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The single-point spectrum operators satisfying Ritt's resolvent condition

by

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Abstract. It is shown that an operator with the properties mentioned in the title does exist in the space $L_p(0,1)$, $1 \le p \le \infty$. The maximal sector for the extended resolvent condition can be prescribed a priori jointly with the corresponding order of the exponential growth of the resolvent in the complementary sector.

Let us recall that *Ritt's condition* for the resolvent $R(\lambda;T) = (T-\lambda I)^{-1}$ of a bounded linear operator T in a complex Banach space X is

$$\|R(\lambda;T)\| \le \frac{C}{|\lambda-1|}, \qquad |\lambda| > 1, \tag{1}$$

where C is a constant, $C \ge 1$. Condition (1) originated in the context of ergodic theory a long time ago [10]. It seems that only the short note [11] was devoted to this topic within the next 40 years. Recently the operators satisfying (1) attracted a new interest due to O. Nevanlinna who showed in [8] that any sectorial extension of (1),

$$\|R(\lambda;T)\| \le \frac{C(\delta)}{|\lambda-1|}, \qquad \lambda \in S_\delta, \tag{2}$$

where

$$S_\delta = \{\lambda : \lambda \ne 1,\ |\arg(\lambda-1)| \le \pi - \delta\}, \qquad 0 \le \delta < \pi/2,$$

implies the power boundedness of T.

It was proven in [6] and [7] independently that the original Ritt condition implies the sectorial extension (2) where, according to [6], $\delta > \arccos q$ and $C(\delta) = (q-\cos\delta)^{-1}$ with $q = C^{-1}$. In view of this result let us consider the maximal sector

$$S(\theta_1,\theta_2) = \{\lambda : \lambda \ne 1,\ \theta_1 < \arg(\lambda-1) < \theta_2\} \tag{3}$$

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such that

$$\sup|\lambda - 1| \cdot \|R(\lambda;T)\| < \infty, \qquad \lambda \in S(\theta_1 + \varepsilon, \theta_2 - \varepsilon) \tag{4}$$

for all ε, $0 < \varepsilon < \frac{1}{2}\theta$, $\theta = \theta_2 - \theta_1$. In (3) the "arg" means the relevant continuous branch.

We call the size θ of the maximal sector $S(\theta_1, \theta_2)$ the *resolvent angle size* of the operator T. Obviously, $\theta \geq \pi$ by (1) but, eventually, $\theta > \pi$ as aforesaid.

The resolvent $R(\lambda;T)$ under consideration does exist on the open sector $S(\theta_1, \theta_2)$ and outside the open unit disk with adjoint point $\lambda = 1$. Accordingly, the spectrum $\sigma(T)$ is contained in the intersection of the complementary closed sector and the open unit disk together with $\lambda = 1$. Conversely, with such a localization of the spectrum, the sectorial version (4) of Ritt's condition implies (1). (See [6] for a precise spectral localization provided by (1).) In particular, the point $\lambda = 1$ may belong to $\sigma(T)$ (and the opposite case is not interesting). Moreover, it may happen that $\sigma(T) = \{1\}$ or, equivalently, $T = I + V$ where $\sigma(V) = \{0\}$, i.e. V is quasinilpotent,

$$\lim_{n\to\infty} \sqrt[n]{\|V^n\|} = 0. \tag{5}$$

The operator $T = I$ (respectively, $V = 0$) is the only case such that $|\lambda - 1| \cdot \|R_\lambda(T)\|$ is bounded on the whole punctured plane $\mathbb{C} \setminus \{1\}$.

It was noted in [9] that it is still unknown whether there exists a quasinilpotent operator $V \neq 0$ such that $T = I + V$ satisfies (1) (the question of J. Zemánek [12]). In the present paper we construct a series of such operators corresponding to a natural scale of convergence rates in (5).

For any quasinilpotent operator V its *Fredholm resolvent* $\Phi(\zeta;V) = (I - \zeta V)^{-1}$ is an entire function,

$$\Phi(\zeta;V) = \sum_{n=0}^{\infty} \zeta^n V^n, \qquad \zeta \in \mathbb{C}. \tag{6}$$

The order of its exponential growth can be called the *order of the operator* V. We denote it by $\omega(V)$. By definition, either

$$\omega(V) = \inf\{\omega : \log\|\Phi(\zeta;V)\| = O(|\zeta|^\omega + 1)\} \tag{7}$$

or $\omega(V) = \infty$ if the set in (7) is empty. By the Liouville theorem, $\omega(V) \geq 0$.

As in the case of scalar entire functions [5, Section 1.3],

$$\omega(V) = \overline{\lim_{n\to\infty}} \frac{\log n}{|\log \sqrt[n]{\|V^n\|}|} \tag{8}$$

(with the agreement $1/0 = \infty,\ 1/\infty = 0$).

It is easy to reformulate the sectorial version of Ritt's condition in terms of $\Phi(\zeta;V)$. Indeed, if $T = I + V$ then $(\lambda - 1)R(\lambda;T) = -\Phi(\zeta;V)$, $\zeta =$

$(\lambda-1)^{-1}$. After the last transformation the sector $S(\theta_1,\theta_2)$ turns into

$$S^0(-\theta_2,-\theta_1)=\{\zeta:\zeta\neq 0,\ -\theta_2<\arg\zeta<-\theta_1\},$$

a sector of the same angle size $\theta=\theta_2-\theta_1$. The condition (4) takes the form

$$\sup\|\Phi(\zeta;V)\|<\infty,\qquad \zeta\in S^0(-\theta_2+\varepsilon,-\theta_1-\varepsilon),\tag{9}$$

for all ε with $0<\varepsilon<\frac{1}{2}\theta$. In this context one can forget T and deal only with V and with $\theta=\theta(V)$, the *resolvent angle size* of the operator V. The constraint $\theta(V)>\pi$ can be omitted.

It immediately follows from the Phragmén–Lindelöf principle and Liouville theorem that if $\omega(V)$ is finite and $\omega(V)(2\pi-\theta(V))<\pi$ then $V=0$. In other words, we have the following

PROPOSITION. *If $V\neq 0$ then*

$$\omega(V)(2\pi-\theta(V))\geq\pi.\tag{10}$$

In particular, the inequality (10) shows that if $\theta(V)>\pi$ then $\omega(V)>1$. Therefore $\omega(V)>1$ under Ritt's condition for $T=I+V$, V quasinilpotent, $V\neq 0$.

THEOREM. *Let $0\leq\gamma<2$. In any space $X=L_p(0,1)$, $1\leq p\leq\infty$, there exists a quasinilpotent operator V_γ such that*

$$\omega(V_\gamma)=1/\gamma,\qquad \theta(V_\gamma)=\pi(2-\gamma).\tag{11}$$

Thus, we see that the equality in (10) can be realized with any finite value $\omega(V)>1/2$ and also with $\omega(V)=\infty$, $\theta(V)=2\pi$.

As a consequence, *in any $L_p(0,1)$ there exists a bounded linear operator $T\neq I$ satisfying Ritt's condition and such that $\sigma(T)=\{1\}$. A fortiori, such an operator exists in any infinite-dimensional Hilbert space.*

Proof of Theorem. For simplicity we restrict ourselves to the case $p=1$. The case $p=\infty$ is similar (even simpler). For $1<p<\infty$, only the standard Hölder inequality technique should be added to the proof.

Let us consider the Riemann–Liouville integral of any fractional order $\alpha>0$,

$$(J^\alpha f)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\,ds.\tag{12}$$

(Γ is the Euler gamma function.) By the Fubini theorem the integral (12) is well defined for $f\in L_1(0,1)$, belongs to the same space and

$$\|J^\alpha f\|\leq\frac{1}{\Gamma(\alpha+1)}\|f\|,\tag{13}$$

so that J^α is a bounded linear operator in $L_1(0,1)$. Combining (13) with the easily calculated value of $\|J^\alpha f\|$ for $f(t) \equiv 1$ we get

$$\frac{1}{\Gamma(\alpha+2)} \le \|J^\alpha\| \le \frac{1}{\Gamma(\alpha+1)}. \tag{14}$$

The family $\{J^\alpha\}$ is a semigroup,

$$J^\alpha J^\beta = J^{\alpha+\beta}, \tag{15}$$

which follows from the equality

$$\int_u^t (t-s)^{\alpha-1}(s-u)^{\beta-1}\, ds = (t-u)^{\alpha+\beta-1} B(\alpha,\beta),$$

where B is the beta function,

$$B(\alpha,\beta) = \int_0^1 v^{\alpha-1}(1-v)^{\beta-1}\, dv = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

For $0 < \gamma < 2$ we set

$$V_\gamma = -J^\gamma. \tag{16}$$

It follows from (15), (14) and the Stirling formula that

$$\sqrt[n]{\|V_\gamma^n\|} = \sqrt[n]{\|J^{\gamma n}\|} \sim (e/\gamma)^\gamma n^{-\gamma} \tag{17}$$

asymptotically as $n \to \infty$. Thus, V_γ is quasinilpotent and $\omega(V_\gamma) = \gamma^{-1}$ according to (8). It remains to prove the second equality from (11). To this end we have to consider the Fredholm resolvent $\Phi(\zeta; V_\gamma)$. By (6), (16) and (15),

$$\Phi(\zeta; V_\gamma) = \sum_{n=0}^\infty (-1)^n \zeta^n J^{\gamma n} = I - \zeta \sum_{n=0}^\infty (-\zeta)^n J^{\gamma n+\gamma}.$$

Hence, $\Phi(\zeta; V_\gamma)$ is the Volterra integral operator,

$$(\Phi(\zeta; V_\gamma) f)(t) = f(t) - \zeta \int_0^t K_\gamma(t-s;\zeta) f(s)\, ds, \tag{18}$$

with the difference kernel defined by

$$K_\gamma(u;\zeta) = u^{\gamma-1} E_{\gamma,\gamma}(-\zeta u^\gamma), \qquad 0 < u \le 1, \tag{19}$$

where

$$E_{\gamma,\gamma}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\gamma n+\gamma)}, \qquad z \in \mathbb{C}. \tag{20}$$

The entire function (20) is a member of the two-parameter family

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \qquad (\alpha, \beta > 0), \tag{21}$$

which was introduced and initially investigated by Mittag-Leffler and Wiman at the very beginning of the 20th century. It is known (see [2, Section 18.1] or [3, p. 134]) that $E_{\alpha,\beta}(z)$ admits the uniform asymptotic expansion

$$E_{\alpha,\beta}(z) \sim -\sum_{k=1}^{\infty} \frac{1}{\Gamma(\beta - \alpha k)} \cdot \frac{1}{z^k} \tag{22}$$

as $|z| \to \infty$ in any sector $|\arg(-z)| \le \pi(1 - \alpha/2) - \varepsilon$, $\varepsilon > 0$, $0 < \alpha < 2$. For $\beta = \alpha$ the first term in (22) vanishes ($\Gamma(0) = \infty$), so that in the above mentioned sector

$$|E_{\alpha,\alpha}(z)| \le \frac{a_\alpha(\varepsilon)}{|z|^2 + 1}, \qquad a_\alpha(\varepsilon) = \text{const} > 0. \tag{23}$$

It follows from (18), (19) and (23) that

$$\begin{aligned} \|\Phi(\zeta; V_\gamma)\| &\le 1 + a_\gamma(\varepsilon) \int_0^1 \frac{|\zeta| u^{\gamma-1}\, du}{(|\zeta| u^\gamma)^2 + 1} \\ &= 1 + a_\gamma(\varepsilon)\gamma^{-1} \int_0^{|\zeta|} \frac{dv}{v^2 + 1} < 1 + \frac{\pi}{2} a_\gamma(\varepsilon)\gamma^{-1} \end{aligned} \tag{24}$$

in the sector $|\arg \zeta| \le \pi(1 - \gamma/2) - \varepsilon$. Thus, $\theta(V_\gamma) \ge \pi(2 - \gamma)$. On the other hand, $\theta(V_\gamma) \le \pi(2 - \gamma)$ by (10) and the equality $\omega(V_\gamma) = \gamma^{-1}$. As a result, $\theta(V_\gamma) = \pi(2 - \gamma)$.

Now let us consider the case $\gamma = 0$ where the previous construction (16) disappears. However, we prove that the operator

$$V_0 f = -\int_0^{\infty} (J^\alpha f)\, d\alpha \tag{25}$$

is such that $\omega(V_0) = \infty$, $\theta(V_0) = 2\pi$.

First of all, note that the vector-valued function $\alpha \mapsto J^\alpha f$ is continuous since

$$\|\Gamma(\alpha) J^\alpha f - \Gamma(\beta) J^\beta f\| \le \|f\| \int_0^1 |u^{\alpha-1} - u^{\beta-1}|\, du.$$

For this reason the linear operator V_0 is well defined and bounded in $L_1(0,1)$,

$$\|V_0\| \le \int_0^{\infty} \frac{d\alpha}{\Gamma(\alpha + 1)} \tag{26}$$

(see (13)).

It is easy to check by induction on $n \in \mathbb{N}$ that

$$V_0^n f = \frac{(-1)^n}{(n-1)!} \int_0^\infty \alpha^{n-1} (J^\alpha f)\, d\alpha.$$

Hence,

$$\frac{1}{(n-1)!} \int_0^\infty \frac{\alpha^{n-1}\, d\alpha}{\Gamma(\alpha+2)} \le \|V_0^n\| \le \frac{1}{(n-1)!} \int_0^\infty \frac{\alpha^{n-1}\, d\alpha}{\Gamma(\alpha+1)},$$

similarly to (14). By the Stirling formula

$$c_n \left(\int_1^\infty \frac{e^{h_n(\alpha)}\, d\alpha}{\alpha^{5/2}} \right)^{1/n} \le \frac{n}{e} \sqrt[n]{\|V_0^n\|} \le C_n \left(\int_1^\infty \frac{e^{h_n(\alpha)}\, d\alpha}{\alpha^{3/2}} \right)^{1/n} \tag{27}$$

where $h_n(\alpha) = (n-\alpha)\log\alpha + \alpha$, and both c_n and C_n tend to 1 as $n \to \infty$.

The unique root α_n of the equation $\alpha \log \alpha = n$ brings $h_n(\alpha)$ to the absolute maximum which is

$$M_n = h_n(\alpha_n) = [\alpha(\log^2 \alpha - \log\alpha + 1)]_{\alpha=\alpha_n}.$$

The upper bound (27) yields

$$\frac{n}{e} \sqrt[n]{\|V_0^n\|} \le C_n' e^{M_n/n}$$

where $C_n' = C_n \sqrt[n]{2} \to 1$ while

$$\frac{M_n}{n} = \frac{M_n}{\alpha_n \log \alpha_n} = \log\alpha_n - 1 + (\log\alpha_n)^{-1} = \log\left(\frac{n}{e\log n}\right) + o(1)$$

since $\alpha_n \sim n/\log n,\ n \to \infty$. Thus,

$$\sqrt[n]{\|V_0^n\|} \le \frac{C_n''}{\log n} \tag{28}$$

where $C_n'' \to 1$.

In the lower bound (27) one can restrict the interval of integration to $[\alpha_n, \alpha_n + 1]$. Since $h_n(\alpha_n + 1) = M_n + o(n)$, we get

$$\sqrt[n]{\|V_0^n\|} \ge \frac{c_n'}{\log n} \tag{29}$$

where $c_n' \to 1$. The bounds (28) and (29) result in

$$\sqrt[n]{\|V_0^n\|} \sim \frac{1}{\log n}. \tag{30}$$

Hence, $\omega(V_0) = \infty$ by (8). It remains to prove that $\theta(V_0) = 2\pi$.

The Fredholm resolvent of V_0 is the Volterra integral operator

$$(\Phi(\zeta; V_0) f)(t) = f(t) - \zeta \int_0^t Q(t-s;\zeta) f(s)\, ds \tag{31}$$

where

$$Q(u;\zeta) = \int_0^\infty \frac{u^{\alpha-1}e^{-\zeta\alpha}\,d\alpha}{\Gamma(\alpha)}, \qquad 0 < u \le 1,\ \zeta \in \mathbb{C}. \tag{32}$$

By (31) it is sufficient to show that

$$\sup_\zeta |\zeta| \int_0^1 |Q(u,\zeta)|\,du < \infty \tag{33}$$

in any sector $|\arg\zeta| \le \pi-\varepsilon,\ 0<\varepsilon<\pi$. For definiteness, let $0 \le \arg\zeta \le \pi-\varepsilon$. Consider the ray $R(\varphi) = \{\alpha \in \mathbb{C} : \arg\alpha = \varphi\}$ with $\varphi = \frac{1}{2}(\pi-\varepsilon) - \arg\zeta$, so that $|\varphi| \le \frac{1}{2}(\pi-\varepsilon)$.

The path of integration in (32) can be changed to $R(\varphi)$ because of the rapid growth of the gamma function in the complex plane but outside of any sectorial neighborhood of the negative axis. (Actually, the Stirling formula remains in force in such a domain, see [1, Section 1.18]). Thus,

$$Q(u;\zeta) = \int_{R(\varphi)} \frac{u^{\alpha-1}e^{-\zeta\alpha}\,d\alpha}{\Gamma(\alpha)}$$

with $\alpha = \varrho e^{i\varphi},\ \varrho \ge 0$. Note that

$$\mathrm{Re}(\zeta\alpha) = \varrho|\zeta|\cos(\varphi + \arg\zeta) = \varrho|\zeta|\sin(\varepsilon/2).$$

Hence,

$$\int_0^1 |Q(u;\zeta)|\,du \le \int_0^\infty \frac{u^{\varrho\cos\varphi-1}e^{-\varrho|\zeta|\sin(\varepsilon/2)}\,d\varrho}{|\Gamma(\varrho e^{i\varphi})|}.$$

Since $\cos\varphi \ge \sin(\varepsilon/2)$, we get

$$|\zeta|\int_0^1 |Q(u;\zeta)|\,du \le |\zeta| \int_0^\infty \frac{e^{-\varrho|\zeta|\sin(\varepsilon/2)}\,d\varrho}{\varrho\sin(\varepsilon/2)|\Gamma(\varrho e^{i\varphi})|}. \tag{34}$$

By substitution $\varrho|\zeta|\sin(\varepsilon/2) = \tau$ the inequality (34) takes the form

$$|\zeta|\int_0^1 |Q(u;\zeta)|\,du \le \frac{1}{\sin^2(\varepsilon/2)} \int_0^\infty \frac{e^{-\tau}\,d\tau}{|\Gamma(\xi\tau+1)|} \tag{35}$$

where $\xi = e^{i\varphi}(|\zeta|\sin(\varepsilon/2))^{-1}$. Now (33) follows from (35) since $\mathrm{Re}\,\xi \ge |\zeta|^{-1} > 0$ and the function $1/\Gamma(z)$ is bounded in the half-plane $\mathrm{Re}\,z > 1$. ∎

Remark 1. The same operators $V_\gamma,\ 0 \le \gamma < 2$, serve for all $L_p(0,1)$, $1 \le p \le \infty$. By the way, all of them are compact. The resulting estimates in (17), (24), (30) and (33) remain in force for all of p.

Remark 2. Part of information we have used can also be found in [4, Section 23.16]. In [3] a well developed theory of Mittag-Leffler type functions and their applications is presented.

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(1) *Editorial note*: See this issue, 113–134.