STUDIA MATHEMATICA 145 (2) (2001)

General Haar systems and greedy approximation

by

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Abstract. We show that each general Haar system is permutatively equivalent in $L^p([0,1])$, $1 < p < \infty$, to a subsequence of the classical (i.e. dyadic) Haar system. As a consequence, each general Haar system is a greedy basis in $L^p([0,1])$, $1 < p < \infty$. In addition, we give an example of a general Haar system whose tensor products are greedy bases in each $L^p([0,1]^d)$, $1 < p < \infty$, $d \in \mathbb{N}$. This is in contrast to [11], where it has been shown that the tensor products of the dyadic Haar system are not greedy bases in $L^p([0,1]^d)$ for $1 < p < \infty$, $p \neq 2$ and $d \geq 2$. We also note that the above-mentioned general Haar system is not permutatively equivalent to the whole dyadic Haar system in any $L^p([0,1])$, $1 < p < \infty$, $p \neq 2$.

1. Introduction. By a *general Haar system* corresponding to a dense sequence $\mathcal{T} = \{t_n : n \geq 0\} \subset [0,1]$, we mean a sequence of orthonormal (in $L^2([0,1])$) functions which are constant on intervals generated by the points of the sequence $\mathcal{T}$—it is constructed analogously to the classical dyadic Haar system, but with the sequence $\mathcal{T}$ used instead of the sequence of dyadic points. (For a more detailed description of general Haar functions, see Section 2.2.)

It has been shown by L. E. Dor and E. Odell [3] (cf. also the monograph [9]) that there are pairs of general Haar systems which are not equivalent in any $L^p([0,1])$, $1 < p < \infty$, $p \neq 2$. They have also asked whether there exist general Haar systems which are not permutatively equivalent in these spaces. (Recall that two basic sequences in a Banach space are called *permutatively equivalent* if one of them is equivalent to some permutation of the other.) In the present paper, we prove that each general Haar system is permutatively equivalent in $L^p([0,1])$, $1 < p < \infty$, to some subsequence of the dyadic Haar system (Theorem 3.2). We also give an example of a general Haar system

2000 *Mathematics Subject Classification*: 41A30, 41A46, 41A63, 46E30.

A part of this work was done when A. Kamont was visiting the Technical University of Luleå and the University of Umeå in November 1999. A. Kamont's stay in Sweden was supported by NFR grant Ö-AH/KG 08685-315.

which is not permutatively equivalent to the whole dyadic Haar system (Theorem 4.6).

The second topic of the paper concerns general Haar systems and their tensor products as greedy bases in $L^p([0,1]^d)$, $1 < p < \infty$, $d \in \mathbb{N}$. Let us recall the concepts of greedy approximation and greedy basis (cf. e.g. [7]).

Let $(X, \|\cdot\|)$ be a real Banach space with a normalized basis $\mathcal{X} = \{x_n : n \in \mathbb{N}\}$ (i.e. $\|x_n\| = 1$). For $x \in X$ with $x = \sum_{n=1}^{\infty} a_n x_n$ and $m \in \mathbb{N}$, consider a subset $G(m,x) \subset \mathbb{N}$ of cardinality m such that

$$\min_{n \in G(m,x)} |a_n| \geq \max_{n \in \mathbb{N} \setminus G(m,x)} |a_n|.$$

(There is some ambiguity in the choice of the set $G(m,x)$, but our considerations do not depend on the particular choice.) Then the mth *greedy approximation of x with respect to the basis $\mathcal{X}$* is defined as

$$\mathcal{G}_m(x, \mathcal{X}) = \sum_{n \in G(m,x)} a_n x_n.$$

In addition, consider the mth *best approximation of x with respect to $\mathcal{X}$*:

$$\sigma_m(x, \mathcal{X}) = \inf_{G \subset \mathbb{N},\, \#G = m} \inf_{c_n \in \mathbb{R}} \Big\| x - \sum_{n \in G} c_n x_n \Big\|.$$

Clearly, $\sigma_m(x, \mathcal{X}) \leq \|x - \mathcal{G}_m(x, \mathcal{X})\|$. Now, the basis $\mathcal{X}$ is called *greedy* if there is a constant $C > 0$, independent of m, such that for each $m \in \mathbb{N}$ and $x \in X$,

$$\|x - \mathcal{G}_m(x, \mathcal{X})\| \leq C \sigma_m(x, \mathcal{X}). \tag{1.1}$$

It has been shown by V. N. Temlyakov [10] that the dyadic Haar system (normalized in $L^p([0,1])$), or any basis equivalent to it in $L^p([0,1])$, is a greedy basis in $L^p([0,1])$, $1 < p < \infty$. S. V. Konyagin and V. N. Temlyakov [7] have given a characterization of greedy bases in Banach spaces, recalled in Section 2.1 below (see also P. Wojtaszczyk [13] for more general results in this direction). As a consequence of the results of [10], [7] and our Theorem 3.2, we deduce that each general Haar system (normalized in $L^p([0,1])$) is a greedy basis in $L^p([0,1])$, $1 < p < \infty$ (Corollary 4.1).

In the d-variate case, one can consider either a "localized" dyadic Haar system (obtained by dyadic scaling and translations of some $2^d - 1$ d-variate step functions given on $[0,1]^d$), or the tensor products of the univariate dyadic Haar system. The "localized" d-variate Haar system is also a greedy basis in $L^p([0,1]^d)$, $1 < p < \infty$, $d \geq 2$ (cf. V. N. Temlyakov [10]). However, concerning the tensor product case, the example presented by V. N. Temlyakov [11] shows that for $p \neq 2$ the bases consisting of the tensor products of the dyadic Haar system are not greedy in $L^p([0,1]^d)$ in any dimension $d \geq 2$. In contrast to this result, in Section 4.3 we give an example of a sequence of points such that the tensor products of the corresponding general Haar system are greedy in $L^p([0,1]^d)$ for each $d \in \mathbb{N}$ and $1 < p < \infty$.

To simplify the notation, by C, C_p etc. we denote constants whose value may be different at each occurrence; $a \sim b$ means that there are constants $c_1, c_2 > 0$ such that $c_1 a \le b \le c_2 a$. For a set A, $\#A$ is the cardinality of A, $|A|$ the Lebesgue measure of A, and χ_A the characteristic function of A.

2. Preliminaries

2.1. *Characterization of greedy bases in Banach spaces.* In what follows, we need the characterization of greedy bases given in [7].

Following [7], we call a normalized basis $\mathcal{X}$ of a Banach space $(X, \|\cdot\|)$ *democratic* if there is a constant $C > 0$ such that for each $m \in \mathbb{N}$ and $P, Q \subset \mathbb{N}$ with $\#P = m = \#Q$,

$$\frac{1}{C}\Big\|\sum_{n\in P} x_n\Big\| \le \Big\|\sum_{n\in Q} x_n\Big\| \le C\Big\|\sum_{n\in P} x_n\Big\|. \tag{2.1}$$

For completeness, let us also recall the definition of unconditional basis: a basis $\mathcal{X}$ of a Banach space $(X, \|\cdot\|)$ is called *unconditional* if there is a constant $C > 0$ such that for each $x \in X$ with $x = \sum_{n=1}^{\infty} a_n x_n$ and for each sequence $\varepsilon = \{\varepsilon_n : n \in \mathbb{N}\}$ with $\varepsilon_n \in \{-1, 1\}$,

$$\Big\|\sum_{n=1}^{\infty} \varepsilon_n a_n x_n\Big\| \le C\Big\|\sum_{n=1}^{\infty} a_n x_n\Big\|. \tag{2.2}$$

Now, the characterization of greedy bases from [7] is as follows:

A normalized basis $\mathcal{X}$ of a Banach space $(X, \|\cdot\|)$ is greedy if and only if it is democratic and unconditional.

Moreover, if a basis $\mathcal{X}$ is unconditional and democratic, then the constant in (1.1) depends only on the constants in (2.1) and (2.2).

2.2. *Definition and basic properties of general Haar systems.* In the following, we consider subintervals $I \subset [0,1]$ of the type $I = [a, b)$ with $0 \le a < b < 1$ or $I = [a, 1]$.

Let $I \subset [0,1]$ be an interval and let I', I'' be subintervals of I such that

$$I = I' \cup I'' \quad \text{and} \quad I' \cap I'' = \emptyset. \tag{2.3}$$

Without loss of generality we may assume that

$$\delta_I = |I'| \le |I''| = \varDelta_I. \tag{2.4}$$

Let $h_I : [0,1] \to \mathbb{R}$ be the unique function such that $\operatorname{supp} h_I = \bar{I}$, $\int_0^1 h_I = 0$, $\int_0^1 h_I^2 = 1$, h_I is constant on I', I'' and $h_I > 0$ on I'. Then

$$h_I = \alpha_I \chi_{I'} - \beta_I \chi_{I''}, \quad \text{where}$$
$$\alpha_I = \sqrt{\frac{\Delta_I}{\delta_I}} \cdot \frac{1}{\sqrt{\delta_I + \Delta_I}}, \quad \beta_I = \sqrt{\frac{\delta_I}{\Delta_I}} \cdot \frac{1}{\sqrt{\delta_I + \Delta_I}}. \tag{2.5}$$

We call h_I the *Haar function corresponding to* $I = I' \cup I''$; clearly, h_I depends on the decomposition $I = I' \cup I''$, but for simplicity it is suppressed in the notation.

Observe that
$$\|h_I\|_\infty = \alpha_I, \quad \text{so} \quad 2^{-1/2}\delta_I^{-1/2} \le \|h_I\|_\infty \le \delta_I^{-1/2},$$
and for $0 < p < \infty$,
$$\|h_I\|_p = \frac{\sqrt{\delta_I \Delta_I}}{\sqrt{\delta_I + \Delta_I}} \big(\delta_I^{1-p} + \Delta_I^{1-p}\big)^{1/p}. \tag{2.6}$$
This implies that
$$2^{-1/2}\delta_I^{1/p-1/2} \le \|h_I\|_p \le 2\delta_I^{1/p-1/2} \quad \text{for } 1 \le p \le \infty. \tag{2.7}$$
We will also need general Haar functions normalized in $L^p([0,1])$:
$$h_{I,p} = \frac{h_I}{\|h_I\|_p} = \alpha_{I,p}\chi_{I'} - \beta_{I,p}\chi_{I''}, \quad 1 \le p \le \infty. \tag{2.8}$$

Now, let $\mathcal{T} = \{t_n : n \ge 0\}$ be a sequence of distinct points from $[0,1]$, dense in $[0,1]$, with $t_0 = 0$, $t_1 = 1$. For $n \ge 1$, let $\mathcal{T}_n = \{t_0, \ldots, t_n\} = \{0 = s_0^{(n)} < s_1^{(n)} < \ldots < s_n^{(n)} = 1\}$. For $n \ge 2$, let I_n be the interval $[s_{i-1}^{(n-1)}, s_i^{n-1)})$, $1 \le i \le n-1$, such that $t_n \in I_n$ (clearly, I_n is unique). Let I_n', I_n'' be intervals obtained by dividing I_n by the point t_n. The *Haar system corresponding to* $\mathcal{T}$ is defined as follows:
$$h_1 = 1, \quad h_n = h_{I_n} \quad \text{with the partition} \quad I_n = I_n' \cup I_n'' \quad \text{for } n \ge 2. \tag{2.9}$$
The Haar system corresponding to $\mathcal{T}$ is denoted by $\mathcal{H}_\mathcal{T}$. For convenience, below we write $\delta_n = \delta_{I_n}$, $\Delta_n = \Delta_{I_n}$ etc.

The classical Haar system (i.e. corresponding to the sequence of dyadic points $\mathcal{D} = \{0, 1, \frac12, \frac14, \frac34, \frac18, \frac38, \frac58, \ldots\}$) is denoted by $H_n, n \ge 1$.

For each $\mathcal{T}$ (dense in $[0,1]$), the corresponding Haar system $\mathcal{H}_\mathcal{T}$ is a complete orthonormal system in $L^2([0,1])$, and it is a monotone basis (i.e. with basis constant 1) in each $L^p([0,1])$, $1 \le p < \infty$. Since $\mathcal{H}_\mathcal{T} = \{h_n : n \ge 1\}$ is a sequence of martingale differences (with the Lebesgue measure on $[0,1]$ as probability measure, and the sequence of σ-fields generated by $\mathcal{H}_\mathcal{T}$), it follows from D. L. Burkholder's results on unconditionality of martingale differences (cf. [1], [2]) that every general Haar system is an unconditional basis in $L^p([0,1])$, $1 < p < \infty$, and
$$\Big\| \sum_{n=1}^\infty \varepsilon_n a_n h_n \Big\|_p \le (p^* - 1) \Big\| \sum_{n=1}^\infty a_n h_n \Big\|_p, \tag{2.10}$$

where $p^* = \max(p, p/(p-1))$, $\varepsilon_n = \pm 1$ and $\{a_n : n \geq 1\}$ are any real coefficients. This fact and Khinchin's inequality imply that for each p with $1 < p < \infty$, there are finite constants $C_p, c_p > 0$ such that for each $\mathcal{T}$ and real coefficients $\{a_n : n \geq 1\}$,

$$c_p \Big\| \Big(\sum_{n=1}^{\infty} a_n^2 h_n^2 \Big)^{1/2} \Big\|_p \leq \Big\| \sum_{n=1}^{\infty} a_n h_n \Big\|_p \leq C_p \Big\| \Big(\sum_{n=1}^{\infty} a_n^2 h_n^2 \Big)^{1/2} \Big\|_p. \tag{2.11}$$

We will need the maximal inequality of C. Fefferman and E. Stein (cf. e.g. Theorem 1 in Chapter II of [12]). Let Mf denote the Hardy–Littlewood maximal function of f. Then for each p with $1 < p < \infty$, there is a constant C_p such that for any sequence $g_n \in L^p([0,1])$,

$$\Big\| \Big(\sum_{n=1}^{\infty} (M g_n)^2 \Big)^{1/2} \Big\|_p \leq C_p \Big\| \Big(\sum_{n=1}^{\infty} g_n^2 \Big)^{1/2} \Big\|_p. \tag{2.12}$$

Now, we state some properties of general Haar functions which are needed later on.

Proposition 2.1. *Let $I \subset [0,1]$ be an interval, and I', I'' subintervals of I satisfying* (2.3), (2.4). *Let α_I, β_I be as in* (2.5). *Then*

$$\beta_I \chi_{I''} \leq 2\alpha_I \cdot M\chi_{I'}. \tag{2.13}$$

Proof. Let $x \in I''$. Then

$$M\chi_{I'}(x) \geq \frac{1}{|I|} \int_I \chi_{I'}(u)\, du = \frac{\delta_I}{\delta_I + \Delta_I},$$

so by the definition of α_I, β_I (cf. (2.5)),

$$2\alpha_I \cdot M\chi_{I'}(x) \geq \frac{2\Delta_I}{\delta_I + \Delta_I} \sqrt{\frac{\delta_I}{\Delta_I}} \cdot \frac{1}{\sqrt{\delta_I + \Delta_I}} \geq \beta_I,$$

which implies (2.13). ∎

As $h_I^2 = \alpha_I^2 \chi_{I'} + \beta_I^2 \chi_{I''}$ (cf. (2.5)), Proposition 2.1 combined with (2.12) implies

Proposition 2.2. *Let $1 < p < \infty$. There exists a constant $c_p > 0$ such that for each sequence $\mathcal{T}$ of points, the corresponding Haar system $\mathcal{H}_{\mathcal{T}} = \{h_n : n \geq 1\}$ and real coefficients $\{a_n : n \geq 1\}$,*

$$c_p \Big\| \Big(\sum_{n=1}^{\infty} a_n^2 h_n^2 \Big)^{1/2} \Big\|_p \leq \Big\| \Big(\sum_{n=1}^{\infty} a_n^2 \alpha_n^2 \chi_{I_n'} \Big)^{1/2} \Big\|_p \leq \Big\| \Big(\sum_{n=1}^{\infty} a_n^2 h_n^2 \Big)^{1/2} \Big\|_p. \tag{2.14}$$

3. General Haar systems and subsequences of the dyadic Haar system. Let $I \subset [0,1]$ be an interval, $I = I' \cup I''$, with subintervals I', I'' as in (2.3), (2.4), and let h_I be as in (2.5). We associate with h_I some function $H_{\pi(I)}$ from the dyadic Haar system. $H_{\pi(I)}$ depends only on h_I, or more precisely, on the partition $I = I' \cup I''$ as in (2.3), (2.4). Thus, for each general Haar system $\mathcal{H}_{\mathcal{T}}$, we obtain a subsequence (and a permutation) of the dyadic Haar system. We are going to show the equivalence of $\mathcal{H}_{\mathcal{T}}$ and this permutation of the subsequence of the dyadic Haar system in each $L^p([0,1])$, $1<p<\infty$.

Let $\nu_I \geq 1$ be the integer such that $1/2^{\nu_I+1} < |I'| \leq 1/2^{\nu_I}$. Then, for some integer η_I with $1 \leq \eta_I \leq 2^{\nu_I+2}$, we have $[(\eta_I-1)/2^{\nu_I+2}, \eta_I/2^{\nu_I+2}) \subset I'$ (in case there are two such η's, we take the smaller one). Now, we associate with h_I the dyadic Haar function with support $[(\eta_I-1)/2^{\nu_I+2}, \eta_I/2^{\nu_I+2}]$, i.e. we take $\pi(I) \in \mathbb{N}$ such that $\operatorname{supp} H_{\pi(I)} = [(\eta_I-1)/2^{\nu_I+2}, \eta_I/2^{\nu_I+2}]$.

Note that

$$\frac{1}{4}\delta_I \leq |\operatorname{supp} H_{\pi(I)}| < \frac{1}{2}\delta_I, \quad \operatorname{supp} H_{\pi(I)} \subset I', \tag{3.1}$$

$$2^{1/2}\delta_I^{-1/2} < |H_{\pi(I)}| \leq 2\delta_I^{-1/2} \quad \text{on } \operatorname{supp} H_{\pi(I)}, \tag{3.2}$$

which implies

$$|H_{\pi(I)}| \leq 2^{3/2}\alpha_I \chi_{I'}, \qquad \alpha_I \chi_{I'} \leq 2^{3/2} \cdot MH_{\pi(I)}. \tag{3.3}$$

Moreover, it follows from (3.1)–(3.2) and (2.7) that

$$\frac{1}{4}\|h_I\|_p \leq \|H_{\pi(I)}\|_p \leq 2^{3/2}\|h_I\|_p \quad \text{for } 1 \leq p \leq \infty. \tag{3.4}$$

Now, for a given sequence $\mathcal{T}$ dense in $[0,1]$, with the corresponding Haar system $\mathcal{H}_{\mathcal{T}} = \{h_n : n \geq 1\}$, consider $\tau : \mathbb{N} \to \mathbb{N}$ given by

$$\tau(1) = 1, \qquad \tau(n) = \pi(I_n) \quad \text{for } n \geq 2. \tag{3.5}$$

It follows from (3.3) and (2.12) that for each p with $1 < p < \infty$ there is a constant $C_p > 0$ (depending only on p) such that for each $\mathcal{T}$ and real coefficients $\{a_n : n \geq 1\}$,

$$2^{-3/2}\Big\|\Big(\sum_{n=1}^{\infty} a_n^2 H_{\tau(n)}^2\Big)^{1/2}\Big\|_p \leq \Big\|\Big(\sum_{n=1}^{\infty} a_n^2 \alpha_n^2 \chi_{I_n'}\Big)^{1/2}\Big\|_p \leq C_p\Big\|\Big(\sum_{n=1}^{\infty} a_n^2 H_{\tau(n)}^2\Big)^{1/2}\Big\|_p.$$

This inequality and Proposition 2.2 imply that there are $C_p, c_p > 0$ (depending only on p, $1 < p < \infty$) such that for each $\mathcal{T}$ and real coefficients $\{a_n : n \geq 1\}$,

$$c_p\Big\|\Big(\sum_{n=1}^{\infty} a_n^2 H_{\tau(n)}^2\Big)^{1/2}\Big\|_p \leq \Big\|\Big(\sum_{n=1}^{\infty} a_n^2 h_n^2\Big)^{1/2}\Big\|_p \leq C_p\Big\|\Big(\sum_{n=1}^{\infty} a_n^2 H_{\tau(n)}^2\Big)^{1/2}\Big\|_p. \tag{3.6}$$

Moreover, the following lemma shows that τ is an injection:

LEMMA 3.1. *Let $\mathcal{T} = \{t_n : n \geq 0\}$ be a dense sequence of points in $[0,1]$. Then for $k, m \geq 2$ with $k \neq m$ we have $\tau(k) \neq \tau(m)$.*

Proof. Note that for $k \neq m$ either $I'_k \cap I'_m = \emptyset$, or $I'_k \subset I'_m$, or $I'_m \subset I'_k$.

If $I'_k \cap I'_m = \emptyset$, then by (3.1) the supports of $H_{\tau(k)}$ and $H_{\tau(m)}$ are disjoint, so $\tau(k) \neq \tau(m)$.

If $I'_k \subset I'_m$ and $I'_k \neq I'_m$, then we have $I_k \subset I'_m$, so $\delta_k = |I'_k| \leq \frac{1}{2}|I_k| \leq \frac{1}{2}|I'_m| = \delta_m/2$. By (3.1),

$$|\operatorname{supp} H_{\tau(k)}| < \frac{\delta_k}{2} \leq \frac{\delta_m}{4} \leq |\operatorname{supp} H_{\tau(m)}|,$$

which implies $\tau(k) \neq \tau(m)$. Clearly, the case $I'_m \subset I'_k$ is symmetric. ∎

As τ is an injection, $\{H_{\tau(n)} : n \geq 1\}$ is a permutation of some subsequence of the dyadic Haar system $\{H_n : n \geq 1\}$. This fact in combination with (3.6) and (2.11) gives

THEOREM 3.2. *Let $\mathcal{T}$ be a dense sequence of points in $[0,1]$, with the corresponding general Haar system $\mathcal{H}_\mathcal{T} = \{h_n : n \geq 1\}$. Then $\{H_{\tau(n)} : n \geq 1\}$ is a permutation of a subsequence of the dyadic Haar system, equivalent to $\mathcal{H}_\mathcal{T}$ in $L^p([0,1])$ for each p, $1 < p < \infty$. Moreover, for each p with $1 < p < \infty$, there are constants $C_p, c_p > 0$ such that for all $\mathcal{T}$ and for all coefficient sequences $\{a_n : n \geq 1\}$ we have*

$$c_p \Big\| \sum_{n=1}^{\infty} a_n H_{\tau(n)} \Big\|_p \leq \Big\| \sum_{n=1}^{\infty} a_n h_n \Big\|_p \leq C_p \Big\| \sum_{n=1}^{\infty} a_n H_{\tau(n)} \Big\|_p .$$

4. General Haar systems and greedy approximation in $L^p([0,1]^d)$, $1 < p < \infty$, $d \in \mathbb{N}$

4.1. *The case of $d = 1$.* The fact that the dyadic Haar system normalized in $L^p([0,1])$, and any basis equivalent to it, is a greedy basis in $L^p([0,1])$ for $1 < p < \infty$ has been proved by V. N. Temlyakov [10] (see also [13] for a simplified proof); moreover, it has been shown that there are constants $C_p, c_p > 0$ (depending only on p) such that for any $m \in \mathbb{N}$ and sequence $n_1 < \ldots < n_m$,

$$c_p m^{1/p} \leq \|H_{n_1,p} + \ldots + H_{n_m,p}\|_p \leq C_p m^{1/p}. \tag{4.1}$$

This inequality, (2.11), (3.4) and Theorem 3.2 imply that for each p with $1 < p < \infty$, there are constants $C_p, c_p > 0$ such that for each sequence $\mathcal{T}$ with the corresponding general Haar system (normalized in $L^p([0,1])$) $\mathcal{H}_{\mathcal{T},p} = \{h_{n,p} : n \geq 1\}$, and for any $m \in \mathbb{N}$ and sequence $n_1 < \ldots < n_m$,

$$c_p m^{1/p} \leq \|h_{n_1,p} + \ldots + h_{n_m,p}\|_p \leq C_p m^{1/p}. \tag{4.2}$$

This means that $\mathcal{H}_{\mathcal{T},p}$ is democratic in $L^p([0,1])$; recall that it is also unconditional (cf. (2.10)). Combining this with the characterization of greedy bases proved in [7], i.e. that a normalized basis in a Banach space is greedy if and only if it is unconditional and democratic (cf. Section 2.1), we get

Corollary 4.1. *Let $\mathcal{T}$ be a dense sequence of points in* $[0,1]$, *and let* $\mathcal{H}_{\mathcal{T}} = \{h_n : n \geq 1\}$ *be the corresponding general Haar system. Then for each* p *with* $1 < p < \infty$, $\mathcal{H}_{\mathcal{T},p} = \{h_{n,p} : n \geq 1\}$ *is a greedy basis in* $L^p([0,1])$. *In addition, for each* p, *there is a constant* C_p *such that for each* $\mathcal{T}$, $f \in L^p([0,1])$ *and* $m \in \mathbb{N}$,

$$\|f - \mathcal{G}_m(f, \mathcal{H}_{\mathcal{T},p})\|_p \leq C_p \sigma_m(f, \mathcal{H}_{\mathcal{T},p}).$$

4.2. *Comments*

Remark 1. In this paper, we discuss only general Haar systems based on intervals and partitions of intervals into intervals, but one can replace in (2.3) and (2.5) intervals I, I', I'' by arbitrary measurable sets A, A', A'' with positive Lebesgue measure, and consider general Haar systems corresponding to sequences $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of sets. More precisely, let $(\Omega, \mathcal{F}, P)$ be a non-atomic probability space, and let $(\mathcal{F}_n, n \in \mathbb{N})$ be an increasing sequence of σ-fields, $\mathcal{F}_n \subset \mathcal{F}$, such that $\mathcal{F}_n$ is generated by a set $\mathcal{A}_n$ of atoms with $\#\mathcal{A}_n = n$ and $\mathcal{A}_{n+1}$ is obtained by splitting one of elements of $\mathcal{A}_n$ into two disjoint subsets, and $\lim_{n\to\infty} \max_{A \in \mathcal{A}_n} P(A) = 0$. Then $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$, and the function $h_{\mathcal{A},n}$ is defined by a formula analogous to (2.5), but with $A = A' \cup A''$ such that $A \in \mathcal{A}_n$ and $A', A'' \in \mathcal{A}_{n+1}$, $A' \cap A'' = \emptyset$. However, one then constructs a sequence $\mathcal{I}_{\mathcal{A}} = \{I_n : n \in \mathbb{N}\}$ of intervals and a measure preserving mapping $\varrho : \Omega \to [0,1]$ such that $\varrho(A_n) = I_n$, $P(A_n) = |I_n|$, $A_{n_1} = A'_n$ iff $I_{n_1} = I'_n$ and $A_{n_2} = A''_n$ iff $I_{n_2} = I''_n$. Note that for $1 \leq p \leq \infty$,

$$\Big\|\sum_{n=1}^{\infty} a_n h_{\mathcal{I}_{\mathcal{A}},n}\Big\|_p = \Big\|\sum_{n=1}^{\infty} a_n h_{\mathcal{A},n}\Big\|_p.$$

Thus, Theorem 3.2 is also valid for Haar systems corresponding to $\mathcal{A}$, with the subsequence and its permutation τ being the same as for $\mathcal{I}_{\mathcal{A}}$. Moreover, it follows that $\{h_{\mathcal{A},n,p} : n \in \mathbb{N}\}$ is a greedy basis in its span in $L^p(\Omega, P)$, $1 < p < \infty$.

Remark 2. L. E. Dor and E. Odell [3] have obtained a characterization of monotone bases in $L^p([0,1])$, $1 < p < \infty$, $p \neq 2$, in terms of general Haar systems (see also Theorem 10.b.4 in [9]). More precisely, for each normalized monotone basis $\{x_n : n \in \mathbb{N}\}$ of $L^p([0,1])$, they have constructed disjoint intervals S_i, $i \in P$, where $P = \mathbb{N}$ or $\#P < \infty$ and $[0,1] = \bigcup_{i\in P} S_i$, and general Haar systems $\mathcal{H}_{\mathcal{T}_i,p}$ on S_i such that there is a one-to-one mapping $T : \{x_n : n \in \mathbb{N}\} \to \bigcup_{i\in P} \mathcal{H}_{\mathcal{T}_i,p}$ which extends linearly to an isometry $T : L^p([0,1]) \to L^p([0,1])$. This characterization allows us to extend Theorem 3.2 to monotone bases—it is enough to assign to $|S_i|^{-1/p}\chi_{S_i}$ the dyadic Haar function (normalized in $L^p([0,1])$) with support being a dyadic interval of maximal length contained in S_i, and to the rest of elements of $\bigcup_{i\in P}\mathcal{H}_{\mathcal{T}_i,p}$ assign dyadic Haar functions according to the procedure π described in Section 3. Clearly, this also implies that each normalized monotone basis in $L^p([0,1])$ is greedy.

REMARK 3. V. N. Temlyakov [10] has proved that the dyadic Haar basis and each basis equivalent to the dyadic Haar basis are greedy in $L^p([0,1])$. P. Wojtaszczyk [13] gives an example of a greedy basis in $L^p([0,1])$, $2 < p < \infty$, which is not equivalent to the dyadic Haar basis; his example uses the isomorphism of $L^p([0,1])$ and $X \oplus L^p([0,1])$, where $2 < p < \infty$ and X is the Rosenthal space. As there exist non-equivalent general Haar systems, in fact, there is an uncountable family of general Haar systems such that any two of them are not equivalent in $L^p([0,1])$, $p \neq 2$ (cf. [3] or Theorem 10.b.10 of [9]), we get other examples of greedy bases in $L^p([0,1])$ not equivalent to the dyadic Haar basis. We do not present in detail the examples from [3], as below we give an example of a general Haar system which is not permutatively equivalent to the dyadic Haar basis in any $L^p([0,1])$, $1 < p < \infty$, $p \neq 2$ (Theorem 4.6). (Recall that the question on the existence of general Haar systems which are not permutatively equivalent was asked e.g. in [3].)

REMARK 4. Recall that the unconditionality of general Haar systems in $L^p([0,1])$, $1 < p < \infty$, is a consequence of the results by D. L. Burkholder on unconditionality of martingale differences in appropriate spaces ([1], [2]). Note that though any sequence of martingale differences is unconditional in an appropriate space $L^p(\Omega, P)$, $1 < p < \infty$, it need not be democratic (and greedy). This can be seen by the following example: take the dyadic Haar system $\mathcal{H}_{\mathcal{D}} = \{H_n : n \geq 1\}$. The sequence $\mathcal{M}$ of martingale differences is obtained from $\mathcal{H}_{\mathcal{D}}$ by replacing blocks of Haar functions $H_{2^{2j}+k}$, $k = 1, \ldots, 2^{2j}$, $j \geq 1$, by the corresponding Rademacher function $R_{2j} = 2^{-j}\sum_{k=1}^{2^{2j}} H_{2^{2j}+k}$; the Haar functions H_n with $n \neq 2^{2j}+k$, $1 \leq k \leq 2^{2j}$, belong to $\mathcal{M}$, and the order of functions is induced by the order in $\mathcal{H}_{\mathcal{D}}$. Note that $\mathcal{M}$ is a sequence of martingale differences (with respect to the sequence of generated σ-fields). Now, taking $1 < p < \infty$, $p \neq 2$, and m normalized elements of $\mathcal{M}$ of the form $H_{n_i,p}$ we have $\|\sum_{i=1}^m H_{n_i,p}\|_p \sim m^{1/p}$ (cf. (4.1)), while for m elements of $\mathcal{M}$ of the form R_{2j_i} (note that $\|R_{2j}\|_p = 1$) we have $\|\sum_{i=1}^m R_{2j_i}\|_p \sim m^{1/2}$, so the sequence $\mathcal{M}$ of martingale differences is not democratic in $L^p([0,1])$ with $p \neq 2$.

REMARK 5. In [6], general Franklin systems (i.e. orthonormal systems of piecewise linear functions corresponding to quasi-dyadic partitions) have been discussed. In particular, it has been proved that if the corresponding sequence of partitions is weakly regular (for the definition we refer to

[6]), then the corresponding Franklin system $\{f_n : n \geq 0\}$ is an unconditional basis in $L^p([0,1])$, $1 < p < \infty$, and the basic sequence $\{f_n : n \geq 1\}$ is equivalent in $L^p([0,1])$, $1 < p < \infty$, to the corresponding Haar system (cf. Theorem 3.1 and Proposition 6.1 of [6]). These results, together with Corollary 4.1 and the characterization of greedy bases recalled in Section 2.1, imply that general Franklin systems (properly normalized) corresponding to weakly regular sequences of partitions are greedy bases in $L^p([0,1])$, $1 < p < \infty$.

4.3. *Tensor products of general Haar systems in* $L^p([0,1]^d)$, $d \geq 2$. Let us now discuss bases in $L^p([0,1]^d)$, $d > 1$, consisting of tensor products of general Haar systems. Let $\mathcal{T}_1, \ldots, \mathcal{T}_d$ be dense sequences of points in $[0,1]$. Consider $\mathcal{H}_{(\mathcal{T}_1,\ldots,\mathcal{T}_d)} = \{h_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}^d\}$, where for $\boldsymbol{n} = (n_1, \ldots, n_d)$, $h_{\boldsymbol{n}} = h_{1,n_1} \otimes \ldots \otimes h_{d,n_d}$ with $h_{i,n} \in \mathcal{H}_{\mathcal{T}_i}$. Note that for each p with $1 < p < \infty$, $\mathcal{H}_{(\mathcal{T}_1,\ldots,\mathcal{T}_d)}$ is an unconditional basis in $L^p([0,1]^d)$. (This follows e.g. from unconditionality of univariate general Haar systems by the method used in [8] in the proof of the multivariate version of the Paley–Littlewood theorem, cf. Chapter 1.5.2 of [8].)

V. N. Temlyakov [11] has shown that for $p \neq 2$, $\mathcal{H}_{(\mathcal{D},\ldots,\mathcal{D}),p}$, i.e. the tensor product of the dyadic Haar basis, normalized in $L^p([0,1]^d)$, is not a greedy basis in $L^p([0,1]^d)$. P. Wojtaszczyk [13] has proved that for $f \in L^p([0,1]^d)$,

$$\|f - \mathcal{G}_m(f, \mathcal{H}_{(\mathcal{D},\ldots,\mathcal{D}),p})\|_p \leq C_{p,d} (\log m)^{(d-1)|1/p-1/2|} \sigma_m(f, \mathcal{H}_{(\mathcal{D},\ldots,\mathcal{D}),p}), \tag{4.3}$$

with the constant $C_{p,d}$ depending only on p, d; this was conjectured by V. N. Temlyakov [11]. It is also known that the factor $(\log m)^{(d-1)|1/p-1/2|}$ is optimal (cf. [11], [13]).

Inequality (4.3) and Theorem 3.2 imply that for each $d \in \mathbb{N}$ and p with $1 < p < \infty$, there is a constant $C_{p,d}$ such that for all sequences $\mathcal{T}_1, \ldots, \mathcal{T}_d$ and $f \in L^p([0,1]^d)$,

$$\|f - \mathcal{G}_m(f, \mathcal{H}_{(\mathcal{T}_1,\ldots,\mathcal{T}_d),p})\|_p \leq C_{p,d} (\log m)^{(d-1)|1/p-1/2|} \sigma_m(f, \mathcal{H}_{(\mathcal{T}_1,\ldots,\mathcal{T}_d),p}). \tag{4.4}$$

Clearly, the factor $(\log m)^{(d-1)|1/p-1/2|}$ is optimal for the class of all tensor products of general Haar systems, but it may not be optimal for a particular choice of $\mathcal{T}_1, \ldots, \mathcal{T}_d$. Indeed, now we show examples of sequences such that the tensor products of the corresponding Haar systems are greedy bases in all $L^p([0,1]^d)$ with $1 < p < \infty$ and $d \geq 2$.

Let $S = \{s_\nu : \nu \in \mathbb{N}\}$ be a strictly increasing sequence of natural numbers; later on, we consider sequences $S_\lambda = \{s_{\lambda,\nu} : \nu \in \mathbb{N}\}$ such that $s_{\lambda,\nu} \sim \lambda^\nu$ with fixed $\lambda > 1$ (cf. (4.7) below). To construct the required Haar systems we use points from uniform partitions of $[0,1]$ with steps $1/2^{s_\nu}$, $\nu \in \mathbb{N}$. Let

$$\mathcal{V}_{S,0} = \{0,1\}, \quad \mathcal{U}_{S,0} = \{0,1\},$$
$$\mathcal{V}_{S,\nu} = \{k/2^{s_\nu} : k = 0,\ldots,2^{s_\nu}\}, \quad \mathcal{U}_{S,\nu} = \mathcal{V}_{S,\nu} \setminus \mathcal{V}_{S,\nu-1} \quad \text{for } \nu \geq 1.$$

Clearly, $\mathcal{U}_{S,\nu_1} \cap \mathcal{U}_{S,\nu_2} = \emptyset$ for $\nu_1 \neq \nu_2$. The sequence $\mathcal{U}_S = \{u_n : n \geq 0\}$ is now defined as the points of the set $\bigcup_{\nu=0}^{\infty} \mathcal{U}_{S,\nu}$, with the following natural order: for each pair $\nu_1 < \nu_2$, the points from $\mathcal{U}_{S,\nu_1}$ precede those from $\mathcal{U}_{S,\nu_2}$, and for each ν, the points in $\mathcal{U}_{S,\nu}$ are in increasing order.

Let J_n be the interval corresponding to u_n in $\mathcal{U}_S$. By the construction of $\mathcal{U}_S$ we have

$$\delta_n = |J'_n| = 2^{-s_\nu} \quad \text{for } u_n \in \mathcal{U}_{S,\nu} \text{ with } \nu \geq 1. \tag{4.5}$$

Moreover,

$$J'_n \cap J'_l = \emptyset \quad \text{for } n,\ l \text{ such that } u_n, u_l \in \mathcal{U}_{S,\nu}. \tag{4.6}$$

We consider the following sequences S:

$$S_\lambda = \{s_{\lambda,\nu} := [\lambda^{\nu+\mu_\lambda}] : \nu \in \mathbb{N}\}, \quad \text{with fixed } \lambda > 1, \tag{4.7}$$

where $[x]$ denotes the integer part of x, and $\mu_\lambda = 0$ if $\lambda \geq 2$, while $\mu_\lambda = [-\log(\lambda-1)/\log(\lambda)] + 1$ in case $1 < \lambda < 2$; this choice of μ_λ guarantees that $s_{\lambda,\nu+1} > s_{\lambda,\nu}$.

We are going to show that for each $\lambda > 1$, $1 < p < \infty$ and $d \geq 2$, the product Haar system $\mathcal{H}_{(\mathcal{U}_{S_\lambda},\ldots,\mathcal{U}_{S_\lambda}),p}$ is a greedy basis in $L^p([0,1]^d)$. We start with the following facts:

FACT 4.2. *Let $\lambda > 1$ and $d \in \mathbb{N}$. For $m \in \mathbb{N}$, let*

$$Z_\lambda(d,m) = \{(\nu_1,\ldots,\nu_d) \in \mathbb{N}^d : m = s_{\lambda,\nu_1} + \ldots + s_{\lambda,\nu_d},\ \nu_1 \leq \ldots \leq \nu_d\}.$$

Then there is a constant $C_{\lambda,d} > 0$ (depending only on λ and d) such that

$$\# Z_\lambda(d,m) \leq C_{\lambda,d} \quad \text{for all } m \in \mathbb{N}.$$

Proof. This is checked by induction on d. It is clear that it holds for $d = 1$ with $C_{\lambda,1} = 1$. Now, let $d > 1$ and suppose that $m \in \mathbb{N}$ admits a representation $m = s_{\lambda,\nu_1} + \ldots + s_{\lambda,\nu_d}$ with $\nu_1 \leq \ldots \leq \nu_d$. If $m = s_{\lambda,l_1} + \ldots + s_{\lambda,l_d}$ with $l_1 \leq \ldots \leq l_d$ is another such representation, then

$$\lambda^{\nu_d+\mu_\lambda} \leq [\lambda^{\nu_d+\mu_\lambda}] + 1 \leq d[\lambda^{l_d+\mu_\lambda}] + 1 \leq (2d+1)\lambda^{l_d+\mu_\lambda},$$

and analogously $\lambda^{l_d+\mu_\lambda} \leq (2d+1)\lambda^{\nu_d+\mu_\lambda}$, which implies

$$|\nu_d - l_d| \leq M_{\lambda,d}, \quad \text{where} \quad M_{\lambda,d} = \left[\frac{\log(2d+1)}{\log(\lambda)}\right],$$

and consequently, by the induction hypothesis,

$$\# Z_\lambda(d,m) \leq \sum_{l_d : |\nu_d - l_d| \leq M_{\lambda,d}} Z_\lambda(d-1, m - s_{\lambda,l_d}) \leq (2M_{\lambda,d}+1)C_{\lambda,d-1}.$$

Thus, it is enough to put $C_{\lambda,d} = (2M_{\lambda,d}+1)C_{\lambda,d-1}$. ∎

FACT 4.3. *Let $1 < p < \infty$ and $d \in \mathbb{N}$. For $j = 1, \ldots, d$, let $\Phi_j = \{\varphi_{j,n} : n \in \mathbb{N}\}$, $\Psi_j = \{\psi_{j,n} : n \in \mathbb{N}\}$ with $\varphi_{j,n}, \psi_{j,n} \in L^p([0,1])$. Assume that for each j there are constants $C_j, c_j > 0$ such that for any sequence $\{a_n : n \in \mathbb{N}\}$ of real coefficients*,

$$c_j \Big\| \Big(\sum_{n\in\mathbb{N}} a_n^2 \psi_{j,n}^2 \Big)^{1/2} \Big\|_{L^p([0,1])} \le \Big\| \Big(\sum_{n\in\mathbb{N}} a_n^2 \varphi_{j,n}^2 \Big)^{1/2} \Big\|_{L^p([0,1])} \le C_j \Big\| \Big(\sum_{n\in\mathbb{N}} a_n^2 \psi_{j,n}^2 \Big)^{1/2} \Big\|_{L^p([0,1])}.$$

For $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$, set $\varphi_{\boldsymbol{n}} = \varphi_{1,n_1} \otimes \ldots \otimes \varphi_{d,n_d}$ and $\psi_{\boldsymbol{n}} = \psi_{1,n_1} \otimes \ldots \otimes \psi_{d,n_d}$. Then, with $c = c_1 \ldots c_d$ and $C = C_1 \ldots C_d$, for each sequence $\{a_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{N}^d\}$ of real coefficients,

$$c \Big\| \Big(\sum_{\boldsymbol{n}\in\mathbb{N}^d} a_{\boldsymbol{n}}^2 \psi_{\boldsymbol{n}}^2 \Big)^{1/2} \Big\|_{L^p([0,1]^d)} \le \Big\| \Big(\sum_{\boldsymbol{n}\in\mathbb{N}^d} a_{\boldsymbol{n}}^2 \varphi_{\boldsymbol{n}}^2 \Big)^{1/2} \Big\|_{L^p([0,1]^d)} \le C \Big\| \Big(\sum_{\boldsymbol{n}\in\mathbb{N}^d} a_{\boldsymbol{n}}^2 \psi_{\boldsymbol{n}}^2 \Big)^{1/2} \Big\|_{L^p([0,1]^d)}.$$

This is an immediate consequence of Fubini's theorem.

For $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$, define

$$R_{\boldsymbol{n}} = J'_{n_1} \times \ldots \times J'_{n_d}, \qquad r_{\boldsymbol{n}} = |R_{\boldsymbol{n}}| = \delta_{n_1} \ldots \delta_{n_d}. \tag{4.8}$$

The following lemma is the main step in the proof of the democracy of $\mathcal{H}_{(\mathcal{U}_{S_\lambda}, \ldots, \mathcal{U}_{S_\lambda}), p}$ in $L^p([0,1]^d)$:

LEMMA 4.4. *Let $\lambda > 1$, $1 < p < \infty$, $d \in \mathbb{N}$, $d \ge 2$. Let S_λ, $\mathcal{U}_{S_\lambda}$, $R_{\boldsymbol{n}}, r_{\boldsymbol{n}}$ be as defined above. Then there are constants $C_p, c_p > 0$, depending only on p, λ, d, such that for each $m \in \mathbb{N}$, $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_m \in \mathbb{N}^d$, $\boldsymbol{n}_i \ne \boldsymbol{n}_j$ for $i \ne j$, and $\boldsymbol{x} \in [0,1]^d$*,

$$c_p \sum_{i=1}^m r_{\boldsymbol{n}_i}^{-1} \chi_{R_{\boldsymbol{n}_i}}(\boldsymbol{x}) \le \Big(\sum_{i=1}^m r_{\boldsymbol{n}_i}^{-2/p} \chi_{R_{\boldsymbol{n}_i}}(\boldsymbol{x}) \Big)^{p/2} \le C_p \sum_{i=1}^m r_{\boldsymbol{n}_i}^{-1} \chi_{R_{\boldsymbol{n}_i}}(\boldsymbol{x}). \tag{4.9}$$

Moreover, let $\mathcal{H}_{(\mathcal{U}_{S_\lambda}, \ldots, \mathcal{U}_{S_\lambda}), p} = \{h_{\lambda:\boldsymbol{n},p} : \boldsymbol{n} \in \mathbb{N}^d\}$ be the system consisting of all tensor products of elements of $\mathcal{H}_{\mathcal{U}_{S_\lambda}, p}$, normalized in $L^p([0,1]^d)$. Then there are constants $D_p, d_p > 0$, depending only on p, λ, d, such that for each $m \in \mathbb{N}$ and $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_m \in \mathbb{N}^d$ with $\boldsymbol{n}_i \ne \boldsymbol{n}_j$ for $i \ne j$,

$$d_p m^{1/p} \le \Big\| \Big(\sum_{i=1}^m h_{\lambda;\boldsymbol{n}_i,p}^2 \Big)^{1/2} \Big\|_{L^p([0,1]^d)} \le D_p m^{1/p}. \tag{4.10}$$

Proof. The proof of (4.9) is similar to the proof of Lemma 9 in [13].

Let $m \in \mathbb{N}$ and $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_m \in \mathbb{N}^d$, $\boldsymbol{n}_i \neq \boldsymbol{n}_j$ for $i \neq j$, be fixed. It is enough to consider $\boldsymbol{x} \in [0,1]^d$ for which $\chi_{R_{\boldsymbol{n}_i}}(\boldsymbol{x}) \neq 0$ for some i, $1 \le i \le m$. For such $\boldsymbol{x}$ and $\mu \in \mathbb{N} \cup \{0\}$, let

$$A(\boldsymbol{x},\mu) = \{i : \boldsymbol{x} \in R_{\boldsymbol{n}_i},\ 1/2^{\mu+1} < r_{\boldsymbol{n}_i} \le 1/2^{\mu}\}.$$

Setting $\boldsymbol{n}_i = (n_{i,1}, \ldots, n_{i,d})$, let $\nu_{i,j}$ be such that $u_{n_{i,j}} \in \mathcal{U}_{S_\lambda,\nu_{i,j}}$. It follows from (4.5) and (4.8) that

$$r_{\boldsymbol{n}_i} = \prod_{j=1}^{d} \delta_{n_{i,j}} = 2^{-(s_{\lambda,\nu_{i,1}} + \ldots + s_{\lambda,\nu_{i,d}})},$$

so

$$A(\boldsymbol{x},\mu) = \{i : \boldsymbol{x} \in R_{\boldsymbol{n}_i},\ \mu = s_{\lambda,\nu_{i,1}} + \ldots + s_{\lambda,\nu_{i,d}}\}.$$

Now, (4.6) and Fact 4.2 imply that there is $b_d > 0$, depending only on λ and d, such that

$$\#A(\boldsymbol{x},\mu) \le b_d. \tag{4.11}$$

Let $\mu_{\boldsymbol{x}} = \max\{\mu \in \mathbb{N} \cup \{0\} : \#A(\boldsymbol{x},\mu) > 0\}$. Then

$$2^{\mu_{\boldsymbol{x}}} \le \Big(\sum_{i=1}^{m} r_{\boldsymbol{n}_i}^{-2/p} \chi_{R_{\boldsymbol{n}_i}}(\boldsymbol{x})\Big)^{p/2} = \Big(\sum_{\mu=0}^{\mu_x} 2^{2\mu/p} \cdot \#A(\boldsymbol{x},\mu)\Big)^{p/2}$$
$$\le b_d^{p/2} \Big(\sum_{\mu=0}^{\mu_x} 2^{2\mu/p}\Big)^{p/2} \le C_p 2^{\mu_{\boldsymbol{x}}}.$$

By similar arguments

$$2^{\mu_{\boldsymbol{x}}} \le \sum_{i=1}^{m} r_{\boldsymbol{n}_i}^{-1} \chi_{R_{\boldsymbol{n}_i}}(\boldsymbol{x}) \le C 2^{\mu_{\boldsymbol{x}}},$$

which gives inequality (4.9).

Inequality (4.9) implies

$$\Big\| \Big(\sum_{i=1}^{m} r_{\boldsymbol{n}_i}^{-2/p} \chi_{R_{\boldsymbol{n}_i}}\Big)^{1/2} \Big\|_p \sim m^{1/p}.$$

This equivalence, combined with Proposition 2.2 (cf. also (2.5), (2.7)) and Fact 4.3, gives

$$\Big\| \Big(\sum_{i=1}^{m} h_{\lambda;\boldsymbol{n}_i,p}^2\Big)^{1/2} \Big\|_{L^p([0,1]^d)} \sim \Big\| \Big(\sum_{i=1}^{m} r_{\boldsymbol{n}_i}^{-2/p} \chi_{R_{\boldsymbol{n}_i}}\Big)^{1/2} \Big\|_p \sim m^{1/p},$$

i.e. inequalities (4.10). ■

THEOREM 4.5. *Let $\lambda > 1$ and let $\mathcal{U}_{S_\lambda}$ be the sequence of points as described in this section. Then for each p with $1 < p < \infty$ and $d \in \mathbb{N}$,*

$\mathcal{H}_{(\mathcal{U}_{S_\lambda},\ldots,\mathcal{U}_{S_\lambda}),p}$, *i.e. the system consisting of all tensor products of the general Haar system corresponding to* $\mathcal{U}_{S_\lambda}$, *normalized in* $L^p([0,1]^d)$, *is a greedy basis in* $L^p([0,1]^d)$.

Proof. By the results of [7], a normalized basis is greedy iff it is unconditional and democratic. Unconditionality of $\mathcal{H}_{(\mathcal{U}_{S_\lambda},\ldots,\mathcal{U}_{S_\lambda}),p}$ in $L^p([0,1]^d)$ follows from unconditionality of general Haar systems in $L^p([0,1])$. Democracy of $\mathcal{H}_{(\mathcal{U}_{S_\lambda},\ldots,\mathcal{U}_{S_\lambda}),p}$ is a consequence of its unconditionality in $L^p([0,1]^d)$ and inequality (4.10) from Lemma 4.4. ■

THEOREM 4.6. *Let* $\lambda > 1$ *and let* $\mathcal{U}_{S_\lambda}$ *be the sequence of points corresponding to* S_λ. *Moreover, let* $\mathcal{D}$ *be the sequence of dyadic points. Then for each* p *with* $1 < p < \infty$, $p \neq 2$, *the corresponding Haar systems* $\mathcal{H}_{\mathcal{U}_{S_\lambda},p}$ *and* $\mathcal{H}_{\mathcal{D},p}$ (*normalized in* $L^p([0,1])$) *are not permutatively equivalent in* $L^p([0,1])$. *Thus, there are general Haar systems which are not permutatively equivalent.*

Proof. Let $\{h_n : n \in \mathbb{N}\}$ and $\{H_n : n \in \mathbb{N}\}$ be the general Haar systems corresponding to $\mathcal{U}_{S_\lambda}$ and to the dyadic Haar system, respectively.

Fix p, $1 < p < \infty$, $p \neq 2$. Suppose that there is a permutation $\xi : \mathbb{N} \to \mathbb{N}$ such that $\{h_{\xi(n),p} : n \in \mathbb{N}\}$ and $\{H_{n,p} : n \in \mathbb{N}\}$ are equivalent in $L^p([0,1])$. Then $h_{\xi(n_1),p} \otimes h_{\xi(n_2),p}$ and $H_{n_1,p} \otimes H_{n_2,p}$ are equivalent in $L^p([0,1]^2)$. However, $\{h_{\xi(n_1),p} \otimes h_{\xi(n_2),p} : n_1, n_2 \in \mathbb{N}\}$ is a democratic basis in $L^p([0,1]^2)$, by its unconditionality and inequality (4.10) from Lemma 4.4. On the other hand, the example presented in [11] shows that $\{H_{n_1,p} \otimes H_{n_2,p} : n_1, n_2 \in \mathbb{N}\}$ is not democratic in $L^p([0,1]^2)$. Thus, these two bases cannot be equivalent in $L^p([0,1]^2)$, and consequently, $\mathcal{H}_{\mathcal{U}_{S_\lambda},p}$ and $\mathcal{H}_{\mathcal{D},p}$ are not permutatively equivalent in $L^p([0,1])$. ■

5. The case of $L^1([0,1])$. Let us discuss the analogues of the results of Section 3 for $p = 1$. Since general Haar systems are bases of $L^1([0,1])$, but they are not unconditional, we are going to examine the equivalence of L^1-norms of the appropriate square functions—more precisely, we are going to prove that inequality (3.6) can be extended to the case $p = 1$. However, as there is no analogue of the Fefferman–Stein maximal inequality for $p = 1$, the method of proof is now different—the proof is based on the method developed by G. G. Gevorkyan and used e.g. in the study of Franklin series in L^p and H^p with $0 < p \leq 1$ (cf. e.g. [5]). Note that the results of Section 3 can also be obtained by this method (for $1 < p < 2$ directly, and then by a duality argument also for $2 < p < \infty$); however, we have decided to present also the argument given in Section 3 because of its simplicity.

THEOREM 5.1. *Let* $\mathcal{T}$ *be a dense sequence of points in* $[0,1]$, *with the corresponding general Haar system* $\mathcal{H}_{\mathcal{T}} = \{h_n : n \in \mathbb{N}\}$. *Let* $\tau : \mathbb{N} \to \mathbb{N}$ *be* *given by formula* (3.5). *Then there are constants* $C, c > 0$, *independent of* $\mathcal{T}$, *such that for each sequence* $\{a_n : n \in \mathbb{N}\}$ *of real coefficients*,

$$c\Big\|\Big(\sum_{n=1}^{\infty} a_n^2 H_{\tau(n)}^2\Big)^{1/2}\Big\|_1 \le \Big\|\Big(\sum_{n=1}^{\infty} a_n^2 h_n^2\Big)^{1/2}\Big\|_1 \le C\Big\|\Big(\sum_{n=1}^{\infty} a_n^2 H_{\tau(n)}^2\Big)^{1/2}\Big\|_1. \tag{5.1}$$

For the proof, the following technical lemma is needed:

LEMMA 5.2. *Let* $\mathcal{T}$ *be a sequence of points. Let* I_{n_i}, $i \in \mathbb{N}$, *be a sequence of intervals corresponding to points* $t_{n_i} \in \mathcal{T}$ *such that* $I_{n_{i+1}} \subset I_{n_i}$. *Let* $M = \min\{i : I_{n_{i+1}} \subset I'_{n_i}\}$ *if such an* i *exists*, *and* $M = \infty$ *otherwise. Then*

$$\sum_{i=1}^{\infty} \delta_{n_i} \le 6 \sum_{i=1}^{M} \delta_{n_i}.$$

In addition, *the intervals* $I'_{n_1}, \ldots, I'_{n_M}$ *are disjoint.*

Proof. First, assume that $M = 1$; clearly, we can also assume that $I_{n_{i+1}} = I'_{n_i}$ or $I_{n_{i+1}} = I''_{n_i}$. Put

$$l_1 = \min\{i \ge 1 : I_{n_{i+1}} = I''_{n_i}\}, \qquad k_1 = \min\{i \ge l_1 : I_{n_{i+1}} = I'_{n_i}\},$$

and for $j \ge 2$,

$$l_j = \min\{i \ge k_{j-1} : I_{n_{i+1}} = I''_{n_i}\}, \qquad k_j = \min\{i \ge l_j : I_{n_{i+1}} = I'_{n_i}\}.$$

As $I_{n_2} = I'_{n_1}$, we have $l_1 \ge 2$ and $|I_{n_{l_1}}| \le \delta_{n_1}$. Moreover, $I_{n_{l_j}} = I'_{n_{l_j - 1}}$, which implies

$$|I_{n_{l_{j+1}}}| = |I'_{n_{l_{j+1}-1}}| \le \frac{1}{2}|I_{n_{l_{j+1}-1}}| \le \frac{1}{2}|I_{n_{l_j}}|, \qquad |I_{n_{l_j}}| \le \frac{1}{2^{j-1}}|I_{n_{l_1}}|. \tag{5.2}$$

For fixed j and $l_j \le i \le k_j$, the intervals I'_{n_i} are disjoint and included in $I_{n_{l_j}}$, so

$$\sum_{i=l_j}^{k_j} \delta_{n_i} \le |I_{n_{l_j}}|. \tag{5.3}$$

For $k_j + 1 \le i \le l_{j+1} - 1$ we have $I_{n_i} = I'_{n_{i-1}}$, so

$$\delta_{n_i} = |I'_{n_i}| \le \frac{1}{2}|I_{n_i}| = \frac{1}{2}\delta_{n_{i-1}} \le \frac{1}{2^{i-k_j}}\delta_{n_{k_j}}, \qquad \sum_{i=k_j+1}^{l_{j+1}-1} \delta_{n_i} \le \delta_{n_{k_j}}. \tag{5.4}$$

By similar arguments we check that

$$\sum_{i=1}^{l_1 - 1} \delta_{n_i} \le 2\delta_{n_1}.$$

This inequality, together with (5.2)–(5.4), gives

$$\sum_{i=1}^{\infty}\delta_{n_i} \le \sum_{i=1}^{l_1-1}\delta_{n_i} + \sum_{j=1}^{\infty}\Big(\sum_{i=l_j}^{k_j}\delta_{n_i} + \sum_{i=k_j+1}^{l_{j+1}-1}\delta_{n_i}\Big) \le 2\delta_{n_1} + 2\sum_{j=1}^{\infty}\sum_{i=l_j}^{k_j}\delta_{n_i}$$
$$\le 2\delta_{n_1} + 2\sum_{j=1}^{\infty}|I_{n_{l_j}}| \le 2\delta_{n_1} + 4|I_{n_{l_1}}| \le 6\delta_{n_1}.$$

If $M > 1$, then the sequence $\{I_{n_i} : i \ge M\}$ is of the type considered above, which implies the required inequality in the general case.

The fact that the intervals $I'_{n_1}, \ldots, I'_{n_M}$ are disjoint is an immediate consequence of the definition of M. ■

Proof of Theorem 5.1. The left inequality in (5.1) is a consequence of inequality (3.3) (cf. also (2.5)).

Now, let us prove the right inequality of (5.1).

For a function f, denote by $M^* f$ the dyadic maximal function of f. In addition, define $D_n = \operatorname{supp} H_n$.

For a fixed sequence $\{a_n : n \in \mathbb{N}\}$ of real coefficients, put

$$S(x) = \sum_{n=1}^{\infty} a_n^2 H_{\tau(n)}^2(x),$$
$$E_r = \{x : S(x) > 2^r\}, \qquad r \in \mathbb{Z},$$
$$B_r = \{x : M^*\chi_{E_r}(x) > 1/2\}, \qquad r \in \mathbb{Z},$$
$$N_r = \{n \in \mathbb{N} : D_{\tau(n)} \subset B_r,\ D_{\tau(n)} \not\subset B_{r+1}\},$$
$$\psi_r(x) = \Big(\sum_{n\in N_r} a_n^2 h_n^2(x)\Big)^{1/2}.$$

Note that for $n \in N_r$ we have $|D_{\tau(n)} \cap E_{r+1}^{\mathrm{c}}| \ge \frac{1}{2}$ and consequently

$$\int_{D_{\tau(n)}\cap E_{r+1}^{\mathrm{c}}} H_{\tau(n)}^2(x)\,dx \ge \frac{1}{2}.$$

Using this inequality we obtain

$$\|\psi_r\|_2^2 = \sum_{n\in N_r} a_n^2 \le 2\sum_{n\in N_r} a_n^2 \int_{D_{\tau(n)}\cap E_{r+1}^{\mathrm{c}}} H_{\tau(n)}^2(x)\,dx$$
$$\le 2\int_{B_r\cap E_{r+1}^{\mathrm{c}}} \sum_{n\in N_r} a_n^2 H_{\tau(n)}^2(x)\,dx \le 2\int_{B_r\cap E_{r+1}^{\mathrm{c}}} S(x)\,dx,$$

which combined with the fact that $S(x) \le 2^{r+1}$ on E_{r+1}^{c} gives

$$\|\psi_r\|_2^2 \le 2^{r+2}|B_r|. \tag{5.5}$$

Now, by the Schwarz inequality we get

$$\int_{B_r} \psi_r(x)\,dx \le |B_r|^{1/2}\Big(\int_{B_r} (\psi_r(x))^2\,dx\Big)^{1/2} \le |B_r|^{1/2}\|\psi_r\|_2,$$

which together with (5.5) gives

$$\int_{B_r} \psi_r(x)\,dx \le 2^{1+r/2}|B_r|. \tag{5.6}$$

Now, we estimate the analogous integral over B_r^{c}. First, note that $\|a_n^2 H_{\tau(n)}^2\|_\infty \le 2^{r+1}$ for $n \in N_r$: if not, then, as $|H_{\tau(n)}|$ is constant on $D_{\tau(n)}$, we would have $D_{\tau(n)} \subset E_{r+1}$ and consequently $D_{\tau(n)} \subset B_{r+1}$, contrary to the definition of N_r. Combining this inequality with (3.2), we get

$$|a_n| \le 2^{r/2}\delta_n^{1/2} \quad \text{for } n \in N_r. \tag{5.7}$$

Moreover, B_r is a union of dyadic intervals. Let $\mathcal{D}_r$ denote the family of maximal dyadic intervals contained in B_r; as the interiors of intervals from $\mathcal{D}_r$ are disjoint, we have

$$B_r = \bigcup_{J\in\mathcal{D}_r} J, \quad |B_r| = \sum_{J\in\mathcal{D}_r} |J|. \tag{5.8}$$

For $J \in \mathcal{D}_r$, let

$$N_{r,J} = \{n \in N_r : D_{\tau(n)} \subset J\}, \quad N_{r,J}^* = \{n \in N_{r,J} : I_n \not\subset J\}.$$

Observe that if $n \in N_{r,J}^*$, then I_n contains either the left or the right endpoint of J; denote by $N_{r,J}^L, N_{r,J}^R$ the respective subsets of $N_{r,J}^*$. Recall that any two intervals I_k, I_l are either disjoint, or one is included in the other. All intervals in $N_{r,J}^L$ and in $N_{r,J}^R$ have a common point, so this is the latter case, and the families $N_{r,J}^L, N_{r,J}^R$ satisfy the assumptions of Lemma 5.2. As $\delta_n \le 4|D_{\tau(n)}|$ (cf. (3.1)) and $D_{\tau(n)} \subset J$ for $n \in N_{r,J}$, Lemma 5.2 implies

$$\sum_{n\in N_{r,J}^*} \delta_n \le \sum_{n\in N_{r,J}^L} \delta_n + \sum_{n\in N_{r,J}^R} \delta_n \le 48|J|. \tag{5.9}$$

Now, using the definitions of ψ_r, $N_{r,J}$, $N_{r,J}^*$, (5.8), (5.7), (5.9) and (2.7) we get

$$\begin{aligned}\int_{B_r^{\mathrm{c}}} \psi_r(x)\,dx &\le \int_{B_r^{\mathrm{c}}} \sum_{n\in N_r} |a_n|\cdot|h_n(x)|\,dx \le \sum_{J\in\mathcal{D}_r}\sum_{n\in N_{r,J}} |a_n| \int_{J^{\mathrm{c}}} |h_n(x)|\,dx \\ &\le \sum_{J\in\mathcal{D}_r}\sum_{n\in N_{r,J}^*} |a_n|\cdot\|h_n\|_1 \le 2^{1+r/2}\sum_{J\in\mathcal{D}_r}\sum_{n\in N_{r,J}^*}\delta_n \\ &\le 96\cdot 2^{r/2}\sum_{J\in\mathcal{D}_r}|J| = 96\cdot 2^{r/2}|B_r|.\end{aligned}$$

The last inequality in combination with (5.6) gives

$$\int_0^1 \psi_r(x)\,dx \le 97 \cdot 2^{r/2}|B_r|.$$

As $\{n \in \mathbb{N} : a_n \neq 0\} \subset \bigcup_{r\in\mathbb{Z}} N_r$, the above inequality, together with the definitions of $S(\cdot)$, E_r, B_r and the fact that M^* is of weak type $(1,1)$, implies

$$\int_0^1 \Big(\sum_{n=1}^{\infty} a_n^2 h_n^2(x)\Big)^{1/2} dx \le \sum_{r\in\mathbb{Z}} \int_0^1 \psi_r(x)\,dx \le 97 \cdot \sum_{r\in\mathbb{Z}} 2^{r/2}|B_r|$$
$$\le C \sum_{r\in\mathbb{Z}} 2^{r/2}|E_r| \le C \int_0^1 S(x)^{1/2}\,dx,$$

and the proof of Theorem 5.1 is complete. ■

For a given sequence $\mathcal{T}$ of points and the corresponding Haar system $\mathcal{H}_{\mathcal{T}} = \{h_n : n \in \mathbb{N}\}$, let $H^1_{\mathcal{T}}$ be the space of those $f = \sum_{n=1}^{\infty} a_n h_n \in L^1([0,1])$ for which the series $\sum_{n=1}^{\infty} a_n h_n$ is unconditionally convergent in $L^1([0,1])$, with the norm

$$\|f\|_{H^1_{\mathcal{T}}} = \sup_{\varepsilon=\{\varepsilon_n : n\in\mathbb{N}\}} \Big\|\sum_{n=1}^{\infty} \varepsilon_n a_n h_n\Big\|_1, \quad \text{where } \varepsilon_n = \pm 1.$$

$H^1_{\mathcal{T}}$ is a Banach space, and it is called a (*martingale*) *Hardy space*. It follows from Khinchin's inequality and inequalities between the L^1-norms of the maximal and square functions for martingales (cf. e.g. Chapter II of [4]) that

$$f \in H^1_{\mathcal{T}} \quad \text{iff} \quad Sf = \Big(\sum_{n=1}^{\infty} a_n^2 h_n^2\Big)^{1/2} \in L^1([0,1]),$$

and moreover $\|f\|_{H^1_{\mathcal{T}}} \sim \|Sf\|_1$. It should be clear that $\mathcal{H}_{\mathcal{T},1}$ is a normalized unconditional basis of $H^1_{\mathcal{T}}$. Moreover, it follows from Lemma 9 of [13] that for each sequence $n_1 < \ldots < n_m$,

$$\Big\|\Big(\sum_{i=1}^{m} H^2_{n_i,1}\Big)^{1/2}\Big\|_1 \sim m,$$

which combined with inequalities (3.4) and Theorem 5.1 implies that $\mathcal{H}_{\mathcal{T},1}$ is democratic in $H^1_{\mathcal{T}}$. Thus, by the characterization of greedy bases recalled in Section 2.1, we have

Corollary 5.3. *For each sequence $\mathcal{T}$, the corresponding general Haar system $\mathcal{H}_{\mathcal{T},1}$, normalized in $L^1([0,1])$, is a greedy basis in the corresponding martingale Hardy space $H^1_{\mathcal{T}}$.*

Let us briefly discuss the case $0 < p < 1$. First, for $0 < p < 1$ the analogue of the equivalence (5.1) of Theorem 5.1 can fail to hold. To see this, note that (2.6) implies

$$2^{-1/2}\delta_I^{1/2}\Delta_I^{1/p-1} \le \|h_I\|_p \le 2^{1/p}\delta_I^{1/2}\Delta_I^{1/p-1} \quad \text{for } 0 < p < 1,$$

while (3.1) and (3.2) give

$$4^{1/2-1/p}\delta_I^{1/p-1/2} \le \|H_{\pi(I)}\|_p \le 2^{1/2-1/p}\delta_I^{1/p-1/2} \quad \text{for } 0 < p < 1,$$

so the analogue of (5.1) cannot hold in general for $p < 1$. Moreover, (5.1) need not hold even if h_n and $H_{\tau(n)}$ are replaced by $h_{n,p} = h_n/\|h_n\|_p$ and $H_{\tau(n),p} = H_{\tau(n)}/\|H_{\tau(n)}\|_p$, respectively (see below).

In [13], the notion of a greedy basis is also considered for quasi-Banach spaces. As in the case $p = 1$, for $0 < p < 1$ and a given sequence $\mathcal{T}$ of points, one could consider the martingale Hardy space $H_{\mathcal{T}}^p$ defined in terms of the corresponding square function, and ask if $\mathcal{H}_{\mathcal{T},p}$ is greedy in $H_{\mathcal{T}}^p$. Using Theorem 4 of [13] one can show that, in contrast to the case of $1 \le p < \infty$, now $\mathcal{H}_{\mathcal{T},p}$ need not be greedy in $H_{\mathcal{T}}^p$. More precisely, one can show that $\mathcal{H}_{\mathcal{T},p}$ is democratic (and, consequently, greedy) in the corresponding martingale Hardy space $H_{\mathcal{T}}^p$ with $0 < p < 1$ if and only if there is $K \in \mathbb{N}$ such that for any collection of intervals $I_{n_1}, \ldots, I_{n_l}$ with $I_{n_1} \supset \ldots \supset I_{n_l}$ and $|I_{n_l}| \ge \frac{1}{2}|I_{n_1}|$, we have $l \le K$. Note that for $\mathcal{T}$ not satisfying this condition, the analogue of (5.1) for $h_{n,p}$ and $H_{\tau(n),p}$ $(0 < p < 1)$ cannot hold, as by Lemma 9 of [13] we have $\|(\sum_{i=1}^m H_{n_i,p}^2)^{1/2}\|_p \sim m^{1/p}$ for each $n_1 < \ldots < n_m$, so $\{H_{\tau(n),p} : n \ge 1\}$ is democratic while $\mathcal{H}_{\mathcal{T},p}$ is not.

Finally, let us remark that there is an example of a sequence $\mathcal{T}$ such that $\mathcal{H}_{\mathcal{T},p}$ is greedy in $H_{\mathcal{T}}^p$, $0 < p < 1$, but the analogue of (5.1) for $h_{n,p}$ and $H_{\tau(n),p}$ with $0 < p < 1$ does not hold; however, this example is technical and it will not be presented here.

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Received July 10, 2000 (4566)