak{L}^A_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times):A\in\mathcal{L}^\dagger(\mathcal{D})\}$ is a bornological Hausdorff space [12, Section 1.2.III].

In conclusion,

$$X_A\in\mathfrak{B}(\mathcal{H}_A)\leftrightarrow\theta_{X_A}\in\mathsf{B}^A(\mathcal{D},\mathcal{D})\leftrightarrow X\in\mathfrak{L}^A_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)$$

are isometric *-isomorphic Banach spaces (we recall that the multiplication of $\mathfrak{B}(\mathcal{H}_A)$ is not preserved) and $\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)$ is a C^*-inductive locally convex space, isomorphic to $\mathsf{B}(\mathcal{D},\mathcal{D})$.

There is, however, something more. Indeed, the pair $(\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times),\mathcal{L}^\dagger(\mathcal{D}))$ is a quasi *-algebra, the products of $X\in\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)$ and $A\in\mathcal{L}^\dagger(\mathcal{D})$ being defined by

$$(XA)\xi=X(A\xi)\quad\text{and}\quad(AX)\xi=\hat{A}(X\xi),\qquad\xi\in\mathcal{D},$$

where $\hat{A}$ denotes the extension of A to $\mathcal{D}^\times$ defined by

$$\langle\hat{A}\xi'\,|\,\eta\rangle=\langle\xi'\,|\,A^\dagger\eta\rangle,\qquad\xi'\in\mathcal{D}^\times,\ \eta\in\mathcal{D}.$$

The conditions of Definition 4.1 are satisfied by taking, as usual,

$$\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times)^+=\{X\in\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times):\langle X\xi\,|\,\xi\rangle\ge0,\ \forall\xi\in\mathcal{D}\}.$$

Hence, $(\mathfrak{L}_{\mathsf{B}}(\mathcal{D},\mathcal{D}^\times),\mathcal{L}^\dagger(\mathcal{D}))$ is a C^*-inductive quasi *-algebra.

Example 5.3. We will show here how to choose a family $\{W_A \in \mathfrak{B}(\mathcal{H}_A) : A \in \mathcal{L}^\dagger(\mathcal{D})\}$ so that the partial multiplication defined in $\mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times)$ by the method of Section 3.6 would reproduce the quasi *-algebra structure of the previous example.

Let $A \in \mathcal{L}^\dagger(\mathcal{D})$. Then $(I + A^*\overline{A})^{-1} \in \mathfrak{B}(\mathcal{H}_A)$, as a simple consequence of the closed graph theorem, and $\|(I + A^*\overline{A})^{-1}\|_{A,A} \le 1$. Moreover, for every $\xi, \eta \in \mathcal{D}$,

$$\begin{aligned}\langle (I + A^*\overline{A})^{-1}\xi \,|\, \eta\rangle_A &= \langle (I + A^*\overline{A})^{1/2}(I + A^*\overline{A})^{-1}\xi \,|\, (I + A^*\overline{A})^{1/2}\eta\rangle \\ &= \langle \xi \,|\, \eta\rangle.\end{aligned}$$

We choose $W = \{W_A : A \in \mathcal{L}^\dagger(\mathcal{D})\}$ with $W_A = (I + A^*\overline{A})^{-1}$. We prove that with this choice, $\mathcal{L}^\dagger(\mathcal{D}) \subset R\mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times)^{(W)} \cap L\mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times)^{(W)}$. Indeed, if $Y \in \mathcal{L}^\dagger(\mathcal{D})$, then $Y \in \mathfrak{L}^A_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times)$ for some $A \in \mathcal{L}^\dagger(\mathcal{D})$, and the operator $Y_A \in \mathfrak{B}(\mathcal{H}_A)$ corresponding to Y satisfies $Y_A \upharpoonright \mathcal{D} = (I + A^\dagger A)Y$. If $X \in \mathfrak{L}_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times)$, then $X \in \mathfrak{L}^S_{\mathsf{B}}(\mathcal{D}, \mathcal{D}^\times)$ for sufficiently large $S \in \mathcal{L}^\dagger(\mathcal{D})$. Therefore, for every $\xi \in \mathcal{D}$,

$$X_T W_T Y_T \xi = X_T (I + T^*\overline{T})^{-1}(I + T^\dagger T)Y\xi = X_T Y \xi, \qquad T \succeq S,$$

with $X_T = \Phi_T^{-1}(X)$.

This implies that $X \cdot Y$ is well-defined and $(X \cdot Y)_T = X_T Y$. Hence

$$\langle (X \cdot Y)_T \xi \,|\, \eta\rangle_T = \langle X_T Y \xi \,|\, \eta\rangle_T = \langle XY\xi \,|\, \eta\rangle.$$

In order to prove that also $Y \cdot X$ is well defined we take into account that Y_A is also equal to the operator $\hat{Y}(I + A^*\overline{A})$ where $\hat{Y}$ denotes the extension of Y to $\mathcal{D}^\times$ defined in Example 5.2. Thus we have

$$Y_T W_T X_T \xi = \hat{Y}(I + T^*\overline{T})(I + T^*\overline{T})^{-1} X_T \xi = \hat{Y} X_T \xi.$$

Hence, for every $\xi, \eta \in \mathcal{D}$,

$$\langle (Y \cdot X)_T \xi \,|\, \eta\rangle_T = \langle \hat{Y} X_T \xi \,|\, \eta\rangle_T = \langle X_T \xi \,|\, Y^\dagger \eta\rangle_T = \langle X\xi \,|\, Y^\dagger \eta\rangle,$$

which proves the statement.

Finally, we notice that if I is the identity map from $\mathcal{D}$ into $\mathcal{D}^\times$, then $I = \{W_A^{-1} : A \in \mathcal{L}^\dagger(\mathcal{D})\}$, as expected from Proposition 3.23.

5.2. Functions and distributions

Example 5.4. Let (X, Σ) be a measurable space and $\mathfrak{M}(X, \Sigma)$ the set of positive measures on (X, Σ). If $\mu, \nu \in \mathfrak{M}(X, \Sigma)$, a natural order is defined by

$$\mu \preceq \nu \;\Leftrightarrow\; \mu(E) \le \nu(E),\ \forall E \in \Sigma.$$

This order makes $\mathfrak{M}(X, \Sigma)$ a directed set.

To fix notation, if $\mu \in \mathfrak{M}(X, \Sigma)$ and f is a measurable function, we denote by $\|f\|^\mu_\infty$ the L^∞-norm with respect to μ and, as usual, we put $L^\infty(X, \mu) = \{f \text{ measurable: } \|f\|^\mu_\infty < \infty\}$. As it is well-known, $L^\infty(X, \mu)$ is a C^*-algebra. If $\mu \preceq \nu$, then $\|f\|_\infty^\nu \le \|f\|_\infty^\mu$, hence $L^\infty(X,\mu) \subset L^\infty(X,\nu)$. If $\mathcal{L}_\infty(X, \mathfrak{M}(X,\Sigma))$ denotes the set union of the spaces $\{L^\infty(X,\mu) : \mu \in \mathfrak{M}(X,\Sigma)\}$, then the corresponding map Φ_μ is the identity and so are all the $J_{\nu\mu}$'s, $\nu \succeq \mu$. Therefore, $\mathcal{L}_\infty(X, \mathfrak{M}(X,\Sigma))$ is a *-algebra and, when endowed with the topology τ_{ind}, a C^*-inductive locally convex space. If $X = \mathbb{R}$ and Σ is the σ-algebra of Borel sets, then $\mathcal{L}_\infty(X, \mathfrak{M}(X,\Sigma))$ coincides with the *-algebra of all Borel measurable functions.

Example 5.5. Let us take as index set the family $\mathcal{K}$ of all compact subsets of the real line, ordered by inclusion. For $K \in \mathcal{K}$, put

$$\mathcal{B}(K) = \{g \in L^1_{\mathrm{loc}}(\mathbb{R}) : g(x) = f(x)\chi_K(x),\ f \in L^\infty_{\mathrm{loc}}(\mathbb{R})\}.$$

Then $\mathcal{B}(K)$ is a C^*-algebra under the norm $\|g\| = \|f\chi_K\|_\infty$. It is easily seen that if $K \subseteq K'$, then $\mathcal{B}(K) \subset \mathcal{B}(K')$. Let, as usual, $\mathcal{D}(\mathbb{R})$ denote the space of test functions and $\mathcal{D}'(\mathbb{R})$ the space of distributions. We define

$$\Phi_K : g \in \mathcal{B}(K) \mapsto T_g \in \mathcal{D}'(\mathbb{R}),$$

where T_g denotes the regular distribution defined by

$$T_g(\varphi) = \int_{\mathbb{R}} g(x)\varphi(x)\,dx, \qquad \varphi \in \mathcal{D}(\mathbb{R}).$$

It is clear from the definition that Φ_K does not preserve multiplication. The embedding $J_{K'K}$ of $\mathcal{B}(K)$ into $\mathcal{B}(K')$ is, in this case, the identity map. The algebraic inductive limit $\mathfrak{A}$ of the system $\{\{\mathcal{B}(K), \Phi_K\} : K \in \mathcal{K}\}$ is the set of distributions T having at least one regular restriction T_K to a compact set K which is defined by a function $g \in \mathcal{B}(K)$. This space is quite large: it contains, in fact, all distributions with compact support. When endowed with the topology τ_{ind}, $\mathfrak{A}$ is a C^*-inductive locally convex space.

References

[1] J.-P. Antoine, A. Inoue and C. Trapani, *Partial *-Algebras and Their Operator Realizations*, Kluwer, Dordrecht, 2002.

[2] J.-P. Antoine and C. Trapani, *Partial Inner Product Spaces—Theory and Applications*, Lecture Notes in Math. 1986, Springer, Berlin, 2009.

[3] G. Bellomonte and C. Trapani, *Rigged Hilbert spaces and contractive families of Hilbert spaces*, Monatsh. Math. (published online October 2010, DOI 10.1007/s00605-010-0249-1).

[4] B. Blackadar and E. Kirchberg, *Generalized inductive limits of C^*-algebras and quasidiagonality*, in: C^*-algebras, J. Cuntz and S. Echterhoff (eds.), Springer, Berlin, 2000, 23–41.

[5] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Vol. I*, Academic Press, New York, 1986.

[6] K.-D. Kürsten, *The completion of the maximal $O_p{}^*$-algebra on a Fréchet domain*, Publ. Res. Inst. Math. Sci., Kyoto Univ. 22 (1986), 151–175.

[7] K.-D. Kürsten, *On algebraic properties of partial algebras*, Rend. Circ. Mat. Palermo Suppl. 56 (1998), 111–122.

[8] K.-D. Kürsten and M. Läuter, *An extreme example concerning factorization products on the Schwartz space* $\mathfrak{S}(\mathbb{R}^n)$, Note Mat. 25 (2005/06), 31–38.

[9] T. W. Palmer, *Banach Algebras and the General Theory of* **-Algebras, Volume 2*, Cambridge Univ. Press, Cambridge, 2001.

[10] S. Sakai, C^**-algebras and* W^**-algebras*, Springer, Berlin, 1971.

[11] H. H. Schaefer, *Topological Vector Spaces*, Springer, Berlin, 1971.

[12] K. Schmüdgen, *Unbounded Operator Algebras and Representation Theory*, Birkhäuser, Basel, 1990.

[13] C. Trapani, **-Representations, seminorms and structure properties of normed quasi* **-algebras*, Studia Math. 186 (2008), 47–75.

[14] C. Trapani and F. Tschinke, *Partial multiplication of operators in rigged Hilbert spaces*, Integral Equations Operator Theory 51 (2005), 583–600.

Giorgia Bellomonte, Camillo Trapani
Dipartimento di Matematica e Informatica
Università di Palermo
I-90123 Palermo, Italy
E-mail: bellomonte@math.unipa.it
trapani@unipa.it

Received September 9, 2010
Revised version October 1, 2010 (6979)