STUDIA MATHEMATICA 145 (2) (2001)

On the relative fundamental solutions for a second order differential operator on the Heisenberg group

by

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Abstract. Let H_n be the $(2n+1)$-dimensional Heisenberg group, let $p, q \geq 1$ be integers satisfying $p+q=n$, and let

$$L = \sum_{j=1}^{p} (X_j^2 + Y_j^2) - \sum_{j=p+1}^{n} (X_j^2 + Y_j^2),$$

where $\{X_1, Y_1, \ldots, X_n, Y_n, T\}$ denotes the standard basis of the Lie algebra of H_n. We compute explicitly a relative fundamental solution for L.

1. Introduction. Let $n \geq 2$ and let $p, q \geq 1$ be a pair of integers such that $p + q = n$. Let H_n be the Heisenberg group defined by $H_n = \mathbb{C}^n \times \mathbb{R}$ with group law $(z,t)(z',t') = \big(z+z', t+t' - \frac{1}{2}\operatorname{Im} B(z,z')\big)$ where $B(z,w) = \sum_{j=1}^{p} z_j \overline{w}_j - \sum_{j=p+1}^{n} z_j \overline{w}_j$. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we write $x = (x', x'')$ with $x' \in \mathbb{R}^p$, $x'' \in \mathbb{R}^q$. So, $\mathbb{R}^{2n}$ can be identified with $\mathbb{C}^n$ via the map $\varphi(x', x'', y', y'') = (x' + iy', x'' - iy'')$, $x', y' \in \mathbb{R}^p$, $x'', y'' \in \mathbb{R}^q$. In this setting the form $-\operatorname{Im} B(z,w)$ agrees with the standard symplectic form on $\mathbb{R}^{2(p+q)}$, and the vector fields

$$X_j = -\frac{1}{2} y_j \frac{\partial}{\partial t} + \frac{\partial}{\partial x_j}, \qquad Y_j = \frac{1}{2} x_j \frac{\partial}{\partial t} + \frac{\partial}{\partial y_j}, \qquad j = 1, \ldots, n,$$

and $T = \partial/\partial t$ form the standard basis for the Lie algebra h_n of H_n. Thus H_n can be viewed as $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ via the map $(x,y,t) \mapsto (\varphi(x,y), t)$. From now on, we will use freely this identification.

As usual we denote by $S(H_n)$ the Schwartz space on H_n and by $S'(H_n)$ the space of corresponding tempered distributions.

Let $L = \sum_{j=1}^{p} (X_j^2 + Y_j^2) - \sum_{j=p+1}^{n} (X_j^2 + Y_j^2)$, so L has a self-adjoint extension (that we still denote by L), as an operator densely defined on $L^2(H_n)$. Let K be the (closed) kernel of L in $L^2(H_n)$ and let π be the

2000 *Mathematics Subject Classification*: Primary 43A80; Secondary 35A08.

Research partially supported by CONICOR, SECYT UNC and CONICET.

orthogonal projection on K. We recall that $\Phi \in S'(H_n)$ is a *relative fundamental solution* for L if

$$L(f * \Phi) = L(f) * \Phi = f - \pi(f) \quad \text{for all } f \in S(H_n).$$

In [M-R1] and [M-R2] it is proved that there exist relative fundamental solutions for a wide class of left invariant second order differential operators that includes L. On the other hand, for the case $q = 0$ (i.e. the case $L = \sum_{j=1}^{n}(X_j^2 + Y_j^2)$), a fundamental solution is computed in [F1].

Consider the natural action of $U(p,q)$ on H_n given by $g \cdot (z,t) = (gz,t)$. So $U(p,q)$ acts on $L^2(H_n)$, $S(H_n)$ and $S'(H_n)$ in the canonical way.

In [G-S] it is proved that there exists a family of tempered $U(p,q)$-invariant distributions $S_{\lambda,k}$, $\lambda \in \mathbb{R} - \{0\}$, $k \in \mathbb{Z}$, satisfying

$$LS_{\lambda,k} = -|\lambda|(2k+p-q)S_{\lambda,k}, \quad iTS_{\lambda,k} = \lambda S_{\lambda,k}. \tag{1.1}$$

It is also proved that the solution space in $S'(H_n)^{U(p,q)}$ of the system (1.1) is one-dimensional (see also [F2] and [H-T]) and that the distributions $S_{\lambda,k}$ defined there satisfy

$$f = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} f * S_{\lambda,k}|\lambda|^n \, d\lambda \tag{1.2}$$

for all $f \in S(H_n)$. The convolution product is given, as usual, by the formula $(f * S_{\lambda,k})(z,t) = \langle S_{\lambda,k}, (L_{(z,t)^{-1}}f)^\vee \rangle$, where we write, for $g : H_n \to \mathbb{C}$,

$$(L_{(z,t)^{-1}}g)(z',t') = g((z,t)^{-1}(z',t')) \quad \text{and} \quad g^\vee((z,t)) = g((z,t)^{-1}).$$

In this setting, it is not hard to show (see Lemma 2.1 below) that Φ defined by

$$\langle \Phi, f \rangle = \sum_{2k+p-q\neq 0} \frac{-1}{2k+p-q} \int_{-\infty}^{\infty} \langle S_{\lambda,k}, f \rangle |\lambda|^{n-1} \, d\lambda \tag{1.3}$$

is a well defined element in $S'(H_n)$ and that Φ is a relative fundamental solution for L.

Let us introduce some notation and recall some known facts. Let $\mathcal{H}$ be the space of functions $\varphi : \mathbb{R} \to \mathbb{C}$ such that $\varphi(\tau) = \varphi_1(\tau) + H(\tau)\varphi_2(\tau)\tau^{n-1}$, where $\varphi_1, \varphi_2 \in S(\mathbb{R})$ and H denotes the Heaviside function, i.e. $H(\tau) = \chi_{(0,\infty)}(\tau)$. It is proved in [T] that $\mathcal{H}$ provided with a suitable topology is a Fréchet space. Also, for $p+q = n$, $p, q \geq 1$, there is given a continuous linear surjective map $N : S(\mathbb{R}^n) \to \mathcal{H}$ such that its adjoint $N' : \mathcal{H}' \to S'(\mathbb{R}^n)^{O(p,q)}$ is a linear homeomorphism onto the space of $O(p,q)$-invariant tempered distributions on $\mathbb{R}^n$. As pointed out in [G-S], this construction also works for the space $S'(\mathbb{C}^n)^{U(p,q)}$, i.e., there exists a continuous linear surjective map $N : S(\mathbb{C}^n) \to \mathcal{H}$ whose adjoint $N' : \mathcal{H}' \to S'(\mathbb{C}^n)^{U(p,q)}$ is a homeomorphism.

From now on, $N : S'(\mathbb{C}^n) \to S'(\mathbb{R})$ will be the operator given by (2.11) of [G-S]. For $f \in S(H_n)$, we write $Nf(\tau, t)$ for $N(f(\cdot, t))(\tau)$.

Our aim in this paper is to obtain a rather explicit description of the relative fundamental solution Φ for L. In Theorem 4.10 we will compute a distribution $\mu \in S'(\mathbb{R}^2)$ satisfying $\langle \Phi, f\rangle = \langle \mu, Nf\rangle$, $f \in S(H_n)$.

Acknowledgments. We express our thanks to Fulvio Ricci, who inspired this work and kindly helped us during its preparation.

2. Notations and preliminaries. For $\lambda \in \mathbb{R} - \{0\}$, let π_λ be the Schrödinger representation on $L^2(\mathbb{R}^n)$ of the Heisenberg group $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ defined by

$$\pi_\lambda(x, y, t)h(\zeta) = e^{-i(\lambda t + \mathrm{sgn}(\lambda)\sqrt{|\lambda|}x\cdot\zeta + \frac{1}{2}\lambda x\cdot y)} h(\zeta + \sqrt{|\lambda|}y).$$

We still denote by π_λ the corresponding representation of $H_n = \mathbb{C}^n \times \mathbb{R}$ (via the map φ of the introduction). For $f \in S(H_n)$, we set

$$\pi_\lambda(f) = \int\limits_{H_n} f(z, t)\pi_\lambda(z, t)^{-1}\, dz\, dt$$

where $dzdt$ means the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$. For $f \in S(H_n)$ and $h_1, h_2 \in L^2(\mathbb{R}^n)$, let $E_\lambda(h_1, h_2)$ be the associated matrix entry given by

$$E_\lambda(h_1, h_2)(z, t) = \langle \pi_\lambda(z, t)h_1, h_2\rangle.$$

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, let h_α be the Hermite function given by

$$h_\alpha(\zeta) = (2^{|\alpha|}\alpha!\sqrt{\pi})^{-n/2} e^{-|\zeta|^2/2} \prod_{j=1}^{n} H_{\alpha_j}(\zeta_j)$$

with $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$ and where

$$H_k(s) = (-1)^k e^{s^2} \frac{d^k}{ds^k}(e^{-s^2})$$

is the kth Hermite polynomial. Finally we also set

$$\|\alpha\| = \sum_{j=1}^{p} \alpha_j - \sum_{j=p+1}^{n} \alpha_j.$$

We also recall that, for nonnegative integers m, k with $k \le m$, the Laguerre polynomials $L_m^k(x)$ are defined by (see e.g. [Sz])

$$L_m^0(t) = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \frac{t^j}{j!}, \qquad L_{m-1}^{\beta+1}(t) = -\frac{d}{dt} L_m^\beta(t).$$

The distributions $S_{\lambda,k}$ are defined in [G-S] by

$$\langle S_{\lambda,k}, f\rangle = \sum_{\|\alpha\| = k} \langle E_\lambda(h_\alpha, h_\alpha), f\rangle.$$

It is proved (see [G-S], Theorem 4.1 and Remarks 4.2, 4.3) that $S_{\lambda,k} = F_{\lambda,k} \otimes e^{-i\lambda t}$, where $F_{\lambda,k} \in S'(\mathbb{C}^n)$ is described, in terms of the Laguerre polynomials and the operator N, by

$$\langle F_{\lambda,k}, g\rangle = \Big\langle (L^0_{k-q+n-1}H)^{(n-1)}, \tau \mapsto \frac{2}{|\lambda|} e^{-\tau/2} Ng\Big(\frac{2}{|\lambda|}\tau\Big)\Big\rangle, \quad g \in S(\mathbb{C}^n), \tag{2.1}$$

for $k \geq q$, $\lambda \neq 0$,

$$\langle F_{\lambda,k}, g\rangle = \Big\langle (L^0_{-k-p+n-1}H)^{(n-1)}, \tau \mapsto \frac{2}{|\lambda|} e^{-\tau/2} Ng\Big(-\frac{2}{|\lambda|}\tau\Big)\Big\rangle, \quad g \in S(\mathbb{C}^n), \tag{2.2}$$

for $k \leq -p$, $\lambda \neq 0$, and

$$\langle F_{\lambda,k}, g\rangle = \sum_{l=0}^{n-2} \Big(\frac{2}{|\lambda|}\Big)^l \frac{1}{|\lambda|} \sum_{\substack{l\leq j\leq n-2\\ j\geq q-k-1}} \frac{(-1)^{n-j}}{2^{n+j-l}} \binom{j}{l} \binom{n+k-q-1}{n-j-2} \langle \delta^l, Ng\rangle, \quad g\in S(\mathbb{C}^n), \tag{2.3}$$

for $-p < k < q$, $\lambda \neq 0$.

The next lemma states that (1.3) gives a relative fundamental solution Φ for L.

Lemma 2.1. *The formula* (1.3) *defines a tempered distribution that satisfies* $L(f * \Phi) = L(f) * \Phi = f - \pi(f)$ *for all* $f \in S(H_n)$.

Proof. We first prove that

$$\sum_{2k+p-q\neq 0} \frac{1}{|2k+p-q|} \int_{-\infty}^{\infty} |\langle S_{\lambda,k}, f\rangle| \cdot |\lambda|^{n-1}\, d\lambda < \infty. \tag{2.4}$$

Indeed, the above series can be written

$$\begin{aligned} &\sum_{2k+p-q\neq 0} \frac{1}{|2k+p-q|} \int_{-\infty}^{\infty} \Big| \sum_{\|\alpha\|=k} \langle E_\lambda(h_\alpha, h_\alpha), f\rangle \Big| |\lambda|^{n-1}\, d\lambda \\ &\quad = \sum_{2k+p-q\neq 0} \frac{1}{|2k+p-q|} \int_{-\infty}^{\infty} \Big| \sum_{\|\alpha\|=k} \langle \pi_\lambda(f) h_\alpha, h_\alpha\rangle \Big| |\lambda|^{n-1}\, d\lambda \\ &\quad \leq \sum_{k\in\mathbb{Z}^n} \int_{-\infty}^{\infty} |\langle \pi_\lambda(f) h_\alpha, h_\alpha\rangle| \cdot |\lambda|^{n-1}\, d\lambda. \end{aligned}$$

Now, proceeding as in [B-D-R], Lemma 4.10, we can use results of [M-R1] and [M-R2] to get (2.4) and so Φ is well defined. Moreover, the bounds given there show that $\Phi \in S'(H_n)$.

We recall that, for $f \in S(H_n)$, $\pi f = 0$ if n is odd and that, for n is even, $\pi f = \int_{-\infty}^{\infty} \langle S_{\lambda,(q-p)/2}, f\rangle |\lambda|^n \, d\lambda$ (see [G-S], Remark 4.8).

In order to see that $Lf * \Phi = f - \pi f$ we write (see, e.g., (1.3) in [G-S]) $L = L_0 + L_1$ where

$$L_0 = \Big(\sum_{j=1}^{p}(x_j^2 + y_j^2) - \sum_{j=p+1}^{n}(x_j^2 + y_j^2)\Big)\frac{\partial^2}{\partial t^2} + \sum_{j=1}^{p}\left(\frac{\partial^2}{\partial {x_j}^2} + \frac{\partial^2}{\partial {y_j}^2}\right) - \sum_{j=p+1}^{n}\left(\frac{\partial^2}{\partial {x_j}^2} + \frac{\partial^2}{\partial {y_j}^2}\right)$$

and

$$L_1 = \frac{\partial}{\partial t}\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right).$$

Now, for $g \in S(H_n)$ we have $L_0(g^\vee) = (L_0(g))^\vee$ and $L_1(g^\vee) = -(L_1(g))^\vee$. Since $(Lf * \Phi)(z,t) = \langle \Phi, (L_{(z,t)^{-1}}(Lf))^\vee\rangle$ and $L = L_0 + L_1$ we get

$$(Lf * \Phi)(z,t) = \langle \Phi, L_0((L_{(z,t)^{-1}}f)^\vee)\rangle - \langle \Phi, L_1((L_{(z,t)^{-1}}f)^\vee)\rangle.$$

But $\Phi \in S'(H_n)^{U(p,q)}$, and hence $L_1\Phi = 0$. Also, L_0 and L_1 are self-adjoint, thus $(Lf * \Phi)(z,t) = \langle \Phi, L((L_{(z,t)^{-1}}f)^\vee)\rangle$. So

$$\begin{aligned}(Lf * \Phi)(z,t) &= \sum_{2k+p-q\neq 0}\frac{-1}{2k+p-q}\int_{-\infty}^{\infty}\langle S_{\lambda,k}, L((L_{(z,t)^{-1}}f)^\vee)\rangle|\lambda|^{n-1}\,d\lambda\\ &= \sum_{2k+p-q\neq 0}\frac{-1}{2k+p-q}\int_{-\infty}^{\infty}\langle LS_{\lambda,k}, (L_{(z,t)^{-1}}f)^\vee\rangle|\lambda|^{n-1}\,d\lambda\\ &= \sum_{2k+p-q\neq 0}\int_{-\infty}^{\infty}\langle S_{\lambda,k}, (L_{(z,t)^{-1}}f)^\vee\rangle|\lambda|^{n}\,d\lambda\\ &= \sum_{2k+p-q\neq 0}\int_{-\infty}^{\infty}(f * S_{\lambda,k})(z,t)|\lambda|^{n}\,d\lambda = (f - \pi f)(z,t)\end{aligned}$$

and thus $Lf * \Phi = f - \pi f$.

On the other hand, using (1.1), (1.2) and the fact that L is a left invariant operator, it is easy to check that $L(f * \Phi) = f - \pi f$ for $f \in S(H_n)$. ■

Let us recall a well known Abel lemma about power series that states that if a numerical series $\sum_{k=0}^{\infty} a_k$ converges to a sum s, then $\lim_{r\to 1^-}\sum_{k=0}^{\infty} r^k a_k = s$.

Taking account of (2.4), we can decompose the series (1.3) as

$$\sum_{k\le -p} + \sum_{-p<k<q} + \sum_{k\ge q}.$$

Now, (2.4), the Lebesgue dominated convergence theorem (applied to the measure space $\mathbb{R}\times\mathbb{Z}$) and the above Abel lemma allow us to write

$$\begin{aligned}\langle \Phi, f\rangle = &\lim_{r\to 1^-}\lim_{\varepsilon\to 0^+}\sum_{k\ge q}\frac{-r^{2k+n-2q}}{2k+p-q}\int_{-\infty}^{\infty} e^{-\varepsilon|\lambda|}\langle S_{\lambda,k}, f\rangle|\lambda|^{n-1}\,d\lambda\\ &+\lim_{r\to 1^-}\lim_{\varepsilon\to 0^+}\sum_{k\le -p}\frac{-r^{-2k+n-2p}}{2k+p-q}\int_{-\infty}^{\infty} e^{-\varepsilon|\lambda|}\langle S_{\lambda,k}, f\rangle|\lambda|^{n-1}\,d\lambda\\ &+\lim_{\varepsilon\to 0^+}\sum_{-p<k<q}\frac{-1}{2k+p-q}\int_{-\infty}^{\infty} e^{-\varepsilon|\lambda|}\langle S_{\lambda,k}, f\rangle|\lambda|^{n-1}\,d\lambda.\end{aligned}$$

We change the summation indices writing $k=k'+q$ and $k=-k'-p$ in the first and second series respectively. Using the fact that $S_{\lambda,k}=F_{\lambda,k}\otimes e^{-i\lambda t}$ and the formulas (2.1) and (2.2) we get

$$\langle \Phi, f\rangle = \langle \Phi_1, f\rangle + \langle \Phi_2, f\rangle \tag{2.5}$$

where $\langle \Phi_1, f\rangle$ and $\langle \Phi_2, f\rangle$ are defined by the convergent expressions

$$\begin{aligned}\langle \Phi_1, f\rangle = &\lim_{r\to 1^-}\lim_{\varepsilon\to 0^+}\sum_{k\ge 0}\frac{-r^{2k+n}}{2k+n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\varepsilon|\lambda|}|\lambda|^{n-1}e^{-i\lambda t}\\ &\times\Big\langle (L^0_{k+n-1}H)^{(n-1)}, \tau\mapsto e^{-\tau/2}\frac{2}{|\lambda|}\Big[Nf\Big(\frac{2}{|\lambda|}\tau, t\Big)-Nf\Big(-\frac{2}{|\lambda|}\tau, t\Big)\Big]\Big\rangle\,dt\,d\lambda\end{aligned}$$

and

$$\langle \Phi_2, f\rangle = \sum_{-p<k<q}\frac{-1}{2k+p-q}\int_{-\infty}^{\infty}\langle S_{\lambda,k}, f\rangle|\lambda|^{n-1}\,d\lambda.$$

Now, a computation gives

$$(L^0_{k+n-1}H)^{(n-1)} = (-1)^{n-1}L^{n-1}_k H + \sum_{j=0}^{n-2}(-1)^{n-j}\binom{k+n-1}{n-j-2}\delta^{(j)}. \tag{2.6}$$

On the other hand,

$$\begin{aligned}&\Big\langle \delta^{(j)}, \tau\mapsto e^{-\tau/2}\Big[Nf\Big(\frac{2}{|\lambda|}\tau, t\Big)-Nf\Big(-\frac{2}{|\lambda|}\tau, t\Big)\Big]\Big\rangle\\ &\qquad = -2\sum_{\substack{0\le l\le j\\ l\ \mathrm{odd}}}\binom{j}{l}\frac{1}{2^{j-l}}\Big(\frac{2}{|\lambda|}\Big)^l\frac{\partial^l Nf}{\partial\tau^l}(0,t).\end{aligned} \tag{2.7}$$

Using (2.6), (2.7) and the fact that for $g\in L^2(0,\infty)$ and $h\in L^2(\mathbb{R})$,

$$\int_0^{\infty} g(\tau)[h(\tau)-h(-\tau)]\,d\tau = \int_{-\infty}^{\infty}\mathrm{sgn}(\tau)g(|\tau|)h(\tau)\,d\tau, \tag{2.8}$$

we obtain

$$\Big\langle (L^0_{k+n-1}H)^{(n-1)}, \tau \mapsto e^{-\tau/2}\frac{2}{|\lambda|}\Big[Nf\Big(\frac{2}{|\lambda|}\tau, t\Big) - Nf\Big(-\frac{2}{|\lambda|}\tau, t\Big)\Big]\Big\rangle$$

$$= (-1)^{n-1}\int_{-\infty}^{\infty} L_k^{n-1}\Big(\frac{|\lambda|}{2}|s|\Big)e^{-|\lambda|\cdot|s|/4}\operatorname{sgn}(s)Nf(s,t)\,ds$$

$$- 2\sum_{\substack{0\le l\le n-2\\ l\ \mathrm{odd}}}\sum_{j=l}^{n-2} b_{k,l}\Big(\frac{2}{|\lambda|}\Big)^{l+1}\frac{\partial^l Nf}{\partial\tau^l}(0,t)$$

where

$$b_{k,l} = \sum_{j=l}^{n-2}\frac{1}{2^{j-l}}\binom{j}{l}(-1)^{n-j}\binom{k+n-1}{n-j-2}. \tag{2.9}$$

For $k \ge 0$ and $0 \le l \le n-2$, we also set

$$a_{k,l} = \frac{1}{l!}\int_0^{\infty} L_k^{n-1}(s)e^{-s/2}s^l\,ds \tag{2.10}$$

and

$$G_f(\tau,t) = Nf(\tau,t) - \sum_{j=0}^{n-2}\frac{\partial^l Nf}{\partial\tau^j}(0,t)\frac{\tau^j}{j!}. \tag{2.11}$$

Let $\Phi_{1,1}$ be the linear functional on $S(H_n)$ defined by

$$\langle\Phi_{1,1}, f\rangle = \lim_{r\to 1^-}\sum_{k\ge 0}\frac{(-1)^n r^{2k+n}}{2k+n}\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\varepsilon|\lambda|}|\lambda|^{n-1}e^{-i\lambda t} \times\int_{-\infty}^{\infty} L_k^{n-1}\Big(\frac{|\lambda|}{2}|\tau|\Big)e^{-|\lambda|\cdot|\tau|/4}\operatorname{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt\,d\lambda. \tag{2.12}$$

We will prove below (Proposition 4.3) that $\langle\Phi_{1,1}, f\rangle$ makes sense, i.e. that the expression (2.12) converges. Assume provisionally this fact. From (2.5) it follows that $\Phi_{1,2} : S(H_n) \to \mathbb{C}$ defined by

$$\langle\Phi_{1,2}, f\rangle = 2\lim_{r\to 1^-}\lim_{\varepsilon\to 0^+}\sum_{k\ge 0}\frac{r^{2k+n}}{2k+n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\varepsilon|\lambda|}|\lambda|^{n-1}e^{-i\lambda t} \times\sum_{\substack{0\le l\le n-2\\ l\ \mathrm{odd}}}\Big(\frac{2}{|\lambda|}\Big)^{l+1}(a_{k,l}+b_{k,l})\frac{\partial^l Nf}{\partial\tau^l}(0,t)\,dt\,d\lambda \tag{2.13}$$

is also well defined and that $\Phi_1 = \Phi_{1,1} + \Phi_{1,2}$ and so $\Phi = \Phi_{1,1} + \Phi_{1,2} + \Phi_2$.

Note that the same arguments that prove that $N : S'(\mathbb{C}^n) \to \mathcal{H}$ is surjective, show that for $f \in S(H_n)$, $Nf(\tau, t)$ agrees with a certain function

$\varphi_1 \in S(\mathbb{R}^2)$ on $\{(\tau,t)\in\mathbb{R}^2 : \tau>0\}$ and with another function $\varphi_2 \in S(\mathbb{R}^2)$ on $\{(\tau,t)\in\mathbb{R}^2 : \tau<0\}$. So, for each polynomial $P(\tau,t)$ and for each pair of nonnegative integers $\alpha=(\alpha_1,\alpha_2)$ we have

$$\sup_{\tau\neq 0,\, t\in\mathbb{R}} |P(\tau,t)D^{\alpha}Nf(\tau,t)| < \infty. \tag{2.14}$$

In order to see that $\Phi_{1,1}$ is well defined, we first note that, for $k\in\mathbb{Z}$ and $\varepsilon>0$, (2.14) implies that the function

$$\operatorname{sgn}(\tau)e^{-\varepsilon|\lambda|-i\lambda t}|\lambda|^{n-1}L_k^{n-1}\left(\frac{|\lambda|}{2}|\tau|\right)e^{-|\lambda|\cdot|\tau|/4}G_f(\tau,t)$$

belongs to $L^1(\mathbb{R}^3, d\tau dt d\lambda)$. On the other hand, by (2.8),

$$\int_{-\infty}^{\infty} e^{-\varepsilon|\lambda|-i\lambda t}L_k^{n-1}\left(\frac{|\lambda|}{2}|\tau|\right)e^{-|\lambda|\cdot|\tau|/4}|\lambda|^{n-1}\,d\lambda$$
$$= 2\operatorname{Re}\int_0^{\infty} e^{-\varepsilon\lambda-i\lambda t}L_k^{n-1}\left(\frac{\lambda}{2}|\tau|\right)e^{-\lambda|\tau|/4}\lambda^{n-1}\,d\lambda$$

and as in (4.9) of [G-S], writing there τ instead of $B(z)$, we find that the value of this integral is

$$2\alpha_k\operatorname{Re}\left(\frac{(|\tau|-4\varepsilon-4it)^k}{(|\tau|+4\varepsilon+4it)^{k+n}}\right)$$

with

$$\alpha_k = 4^n(n-1)!(-1)^k\binom{k+n-1}{k}. \tag{2.15}$$

Thus, to prove that (2.12) converges, we must show the convergence of

$$\lim_{r\to 1^-}\sum_{k\geq 0}(-1)^n\frac{r^{2k+n}}{2k+n}\lim_{\varepsilon\to 0^+}2\alpha_k\int_{\mathbb{R}^2}\operatorname{Re}\left(\frac{(|\tau|-4\varepsilon-4it)^k}{(|\tau|+4\varepsilon+4it)^{k+n}}\right) \times\operatorname{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt. \tag{2.16}$$

We will need the following

Lemma 2.2. *The function*

$$\frac{1}{(\tau^2+16t^2)^{n/2}}G_f(\tau,t)$$

is integrable on $\mathbb{R}^2$.

Proof. Let $B=\{(\tau,t):\tau^2+16t^2<1\}$. From (2.14) and the familiar expression for the remainder of a Taylor development, it follows that there exists a positive constant c such that for $(\tau,t)\in B$,

$$\left|\frac{1}{(\tau^2+16t^2)^{n/2}}G_f(\tau,t)\right| \leq \frac{c}{(|\tau|^2+16t^2)^{n/2}}|\tau|^{n-1}$$

and a change to polar coordinates shows that $|\tau|^{n-1}(|\tau|^2+16t^2)^{-n/2}$ is integrable on B.

On the other hand, for $(\tau,t)\in\mathbb{R}^2-B$ we have

$$\left|\frac{1}{(\tau^2+16t^2)^{n/2}}G_f(\tau,t)\right| \le \left[|Nf(\tau,t)|+\sum_{j=1}^{n-2}\frac{|\tau|^j}{j!}\left|\frac{\partial^j Nf}{\partial\tau^j}(0,t)\right|\right]\frac{1}{(|\tau|^2+16t^2)^{n/2}}.$$

Let $A=\{(\tau,t):|t|\le 1/2\}\cap(\mathbb{R}^2-B)$. For $(\tau,t)\in A$, we have $|\tau|\ge 1/2$ and so, by (2.14), for $1\le j\le n-2$ we have

$$\int_A\left|\frac{\partial^j Nf}{\partial\tau^j}(0,t)\right|\frac{|\tau|^j}{(|\tau|^2+16t^2)^{n/2}}\,d\tau\,dt \le \int_{-1/2}^{1/2}\left|\frac{\partial^j Nf}{\partial\tau^j}(0,t)\right|dt\int_{1/2}^{\infty}\frac{1}{\tau^2}\,d\tau<\infty.$$

The same argument shows that the analogous integral with $\partial^j Nf(0,t)/\partial\tau^j$ replaced by $Nf(\tau,t)$ is finite. Now, let $A'=\{(\tau,t):|t|>1/2\}\cap(\mathbb{R}^2-B)$. For $(\tau,t)\in A'$ and $0\le j\le n-2$ we have

$$|\tau|^j(|\tau|^2+16t^2)^{-n/2}\le|\tau|^j(|\tau|^2+4)^{-n/2}$$

and by (2.14), $|G_f(\tau,t)|\le c(1+t^2)^{-1}$. Thus

$$\int_{A'}(|\tau|^2+16t^2)^{-n/2}|G_f(\tau,t)|\,d\tau\,dt<\infty. \ \blacksquare$$

Lemma 2.3.

$$\lim_{\varepsilon\to0^+}\int_{\mathbb{R}^2}\operatorname{Re}\left(\frac{(|\tau|-4\varepsilon-4it)^k}{(|\tau|+4\varepsilon+4it)^{k+n}}\right)\operatorname{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt = \int_{\mathbb{R}^2}\frac{1}{(\tau^2+16t^2)^{n/2}}\operatorname{Re}\left(\left(\frac{|\tau|-4it}{(\tau^2+16t^2)^{1/2}}\right)^{2k+n}\right)\operatorname{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt.$$

Proof. This follows from Lemma 2.2 and the Lebesgue dominated convergence theorem. ■

So, to see that $\langle\Phi_{1,1},f\rangle$ is well defined for $f\in S(H_n)$, we need to study the convergence of

$$\lim_{r\to1^-}\sum_{k\ge0}(-1)^n\frac{r^{2k+n}}{2k+n}\alpha_k\int_{\mathbb{R}^2}\frac{1}{(\tau^2+16t^2)^{n/2}} \times 2\operatorname{Re}\left(\left(\frac{|\tau|-4it}{(\tau^2+16t^2)^{1/2}}\right)^{2k+n}\right)\operatorname{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt$$

with α_k given by (2.15). We decompose this integral as the sum of integrals over the right and left halfplanes. Taking polar coordinates $\tau-4it=\varrho e^{i\theta}$, we can see that the above expression agrees with

$$2(-1)^n \lim_{r\to 1^-} \operatorname{Re}\left(\sum_{k\ge 0} \frac{r^{2k+n}}{2k+n}\alpha_k \int_0^\infty\int_0^{2\pi} e^{i(2k+n)\theta} K_f(\varrho,\theta)\, d\theta\, d\varrho\right) \tag{2.17}$$

where $K_f(\varrho,\theta)$ is the function defined for n odd by

$$K_f(\varrho,\theta) = \varrho^{1-n} G_f\left(\varrho\cos(\theta), -\frac{1}{4}\varrho\sin(\theta)\right) \tag{2.18}$$

and for n even by

$$K_f(\varrho,\theta) = \operatorname{sgn}(\cos(\theta))\varrho^{1-n} G_f\left(\varrho\cos(\theta), -\frac{1}{4}\varrho\sin(\theta)\right). \tag{2.19}$$

Note that, for $0 < r < 1$, from Lemma 2.2, it follows that

$$\sum_{k\ge 0} \frac{r^{2k+n}}{2k+n}|\alpha_k| \int_0^\infty\int_0^{2\pi} |K_f(\varrho,\theta)|\, d\varrho\, d\theta < \infty,$$

so (2.17) can be written as

$$2(-1)^n \lim_{r\to 1^-} \operatorname{Re} \int_0^\infty \left[\sum_{k\ge 0} \frac{r^{2k+n}}{2k+n}\alpha_k \int_0^{2\pi} e^{i(2k+n)\theta} K_f(\varrho,\theta)\, d\theta\right] d\varrho. \tag{2.20}$$

Recall that $\alpha_k = 4^n(n-1)!(-1)^k\binom{k+n-1}{k}$. So, in order to study (2.20), in the next section we will study the distributions defined, for $0 \le r < 1$ and $\theta \in \mathbb{R}$, by

$$\Psi_r(\theta) = \operatorname{Re}\left[\sum_{k\ge 0} \frac{r^{2k+n}}{2k+n}(-1)^k \binom{k+n-1}{k} e^{i(2k+n)\theta}\right]. \tag{2.21}$$

3. Computation of $\lim_{r\to 1^-}\Psi_r$**.** Let Ψ_r be defined by (2.21). Thus

$$\frac{d\Psi_r}{d\theta} = \operatorname{Re}\left[i\sum_{k\ge 0} r^{2k+n}(-1)^k \binom{k+n-1}{k} e^{i(2k+n)\theta}\right]. \tag{3.1}$$

Now, we take $t = 0$, $z = -r^2 e^{2i\theta}$ in the generating identity for the Laguerre polynomials

$$\sum_{k\ge 0} L_k^{n-1}(t) z^k = (1-z)^{-n} e^{-zt/(1-z)}. \tag{3.2}$$

Using the fact that $L_k^{n-1}(0) = \binom{k+n-1}{k}$ we obtain

$$\frac{d\Psi_r}{d\theta} = -\operatorname{Im}\left(\left(\frac{re^{i\theta}}{1+r^2e^{2i\theta}}\right)^n\right). \tag{3.3}$$

Note that the limit $\lim_{r\to 1^-} \Psi_r(0)$ exists. Indeed,

$$\Psi_r(0) = \sum_{k\geq 0} \frac{r^{2k+n}}{2k+n}(-1)^k \binom{k+n-1}{k} \quad \text{for } 0 \leq r < 1.$$

So, a computation shows that $d\Psi_r(0)/dr = r^{n-1}/(1+r^2)^n$ for $0 \leq r < 1$. Also, $\Psi_0(0) = 0$ and thus

$$\lim_{r\to 1^-} \Psi_r(0) = \int_0^1 \sigma^{n-1}(1+\sigma^2)^{-n}\, d\sigma. \tag{3.4}$$

We also have

PROPOSITION 3.1. *If* $0 < \delta < \pi/2$ *then there exists a positive constant* c *such that* $|\Psi_r(\theta) - \Psi_r(0)| \leq c\theta^2(1-r)$ *for* $-\delta < \theta < \delta$ *and* $0 < r < 1$.

Proof. From (3.3), we have

$$|\Psi_r(\theta) - \Psi_r(0)| \leq \int_0^\theta \left| \operatorname{Im}\left(\left(\frac{re^{i\sigma}}{1+r^2e^{2i\sigma}}\right)^n\right)\right| d\sigma. \tag{3.5}$$

Now,

$$\operatorname{Im}\left(\left(\frac{re^{i\sigma}}{1+r^2e^{2i\sigma}}\right)^n\right) = \frac{\operatorname{Im}(r^n e^{in\sigma}(1+r^2e^{-2i\sigma})^n)}{(1+2r^2\cos(2\sigma)+r^4)^n} \tag{3.6}$$

$$= r^n \frac{\sin(n\sigma)\sum_{j=0}^n \binom{n}{j} r^{2j}\cos(2j\sigma) - \cos(n\sigma)\sum_{j=1}^n \binom{n}{j} r^{2j}\sin(2j\sigma)}{(1+2r^2\cos(2\sigma)+r^4)^n}.$$

An induction on n shows that $\operatorname{Im}\left(\left(\frac{e^{i\sigma}}{1+e^{2i\sigma}}\right)^n\right) = 0$ for $\sigma \notin \pi/2 + \pi\mathbb{Z}$. Also, the denominator of (3.6) is bounded away from 0, and so the numerator is a polynomial expression in r that vanishes at $r = 1$ for all σ whose coefficients are trigonometrical polynomials divisible by $\sin(\sigma)$. So the numerator can be written as $(1-r)\sin(\sigma)Q(r, \cos(\sigma), \sin(\sigma)))$ where Q is a polynomial in its arguments. It follows that there exists a positive constant c such that

$$\left| \operatorname{Im}\left(\left(\frac{re^{i\sigma}}{1+r^2e^{2i\sigma}}\right)^n\right)\right| \leq c|\sin(\sigma)|(1-r)$$

for $0 \leq r < 1$ and $|\theta| < \delta$ and so (3.5) gives the lemma. ■

REMARK 3.2. If $0 < \delta < \pi/2$, the same argument shows that $|\Psi_r(\theta) - \Psi_r(\pi)| \leq \text{const}\,(\theta-\pi)^2(1-r)$ for $-\delta < \theta - \pi < \delta$ and $0 < r < 1$.

Let $\mathcal{V}$ be the vector space of functions $g \in C^{n-2}(S^1)$ satisfying $(\partial/\partial\theta)^{n-1}g \in L^\infty(S^1)$. For $h : S^1 \to \mathbb{C}$ and $g \in \mathcal{V}$ we write $\langle h, g\rangle$ for $\int_0^{2\pi} h(e^{i\theta})g(e^{i\theta})\, d\theta$ whenever the integral makes sense.

For $g \in \mathcal{V}$ let

$$(3.7)\qquad \langle \Psi, g\rangle = \operatorname{Re}\sum_{k\geq 0} \frac{1}{2k+n}(-1)^k \binom{k+n-1}{k} \int_{-\pi}^{\pi} e^{i(2k+n)\theta} g(e^{i\theta})\, d\theta.$$

Let $\widetilde{H} : S^1 \to \mathbb{R}$ be the function defined by $\widetilde{H}(e^{i\theta}) = H(\cos(\theta))$. Also, for $s \in \mathbb{R}$, let $\delta_s : \mathcal{V} \to \mathbb{R}$ be the distribution given by $\langle \delta_s, g\rangle = g(e^{is})$.

PROPOSITION 3.3. *For all $g \in \mathcal{V}$ the series* (3.7) *converges. Moreover*:

(1) *If n is even, then there exists a constant c_0 and a polynomial Q_{n-2} of degree $n-2$ such that for all $g \in \mathcal{V}$,*

$$\langle \Psi, g\rangle = -c_0\langle 1, g\rangle + \left\langle Q_{n-2}\left(\frac{\partial}{\partial\theta}\right)(\delta_{\pi/2} + \delta_{-\pi/2}), g\right\rangle.$$

(2) *If n is odd, then there exist two constants c_0 and d_0 and a polynomial Q_{n-2} of degree $n-2$ such that for all $g \in \mathcal{V}$,*

$$\langle \Psi, g\rangle = \langle -c_0 + d_0\widetilde{H}, g\rangle + \left\langle Q_{n-2}\left(\frac{\partial}{\partial\theta}\right)(\delta_{\pi/2} - \delta_{-\pi/2}), g\right\rangle.$$

Proof. For $g \in \mathcal{V}$, the convergence of (3.7) follows from the fact that $\frac{1}{2k+n}\binom{k+n-1}{k} = O(k^{n-2})$ and $|\langle g, e^{i\kappa\theta}\rangle| \leq |k|^{-(n-1)}|\langle g^{(n-1)}, e^{i\kappa\theta}\rangle|$ with $g^{(n-1)} \in L^2(S^1)$.

The familiar Fourier expansion for a function $g \in \mathcal{V}$ can be read as $\delta_s = \sum_{k\in\mathbb{Z}} e^{iks}\phi_k$, where $\phi_k(e^{i\theta}) = e^{ik\theta}$. Then a computation gives

$$\frac{1}{2}(\delta_{\pi/2} + \delta_{-\pi/2}) = \sum_{j\in\mathbb{Z}}(-1)^j e^{i2j\theta}, \qquad \frac{1}{2}(\delta_{\pi/2} - \delta_{-\pi/2}) = \sum_{j\in\mathbb{Z}}(-1)^j e^{i(2j+1)\theta}.$$

From (3.7) we have

$$(3.8)\qquad \Psi' = \operatorname{Re}\sum_{k\geq 0}(-1)^k \frac{(k+n-1)\ldots(k+1)}{(n-1)!}\, ie^{i(2k+n)\theta}.$$

Assume n is even. A change of the summation index in (3.8) gives

$$\Psi' = \operatorname{Re}\sum_{j\geq n/2}(-1)^{j+n/2}\frac{(2j-n+2)\ldots(2j+n-2)}{2^{n-1}(n-1)!}\, ie^{i2j\theta}.$$

On the other hand, the change $2k+n = -2j$ of the summation index in (3.7) also gives

$$\Psi' = \operatorname{Re}\sum_{j\leq n/2}(-1)^{j+n/2}\frac{(2j-n+2)\ldots(2j+n-2)}{2^{n-1}(n-1)!}\, ie^{i2j\theta},$$

and, since n is even, we also have $(2j+n-2)\dots(2j-n+2)=0$ for $-n/2<j<n/2$. Thus

$$\Psi' = \frac{1}{2}\operatorname{Re}\sum_{k\in\mathbb{Z}}(-1)^{k+n/2}\frac{(2k-n+2)\dots(2k+n-2)}{2^{n-1}(n-1)!}\, ie^{i2k\theta}.$$

Let P_{n-1} the polynomial of degree $n-1$ given by

$$P_{n-1}(s) = \frac{(2s-n+2)\dots(2s+n-2)}{2^n(n-1)!}, \qquad s\in\mathbb{R},$$

and let $\Theta = \sum_{k\in\mathbb{Z}}(-1)^{k+n/2}P_{n-1}(k)ie^{i2k\theta}$. Thus $\Psi' = \operatorname{Re}\Theta$. Since n is even, we have $P_{n-1}(0)=0$ and so $P_{n-1}(s)=s\widetilde{Q}_{n-2}(s)$ for some polynomial $\widetilde{Q}_{n-2}$ of degree $n-2$. Now, $k^l e^{i2k\theta} = \left(\frac{1}{2i}\frac{\partial}{\partial\theta}\right)^l e^{i2k\theta}$, $l\geq 0$, and so

$$\Theta = \frac{1}{2i}\frac{\partial}{\partial\theta}\left(i\sum_{k\in\mathbb{Z}}(-1)^{k+n/2}\widetilde{Q}_{n-2}\left(\frac{1}{2i}\frac{\partial}{\partial\theta}\right)e^{i2k\theta}\right).$$

Then, taking account of $\frac{1}{2}(\delta_{\pi/2}+\delta_{-\pi/2}) = \sum_{k\in\mathbb{Z}}(-1)^k e^{i2k\theta}$, we obtain

$$\langle\Psi', g\rangle = \operatorname{Re}\langle\Theta, g\rangle = \left\langle \frac{\partial}{\partial\theta}Q_{n-2}\left(\frac{\partial}{\partial\theta}\right)(\delta_{\pi/2}+\delta_{-\pi/2}), g\right\rangle$$

for some polynomial Q_{n-2} of degree $n-2$ with real coefficients. So

$$\Psi = -c_0 + Q_{n-2}\left(\frac{\partial}{\partial\theta}\right)(\delta_{\pi/2}+\delta_{-\pi/2})$$

for some constant c_0. This ends the proof for the case of n even.

Assume now that n is odd. By (3.7) we now have

$$\Psi' = \operatorname{Re}\sum_{2j+1\geq n}(-1)^{j+(n-1)/2}\frac{(2j+n-1)\dots(2j-n+3)}{2^{n-1}(n-1)!}\, ie^{i(2j+1)\theta}.$$

Thus we get, as above,

$$\Psi' = \frac{1}{2}\operatorname{Re}\sum_{k\in\mathbb{Z}}(-1)^{j+(n-1)/2}\frac{(2j+n-1)\dots(2j-n+3)}{2^{n-1}(n-1)!}\, ie^{i(2j+1)\theta}.$$

Let P_{n-1} be the polynomial given by

$$P_{n-1}(2s+1) = \frac{(2s+n-1)\dots(2s-n+3)}{2^n(n-1)!}, \qquad s\in\mathbb{R},$$

and let $\Theta = \sum_{k\in\mathbb{Z}}(-1)^{k+(n-1)/2}P_{n-1}(2k+1)ie^{i(2k+1)\theta}$. Thus $\Psi' = \operatorname{Re}\Theta$. Now

$$\Theta = \sum_{k\in\mathbb{Z}}(-1)^{k+(n-1)/2}P_{n-1}\left(\frac{1}{2i}\frac{\partial}{\partial\theta}\right)ie^{i2k\theta}.$$

But $\frac{1}{2}(\delta_{\pi/2}-\delta_{-\pi/2})=\sum_{k\in\mathbb{Z}}(-1)^k e^{i(2k+1)\theta}$ and hence

$$\langle \Psi', g\rangle = \mathrm{Re}\langle \Theta, g\rangle = \left\langle \widetilde{P}_{n-1}\left(\frac{\partial}{\partial\theta}\right)(\delta_{\pi/2}-\delta_{-\pi/2}), g\right\rangle$$

for some polynomial $\widetilde{P}_{n-1}$ of degree $n-1$ with real coefficients. Thus,

$$\langle \Psi', g\rangle = -d_0(\delta_{\pi/2}-\delta_{-\pi/2}) + \frac{\partial}{\partial\theta}Q_{n-2}\left(\frac{\partial}{\partial\theta}\right)(\delta_{\pi/2}-\delta_{-\pi/2})$$

for some polynomial Q_{n-2} of degree $n-2$ and some constant d_0. Now, $\delta_{\pi/2}-\delta_{-\pi/2} = -\frac{\partial}{\partial\theta}\widetilde{H}$, and so

$$\Psi' = \frac{\partial}{\partial\theta}Q_{n-2}\left(\frac{\partial}{\partial\theta}\right)(\delta_{\pi/2}-\delta_{-\pi/2}) + d_0\frac{\partial}{\partial\theta}\widetilde{H}.$$

Thus $\Psi = -c_0 + d_0\widetilde{H} + Q_{n-2}(\partial/\partial\theta)(\delta_{\pi/2}-\delta_{-\pi/2})$ for some constant c_0. ∎

Corollary 3.4. $\lim_{r\to 1^-}\langle \Psi_r, g\rangle = \langle \Psi, g\rangle$ *for all* $g\in\mathcal{V}$.

Proof. Follows from the Abel lemma and Proposition 3.3. ∎

Proposition 3.5. *If n is even and if c_0 is the constant of Proposition* 3.3 *then* $c_0 = -\lim_{r\to 1^-}\Psi_r(0)$.

If n is odd and if c_0, d_0 are the constants of Proposition 3.3 *then* $c_0 = \lim_{r\to 1^-}\Psi_r(0)$ *and* $d_0 = 2c_0$.

Proof. Assume that n is odd. Let $h\in C_c^\infty(\mathbb{R})$ be such that $\mathrm{supp}(h)\subset(-\pi/4,\pi/4)$, $h\ge 0$ and $\int h = 1$, and let Q_{n-2} be the polynomial given by Proposition 3.3. Thus $Q_{n-2}(\partial/\partial\theta)(\delta_{\pi/2}\pm\delta_{-\pi/2})(h)=0$ and so

$$\lim_{r\to 1^-}\langle \Psi_r, h\rangle = \langle -c_0 + d_0\widetilde{H}, h\rangle = -c_0 + d_0.$$

On the other hand, since $\int h = 1$ we have

$$\lim_{r\to 1^-}\langle \Psi_r, h\rangle = \lim_{r\to 1^-}\left[\int(\Psi_r(\theta)-\Psi_r(0))h(\theta)\,d\theta + \Psi_r(0)\right]$$

and by Proposition 3.1,

$$\left|\int(\Psi_r(\theta)-\Psi_r(0))h(\theta)\,d\theta\right| \le \left|\int_{|\theta|<1/2}|\Psi_r(\theta)-\Psi_r(0)|h(\theta)\,d\theta\right|$$
$$\le \mathrm{const}\,(1-r)\left|\int_{|\theta|<1/2}\theta^2 h(\theta)\,d\theta\right| \le \mathrm{const}\,(1-r)$$

and so $-c_0+d_0 = \lim_{r\to 1^-}\Psi_r(0)$. Similarly, taking $h\in C_c^\infty(\mathbb{R})$ such that $\mathrm{supp}(h)\subset(\pi/2,3\pi/2)$ with $h\ge 0$ and $\int h = 1$ we get $d_0 = \lim_{r\to 1^-}\Psi_r(\pi)$. Since $\Psi_r(\pi) = -\Psi_r(0)$ if n is odd, the lemma follows in this case. The proof for n even follows the same lines. ∎

4. The main result

REMARK 4.1. Let $K_f(\varrho,\theta)$ be defined by (2.18) and (2.19). A computation shows that $K_f(\varrho,\cdot)\in C^{n-2}(\mathbb{R})$ for each $\varrho\in(0,\infty)$. Moreover, for such a ϱ, $d^{n-1}K_f(\varrho,\theta)/d\theta^{n-1}$ is well defined for $\theta\notin\pi/2+\mathbb{Z}$,

$$\frac{d^j K_f(\varrho,\cdot)}{d\theta^j}\Big(\pm\frac{\pi}{2}\Big)=0\qquad\text{for } j=0,\dots,n-2$$

and $d^{n-1}K_f(\varrho,\theta)/d\theta^{n-1}\in L^\infty(\mathbb{R})$. We also need the following facts.

Let $A=[\delta,\pi-\delta]\cup[\pi+\delta,2\pi-\delta]$ with $0<\delta<\pi/2$ and let $\widetilde{K}_f:A\to\mathbb{R}$ be defined by $\widetilde{K}_f(\theta)=\int_1^\infty K_f(\varrho,\theta)\,d\varrho$. Then a computation using (2.14) and the Lebesgue dominated convergence theorem show that $\widetilde{K}_f\in C^{n-1}(A)$. Moreover $\widetilde{K}_f^{(j)}(\pm\pi/2)=0$ for $0\le j\le n-2$. The same argument gives that the function $\int_0^1 K_f(\varrho,\cdot)\,d\varrho$ belongs to $\mathcal{V}$ with its $n-2$ first derivatives vanishing at $\pm\pi/2$.

PROPOSITION 4.2. *Let α_k be given by* (2.15) *and let c_0 be as in Proposition* 3.3. *Then*

$$\lim_{r\to1^-}\operatorname{Re}\int_0^\infty\Big[\sum_{k\ge0}\frac{r^{2k+n}}{2k+n}\alpha_k\int_0^{2\pi}e^{i(2k+n)\theta}K_f(\varrho,\theta)\,d\theta\Big]\,d\varrho \tag{4.1}$$
$$=4^n(n-1)!c_0\int_{\mathbb{R}^2}\frac{1}{(\tau^2+16t^2)^{n/2}}\operatorname{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt.$$

Proof. Assume n is odd. Since

$$c_0\int_{\mathbb{R}^2}\frac{1}{(\tau^2+16t^2)^{n/2}}G_f(\tau,t)\,d\tau\,dt$$
$$=\int_{\mathbb{R}^2}\frac{1}{(\tau^2+16t^2)^{n/2}}(-c_0+d_0H(\tau))\operatorname{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt$$
$$=\int_0^\infty\int_0^{2\pi}(-c_0+d_0\widetilde{H}(\theta))K_f(\varrho,\theta)\,d\theta\,d\varrho$$

it is enough to prove that

$$\lim_{r\to1^-}\int_0^\infty\langle\Psi_r,K_f(\varrho,\cdot)\rangle d\varrho=\int_0^\infty\langle -c_0+d_0\widetilde{H},K_f(\varrho,\cdot)\rangle\,d\varrho. \tag{4.2}$$

In order to see this, we decompose the integral as $\int_0^1+\int_1^\infty$. To study the second term of this sum, we pick $0<\delta<\pi/4$. We first show that

$$\lim_{r\to1^-}\int_1^\infty\int_{|\theta|<\delta}(\Psi_r(\theta)-(-c_0+d_0\widetilde{H}(\theta)))K_f(\varrho,\theta)\,d\theta\,d\varrho=0. \tag{4.3}$$

To see this, we decompose the integral in (4.3) as

$$(4.4)\qquad \int_1^\infty \int_{|\theta|<\delta} (\Psi_r(\theta) - \Psi_r(0)) K_f(\varrho,\theta)\, d\theta\, d\varrho + (\Psi_r(0) - (-c_0 + d_0)) \int_1^\infty \int_{|\theta|<\delta} K_f(\varrho,\theta)\, d\theta\, d\varrho.$$

From the definition of $K_f(\varrho,\theta)$ and (2.14) we have $|K_f(\varrho,\theta)| \le c\varrho^{-2}|\sin(\theta)|^{-1}$ for some constant c and for all $\varrho > 1$, $|\theta| < \delta$. Then by Proposition 3.1,

$$(4.5)\qquad \Big| \int_1^\infty \int_{|\theta|<\delta} (\Psi_r(\theta) - \Psi_r(0)) K_f(\varrho,\theta)\, d\theta\, d\varrho \Big| \le \operatorname{const}(1-r) \int_1^\infty \int_{|\theta|<\delta} \frac{1}{\varrho^2 |\sin(\theta)|} \theta^2 d\theta \le \operatorname{const}(1-r).$$

It is also clear that the second term of the sum (4.4) converges to 0 as r tends to 1, so (4.3) holds. Replacing $\Psi_r(0)$ by $\Psi_r(\pi)$ and $-c_0 + d_0$ by d_0 in the above proof we also obtain the analogue of (4.3) where the integration domain $|\theta| < \delta$ is replaced by $|\theta - \pi| < \delta$.

On the other hand, we also have

$$(4.6)\qquad \lim_{r\to 1^-} \int_1^\infty \int_{(\delta,\pi-\delta)\cup(\pi+\delta,-\delta)} (\Psi_r(\theta) - (-c_0 + d_0 \widetilde{H}(\theta))) K_f(\varrho,\theta)\, d\theta\, d\varrho = 0.$$

Indeed, let A and $\widetilde{K}_f$ be as in Remark 4.1. Let K_f^* be an extension of $\widetilde{K}_f$ belonging to $\mathcal{V}$. Then (4.6) follows from the facts that $\lim_{r\to 1^-} \langle \Psi_r, K_f^* \rangle = \langle \Psi, K_f^* \rangle$ and that for some positive constant and all $\theta \in (-\delta,\delta) \cup (\pi-\delta, \pi+\delta)$ we have $|\Psi_r(\theta) - (-c_0 + d_0 \widetilde{H}(\theta))| \le \operatorname{const}(1-r)\delta^2$.

Finally, since $\int_0^1 K_f(\varrho,\cdot)\, d\varrho$ belongs to $\mathcal{V}$ and since for $0 \le r < 1$ the function $\Psi_r(\theta) K_f(\varrho,\theta)$ is integrable on $(0,2\pi)\times(0,1)$ we have

$$\lim_{r\to 1^-} \int_0^1 \langle \Psi_r, K_f(\varrho,\cdot)\rangle\, d\varrho = \lim_{r\to 1^-} \Big\langle \Psi_r, \int_0^1 K_f(\varrho,\cdot)\, d\varrho \Big\rangle = \int_0^{2\pi} (c_0 + d_0 \widetilde{H}(\theta)) \int_0^1 K_f(\varrho,\theta)\, d\varrho\, d\theta$$

and the lemma follows for n odd.

The proof for n even is similar with $-c_0 + d_0 \widetilde{H}(\theta)$ replaced by $-c_0$ in the above argument. ∎

PROPOSITION 4.3. *$\langle \Phi_{1,1}, f\rangle$ is well defined for $f \in S(H_n)$ and*

$$\langle \Phi_{1,1}, f\rangle = 4^n(n-1)!c_0 \int_{\mathbb{R}^2} \frac{1}{(\tau^2+16t^2)^{n/2}} \times \operatorname{sgn}(\tau)\Big(Nf(\tau,t) - \sum_{j=0}^{n-2} \frac{\tau^j}{j!}\cdot\frac{\partial^j Nf}{\partial\tau^j}(0,t)\Big)\,d\tau\,dt \tag{4.8}$$

where c_0 is the constant of Proposition 3.3.

Proof. The right member of (4.8) is finite by Lemma 2.3. On the other hand, the expression (2.12) that defines $\langle \Phi_{1,1}, f\rangle$ agrees, term to term, with (2.16) and so, by Lemma 2.3, also agrees with (2.17). Thus the corollary follows from Proposition 4.2. ∎

So $\langle \Phi_{1,2}, f\rangle$ is also well defined for $f \in S(H_n)$. Our next step will be to compute $\langle \Phi_{1,2}, f\rangle$.

For $\varepsilon > 0$, $l \in \mathbb{N}$ and $f \in S(H_n)$ we set

$$d_{\varepsilon,l,f} = \int_{\mathbb{R}^2} e^{-\varepsilon|\lambda|}|\lambda|^{n-2-l}e^{-i\lambda t}\frac{\partial^l Nf}{\partial\tau^l}(0,t)\,dt\,d\lambda. \tag{4.9}$$

So, (2.13) reads

$$\langle \Phi_{1,2}, f\rangle = \lim_{r\to1^-}\lim_{\varepsilon\to0^+}\sum_{k\ge0}\sum_{\substack{l\,\mathrm{odd}\\0\le l\le n-2}} 2^{l+2}\frac{r^{2k+n}}{2k+n}(a_{k,l}+b_{k,l})d_{\varepsilon,l,f} \tag{4.10}$$

with $a_{k,l}$ and $b_{k,l}$ defined by (2.10) and (2.9) respectively.

We will need the following

LEMMA 4.4. *Let $a_{k,l}$ be defined by* (2.10). *Then*

$$a_{k,l} = (-1)^k\sum_{j=1}^{l+1}\frac{1}{2^{n-l-j-1}}\binom{n-j-1}{l-j+1}\binom{j+k-1}{k} + \sum_{j=1}^{n-l-1}\frac{1}{2^{n-l-j-1}}\binom{n-j-1}{l}\binom{j+k-1}{k}.$$

Proof. From the generating identity (3.2) for the Laguerre polynomials we can write, for $0 \le r < 1$,

$$\sum_{k=0}^{\infty} r^k\frac{1}{l!}\int_0^\infty L_k^{n-1}(\tau)e^{-\tau/2}\tau^l\,d\tau = \frac{1}{l!}\cdot\frac{1}{(1-r)^n}\int_0^\infty \tau^l e^{-(r+1)\tau/(2(1-r))}d\tau = \frac{2^{l+1}}{(1-r)^{n-l-1}(1+r)^{l+1}}. \tag{4.11}$$

So $a_{k,l}$ is the coefficient of r^k in the power expansion at the origin of (4.11). We decompose (4.11) in simple fractions

$$\frac{2^{l+1}}{(1-r)^{n-l-1}(1+r)^{l+1}} = \sum_{j=1}^{n-l-1} \frac{A_j}{(1-r)^j} + \sum_{j=1}^{l+1} \frac{B_j}{(1+r)^j}.$$

Thus, by the residue theorem, we have, for a small positive $\varrho > 0$,

$$A_j = -\frac{2^{l+1}}{2\pi i} \int_{|z-1|=\varrho} \frac{1}{(1-z)^{n-l-j}(1+z)^{l+1}}\, dz.$$

So

$$A_j = -\frac{2^{l+1}(-1)^{n-l-j}}{(n-l-j-1)!}\left(\frac{d}{dz}\right)^{n-l-j-1}_{|z=1}((1+z)^{-l-1}) = \frac{1}{2^{n-l-j-1}}\binom{n-j-1}{l}.$$

Similarly,

$$\begin{aligned} B_j &= \frac{2^{l+1}}{2\pi i} \int_{|z+1|=\varrho} \frac{1}{(1-z)^{n-l-1}(1+z)^{l-j+2}}\, dz \\ &= \frac{2^{l+1}}{(l+1-j)!}\left(\frac{d}{dz}\right)^{l+1-j}_{|z=-1}((1-z)^{l+1-n}) = \frac{1}{2^{n-l-j-1}}\binom{n-j-1}{n-l-2}. \end{aligned}$$

Since, for $m \in \mathbb{N} \cup \{0\}$, $\binom{m+k-1}{k}$ is the coefficient of r^k in the power expansion at the origin of $(1-r)^{-m}$, the lemma follows. ∎

For $T \in S'(\mathbb{R})$, we write $\widehat{T}$ for its Fourier transform and, as before, for $g : \mathbb{R} \to \mathbb{C}$ we set $g^{\vee}(t) = g(-t)$.

LEMMA 4.5. *Assume* $0 \le l \le n-2$. *For* $\varepsilon > 0$ *and* $f \in S(H_n)$ *let* $d_{\varepsilon,l,f}$ *be defined by* (4.9). *Then, for* $n-l$ *odd,*

$$\lim_{\varepsilon\to 0^+} d_{\varepsilon,l,f} = 2(-1)^{(n-l-1)/2}\left\langle \text{p.v.}\left(\frac{1}{t}\right), \frac{\partial^{n-2}Nf}{\partial\tau^l\partial\lambda^{n-l-2}}(0,\cdot)\right\rangle,$$

and for $n-l$ *even,*

$$\lim_{\varepsilon\to 0^+} d_{\varepsilon,l,f} = 2(-1)^{(n-l-2)/2}\frac{\partial^{n-2}Nf}{\partial\tau^l\partial\lambda^{n-l-2}}(0,0).$$

Proof. Let $g(\lambda) = e^{-\varepsilon|\lambda|}|\lambda|^{n-2-l}$. Since

$$d_{\varepsilon,l,f} = \int_{\mathbb{R}} e^{-\varepsilon|\lambda|}|\lambda|^{n-2-l}\left(\frac{\partial^l Nf}{\partial\tau^l}(0,\cdot)\right)^{\wedge}(\lambda)\, d\lambda,$$

a computation of $\widehat{g}$ and the properties of the Fourier transform give the assertion. ∎

REMARK 4.6. For $j \in \mathbb{N}$ and $0 \le r < 1$, let

$$h_j(r) = \sum_{k \ge 0} \frac{r^{2k+n}}{2k+n} (-1)^k \binom{j+k-1}{k}.$$

Then $\lim_{r\to 1^-} h_j(r) = \int_0^1 r^{j-1}(1+r^2)^{-j}\, dr$. Indeed, $h_j(0) = 0$ and proceeding as at the beginning of Section 3, we now get $h_j'(r) = r^{j-1}/(1+r^2)^j$.

REMARK 4.7. Note that if P is a nonzero polynomial then

$$\lim_{r\to 1^-} \sum_{k\ge 0} P(k) \frac{r^{2k+n}}{2k+n} = \pm\infty.$$

Indeed, without loss of generality we can assume that the leading term of P has a positive coefficient. Then, for k large enough, $P(k)$ is greater than a positive constant and the assertion follows.

For $1 \le j \le n-1$, we set

$$c_j = \int_0^1 \frac{r^{j-1}}{(1+r^2)^j}\, dr. \tag{4.12}$$

PROPOSITION 4.8. *For n even we have*

$$\langle \Phi_{1,2}, f\rangle = \sum_{\substack{1\le l\le n-2\\ l\ odd}} 2^{2l+3-n}(-1)^{(n-l+1)/2} \times \sum_{j=1}^{l+1} 2^j c_j \binom{n-j-1}{l-j+1} \left\langle \mathrm{p.v.}\left(\frac{1}{t}\right), \frac{\partial^{n-2} Nf}{\partial\tau^l \partial\lambda^{n-l-2}}(0,\cdot)\right\rangle \tag{4.13}$$

and for n odd we have

$$\langle \Phi_{1,2}, f\rangle = \sum_{\substack{1\le l\le n-2\\ l\ odd}} 2^{2l+3-n}(-1)^{(n-l+2)/2} \times \sum_{j=1}^{l+1} 2^j c_j \binom{n-j-1}{l-j+1} \frac{\partial^{n-2} Nf}{\partial\tau^l \partial\lambda^{n-l-2}}(0,0). \tag{4.14}$$

Proof. Let $a_{k,l}$ and $b_{k,l}$ be defined by (2.10) and (2.9) respectively. We write $a_{k,l} = a'_{k,l} + a''_{k,l}$, with

$$a'_{k,l} = (-1)^k \sum_{j=1}^{l+1} \frac{1}{2^{n-l-j-1}} \binom{n-j-1}{l-j+1} \binom{j+k-1}{k},$$

$$a''_{k,l} = \sum_{j=1}^{n-l-1} \frac{1}{2^{n-l-j-1}} \binom{n-j-1}{l} \binom{j+k-1}{k}.$$

For $\varepsilon > 0$, $0 \le l \le n-2$ and $f \in S(H_n)$ let $d_{\varepsilon,l,f}$ be defined by (4.9). We set

$$\langle \Phi^*_{1,2}, f\rangle = \lim_{r\to 1^-}\lim_{\varepsilon\to 0^+}\sum_{k\ge 0}\sum_{\substack{0\le l\le n-2\\ l\,\mathrm{odd}}}\frac{r^{2k+n}}{2k+n}\,2^{l+2}d_{\varepsilon,l,f}a'_{k,l}.$$

Note that, by Lemma 4.5 and Remark 4.6, $\langle \Phi^*_{1,2}, f\rangle$ is well defined, and moreover

$$\langle \Phi^*_{1,2}, f\rangle = \sum_{\substack{0\le l\le n-2\\ l\,\mathrm{odd}}} 2^{l+2}\lim_{\varepsilon\to 0^+} d_{\varepsilon,l,f}\sum_{j=1}^{l+1}\frac{1}{2^{n-l-j-1}}\binom{n-j-1}{l-j+1}c_j. \tag{4.15}$$

So, the functional $\Phi^{**}_{1,2}$ defined by

$$\langle \Phi^{**}_{1,2}, f\rangle = \lim_{r\to 1^-}\lim_{\varepsilon\to 0^+}\sum_{k\ge 0}\sum_{\substack{0\le l\le n-2\\ l\,\mathrm{odd}}}2^{l+2}\frac{r^{2k+n}}{2k+n}\,d_{\varepsilon,l,f}(a''_{k,l}+b_{k,l})$$

is also well defined. Now, given $\varphi \in S(\mathbb{R})$, it is easy to see that there exists $f \in S(H_n)$ such that $\partial^m Nf(0,t)/\partial\tau^m = \delta_{l,m}\varphi(t)$ for $m = 0,\ldots,n-2$, $t \in \mathbb{R}$. Indeed, pick $g \in S(\mathbb{R})$ such that $g^{(j)}(0) = \delta_{j,l}$ for $j = 0,1,\ldots,n-2$. Since $N : S(\mathbb{C}^n) \to \mathcal{H}$ is surjective, there exists $\widetilde{g} \in S(\mathbb{C}^n)$ such that $N\widetilde{g} = g$. Now, $f(z,t) = \widetilde{g}(z)\varphi(t)$ has the desired property. So for each l odd with $0 \le l \le n-2$ the limit

$$\lim_{r\to 1^-}\sum_{k\ge 0}\frac{r^{2k+n}}{2k+n}(a''_{k,l}+b_{k,l})$$

exists and is finite. But $a''_{k,l}+b_{k,l}$ is a polynomial in k, so Remark 4.7 implies that $a''_{k,l}+b_{k,l}$ vanishes identically at k and so $\langle \Phi^{**}_{1,2}, f\rangle = 0$. Now, (4.15) and Lemma 4.5 give the assertion. ∎

Remark 4.9. Let Φ_2 be as in (2.5). Then

$$\langle \Phi_2, f\rangle = \sum_{l=0}^{n-2}\sum_{\substack{1\le k\le n-1\\ k\ne n/2}}\frac{1}{n-2k} \times \sum_{\max(n-k-1,l)\le j\le n-2}\binom{j}{l}\binom{k-1}{n-j-2}2^{2l+3-n-j}\langle T_l, f\rangle \tag{4.16}$$

where

$$\langle T_l, f\rangle = (-1)^{(n-l-1+2j)/2}\left\langle \mathrm{p.v.}\left(\frac{1}{t}\right), \frac{\partial^{n-2}Nf}{\partial\tau^l\partial t^{n-l-2}}(0,\cdot)\right\rangle$$

if $n-l$ is odd, and

$$\langle T_l, f\rangle = (-1)^{(n-l+2j)/2} \frac{\partial^{n-2} Nf}{\partial\tau^l \partial t^{n-l-2}}(0,0)$$

if $n-l$ is even.

Indeed, to see (4.16), we recall that $S_{\lambda,k} = F_{\lambda,k} \otimes e^{-i\lambda t}$. Then a change of indices in (2.3) gives

$$\langle \Phi_2, f\rangle = \sum_{l=0}^{n-2} \sum_{\substack{1\le k\le n-1\\ k\ne n/2}} \frac{1}{n-2k} \times \sum_{\max(n-k-1,l)\le j\le n-2} (-1)^{n-j+l} \binom{j}{l} \binom{k-1}{n-j-2} 2^{2l+2-n-j} d_{\varepsilon,l,f}$$

so the assertion follows from Lemma 4.5. ∎

Now, we can state

THEOREM 4.10. *There exist constants* $c_0, e_0, \ldots, e_{n-2}, e'_0, \ldots, e'_{n-2}$ *such that*, *for* $f \in S(H_n)$,

$$\langle \Phi, f\rangle = 4^n(n-1)!c_0 \int \frac{\operatorname{sgn}(\tau)}{(\tau^2+16t^2)^{n/2}} \Big(Nf(\tau,t) - \sum_{l=0}^{n-2} \frac{\tau^l}{l!} \cdot \frac{\partial^l Nf}{\partial \tau^l}(0,t)\Big)\, d\tau\, dt + \sum_{l=0}^{n-2} e_l \Big\langle \text{p.v.}\Big(\frac{1}{t}\Big), \frac{\partial^{n-2} Nf}{\partial\tau^l \partial t^{n-2-l}}(0,\cdot)\Big\rangle + \sum_{l=0}^{n-2} e'_l \frac{\partial^{n-2} Nf}{\partial\tau^l \partial t^{n-2-l}}(0,0).$$

Proof. Since $\Phi = \Phi_{1,1} + \Phi_{1,2} + \Phi_2$, the theorem follows from Propositions 4.3, 4.8 and 4.9. ∎

Finally, we remark that the constants $c_0, e_0, \ldots, e_{n-2}, e'_0, \ldots, e'_{n-2}$ in the above theorem can be explicitly computed; in fact, the value of c_0 is found from Proposition 4.3 and (3.4) and the values of $e_0, \ldots, e_{n-2}, e'_0, \ldots, e'_{n-2}$ are found from (4.13), (4.14) and (4.16).

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Received March 27, 2000
Revised version September 28, 2000 (4501)