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The non-pluripolarity of compact sets in complex spaces and the property (LB^∞) for the space of germs of holomorphic functions

by

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Abstract. The aim of this paper is to establish the equivalence between the non-pluripolarity of a compact set in a complex space and the property (LB^∞) for the dual space of the space of germs of holomorphic functions on that compact set.

1. Introduction. Let E be a Fréchet space with the topology defined by an increasing system $\{\|\cdot\|_k\}_{k\ge 1}$ of seminorms. For each $k \ge 1$ put

$$\|u\|_k^* = \sup\{|u(x)| : \|x\|_k \le 1\}$$

where $u \in E^*$, the topological dual space of E.

We say that E has the *property* $(\widetilde{\Omega})$ if

$$\forall p\ \exists q, d > 0\ \forall k\ \exists C > 0 : \quad \|u\|_q^{*1+d} \le C\|u\|_k^*\|u\|_p^{*d}, \quad \forall u \in E^*,$$

and has the *property* (LB^∞) if

$$\forall\{\varrho_n\}\uparrow +\infty\ \forall p\ \exists q\ \forall n_0\ \exists N_0, C > 0\ \forall u \in E^*\ \exists n_0 \le k \le N_0 :$$
$$\|u\|_q^{*1+\varrho_k} \le C\|u\|_k^*\|u\|_p^{*\varrho_k}.$$

We then write $E \in (\widetilde{\Omega})$ (resp. $E \in (\mathrm{LB}^\infty)$).

The above properties and many others were introduced and investigated by Vogt (for example, see [11], [14]). One of the first problems raised here is to find conditions under which a Fréchet space has the property (LB^∞) or $(\widetilde{\Omega})$. In [5], S. Dineen, R. Meise and D. Vogt have shown that a nuclear Fréchet space E has the property $(\widetilde{\Omega})$ if and only if E contains a bounded subset which is not uniformly polar. In [11] they have obtained a holomorphic characterization of nuclear Fréchet spaces E with $(\widetilde{\Omega})$ which is related to holomorphic extendability. Another problem considered here is the following. In [14] Vogt proved that the property $(\widetilde{\Omega})$ implies the property (LB^∞). Also in [14] by an example in the space $\Lambda(B)$ of Köthe sequences Vogt showed

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that the class (LB^∞) is strictly larger than $(\widetilde{\Omega})$. Next, in [11] the following question was raised: in what Fréchet spaces the property (LB^∞) implies $(\widetilde{\Omega})$? Our paper concerns the above two problems.

Let K be a compact set in a Stein space X. We denote by $H(K)$ the space of germs of holomorphic functions on K. The first main result of this paper establishes the relationship between the property (LB^∞) on $[H(K)]^*_\beta$ and the non-pluripolarity of K. Actually the following theorem is proved:

Theorem 3.1. *Let K be a compact subset in a Stein space X. Then the following conditions are equivalent*:

(i) $[H(K)]^*_\beta \in (\mathrm{LB}^\infty)$.

(ii) *$K \cap Z$ is not pluripolar in Z for every irreducible branch Z of every neighbourhood U of K in X with $K \cap Z \neq \emptyset$.*

As a consequence of the above theorem we obtain the following second main result.

Corollary 3.2. *Let K be a compact subset in a Stein space X. Then the following statements are equivalent*:

(i) $[H(K)]^*_\beta \in (\widetilde{\Omega})$.

(ii) $[H(K)]^*_\beta \in (\mathrm{LB}^\infty)$.

(iii) *$K \cap Z$ is not pluripolar in Z for every irreducible branch Z of every neighbourhood U of K in X with $K \cap Z \neq \emptyset$.*

(iv) *$w^*(x, K \cap Z, Z) < 1$ for every $x \in Z$ and every irreducible branch Z of every neighbourhood U of K in X with $K \cap Z \neq \emptyset$.*

Here we remark that Siciak's relative extremal function $w^*(\cdot, K \cap Z, Z)$ helps us in proving the implication (ii)$\Rightarrow$(i) in the above corollary.

Besides the introduction, the paper contains two sections. In the second we recall some definitions and fix the notations. The third is devoted to proving the main results of the paper.

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2. Preliminaries

2.1. *Plurisubharmonic functions on complex spaces.* Let X be a complex space and $\varphi : X \to [-\infty, +\infty)$ an upper semicontinuous function on X. We say that φ is *plurisubharmonic* on X if for every $a \in X$ and every local embedding h of a neighbourhood of a in X into a neighbourhood W of $h(a)$ in $\mathbb{C}^n$ there exists a plurisubharmonic function ψ on W such that $\psi \circ h$ coincides with φ on a neighbourhood of a. In [7] Fornæss and Narasimhan proved that an upper semicontinuous function $\varphi : X \to [-\infty, +\infty)$ is plurisubharmonic on X if and only if $\varphi \circ h$ is subharmonic for every holomorphic map $h : \Delta \to X$ of the unit disc $\Delta = \{t \in \mathbb{C} : |t| < 1\}$ into X.

A subset K of X is said to be *pluripolar* if for every $x \in K$ there exists a neighbourhood U of x in X and a plurisubharmonic function φ on U such that $\varphi|_{U \cap K} = -\infty$ and $\varphi \not\equiv -\infty$ on every irreducible branch of U.

According to Bedford [1], the Josefson theorem [10] is true in the case of Stein spaces, i.e. $K \subset X$ is pluripolar if and only if there exists a plurisubharmonic function φ on X such that $\varphi|_K = -\infty$ and $\varphi \not\equiv -\infty$ on every irreducible branch Z of X with $K \cap Z \neq \emptyset$.

2.2. *Extremal plurisubharmonic functions.* Let K be a compact set in a complex space X and U a neighbourhood of K in X. Put

$$\mathcal{U}(K, U) = \{\varphi \in \mathrm{PSH}(U) : \varphi|_K \le 0,\ \varphi \le 1 \text{ on } U\}$$

where $\mathrm{PSH}(U)$ denotes the cone of plurisubharmonic functions on U.

For $x \in U$ define

$$\begin{aligned} w(x, K, U) &= \sup\{\varphi(x) : \varphi \in \mathcal{U}(K, U)\}, \\ w^*(x, K, U) &= \limsup_{x' \to x} w(x', K, U), \qquad x \in U. \end{aligned}$$

It follows that $w^*(\cdot, K, U)$ is weakly plurisubharmonic on U, i.e. $w^*(\cdot, K, U)$ is upper semicontinuous and plurisubharmonic at every regular point of U (see [3]).

2.3. *Holomorphic functions.* Let E and F be locally convex spaces and D a non-empty open subset of E. A function $f : D \to F$ is called *holomorphic* if f is continuous and Gateaux-holomorphic. We denote by $H(D, F)$ the space of F-valued holomorphic functions on D equipped with the compact-open topology. Instead of $H(D, \mathbb{C})$ we write $H(D)$.

Now assume that K is a compact subset of E and let $H(K)$ denote the space of germs of holomorphic functions on K. This space is equipped with the inductive limit topology

$$H(K) = \liminf_{U \downarrow K} H^\infty(U)$$

where U ranges over all neighbourhoods of K in E and $H^\infty(U)$ denotes the Banach space of bounded holomorphic functions on U.

If K is a compact subset in a Stein space X, then by [2], $H(K)$ is a (DFN)-space.

For details concerning holomorphic functions and germs of holomorphic functions we refer to [2], [4] and [13].

3. The non-pluripolarity of a compact subset K in a Stein space and the properties (LB^∞) and $(\widetilde{\Omega})$ on $[H(K)]^*_\beta$. The main result of this section is the following.

3.1. Theorem. *Let K be a compact subset in a Stein space X. Then the following conditions are equivalent*:

(i) $[H(K)]^*_\beta \in (\mathrm{LB}^\infty)$.

(ii) *$K \cap Z$ is not pluripolar in Z for every irreducible branch Z of every neighbourhood U of K in X with $K \cap Z \neq \emptyset$.*

For the proof of Theorem 3.1 we need the following

3.2. Proposition. *Let $\Theta : X \to Y$ be a finite proper holomorphic surjection between Stein spaces and K a compact subset in Y. Then $[H(K)]^*_\beta \in (\mathrm{LB}^\infty)$ if and only if $[H(\Theta^{-1}(K))]^*_\beta \in (\mathrm{LB}^\infty)$.*

Proof. Sufficiency. Let $[H(\Theta^{-1}(K))]^*_\beta \in (\mathrm{LB}^\infty)$. Choose a decreasing neighbourhood basis $\{U_k\}_{k\geq 1}$ of K in Y. Since Θ is a proper holomorphic surjection, it follows that $\{V_k = \Theta^{-1}(U_k)\}_{k\geq 1}$ forms a neighbourhood basis of $\Theta^{-1}(K)$ in X (see [8]). Let $\{\varrho_n\}\uparrow +\infty$ and $p \geq 1$ be given. In view of the hypothesis we can find $q \geq p$ such that (see [2])

$\forall n_0 \geq q\ \exists N_0 \geq n_0,\ C > 0\ \forall f \in ([H(\Theta^{-1}(K))]^*_\beta)^*_\beta = H(\Theta^{-1}(K))$ $\exists k \in [n_0, N_0]$:

$$\|f\|_q^{*1+\varrho_k} \leq C\|f\|_k^*\|f\|_p^{*\varrho_k}$$

where

$$\|f\|_q^* = \sup\{|f(x)| : x \in V_q\} = \|f\|_q, \qquad \|f\|_k^* = \|f\|_k, \qquad \|f\|_p^* = \|f\|_p.$$

This shows that for every $g \in ([H(K)]^*_\beta)^*_\beta = H(K)$ (see [2]), there exists $k \in [n_0, N_0]$ such that

$$\|g\|_q^{*1+\varrho_k} \leq C\|g\|_k^*\|g\|_p^{*\varrho_k}$$

where

$$\|g\|_q^* = \sup\{|g(y)| : y \in U_q\}$$

and similarly for $\|g\|_k^*$ and $\|g\|_p^*$. Hence, by [14], $[H(K)]^*_\beta \in (\mathrm{LB}^\infty)$.

Necessity. Assume that $[H(K)]^*_\beta \in (\mathrm{LB}^\infty)$.

(i) First we consider the case where $\Theta : X \to Y$ is the normalization of Y. Put

$$\mathcal{S}_y = \{h \in H_{Y,y} : h(\Theta_* H_X)_y \subset H_{Y,y}\}$$

for each $y \in Y$, where $\Theta_* H_X$ is the direct image sheaf of the sheaf H_X. Since Θ is finite proper holomorphic, it follows that $\Theta_* H_X$ is H_Y-coherent. Hence $\mathcal{S}$ is an H_Y-coherent sheaf. In view of the Steinness of Y and by Cartan's Theorem B (see [9]) we can find $h \in H^0(Y, \mathcal{S})$ such that $h \not\equiv 0$ on every irreducible branch of Y. It follows that $\sigma \mapsto h\sigma$ defines an isomorphism between $H(\Theta^{-1}(K))$ and $h[H(\Theta^{-1}(K))] \subset H(K)$. Hence, $[H(\Theta^{-1}(K))]^*_\beta$ is a quotient space of $[H(K)]^*_\beta$. This shows that $[H(\Theta^{-1}(K))]^*_\beta \in (\mathrm{LB}^\infty)$.

(ii) Now assume that Y is a normal space. By the integrity lemma [6], Θ is a branched covering. Let n be the order of Θ. Then $H(\Theta^{-1}(K))$ is an integral extension of order n of $H(K)$.

The integrity lemma [6] implies that there exist continuous polynomials $p_0, p_1, \ldots, p_{n-1}$ on $H(\Theta^{-1}(K))$ with values in $H(K)$ such that

$$f^n + p_{n-1}(f)f^{n-1} + \ldots + p_0(f) = 0$$

for $f \in H(\Theta^{-1}(K))$.

In order to show that $[H(\Theta^{-1}(K))]^*_\beta \in (\mathrm{LB}^\infty)$, by [14] it suffices to prove that every continuous linear map $T : [H(\mathbb{C})]^*_\beta \to H(\Theta^{-1}(K))$ is bounded on some neighbourhood of $0 \in [H(\mathbb{C})]^*_\beta$. Since $p_j \circ T$ are continuous polynomials on $[H(\mathbb{C})]^*_\beta$ with values in $H(K)$ and $[H(K)]^*_\beta \in (\mathrm{LB}^\infty)$, by Vogt's above-mentioned result [14] we infer that these polynomials are bounded on a neighbourhood W of $0 \in [H(\mathbb{C})]^*_\beta$. Hence from the relation

$$[T(\mu)]^n + p_{n-1}[T(\mu)][T(\mu)]^{n-1} + \ldots + p_0[T(\mu)] \equiv 0$$

for $\mu \in [H(\mathbb{C})]^*_\beta$ it follows that T is bounded on W.

(iii) General case: Consider the commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\Theta}"] \arrow[d, "\widetilde{\nu}"'] & \widetilde{Y} \arrow[d, "\nu"] \\ X \arrow[r, "\Theta"'] & Y \end{tikzcd}$$

where $\nu : \widetilde{Y} \to Y$ is the normalization of Y and $\widetilde{X} = X \times_Y \widetilde{Y}$ is the fibre product of X and $\widetilde{Y}$ over Y, and $\widetilde{\Theta}$ and $\widetilde{\nu}$ are the canonical projections. Note that $\widetilde{\Theta}$ and $\widetilde{\nu}$ are finite proper holomorphic surjections.

By (i), $[H(\nu^{-1}(K))]^*_\beta \in (\mathrm{LB}^\infty)$, and by (ii), $[H(\widetilde{\Theta}^{-1}\nu^{-1}(K))]^*_\beta \in (\mathrm{LB}^\infty)$. But $\widetilde{\Theta}^{-1}\nu^{-1}(K) = \widetilde{\nu}^{-1}\Theta^{-1}(K)$. Hence $[H(\widetilde{\nu}^{-1}\Theta^{-1}(K))]^*_\beta \in (\mathrm{LB}^\infty)$. From the sufficiency part it follows that $[H(\Theta^{-1}(K))]^*_\beta \in (\mathrm{LB}^\infty)$.

Now we need the following lemma:

3.3. Lemma. *Let K be a compact set in a complex space X such that $[H(K)]^*_\beta$ has the property* (LB$^\infty$). *Then K is a set of uniqueness*, *i.e. if $f \in H(K)$ and $f|_K = 0$ then $f = 0$ on some neighbourhood of K.*

Proof. Let $\{U_n\}$ be a decreasing neighbourhood basis of K in X. Given $f \in H(K)$ with $f|_K = 0$, choose $p \geq 1$ such that $f \in H^\infty(U_p)$. For each $n \geq p$, put

$$\varepsilon_n = \|f\|_n = \sup\{|f(z)| : z \in U_n\}.$$

Then $\{\varepsilon_n\} \downarrow 0$. Using the property (LB$^\infty$) of $[H(K)]^*_\beta$ for the sequence

$$\{\varrho_n = \sqrt{-\log \varepsilon_n}\,\} \uparrow +\infty$$

and for the above p we find a $q \geq p$ such that for each $j = 1, 2, \ldots,$ there exist $N_j > j$, $C_j > 0$ such that for each m there exists k_m with $j \leq k_m \leq N_j$ such that

$$\|f^m\|_q^{1+\varrho_{k_m}} \leq C_j \|f^m\|_{k_m} \|f^m\|_p^{\varrho_{k_m}}.$$

This yields

$$\|f\|_q^{1+\varrho_{k_m}} \leq C_j^{1/m} \|f\|_{k_m} \|f\|_p^{\varrho_{k_m}}.$$

Choose k_j with $j \leq k_j \leq N_j$ such that

$$\#\{m : k_m = k_j\} = \infty.$$

Then

$$\|f\|_q \leq \|f\|_{k_j}^{1/(1+\varrho_{k_j})} \|f\|_p^{\varrho_{k_j}/(1+\varrho_{k_j})} \leq \varepsilon_{k_j}^{1/(1+\varrho_{k_j})} \varepsilon_p^{\varrho_{k_j}/(1+\varrho_{k_j})}.$$

Letting $j \to \infty$ we have $\varepsilon_{k_j}^{1/(1+\varrho_{k_j})} \to 0$. Hence $f|_{V_q} = 0$. The lemma is proved.

Proof of Theorem 3.1. (i)$\Rightarrow$(ii). Assume that $[H(K)]^*_\beta$ has the property (LB$^\infty$) and that $K \cap Z$ is pluripolar for some irreducible branch Z of a neighbourhood U of K in X with $K \cap Z \neq \emptyset$. Consider the normalization $\nu : \widetilde{X} \to X$ of X. Choose an irreducible branch $\widetilde{Z}$ of $\nu^{-1}(U)$ such that $\nu(\widetilde{Z}) = Z$. By the normality of $\nu^{-1}(U)$, it follows that $\widetilde{Z}$ is a connected component of $\nu^{-1}(U)$ and, hence, by Proposition 3.2, $[H(\nu^{-1}(K) \cap \widetilde{Z})]^*_\beta$ has the property (LB$^\infty$). Moreover, $E = \nu^{-1}(K) \cap \widetilde{Z}$ is pluripolar in $\widetilde{X}$. Choose a plurisubharmonic function φ on $\widetilde{X}$ such that

$$\varphi|_E = -\infty \quad \text{and} \quad \varphi \not\equiv -\infty \text{ on every irreducible branch of } \widetilde{X}.$$

Let W be a neighbourhood of E in $\widetilde{X}$ such that there exists a finite proper holomorphic map Θ from W onto the unit polydisc $\Delta^n \subset \mathbb{C}^n$, $n = \dim X$. For each $z \in \Delta^n \setminus S(\Theta)$ put

$$\widetilde{\varphi}(z) = \sum_{\Theta(x)=z} \varphi(x),$$

where $S(\Theta)$ denotes the branched locus of Θ.

Then $\widetilde{\varphi}$ is a plurisubharmonic function on $\Delta^n \setminus S(\Theta)$ and locally bounded above on Δ^n. The function $\widehat{\varphi}$ defined by the formula

$$\widehat{\varphi}(z) = \begin{cases} \widetilde{\varphi}(z) & \text{for } z \in \Delta^n \setminus S(\Theta), \\ \limsup\limits_{\substack{z' \to z \\ z' \in \Delta^n \setminus S(\Theta)}} \widetilde{\varphi}(z') & \text{for } z \in S(\Theta), \end{cases}$$

is plurisubharmonic on Δ^n. This function is equal to $-\infty$ on $\Theta(E)$. Indeed, let $z_0 \in \Theta(E)$. Write

$$\Theta^{-1}(z_0) = \{x^1, \ldots, x^q, x^{q+1}, \ldots, x^p\}$$

where $\varphi(x^j) = -\infty$ for $1 \le j \le q$ and $\varphi(x^j) \ne -\infty$ for $q+1 \le j \le p$. Let $M > 0$ be arbitrary. Since Θ is a finite proper holomorphic surjection, it follows that there exists a neighbourhood V of z_0 in Δ^n such that $\Theta^{-1}(V) \subset \bigcup_{j=1}^p U_j$, where the U_j are disjoint neighbourhoods of the points x^j, $1 \le j \le p$, respectively. Moreover, by the upper semicontinuity of φ we can assume that

$$\varphi(x) < \begin{cases} -M & \text{for } x \in U_j,\ 1 \le j \le q, \\ \varphi(x^j) + 1 & \text{for } x \in U_j,\ q+1 \le j \le p. \end{cases}$$

This shows that for $z' \in V \setminus S(\Theta)$ we have

$$\begin{aligned} \widehat{\varphi}(z') = \widetilde{\varphi}(z') &= \sum \Big\{\varphi(x') : \Theta(x') = z',\ x' \in \bigcup_{j=1}^q U_j\Big\} \\ &\quad + \sum \Big\{\varphi(x') : \Theta(x') = z',\ x' \in \bigcup_{j=q+1}^p U_j\Big\} \\ &< -M + (p-q) \max_{q+1 \le j \le p} (\varphi(x^j) + 1). \end{aligned}$$

Hence, $\widehat{\varphi}(z_0) = -\infty$.

Consider the Hartogs domain $\Omega_{\widehat{\varphi}}$ given by

$$\Omega_{\widehat{\varphi}} = \{(z, \lambda) \in \Delta^n \times \mathbb{C} : |\lambda| < e^{-\widehat{\varphi}(z)}\}.$$

Since $\Omega_{\widehat{\varphi}}$ is pseudoconvex, there exists $f \in H(\Omega_{\widehat{\varphi}})$ such that $\Omega_{\widehat{\varphi}}$ is the domain of existence of f (see [9]).

Write the Hartogs expansion of f on $\Omega_{\widehat{\varphi}}$:

$$f(z, \lambda) = \sum_{k \ge 0} f_k(z) \lambda^k$$

where

$$f_k(z) = \frac{1}{2\pi i} \int\limits_{|t| = e^{-\widehat{\varphi}(z) - \delta}} \frac{f(z,t)}{t^{k+1}}\, dt, \qquad \delta > 0.$$

In view of the upper semicontinuity of $\widehat{\varphi}$ it follows that the f_k are holomorphic on Δ^n. Consider the continuous linear map $T : [H(E)]^*_\beta \to H(\mathbb{C})$ given by

$$(T\mu)(\lambda) = \mu(f(\Theta(\cdot), \lambda))$$

for $\mu \in [H(E)]^*_\beta$ and $\lambda \in \mathbb{C}$. Since $H(E) = \liminf_{V \downarrow E} H^\infty(V)$ and $[H(E)]^*_\beta \in$ (LB$^\infty$), by [4], we can find a neighbourhood V of E in $\widetilde{Z}$ such that

$$T^* : [H(\mathbb{C})]^*_\beta \to ([H(E)]^*_\beta)^*_\beta = H(E)$$

(see [2]) maps $[H(\mathbb{C})]^*_\beta$ continuously into $H^\infty(V)$. This shows that f extends holomorphically to $\Theta(V)\times\mathbb{C}$. Hence $\widehat{\varphi}|_{\Theta(V)} = -\infty$. However, since $\Theta : W \to \Delta^n$ is an analytic covering and $\widehat{\varphi}|_{\Theta(V)} = -\infty$, it follows that $\varphi = -\infty$ on $\widetilde{Z}$. This is impossible.

(ii)⇒(i). By Vogt [14] it suffices to show that every continuous linear map $T : [H(\mathbb{C})]^*_\beta \to H(K)$ is bounded on a neighbourhood of $0 \in [H(\mathbb{C})]^*_\beta$. First consider the function $f : K \times \mathbb{C} \to \mathbb{C}$ given by

$$f(x,\lambda) = T(\delta_\lambda)(x) \quad \text{for } x \in K,\ \lambda \in \mathbb{C}$$

where δ_λ is the Dirac functional associated to λ. For each $n \geq 1$, put

$$A_n = \{\lambda \in \mathbb{C} : f^\lambda \in H^\infty(U_n),\ \|f^\lambda\|_{U_n} \leq n\}$$

where $\{U_n\}$ is a decreasing neighbourhood basis of K and $f^\lambda(x) = f(x,\lambda)$ for $x \in U_n$. The Montel theorem and the continuity of T imply that A_n is closed in $\mathbb{C}$ for every $n \geq 1$. We see that $\mathbb{C} = \bigcup_{n\geq 1} A_n$. The Baire theorem shows that there exists n_0 such that $\mathring{A}_{n_0} = \operatorname{Int} A_{n_0} \neq \emptyset$. Since K is a compact subset contained in U_{n_0}, it meets only finitely many irreducible branches of U_{n_0}. Hence, we may assume that U_{n_0} has only finitely many irreducible branches, all meeting K. By the hypothesis each branch meets K in a non-pluripolar subset.

Take a hypersurface H in X containing the singular locus $S(X)$ of X. Let $Y = X \setminus H$. Since H is a hypersurface and X is a Stein space, it follows that Y is a Stein manifold. Put $Y_0 = U_{n_0} \setminus H$. Then Y_0 is an open subset of Y. Since each irreducible branch of U_{n_0} is the closure of a connected component of Y_0, Y_0 has only finitely many connected components. At the same time, K intersects each connected component in a non-pluripolar subset. Let $\{Z_i\}_{i=1}$ be the connected components of Y_0. For each $i = 1,\dots,m$, write $K \cap Z_i$ as a countable union of compact subsets of Z_i. Since $K \cap Z_i$ is not pluripolar, it follows that for each $i = 1,\dots,m$ we can find a non-pluripolar compact subset $E_i \subset K \cap Z_i$. Let $E = \bigcup_{i=1}^m E_i$. Then E is a compact subset of Y_0 and $E \cap Z_i = E_i$ is not pluripolar in Z_i, $i = 1,\dots,m$. Since the holomorphic functions on Y_0 separate points of Y_0, Y_0 can be holomorphically embedded as an open subset of $\widehat{Y}_0$, the envelope of holomorphy of Y_0. Hence E is a compact subset of $\widehat{Y}_0$. Let

$$T_1 = T^*|_{[H(E)]^*_\beta} : [H(E)]^*_\beta \to H(\mathbb{C})$$

and let

$$T_1^* = [H(\mathbb{C})]^*_\beta \to ([H(E)]^*_\beta)^*_\beta = H(E)$$

be the conjugate map of T_1.

Consider the separately holomorphic function $g : (E\times\mathbb{C}) \cup (\widehat{Y}_0 \times \mathring{A}_{n_0}) \to \mathbb{C}$ given by

$$g(x,\lambda)=\begin{cases} T_1^*(\delta_\lambda)(x) & \text{for } (x,\lambda)\in E\times\mathbb{C},\\ \widehat{f^\lambda}(x) & \text{for } (x,\lambda)\in \widehat{Y}_0\times \mathring{A}_{n_0},\end{cases}$$

where $\widehat{f^\lambda}$ is the holomorphic extension of f^λ to $\widehat{Y}_0$ for $\lambda\in\mathring{A}_{n_0}$. Using a result of N. T. Van and A. Zeriahi [12] we see that g extends to a holomorphic function $\widehat{g}:\widehat{Y}_0\times\mathbb{C}\to\mathbb{C}$. Then $h=\widehat{g}\cdot(e\times \mathrm{id}):Y_0\times\mathbb{C}\to\mathbb{C}$ is holomorphic, where $e:Y_0\to\widehat{Y}_0$ is the canonical holomorphic embedding. Consider the holomorphic function $h_1:\mathbb{C}\to H(Y_0)$ induced by h. Since $h_1(\mathring{A}_{n_0})\subset H(U_{n_0})$ and $H(U_{n_0})$ is contained as a closed subspace in $H(Y_0)$, and from the uniqueness of holomorphic functions, we get $h_1(\mathbb{C})\subset H(U_{n_0})$. Hence, f extends to a holomorphic function $\widehat{f}:U_{n_0}\times\mathbb{C}\to\mathbb{C}$ such that $\widehat{f}(\cdot,\lambda)\in H\big(U_{n_0}\big)$ for all $\lambda\in\mathbb{C}$. By shrinking U_{n_0} we derive that

$$\sup_{\substack{x\in U_{n_0}\\ \lambda\in L}} |\widehat{f}(x,\lambda)|\le C_L$$

for each compact subset $L\subset\mathbb{C}$.

Hence, we can define a continuous linear map $S:[H^\infty(U_{n_0})]^*\to H(\mathbb{C})$ by

$$S(\mu)(\lambda)=\mu(\widehat{f}(\cdot,\lambda))$$

for $\lambda\in\mathbb{C}$.

The condition (ii) implies that K is a subset of uniqueness and, hence, $\operatorname{span}\delta(K)$ is weakly dense in $[H(K)]^*_\beta$. By [3], $[H(K)]^*_\beta$ is reflexive and it follows that $\operatorname{span}\delta(K)$ is dense in $[H(K)]^*_\beta$ where $\delta:K\to[H(K)]^*_\beta$ is defined by $\delta(x)(\varphi)=\varphi(x)$, $x\in K$, $\varphi\in H(K)$. Let $T^*:[H(K)]^*_\beta\to H(\mathbb{C})$ be the conjugate map of T. Now we have

$$\begin{aligned} T^*\Big(\sum_{j=1}^m\lambda_j\delta_{z_j}\Big)(\lambda) &= \sum_{j=1}^m\lambda_jT^*(\delta_{z_j})(\lambda)=\sum_{j=1}^m\lambda_jf(z_j,\lambda)\\ &=\sum_{j=1}^m\lambda_j\widehat{f}(z_j,\lambda)=\sum_{j=1}^m\lambda_jS(\delta_{z_j})(\lambda)=S\Big(\sum_{j=1}^m\lambda_j\delta_{z_j}\Big)(\lambda)\end{aligned}$$

for $\lambda\in\mathbb{C}$, $\{z_j\}\subset K$, $1\le j\le m$.

Hence, $S|_{[H(K)]^*_\beta}=T^*$ and T^* is bounded on a neighbourhood of $0\in[H(K)]^*_\beta$. Then from the reflexivity of $H(K)$ and $H(\mathbb{C})$ and, hence, $T=T^{**}$, as well as from the definition of the strong topologies β on $[H(K)]^*$ and $[H(\mathbb{C})]^*$ and the equality $(T^*(U))^0=T^{-1}(U^0)$ which holds for every neighbourhood U of $0\in[H(K)]^*_\beta$, where U^0 denotes the polar of U, it follows that T is bounded on a neighbourhood of $0\in[H(\mathbb{C})]^*_\beta$. Hence $[H(K)]^*_\beta\in$ (LB$^\infty$). Theorem 3.1 is proved.

Now we obtain the following corollary.

3.2. COROLLARY. *Let K be a compact subset in a Stein space X. Then the following statements are equivalent*:

(i) $[H(K)]^*_\beta \in (\widetilde{\Omega})$.

(ii) $[H(K)]^*_\beta \in (\mathrm{LB}^\infty)$.

(iii) *$K \cap Z$ is not pluripolar in Z for every irreducible branch Z of every neighbourhood U of K in X with $K \cap Z \neq \emptyset$.*

(iv) *$w^*(x, K \cap Z, Z) < 1$ for every $x \in Z$ and every irreducible branch Z of every neighbourhood U of K in Z with $K \cap Z \neq \emptyset$.*

Proof. (i)⇒(ii) follows from Vogt's results [14]; (ii)⇒(iii) by Theorem 3.1.

(iii)⇒(iv). Assume that there exists a neighbourhood U of K in X and an irreducible branch Z of U such that

$$w^*(x, K \cap Z, Z) = 1 \quad \text{ for } x \in Z.$$

Hence

$$w^*(x, \nu^{-1}(K) \cap \widetilde{Z}, \widetilde{Z}) = 1 \quad \text{ for } x \in \widetilde{Z}$$

where $\nu : \widetilde{X} \to X$ is the normalization of X and $\widetilde{Z}$ is an irreducible branch of $\nu^{-1}(U)$ such that $\nu(\widetilde{Z}) = Z$.

On the other hand, since the set

$$\{x \in \widetilde{Z} : w(x, \nu^{-1}(K) \cap \widetilde{Z}, \widetilde{Z}) < w^*(x, \nu^{-1}(K) \cap \widetilde{Z}, \widetilde{Z})\}$$

is pluripolar, we can find $\xi \in \widetilde{Z}$ such that for every $k \in \mathbb{N}$ we can find $\varphi_k \in \mathcal{U}(\nu^{-1}(K) \cap \widetilde{Z}, \widetilde{Z})$ satisfying $\varphi_k = 0$ on $\nu^{-1}(K) \cap \widetilde{Z}$, $\varphi_k \leq 1$ on $\widetilde{Z}$ and $\varphi_k(\xi) \geq 1 - 2^{-k}$. We claim that the function

$$\varphi(x) = \sum_{k \geq 1} (\varphi_k(x) - 1), \quad x \in \widetilde{Z},$$

is a plurisubharmonic function on $\widetilde{Z}$ and $\varphi = -\infty$ on $\nu^{-1}(K) \cap \widetilde{Z}$. Indeed, since φ is the limit of the decreasing sequence of its partial sums which are plurisubharmonic and $\varphi(\xi) \geq -1$, it follows that $\varphi \in \mathrm{PSH}(\widetilde{Z})$. Obviously, $\varphi = -\infty$ on $\nu^{-1}(K) \cap \widetilde{Z}$. By (ii)⇒(iii) we derive that $[H(\nu^{-1}(K) \cap \widetilde{Z})]^*_\beta$ does not have (LB^∞). Proposition 3.2 also yields that $[H(K \cap Z)]^*_\beta$ does not have (LB^∞). Since (ii)⇔(iii), it follows that $K \cap Z$ is pluripolar. That contradicts the hypothesis (iii).

(iv)⇒(i). Let U be a neighbourhood of K in X. By hypothesis, $w^*(x, K, U) < 1$ for $x \in K$. From the upper semicontinuity of $w^*(\cdot, K, U)$ we can find a neighbourhood V of K in U such that

$$d = \sup\{w^*(x, K, U) : x \in V\} < 1.$$

Let $f \in H^\infty(U)$. Then for each neighbourhood W of K with $K \subset W \subset V$ the function

$$\frac{\log|f(z)| - \log\|f\|_W}{\log\|f\|_U - \log\|f\|_W}, \qquad z \in U,$$

belongs to $\mathcal{U}(K, U)$. Hence for $z \in V$ we have

$$\frac{\log|f(z)| - \log\|f\|_W}{\log\|f\|_U - \log\|f\|_W} \le d.$$

This shows that

$$\|f\|_V \le \|f\|_W^{1-d}\|f\|_U^d$$

and, hence, $[H(K)]^*_\beta$ has the property $(\widetilde{\Omega})$.

References

[1] E. Bedford, *The operator* $(dd^c)^n$ *on complex spaces*, in: Séminaire "Lelong–Skoda", Lecture Notes in Math. 919, Springer, 1981, 294–323.

[2] K. D. Bierstedt and R. Meise, *Nuclearity and the Schwartz property in the theory of holomorphic functions on metrizable locally convex spaces*, in: Infinite Dimensional Holomorphy and Applications, M. C. Matos (ed.), North-Holland Math. Stud. 12, North-Holland, 1977, 93–129.

[3] J. P. Demailly, *Mesures de Monge–Ampère et caractérisation géométrique des variétés algébriques affines*, Mém. Soc. Math. France 19 (1985).

[4] S. Dineen, *Complex Analysis in Locally Convex Spaces*, North-Holland Math. Stud. 57, North-Holland, 1981.

[5] S. Dineen, R. Meise and D. Vogt, *Characterization of nuclear Fréchet spaces in which every bounded set is polar*, Bull. Soc. Math. France 112 (1984), 41–68.

[6] G. Fischer, *Complex Analytic Geometry*, Lecture Notes in Math. 538, Springer, 1976.

[7] J. E. Fornæss and R. Narasimhan, *The Levi problem on complex spaces with singularities*, Math. Ann. 284 (1980), 47–72.

[8] H. Grauert and R. Remmert, *Coherent Analytic Sheaves*, Springer, 1984.

[9] R. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, NJ, 1965.

[10] B. Josefson, *On the equivalence between locally polar and globally polar sets for plurisubharmonic functions on* $\mathbb{C}^n$, Ark. Mat. 16 (1978), 109–115.

[11] R. Meise and D. Vogt, *Holomorphic functions of uniformly bounded type on nuclear Fréchet spaces*, Studia Math. 83 (1986), 117–166.

[12] Nguyen Thanh Van et A. Zeriahi, *Une extension du théorème de Hartogs sur les fonctions séparément analytiques*, in: Analyse Complexe Multivariable, Récents Développements, A. Meril (ed.), Editel, Rende, 1991, 183–194.

[13] P. Novérraz, *Pseudoconvexité, Convexité Polynomiale et Domaines d'Holomorphie en Dimension Infinie*, North-Holland Math. Stud. 3, North-Holland, 1973.

[14] D. Vogt, *Fréchetträume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. Reine Angew. Math. 345 (1983), 182–200.

[15] D. Vogt, *Eine Charakterisierung der Potenzreihenräume von endlichem Typ und ihre Folgerungen*, Manuscripta Math. 37 (1982), 269–301.

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