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Uniqueness of unconditional basis of $\ell_p(c_0)$ and $\ell_p(\ell_2)$, $0 < p < 1$

by

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Abstract. We prove that the quasi-Banach spaces $\ell_p(c_0)$ and $\ell_p(\ell_2)$ $(0 < p < 1)$ have a unique unconditional basis up to permutation. Bourgain, Casazza, Lindenstrauss and Tzafriri have previously proved that the same is true for the respective Banach envelopes $\ell_1(c_0)$ and $\ell_1(\ell_2)$. They used duality techniques which are not available in the non-locally convex case.

1. Introduction. Suppose that X is a quasi-Banach space (in particular, a Banach space) with a quasi-norm $\|\cdot\|$ and a normalized unconditional basis $(e_n)_{n=1}^{\infty}$, i.e. $\|e_n\| = 1$ for all $n \in \mathbb{N}$. We say that X has a *unique unconditional basis up to permutation* if whenever $(x_n)_{n=1}^{\infty}$ is another normalized unconditional basis of X, then there is a permutation π of $\mathbb{N}$ so that $(x_n)_{n=1}^{\infty}$ is *equivalent* to $(e_{\pi(n)})_{n=1}^{\infty}$, that is, there is a constant D so that

$$D^{-1}\Big\|\sum_{i=1}^{n} a_i x_i\Big\| \le \Big\|\sum_{i=1}^{n} a_i e_{\pi(i)}\Big\| \le D\Big\|\sum_{i=1}^{n} a_i x_i\Big\|$$

for any choice of scalars $(a_i)_{i=1}^{n}$ and every $n \in \mathbb{N}$. In that case, we will write $(x_n)_{n\in\mathbb{N}} \sim (e_{\pi(n)})_{n\in\mathbb{N}}$.

In [7], the authors showed that ℓ_2 has a unique unconditional basis. Lindenstrauss and Pełczyński ([9]) proved that the same holds for c_0 and ℓ_1. Lindenstrauss and Zippin ([11]) showed that c_0, ℓ_1 and ℓ_2 are the only Banach spaces with this property, which indicates that in the context of Banach spaces it is quite exceptional for a space to have a unique unconditional basis. The situation for quasi-Banach spaces which are not Banach is quite different. Kalton ([3]) showed that a wide class of non-locally convex Orlicz sequence spaces, including ℓ_p for $0 < p < 1$, have a unique unconditional basis. Edelstein and Wojtaszczyk proved in [2] that finite direct sums of c_0, ℓ_1 and ℓ_2 have a unique unconditional basis up to permutation. Bourgain, Casazza, Lindenstrauss and Tzafriri studied in [1] infinite direct

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sums of these spaces, showing that in $c_0(\ell_1)$, $c_0(\ell_2)$, $\ell_1(c_0)$ and $\ell_1(\ell_2)$ the canonical unit vector basis is unique up to permutation, while the result is not true for $\ell_2(c_0)$ and $\ell_2(\ell_1)$. This was the motivation to ask about the uniqueness of unconditional basis up to permutation of the quasi-Banach spaces $c_0(\ell_p)$, $\ell_p(c_0)$ and $\ell_p(\ell_2)$ $(0 < p < 1)$, whose Banach envelopes $c_0(\ell_1)$, $\ell_1(c_0)$ and $\ell_1(\ell_2)$, respectively, were the ones on the previous list which had that property. Leránoz proved in [8] that in $c_0(\ell_p)$, $0 < p < 1$, all normalized unconditional bases are equivalent up to permutation to the canonical basis.

In Sections 2 and 3 we prove that the same result is true for the spaces $\ell_p(c_0)$ and $\ell_p(\ell_2)$, respectively.

We recall that if X is a quasi-Banach space whose dual separates points, then the gauge functional of the convex hull of the closed unit ball of X is a norm on X; we will denote it by $\|\cdot\|_{\mathrm{c}}$. The Banach space $\widehat{X}$ resulting from the completion of $(X, \|\cdot\|_{\mathrm{c}})$ is called the *Banach envelope* of X (see [6] and [4]). The Banach envelope has the property that every continuous linear operator from X into a Banach space extends to $\widehat{X}$ with preservation of norm. In particular, the dual of $\widehat{X}$ is X^*. If $(e_n)_{n=1}^{\infty}$ is a K-unconditional basis X, then it is also a K-unconditional basis of $\widehat{X}$, and

$$K^{-1}\|e_n\| \le \|e_n\|_{\mathrm{c}} \le \|e_n\| \quad \text{for all } n \in \mathbb{N}. \tag{1.1}$$

A quasi-Banach lattice X is said to be *p-convex*, where $0 < p < \infty$, if there is a constant $C > 0$ such that for any $x_1, \dots, x_n \in X$ and $n \in \mathbb{N}$, we have

$$\Big\|\Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p}\Big\| \le C\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{1/p}.$$

The procedure to define the element $(\sum_{i=1}^n |x_i|^p)^{1/p} \in X$ is exactly the same as in Banach lattices (see [10], pp. 40–41).

2. Uniqueness of unconditional basis of $\ell_p(c_0)$, $0 < p < 1$. For $0 < p \le 1$ fixed,

$$\ell_p(c_0) = \{(\overline{x}_l)_{l=1}^{\infty} : \overline{x}_l \in c_0 \text{ for each } l \text{ and } (\|\overline{x}_l\|_\infty)_{l=1}^{\infty} \in \ell_p\}.$$

This space endowed with the p-norm

$$\|(\overline{x}_l)_l\|_p = \Big(\sum_{l=1}^{\infty} \|\overline{x}_l\|_\infty^p\Big)^{1/p}$$

is a p-Banach space.

For each $l \in \mathbb{N}$, we can write $\overline{x}_l = (x_{l1}, x_{l2}, \dots, x_{lk}, \dots) \in c_0$, and then identify $\ell_p(c_0)$ with the space of infinite matrices $(x_{lk})_{l,k=1}^{\infty}$ satisfying

$x_{lk} \to 0$ for all $l \in \mathbb{N}$ as $k \to \infty$, and

$$\|(x_{lk})_{l,k}\|_p = \Big(\sum_{l=1}^{\infty} \sup_k |x_{lk}|^p\Big)^{1/p} < \infty.$$

The dual space of $\ell_p(c_0)$ can be identified with $\ell_\infty(\ell_1)$, where $\ell_\infty(\ell_1)$ is the Banach space of infinite matrices $a = (a_{lk})_{l,k=1}^{\infty}$ such that

$$\|a\| = \sup_l \sum_{k=1}^{\infty} |a_{lk}| < \infty,$$

and the Banach envelope of $(\ell_p(c_0), \|\cdot\|_p)$ is $(\ell_1(c_0), \|\cdot\|_1)$.

$\|\cdot\|$ will denote without confusion both the quasi-norm in $\ell_p(c_0)$ and the norm in the dual $\ell_\infty(\ell_1)$, and $\|\cdot\|_{\mathrm{c}}$ will denote the norm in the Banach envelope $\ell_1(c_0)$.

The spaces $\ell_p(c_0)$ $(0 < p \le 1)$ have a canonical 1-unconditional basis of unit vectors that we will denote by $(e_{lk})_{l,k=1}^{\infty}$. The (l,k) coordinate of $e_{l_0 k_0}$ is 1 if $l = l_0$ and $k = k_0$, and 0 otherwise. The lattice structure induced by the canonical basis in $\ell_p(c_0)$ $(0 < p \le 1)$ is p-convex.

Suppose Q is a bounded linear projection from $\ell_p(c_0)$ onto a subspace X with normalized K-unconditional basis $(x_n)_{n=1}^{\eta}$ (the symbol η can denote either a positive integer or ∞). The sequences in $\ell_\infty(\ell_1)$ of the biorthogonal linear functionals associated with the unconditional bases (e_{lk}) and (x_n) are denoted by (e_{lk}^*) and (x_n^*) respectively. From now on, for abbreviation, we will write

$$e_{lk}^*(x_n) = b_{lk}^n \quad \text{and} \quad x_n^*(e_{lk}) = a_{lk}^n.$$

Then

$$\|x_n^*\| = \sup_l \sum_{k=1}^{\infty} |a_{lk}^n| \le K\|Q\|. \tag{2.1}$$

We also recall that $(x_n)_{n=1}^{\eta}$ is a K-unconditional basis of $\widehat{X}$, the Banach envelope of X, which is complemented in $\ell_1(c_0)$, and that from (2.1) we easily obtain

$$(\|Q\|K)^{-1} \le \|x_n\|_{\mathrm{c}} \le 1 \quad \text{for all } n = 1, \ldots, \eta.$$

In this section, we prove the next theorem:

THEOREM 2.1. *Suppose $0 < p < 1$. Let Q be a bounded linear projection from $\ell_p(c_0)$ onto a subspace X with a normalized K-unconditional basis $(x_n)_{n=1}^{\eta}$. Then there exist constants Δ_1, Δ_2 and a partition of $\{1, \ldots, \eta\}$ into mutually disjoint subsets $(L_i)_{i=1}^{I}$ so that*

$$\Delta_1 \Big(\sum_{i=1}^{I} \sup_{n \in L_i} |a_n|^p\Big)^{1/p} \overset{(2)}{\le} \Big\|\sum_{n=1}^{\eta} a_n x_n\Big\| \overset{(1)}{\le} \Delta_2 \Big(\sum_{i=1}^{I} \sup_{n \in L_i} |a_n|^p\Big)^{1/p}$$

for any choice of scalars $(a_n)_n$.

Bourgain, Casazza, Lindenstrauss and Tzafriri proved the analogue of the previous result for the Banach envelope $\ell_1(c_0)$:

Theorem 2.2 (Corollary 4.8 of [1]). *Let Q be a bounded linear projection from $\ell_1(c_0)$ onto a subspace Z with a normalized K-unconditional basis $(z_n)_{n=1}^{\eta}$. Then there exist a constant Δ, depending only on K and $\|Q\|$, and a partition of the integers $\{1,\dots,\eta\}$ into mutually disjoint subsets $(B_j)_{j=1}^{J}$ such that*

$$\Delta^{-1}\sum_{j=1}^{J}\sup_{n\in B_j}|a_n| \le \Big\|\sum_{n=1}^{\eta}a_n z_n\Big\|_{\mathrm{c}} \le \Delta\sum_{j=1}^{J}\sup_{n\in B_j}|a_n| \tag{2.2}$$

for any choice of scalars $(a_n)_n$. In particular, $\ell_1(c_0)$ has a unique normalized unconditional basis up to permutation.

The proof of Theorem 2.2 uses duality techniques which are not available in the non-locally convex case. We will prove Theorem 2.1 in two parts, corresponding to each one of the inequalities (1) and (2).

The proof of Theorem 2.1(1) is based on two deep results: Theorem 2.2 itself and the following theorem due to Kalton.

Lemma 2.3 (Theorem 3.3 of [4]). *Let Y be a p-convex quasi-Banach lattice with unconditional basis such that Y^* has finite cotype. Then there exists a constant A, depending only on p and the cotype constant of Y^*, such that*

$$\|y\|_Y \le A\|y\|_{\mathrm{c}}$$

for every $y\in Y$. (In particular, Y is isomorphic to its Banach envelope.)

Proof of Theorem 2.1(1). The Banach envelope $\widehat{X}$ of X is a complemented subspace of $\ell_1(c_0)$ and $(x_n)_{n=1}^{\eta}$ is a K-unconditional basis of $\widehat{X}$, equivalent in $\ell_1(c_0)$ to the normalized basis $(x_n/\|x_n\|_{\mathrm{c}})_{n=1}^{\eta}$. Therefore, Theorem 2.2 applies, hence there exist a constant D, depending only on K and $\|Q\|$, and a partition of $\{1,\dots,\eta\}$ into disjoint subsets $(B_j)_{j=1}^{J}$ so that (2.2) holds. We will see that this is the partition $(L_i)_{i=1}^{I}$ stated in Theorem 2.1.

For each $j\in\{1,\dots,J\}$ let X_j be the closed linear span in $\ell_p(c_0)$ of $\{x_n : n\in B_j\}$. The Banach envelope $\widehat{X}_j$ of X_j is the closed linear span in $\ell_1(c_0)$ of $\{x_n : n\in B_j\}$. By (2.2) applied to each fixed j, we obtain

$$\Delta^{-1}\sup_{n\in B_j}|a_n| \le \Big\|\sum_{n\in B_j}a_n x_n\Big\|_{\mathrm{c}} \le \Delta\sup_{n\in B_j}|a_n|, \tag{2.3}$$

for any scalars $(a_n)_n$. That is, $(x_n)_{n\in B_j}$ is Δ-equivalent (in $\ell_1(c_0)$) to $(e_n)_{n\in B_j}$, where $(e_n)_{n=1}^{\infty}$ denotes the canonical basis of c_0, with the equivalence constant Δ independent of j. Hence, X_j^* $(=\widehat{X}_j^*\simeq_{\Delta'}\ell_1^{(|B_j|)})$ has cotype 2.

By Lemma 2.3 applied to X_j (p-convex quasi-Banach lattice with an unconditional basis $(x_n)_{n\in B_j}$, so that X_j^* has cotype 2) there exists a constant A (not depending on j) so that

$$\Big\|\sum_{n\in B_j} a_n x_n\Big\| \le A\Big\|\sum_{n\in B_j} a_n x_n\Big\|_{\mathrm{c}} \tag{2.4}$$

for any choice of scalars $(a_n)_n$.

By the p-subadditivity of $\|\cdot\|$,

$$\Big\|\sum_{n=1}^{\eta} a_n x_n\Big\|^p = \Big\|\sum_{j=1}^{J}\sum_{n\in B_j} a_n x_n\Big\|^p \le \sum_{j=1}^{J}\Big\|\sum_{n\in B_j} a_n x_n\Big\|^p. \tag{2.5}$$

Combining (2.3), (2.4) and (2.5) we get

$$\Big\|\sum_{n=1}^{\eta} a_n x_n\Big\|^p \le A^p\Delta^p\sum_{j=1}^{J}\sup_{n\in B_j}|a_n|^p$$

for any scalars (a_n), so the inequality (1) holds with $\Delta_2 = A\Delta$.

The proof of Theorem 2.1(2) relies on the next four lemmas.

The following technique was introduced by N. Kalton to prove the uniqueness of unconditional basis in non-locally convex Orlicz sequence spaces (cf. [3]), and was used to prove the uniqueness of unconditional basis up to permutation of $c_0(\ell_p)$ $(0<p<1)$ (see [8]).

Lemma 2.4 (cf. Theorem 2.3 of [5]). *Let X be a p-convex quasi-Banach lattice $(0<p<1)$ with a normalized unconditional basis $(e_n)_{n=1}^{\infty}$; let Q be a bounded linear projection from X onto a subspace Z with a normalized unconditional basis $(x_n)_{n\in S}$ $(S\subset\mathbb{N})$. Let $(e_n^*)_{n=1}^{\infty}$ and $(x_n^*)_{n\in S}$ be the sequences of biorthogonal linear functionals associated with $(e_n)_{n=1}^{\infty}$ and $(x_n)_{n\in S}$ respectively, i.e.*

$$x=\sum_{n=1}^{\infty} e_n^*(x)e_n \quad \textit{and} \quad Q(x)=\sum_{n\in S} x_n^*(x)x_n$$

for all $x\in X$. Suppose that there is a constant $\beta>0$ and an injective map $\sigma: S\to\mathbb{N}$ so that

$$|e_{\sigma(n)}^*(x_n)|\ge\beta \quad \textit{and} \quad |x_n^*(e_\sigma(n))|\ge\beta$$

for all $n\in S$. Then there exist positive constants ϱ, ϱ' so that

$$\varrho\Big\|\sum_{n\in S}\alpha_n e_{\sigma(n)}\Big\| \le \Big\|\sum_{n\in S}\alpha_n x_n\Big\| \le \varrho'\Big\|\sum_{n\in S}\alpha_n e_{\sigma(n)}\Big\|$$

for any scalars (α_n).

The following result is an elementary remark which nevertheless becomes fundamental when we want to apply Lemma 2.4 to those quasi-Banach spaces in which ℓ_p $(0 < p < 1)$ is involved.

LEMMA 2.5 (Lemma 2.2 of [8]). *Fix* $0 < p < 1$. *For any* $\varepsilon > 0$ *there exist* $C_\varepsilon > 0$ *so that*

$$\sum_{n=1}^{N} |\alpha_n| \le C_\varepsilon \sup_n |\alpha_n| + \varepsilon \Big(\sum_{n=1}^{N} |\alpha_n|^p \Big)^{1/p}$$

for any choice of scalars $(\alpha_n)_{n=1}^N$ *and* $N \in \mathbb{N}$.

The next result is a "patching lemma":

LEMMA 2.6. *Suppose* $\{\Lambda_i : i = 1, \ldots, N\}$ *is a partition of* $\{1, \ldots, J\}$ *and that for each* $j = 1, \ldots, J$, $\{\Omega_j^m : m = 1, \ldots, M\}$ *is a partition of* B_j. *Suppose there is a constant* $\varrho > 0$ *so that for each* i *and* m

$$\Big\| \sum_{n\in\Omega_j^m,\, j\in\Lambda_i} a_n x_n \Big\| = \Big\| \sum_{j\in\Lambda_i} \sum_{n\in\Omega_j^m} a_n x_n \Big\| \ge \varrho \Big(\sum_{j\in\Lambda_i} \sup_{n\in\Omega_j^m} |a_n|^p \Big)^{1/p}$$

for any sequence of scalars (a_n). *Further suppose that* M, N *and* ϱ *depend only on* K *and* $\|Q\|$. *Then there exists a constant* $\Gamma > 0$ *so that*

$$\Big\| \sum_{n=1}^{\eta} a_n x_n \Big\| = \Big\| \sum_{j=1}^{J} \sum_{n\in B_j} a_n x_n \Big\| \ge \Gamma \Big(\sum_{j=1}^{J} \sup_{n\in B_j} |a_n|^p \Big)^{1/p}$$

for any sequence of scalars (a_n).

Proof. By the unconditionality of the basis $(x_n)_{n=1}^{\eta}$ we have

$$\Big\| \sum_{j\in\Lambda_i} \sum_{n\in\Omega_j^m} a_n x_n \Big\| \le K \Big\| \sum_{j=1}^{J} \sum_{n\in B_j} a_n x_n \Big\| = K \Big\| \sum_{n=1}^{\eta} a_n x_n \Big\|,$$

for each i and m fixed. Then, by the hypothesis,

$$\Big\| \sum_{n=1}^{\eta} a_n x_n \Big\|^p \ge K^{-p} \varrho^p \sum_{j\in\Lambda_i} \sup_{n\in\Omega_j^m} |a_n|^p;$$

therefore

$$\Big\| \sum_{n=1}^{\eta} a_n x_n \Big\|^p \ge K^{-p} \varrho^p \sup_{1\le i\le N} \sup_{1\le m\le M} \sum_{j\in\Lambda_i} \sup_{n\in\Omega_j^m} |a_n|^p.$$

Now,

$$\sup_{1\le i\le N}\Big(\sup_{1\le m\le M}\sum_{j\in\Lambda_i}\sup_{n\in\Omega_j^m}|a_n|^p\Big)\ge\frac{1}{N}\sum_{i=1}^{N}\Big(\sup_{1\le m\le M}\sum_{j\in\Lambda_i}\sup_{n\in\Omega_j^m}|a_n|^p\Big)$$
$$\ge\frac{1}{NM}\sum_{i=1}^{N}\sum_{m=1}^{M}\sum_{j\in\Lambda_i}\sup_{n\in\Omega_j^m}|a_n|^p=\frac{1}{NM}\sum_{i=1}^{N}\sum_{j\in\Lambda_i}\sum_{m=1}^{M}\sup_{n\in\Omega_j^m}|a_n|^p$$
$$\ge\frac{1}{NM}\sum_{i=1}^{N}\sum_{j\in\Lambda_i}\sup_{1\le m\le M}\sup_{n\in\Omega_j^m}|a_n|^p=\frac{1}{NM}\sum_{j=1}^{J}\sup_{n\in B_j}|a_n|^p.$$

Hence

$$\Big\|\sum_{n=1}^{\eta}a_nx_n\Big\|\ge\frac{\varrho}{KM^{1/p}N^{1/p}}\Big(\sum_{j=1}^{J}\sup_{n\in B_j}|a_n|^p\Big)^{1/p}.$$

The following is a “counting lemma”.

LEMMA 2.7. *Let $\{F_j : j\in\mathbb{N}\}$ be a family of sets so that*:

(i) $|F_j|\le r$ *for all* $j\in\mathbb{N}$,
(ii) $|\{j\in\mathbb{N} : \bar{l}\in F_j\}|\le M$ *for every* $\bar{l}$,

for some constants r, M independent of j. Then there exists a partition of $\mathbb{N}$ into $N\le rM$ subsets

$$\mathbb{N}=S_1\cup\ldots\cup S_N$$

so that $F_{j_1}\cap F_{j_2}=\emptyset$ for any $j_1\ne j_2\in S_i$, $0\le i\le N$.

Proof. From the hypotheses it is easy to deduce that, for each $j_0\in\mathbb{N}$ fixed,

$$|\{j\in\mathbb{N} : F_j\cap F_{j_0}\ne\emptyset\}|\le rM.$$

We will prove the existence of a partition $\mathbb{N}=S_1\cup\ldots\cup S_N$ satisfying the assertion by building the sets $S_1,S_2,\ldots$ recurrently, as follows:

• We put the element $j_0=1$ into a set, namely S_1.

• Now, for $j_0=2$, if $F_2\cap F_1=\emptyset$ then we put the element 2 into the set S_1 (because we still do not have any needs of taking a new set in order for the lemma to be satisfied), but if $F_2\cap F_1\ne\emptyset$ then we must put the element 2 into a new set, namely S_2.

• Let $j_0>2$.

If $F_{j_0}\cap F_j=\emptyset$ for all $j\in S_1$ ($j<j_0$, of course) then $j_0\in S_1$.

If $F_{j_0}\cap F_j\ne\emptyset$ for some $j\in S_1$ then $j_0\notin S_1$. Now, if $F_{j_0}\cap F_j=\emptyset$ for all $j\in S_2$ then $j_0\in S_2$.

. . .

If $F_{j_0} \cap F_j \neq \emptyset$ for some $j \in S_1$ $(j < j_0)$, $F_{j_0} \cap F_j \neq \emptyset$ for some $j \in S_2$ $(j < j_0)$, ..., $F_{j_0} \cap F_j \neq \emptyset$ for some $j \in S_{k-1}$ $(j < j_0)$, then we put j_0 into S_k, where

$$k = \min\{m \in \mathbb{N} : \text{there is no } j \in S_m \ (j < k) \text{ such that } F_{j_0} \cap F_j \neq \emptyset\}.$$

Since $|\{j \in \mathbb{N} : F_j \cap F_{j_0} \neq \emptyset\}| \leq rM$ it follows that

$$|\{j \in \mathbb{N},\, j < j_0 : F_j \cap F_{j_0} \neq \emptyset\}| \leq rM,$$

so k is finite and uniformly bounded by rM, the number of sets we need, at most, in order to distribute all j's according to the lemma.

Proof of Theorem 2.1(2). For each $n \in B_j$, $j \in \{1, \ldots, J\}$, we have

$$1 = x_n^*(x_n) = x_n^*\Big(\sum_{l,k=1}^{\infty} b_{lk}^n e_{lk}\Big) = \sum_{l,k=1}^{\infty} b_{lk}^n a_{lk}^n \leq \sum_{l=1}^{\infty}\Big(\sum_{k=1}^{\infty} |a_{lk}^n b_{lk}^n|\Big).$$

On the other hand, since $\|x_n\| = 1$ and $\|x_n^*\| \leq K\|Q\|$,

$$\sum_{l=1}^{\infty}\Big(\sum_{k=1}^{\infty} |a_{lk}^n b_{lk}^n|\Big)^p \leq \sum_{l=1}^{\infty} \sup_k |b_{lk}^n|^p \Big(\sum_{k=1}^{\infty} |a_{lk}^n|\Big)^p \leq \Big(\sup_l \sum_{k=1}^{\infty} |a_{lk}^n|\Big)^p \sum_{l=1}^{\infty} \sup_k |b_{lk}^n|^p$$
$$= \|x_n^*\|^p \|x_n\|^p \leq K^p \|Q\|^p.$$

From Lemma 2.5 applied to the sequence $(\sum_{k=1}^{\infty} |a_{lk}^n b_{lk}^n|)_{l\in\mathbb{N}}$, with $\varepsilon = 1/(2K\|Q\|)$, there is a constant $C = C(\varepsilon)$ so that

$$1 \leq \sum_{l=1}^{\infty}\Big(\sum_{k=1}^{\infty} |a_{lk}^n b_{lk}^n|\Big) \leq C \sup_l \sum_{k=1}^{\infty} |a_{lk}^n b_{lk}^n| + \frac{1}{2}.$$

Then

$$\sup_l \sum_{k=1}^{\infty} |a_{lk}^n b_{lk}^n| \geq \frac{1}{2C},$$

and therefore, there exists $l = l(n)$ so that $\sum_{k=1}^{\infty} |a_{lk}^n b_{lk}^n| > 1/(4C)$.

Thus, we can define a function $\{1, \ldots, \eta\} \to \mathbb{N}, n \mapsto l_n$, so that

$$\sum_{k=1}^{\infty} |a_{l_n k}^n b_{l_n k}^n| > \frac{1}{4C}.$$

Let us remark that

$$\frac{1}{4C} < \sum_{k=1}^{\infty} |a_{l_n k}^n b_{l_n k}^n| \leq \sup_k |b_{l_n k}^n| \sum_{k=1}^{\infty} |a_{l_n k}^n|,$$

and, in particular, we have

$$\sup_k |b_{l_n k}^n| > \frac{1}{4CK\|Q\|} \quad \text{and} \quad \sum_{k=1}^{\infty} |a_{l_n k}^n| > \frac{1}{4C}$$

for each $n \in \{1, \ldots, \eta\}$.

We want the correspondence $\{1,\ldots,\eta\}\to\mathbb{N}$, $n\mapsto l_n$, to be injective on j, that is, to all n's belonging to B_j (j fixed) should correspond the same l, and to n's in different B_j's should correspond different l's. We will see that we are not essentially far from this situation.

For each given $j\in\{1,\ldots,J\}$, we define

$$F_j := \{l_n : n\in B_j\}.$$

Let us first see that $|F_j|$ is uniformly bounded on j. Fix $j\in\{1,\ldots,J\}$ and suppose that $l_1,\ldots,l_r$ are different elements in F_j, that is, there exist $n_1,\ldots,n_r\in B_j$ so that

$$\sup_k |b^{n_i}_{l_i k}| > \frac{1}{4CK\|Q\|}, \qquad i=1,\ldots,r.$$

Then, by Banach lattice estimates (Theorem 1.d.6 of [10]),

$$\begin{aligned}
1 &\overset{\Delta}{\sim} \|x_{n_1}+\ldots+x_{n_r}\|_{\mathrm{c}} \overset{D}{\sim} \|(|x_{n_1}|^2+\ldots+|x_{n_r}|^2)^{1/2}\|_{\mathrm{c}}\\
&= \sum_{l=1}^{\infty}\sup_k\Big(\sum_{i=1}^{r}|b^{n_i}_{lk}|^2\Big)^{1/2} \ge \sum_{s=1}^{r}\sup_k\Big(\sum_{i=1}^{r}|b^{n_i}_{l_s k}|^2\Big)^{1/2} \ge \sum_{i=1}^{r}\sup_k |b^{n_i}_{l_i k}|\\
&\ge r\frac{1}{4CK}.
\end{aligned}$$

Thus, for j fixed, there is a partition of the set B_j,

$$B_j = B_j^{(1)}\cup\ldots\cup B_j^{(r)},$$

in such a way that to all n's belonging to one of these subsets $B_j^{(i)}$ corresponds the same l, i.e. $l_n =: l_{ji}$ for any $n\in B_j^{(i)}$, and $i=1,\ldots,r$. Furthermore, $r\le 4CDK\Delta\|Q\|$ (a constant that does not depend on j) for all $j\in\{1,\ldots,J\}$.

Now, for each fixed $\bar l$, we will see that $\bar l\in F_j$ for, at most, a finite and uniformly bounded number of j's. Suppose that there are M different j's, $j_1,\ldots,j_M$, such that $\bar l\in F_{j_1}\cap\ldots\cap F_{j_M}$, i.e. there is $n_i\in B_{j_i}$ so that

$$\sum_{k=1}^{\infty}|a^{n_i}_{\bar l k}| > \frac{1}{4C}, \qquad i=1,\ldots,M.$$

Then, combining Banach lattice estimates and the triangular inequality of the ℓ_2 norm, we have

$$\begin{aligned}
1 &\overset{\Delta'}{\sim} \|x^*_{n_1}+\ldots+x^*_{n_M}\| \ge (\sqrt{2}\,K)^{-1}\|(|x^*_{n_1}|^2+\ldots+|x^*_{n_M}|^2)^{1/2}\|\\
&= (\sqrt{2}\,K)^{-1}\sup_l\sum_{k=1}^{\infty}\Big(\sum_{m=1}^{M}|a^{n_m}_{lk}|^2\Big)^{1/2} \ge (\sqrt{2}\,K)^{-1}\sum_{k=1}^{\infty}\Big(\sum_{m=1}^{M}|a^{n_m}_{\bar l k}|^2\Big)^{1/2}\\
&\ge (\sqrt{2}\,K)^{-1}\Big(\sum_{m=1}^{M}\Big(\sum_{k=1}^{\infty}|a^{n_m}_{\bar l k}|\Big)^2\Big)^{1/2} \ge M^{1/2}\frac{1}{4\sqrt{2}\,CK}.
\end{aligned}$$

Therefore, $M \le (4\sqrt{2}\,CK\Delta')^2$ (a constant that does not depend on $\bar{l}$). Lemma 2.7 will allow us to split the set $\{1,\dots,J\}$ into subsets in such a way that the correspondence $j \mapsto l_j$ will be injective in each one of them.

Combining Lemma 2.7 and the partitions of B_j, we obtain a partition of $\{1,\dots,J\}$ into at most $N \le rM$ subsets,

$$\{1,\dots,J\} = S_1 \cup \ldots \cup S_N,$$

and a function

$$\sigma : \{(j,i) : j \in \{1,\dots,J\},\ i \in \{1,\dots,r\}\} \to \mathbb{N}, \qquad \sigma(j,i) = l_{ji},$$

so that

$$\sup_k |b^n_{\sigma(j,i)k}| > \frac{1}{4CK}$$

for each $n \in B_j^{(i)}$. Furthermore, given $j_1 \ne j_2 \in S_m$, we have $\sigma(j_1,i_1) \ne \sigma(j_2,i_2)$ for any $i_1,i_2 \in \{1,\dots,r\}$ and $1 \le m \le N$. Therefore, for each $n \in B_j^{(i)}$ there exists k_n so that

$$|b^n_{\sigma(j,i)k_n}| > \frac{1}{4CK\|Q\|}.$$

We want the correspondences

$$\nu_j^i : B_j^{(i)} \to \mathbb{N}, \qquad n \mapsto \nu_j^i(n) = k_n,$$

to be injective on each $B_j^{(i)}$ for all (j,i). Let us see that we are not essentially far from this situation by proving that the number of n's belonging to the same $B_j^{(i)}$ to which can correspond the same $\bar{k}$ is at most finite and uniformly bounded.

Indeed, for (j,i) fixed, suppose there are different $n_1,\dots,n_I \in B_j^{(i)}$ so that $k_{n_1} = \ldots = k_{n_I} = \bar{k}$, i.e.

$$|b^{n_m}_{\sigma(j,i)\bar{k}}| > \frac{1}{4CK}, \qquad m = 1,\dots,I.$$

Then

$$\begin{aligned} 1 &\overset{\Delta}{\sim} \|x_{n_1} + \ldots + x_{n_I}\|_{\ell_1(c_0)} \overset{D}{\sim} \|(|x_{n_1}|^2 + \ldots + |x_{n_I}|^2)^{1/2}\|_{\ell_1(c_0)} \\ &= \sum_{l=1}^{\infty} \sup_k \Big(\sum_{m=1}^{I} |b^{n_m}_{lk}|^2\Big)^{1/2} \ge \sup_k \Big(\sum_{m=1}^{I} |b^{n_m}_{\sigma(j,i)k}|^2\Big)^{1/2} \\ &\ge \Big(\sum_{m=1}^{I} |b^{n_m}_{\sigma(j,i)\bar{k}}|^2\Big)^{1/2} \ge I^{1/2}\frac{1}{4CK\|Q\|}; \end{aligned}$$

therefore $I \le (4CKD\Delta)^2$, a constant that depends on neither (j,i) nor $\bar{k}$.

Hence, for each (j,i) there is a partition of $B_j^{(i)}$ into at most I subsets,

$$B_j^{(i)} = R_j^{i1} \cup \ldots \cup R_j^{iI},$$

and a function $\nu_j^i : B_j^{(i)} \to \mathbb{N}$ whose restriction to each $R_j^{i\,t}$ is injective and such that

$$|b^n_{\sigma(j,i)\nu_j^i(n)}| > \frac{1}{4CK}$$

for all $n \in B_j^{(i)}$. In this way, for any $1 \le m \le N$, $1 \le i \le r$, $1 \le t \le I$ fixed we have injective functions

$$\pi^t_{m,i} : \bigcup_{j\in S_m} R_j^{i\,t} \to \mathbb{N}\times\mathbb{N}, \qquad n \mapsto \pi^t_{m,i}(n) = (l_{\sigma(j,i)}, \nu_j^i(n)),$$

so that

$$|b^n_{\pi^t_{m,i}(n)}| > \frac{1}{4CK\|Q\|}.$$

By Lemma 2.4, there is a constant $\varrho > 0$ (independent of m, i, t) so that, for any $1 \le m \le N$, $1 \le i \le r$, $1 \le t \le I$ given we have

$$\Big\| \sum_{j\in S_m} \sum_{n\in R_j^{i\,t}} a_n x_n \Big\| \ge \varrho \Big\| \sum_{j\in S_m} \sum_{n\in R_j^{it}} a_n e_{\pi^t_{m,i}(n)} \Big\| = \varrho \Big\| \sum_{j\in S_m} \sum_{n\in R_j^{it}} a_n e_{l_{ji} k_n} \Big\|$$
$$= \varrho \Big(\sum_{j\in S_m} \sup_{n\in R_j^{it}} |a_n|^p \Big)^{1/p}$$

for any scalars (a_n). Now, the result follows from Lemma 2.6.

As a consequence we get the following infinite-dimensional results:

THEOREM 2.8. *Every normalized unconditional basis of an infinite-dimensional complemented subspace of $\ell_p(c_0)$ $(0 < p < 1)$ is equivalent to a permutation of the unit vector basis of one of the following spaces*: ℓ_p, c_0, $\ell_p \oplus c_0$, $\ell_p(\ell_\infty^n)_{n=1}^\infty$, $c_0 \oplus \ell_p(\ell_\infty^n)_{n=1}^\infty$, $\ell_p(c_0)$.

THEOREM 2.9. *The following quasi-Banach spaces have a unique unconditional basis up to permutation*: $\ell_p \oplus c_0$, $\ell_p(\ell_\infty^n)_{n=1}^\infty$, $c_0 \oplus \ell_p(\ell_\infty^n)_{n=1}^\infty$, $\ell_p(c_0)$.

3. Uniqueness of unconditional basis of $\ell_p(\ell_2)$, $0 < p < 1$. Let $\ell_p(\ell_2)$ $(0 < p \le 1)$ be the space of infinite matrices $(x_{lk})_{l,k=1}^\infty$ satisfying

$$\|(x_{lk})_{l,k}\|_p = \Big(\sum_{l=1}^\infty \Big(\sum_{k=1}^\infty |x_{lk}|^2 \Big)^{p/2} \Big)^{1/p} < \infty.$$

Then $\|\cdot\|_p$ is a p-norm and $(\ell_p(\ell_2), \|\cdot\|_p)$ is a p-Banach space; in particular, $(\ell_1(\ell_2), \|\cdot\|_1)$ is a Banach space, the Banach envelope of $(\ell_p(\ell_2), \|\cdot\|_p)$.

The dual space of $\ell_p(\ell_2)$ can be identified with $\ell_\infty(\ell_2)$, where $\ell_\infty(\ell_2)$ is the Banach space of infinite matrices $a = (a_{lk})_{l,k=1}^\infty$ satisfying

$$\|a\| = \sup_l \Big(\sum_{k=1}^\infty |a_{lk}|^2 \Big)^{1/2} < \infty.$$

We will denote by $\|\cdot\|$ without confusion both the quasi-norm in $\ell_p(\ell_2)$ and the norm in the dual $\ell_\infty(\ell_2)$, and $\|\cdot\|_c$ will denote the norm in the Banach envelope $\ell_1(\ell_2)$.

The spaces $\ell_p(\ell_2)$ $(0 < p \le 1)$ have a canonical 1-unconditional basis of unit vectors that we will denote by $(e_{lk})_{l,k=1}^{\infty}$. The (l,k) coordinate of $e_{l_0 k_0}$ is 1 if $l = l_0$ and $k = k_0$, and 0 otherwise.

As in the previous section, if $(x_n)_{n=1}^{\eta}$ is a complemented normalized unconditional basic sequence in $\ell_p(\ell_2)$, we will write, for abbreviation,

$$e_{lk}^*(x_n) = b_{lk}^n \quad \text{and} \quad x_n^*(e_{lk}) = a_{lk}^n.$$

Then

$$\|x_n^*\| = \sup_l \Big(\sum_{k=1}^{\infty} |a_{lk}^n|^2 \Big)^{1/2} \le K\|Q\|,$$

where Q is the projection from $\ell_p(\ell_2)$ onto the closed linear span of $(x_n)_{n=1}^{\eta}$.

The lattice structure induced by the canonical basis in $\ell_p(\ell_2)$ $(0 < p \le 1)$ is p-convex.

Bourgain, Casazza, Lindenstrauss and Tzafriri proved:

Theorem 3.1 (Theorem 2.2 of [1]). *Let Q be a bounded linear projection from $\ell_1(\ell_2)$ onto a subspace Z which has a normalized K-unconditional basis $(z_n)_{n=1}^{\eta}$. Then there exist a constant Δ and a partition $(B_j)_{j=1}^{J}$ of the integers $\{1,\dots,\eta\}$ into mutually disjoint subsets so that*

$$\Delta^{-1} \sum_{j=1}^{J} \Big(\sum_{n\in B_j} |a_n|^2 \Big)^{1/2} \le \Big\| \sum_{n=1}^{\eta} a_n z_n \Big\| \le \Delta \sum_{j=1}^{J} \Big(\sum_{n\in B_j} |a_n|^2 \Big)^{1/2} \tag{3.1}$$

for any choice of scalars $(a_n)_n$. In particular, $\ell_1(\ell_2)$ has a unique normalized unconditional basis up to permutation.

This was the motivation for the following result:

Theorem 3.2. *Suppose $0 < p < 1$. Let Q be a bounded linear projection from $\ell_p(\ell_2)$ onto a subspace X with a normalized K-inconditional basis $(x_n)_{n=1}^{\eta}$. Then there exist constants Γ_1, Γ_2 (which depend only on K and $\|Q\|$) and a partition of $\{1,\dots,\eta\}$ into mutually disjoint subsets $(L_i)_{i=1}^{I}$ so that*

$$\Gamma_1 \Big(\sum_{i=1}^{I} \Big(\sum_{n\in L_i} |a_n|^2 \Big)^{p/2} \Big)^{1/p} \overset{(1)}{\le} \Big\| \sum_{n=1}^{\eta} a_n x_n \Big\| \overset{(2)}{\le} \Gamma_2 \Big(\sum_{i=1}^{I} \Big(\sum_{n\in L_i} |a_n|^2 \Big)^{p/2} \Big)^{1/p}$$

for any scalars $(a_n)_{n=1}^{\eta}$.

The proof of Theorem 3.2 is completely analogous to that of Theorem 2.1, and uses essentially the same lemmas. As we did in the previous case, we will prove Theorem 3.2 in two parts, corresponding to each one of the inequalities (1) and (2).

The proof of Theorem 3.2(1) is based on Theorem 3.1 (the analogue of Theorem 2.2) and Lemma 2.3.

Proof of Theorem 3.2(1). The Banach envelope $\widehat{X}$ of X is a complemented subspace of $\ell_1(\ell_2)$ and $(x_n)_{n=1}^{\eta}$ is a K-unconditional basis of $\widehat{X}$, equivalent in $\ell_1(\ell_2)$ to the normalized basis $(x_n/\|x_n\|_{\mathrm{c}})_{n=1}^{\eta}$. Therefore, Theorem 3.2 applies, hence there exist a constant Δ, depending only on K and $\|Q\|$, and a partition of $\{1,\dots,\eta\}$ into disjoint subsets $(B_j)_{j=1}^{J}$ so that (3.1) holds. We will see that this is the partition $(L_i)_{i=1}^{I}$ stated in Theorem 3.2.

For each $j\in\{1,\dots,J\}$ let X_j be the closed linear span in $\ell_p(\ell_2)$ of $\{x_n : n\in B_j\}$. The Banach envelope $\widehat{X}_j$ of X_j is the closed linear span in $\ell_1(\ell_2)$ of $\{x_n : n\in B_j\}$.

By (3.1) applied to each fixed j, we obtain

$$\Delta^{-1}\Big(\sum_{n\in B_j}|a_n|^2\Big)^{1/2}\le\Big\|\sum_{n\in B_j}a_nx_n\Big\|_{\mathrm{c}}\le\Delta\Big(\sum_{n\in B_j}|a_n|^2\Big)^{1/2}$$

for any j and scalars $(a_n)_n$. That is, $(x_n)_{n\in B_j}$ is Δ-equivalent (in $\ell_1(\ell_2)$) to $(e_n)_{n\in B_j}$, where $(e_n)_{n=1}^{\infty}$ denotes the canonical basis of ℓ_2, and the equivalence constant Δ is independent of j.

Thus, X_j^* $(=\widehat{X}_j^*\simeq_{\Delta'}\ell_2^{(|B_j|)})$ has cotype 2.

Hence, the same arguments used in Theorem 2.1(1) lead us to

$$\begin{aligned}\Big\|\sum_{n=1}^{\eta}a_nx_n\Big\|^p&=\Big\|\sum_{j=1}^{J}\sum_{n\in B_j}a_nx_n\Big\|^p\le\sum_{j=1}^{J}\Big\|\sum_{n\in B_j}a_nx_n\Big\|^p\\&\le A^p\sum_{j=1}^{J}\Big\|\sum_{n\in B_j}a_nx_n\Big\|_{\mathrm{c}}^p\le A^p\Delta^p\sum_{j=1}^{J}\Big(\sum_{n\in B_j}|a_n|^2\Big)^{p/2}\end{aligned}$$

for any sequence of scalars (a_n). Therefore, the inequality (1) holds with $\Gamma_2=A\Delta$.

The proof of Theorem 3.2(2) relies on Lemmas 2.4, 2.5, 2.7 and 3.3, an analogue of Lemma 2.6.

LEMMA 3.3. *Suppose $\{S_m : m=1,\dots,N\}$ is a partition of the set $\{1,\dots,J\}$ and that for each $j=1,\dots,J$, $\{\Omega_j^i : i=1,\dots,r\}$ is a partition of B_j. Suppose there is a constant $\varrho>0$ so that for each i and m,*

$$\Big\|\sum_{j\in S_m}\sum_{n\in\Omega_j^i}a_nx_n\Big\|\ge\varrho\Big(\sum_{j\in S_m}\Big(\sum_{n\in\Omega_j^i}|a_n|^2\Big)^{p/2}\Big)^{1/p}$$

for any sequence of scalars (a_n). Further suppose that N, r and ϱ depend only on K and $\|Q\|$. Then there exists a constant $\Gamma' > 0$ so that

$$\Big\|\sum_{n=1}^{\eta} a_n x_n\Big\| \geq \Gamma' \Big(\sum_{j=1}^{J}\Big(\sum_{n\in B_j}|a_n|^2\Big)^{p/2}\Big)^{1/p}$$

for any sequence of scalars (a_n).

Proof. For every $m \in \{1,\dots,N\}$ and $i \in \{1,\dots,r\}$ fixed, by the unconditionality of the basis $(x_n)_{n=1}^{\eta}$ we have

$$\Big\|\sum_{j\in S_m}\sum_{n\in\Omega_j^i} a_n x_n\Big\| \leq K\Big\|\sum_{j=1}^{J}\sum_{n\in B_j} a_n x_n\Big\| = K\Big\|\sum_{n=1}^{\eta} a_n x_n\Big\|.$$

Raising to the pth power and using the hypothesis, we get

$$\Big\|\sum_{n=1}^{\eta} a_n x_n\Big\|^p \geq K^{-p}\Big\|\sum_{j\in S_m}\sum_{n\in\Omega_j^i} a_n x_n\Big\|^p \geq K^{-p}\varrho^p \sum_{j\in S_m}\Big(\sum_{n\in\Omega_j^i}|a_n|^2\Big)^{p/2}.$$

Therefore,

$$\Big\|\sum_{n=1}^{\eta} a_n x_n\Big\|^p \geq K^{-p}\varrho^p \sup_{1\leq m\leq N}\sup_{1\leq i\leq r}\sum_{j\in S_m}\Big(\sum_{n\in\Omega_j^i}|a_n|^2\Big)^{p/2}.$$

Observe that,

$$\sup_{1\leq m\leq N}\sup_{1\leq i\leq r}\sum_{j\in S_m}\Big(\sum_{n\in\Omega_j^i}|a_n|^2\Big)^{p/2} \geq \frac{1}{N}\sum_{m=1}^{N}\sup_{1\leq i\leq r}\sum_{j\in S_m}\Big(\sum_{n\in\Omega_j^i}|a_n|^2\Big)^{p/2}$$

$$\geq \frac{1}{Nr}\sum_{m=1}^{N}\sum_{i=1}^{r}\sum_{j\in S_m}\Big(\sum_{n\in\Omega_j^i}|a_n|^2\Big)^{p/2} = \frac{1}{Nr}\sum_{m=1}^{N}\sum_{j\in S_m}\sum_{i=1}^{r}\Big(\sum_{n\in\Omega_j^i}|a_n|^2\Big)^{p/2}$$

$$\geq \frac{1}{Nr}\sum_{m=1}^{N}\sum_{j\in S_m}\Big(\sum_{i=1}^{r}\sum_{n\in\Omega_j^i}|a_n|^2\Big)^{p/2} = \frac{1}{Nr}\sum_{j=1}^{J}\Big(\sum_{n\in B_j}|a_n|^2\Big)^{p/2}.$$

Then

$$\Big\|\sum_{n=1}^{\eta} a_n x_n\Big\| \geq \frac{\varrho}{KN^{1/p}r^{1/p}}\Big(\sum_{j=1}^{J}\Big(\sum_{n\in B_j}|a_n|^2\Big)^{p/2}\Big)^{1/p}.$$

Proof of Theorem 3.2(2). For each $j \in \{1,\dots,J\}$, and $n \in B_j$, we have

$$1 = x_n^*(x_n) = x_n^*\Big(\sum_{l,k=1}^{\infty} b_{lk}^n e_{lk}\Big) = \sum_{l,k=1}^{\infty} b_{lk}^n a_{lk}^n \leq \sum_{l=1}^{\infty}\Big(\sum_{k=1}^{\infty}|a_{lk}^n b_{lk}^n|\Big).$$

On the other hand, for each $l \in \mathbb{N}$, by Hölder's inequality,

$$\sum_{k=1}^{\infty} |a_{lk}^n b_{lk}^n| \leq \Big(\sum_{k=1}^{\infty} |a_{lk}^n|^2\Big)^{1/2} \Big(\sum_{k=1}^{\infty} |b_{lk}^n|^2\Big)^{1/2}$$
$$\leq \Big(\sum_{k=1}^{\infty} |b_{lk}^n|^2\Big)^{1/2} \sup_l \Big(\sum_{k=1}^{\infty} |a_{lk}^n|^2\Big)^{1/2}.$$

Then, since $\|x_n\| = 1$, $\|x_n^*\| \leq K\|Q\|$, we obtain

$$\sum_{l=1}^{\infty} \Big(\sum_{k=1}^{\infty} |a_{lk}^n b_{lk}^n|\Big)^p \leq \sup_l \Big(\sum_{k=1}^{\infty} |a_{lk}^n|^2\Big)^{p/2} \sum_{l=1}^{\infty} \Big(\sum_{k=1}^{\infty} |b_{lk}^n|^2\Big)^{p/2}$$
$$= \|x_n^*\|^p \, \|x_n\|^p \leq K^p \|Q\|^p.$$

From Lemma 2.5 applied to the sequence $(\sum_{k=1}^{\infty} |a_{lk}^n b_{lk}^n|)_{l\in\mathbb{N}}$, with $\varepsilon = 1/(2K\|Q\|)$, there is a constant $C = C(\varepsilon)$ so that

$$1 \leq \sum_{l=1}^{\infty} \Big(\sum_{k=1}^{\infty} |a_{lk}^n b_{lk}^n|\Big) \leq C \sup_l \sum_{k=1}^{\infty} |a_{lk}^n b_{lk}^n| + \frac{1}{2}.$$

Thus,

$$\sup_l \sum_{k=1}^{\infty} |a_{lk}^n b_{lk}^n| \geq \frac{1}{2C},$$

and therefore, there exists $l = l(n)$ so that $\sum_{k=1}^{\infty} |a_{lk}^n b_{lk}^n| > 1/(4C)$.

So, we can define a function $\{1, \ldots, \eta\} \to \mathbb{N},\ n \mapsto l_n$, so that

$$\sum_{k=1}^{\infty} |a_{l_n k}^n b_{l_n k}^n| > \frac{1}{4C}$$

for any $n \in B_j$, $j \in \{1, \ldots, J\}$. Let us remark that for each n,

$$\frac{1}{4C} < \sum_{k=1}^{\infty} |a_{l_n k}^n b_{l_n k}^n| \leq \Big(\sum_{k=1}^{\infty} |a_{l_n k}^n|^2\Big)^{1/2} \Big(\sum_{k=1}^{\infty} |b_{l_n k}^n|^2\Big)^{1/2};$$

then, in particular,

$$\Big(\sum_{k=1}^{\infty} |b_{l_n k}^n|^2\Big)^{1/2} > \frac{1}{4CK\|Q\|} \quad \text{and} \quad \Big(\sum_{k=1}^{\infty} |a_{l_n k}^n|^2\Big)^{1/2} > \frac{1}{4C}.$$

For each $j \in \{1, \ldots, J\}$, we define $F_j := \{l_n : n \in B_j\}$. Let us first see that $|F_j|$ is uniformly bounded in j: Fix $j \in \{1, \ldots, J\}$ and suppose that $l_1, \ldots, l_r$ are different elements in F_j, i.e., there exist $n_1, \ldots, n_r \in B_j$ so that

$$\Big(\sum_{k=1}^{\infty} |b_{l_i k}^{n_i}|^2\Big)^{1/2} > \frac{1}{4CK\|Q\|}, \qquad i = 1, \ldots, r.$$

Then, by Banach lattice estimates (Theorem 1.d.6 of [10]),

$$r^{1/2} \overset{\Delta}{\sim} \|x_{n_1} + \ldots + x_{n_r}\|_{\mathrm{c}} \overset{D}{\sim} \|(|x_{n_1}|^2 + \ldots + |x_{n_r}|^2)^{1/2}\|_{\mathrm{c}}$$
$$= \sum_{l=1}^{\infty} \Big(\sum_{k=1}^{\infty} \sum_{i=1}^{r} |b_{lk}^{n_i}|^2 \Big)^{1/2} \ge \sum_{i=1}^{r} \Big(\sum_{k=1}^{\infty} |b_{l_i k}^{n_i}|^2 \Big)^{1/2} \ge r \frac{1}{4CK\|Q\|}.$$

Thus, for each $j \in \{1, \ldots, J\}$ there is a partition of the set B_j,

$$B_j = B_j^{(1)} \cup \ldots \cup B_j^{(r)},$$

in such a way that $l_n =: l_{ji}$ for any $n \in B_j^{(i)}$. Furthermore, $r \le (4CDK\Delta\|Q\|)^2$ (a constant that does not depend on j) for all $j = 1, \ldots, J$.

Now, for each fixed $\bar{l}$, we will see that $\bar{l} \in F_j$ for, at most, a finite and uniformly bounded number of j's. Suppose that there are M different j's, $j_1, \ldots, j_M$, such that $\bar{l} \in F_{j_1} \cap \ldots \cap F_{j_M}$, i.e. there are $n_i \in B_{j_i}$ so that

$$\Big(\sum_{k=1}^{\infty} |a_{\bar{l}k}^{n_i}|^2 \Big)^{1/2} > \frac{1}{4C}, \qquad i = 1, \ldots, M.$$

Then, by Banach lattice estimates,

$$1 \overset{\Delta'}{\sim} \|x_{n_1}^* + \ldots + x_{n_M}^*\| \ge (\sqrt{2}\,K)^{-1} \|(|x_{n_1}^*|^2 + \ldots + |x_{n_M}^*|^2)^{1/2}\|$$
$$= (\sqrt{2}\,K)^{-1} \sup_l \Big(\sum_{k=1}^{\infty} \sum_{m=1}^{M} |a_{lk}^{n_m}|^2 \Big)^{1/2} \ge (\sqrt{2}\,K)^{-1} \Big(\sum_{k=1}^{\infty} \sum_{m=1}^{M} |a_{\bar{l}k}^{n_m}|^2 \Big)^{1/2}$$
$$\ge M^{1/2} \frac{1}{4\sqrt{2}\,CK}.$$

Therefore, $M \le (4\sqrt{2}\,CK\Delta')^2$ (a constant that does not depend on $\bar{l}$).

Combining Lemma 2.7 and the partitions of the sets B_j, we get a partition of $\{1, \ldots, J\}$ into at most $N \le rM$ subsets $S_1, \ldots, S_N$, and a function

$$\sigma : \{(j,i) : j \in \{1, \ldots, J\},\ i \in \{1, \ldots, r\}\} \to \mathbb{N}, \qquad (j,i) \mapsto \sigma(j,i) = l_{ji},$$

so that

$$\Big(\sum_{k=1}^{\infty} |b_{\sigma(j,i)k}^{n}|^2 \Big)^{1/2} > \frac{1}{4CK\|Q\|}$$

for each $n \in B_j^{(i)}$. Furthermore, given $j_1 \ne j_2 \in S_m$, we have $\sigma(j_1, i_1) \ne \sigma(j_2, i_2)$ for any $i_1, i_2 \in \{1, \ldots, r\}$ and $1 \le m \le N$.

Now, for each $1 \le m \le N$ and $1 \le i \le r$ fixed, by quasi-Banach lattice estimates (Proposition 2.1 of [5]),

$$\Big\|\sum_{j\in S_m}\sum_{n\in B_j^{(i)}} a_n x_n\Big\|^p \overset{B}{\sim} \Big\|\Big(\sum_{j\in S_m}\sum_{n\in B_j^{(i)}} |a_n x_n|^2\Big)^{1/2}\Big\|^p$$

$$= \sum_{l=1}^{\infty}\Big(\sum_{k=1}^{\infty}\sum_{j\in S_m}\sum_{n\in B_j^{(i)}} |a_n|^2|b_{lk}^n|^2\Big)^{p/2} \ge \sum_{j\in S_m}\Big(\sum_{k=1}^{\infty}\sum_{n\in B_j^{(i)}} |a_n|^2|b_{l_{ji}k}^n|^2\Big)^{p/2}$$

$$= \sum_{j\in S_m}\Big(\sum_{n\in B_j^{(i)}} |a_n|^2\sum_{k=1}^{\infty}|b_{l_{ji}k}^n|^2\Big)^{p/2} \ge \frac{1}{(4CK\|Q\|)^p}\sum_{j\in S_m}\Big(\sum_{n\in B_j^{(i)}} |a_n|^2\Big)^{p/2}$$

for any scalars (a_n).

Hence, the inequality (2) follows from Lemma 3.3.

As a consequence we get the following results:

THEOREM 3.4. *Every normalized unconditional basis of an infinite-dimensional complemented subspace of* $\ell_p(\ell_2)$ $(0 < p < 1)$ *is equivalent to a permutation of the unit vector basis of one of the following spaces*: ℓ_p, ℓ_2, $\ell_p \oplus \ell_2$, $\ell_p(\ell_2^n)_{n=1}^{\infty}$, $\ell_2 \oplus \ell_p(\ell_2^n)_{n=1}^{\infty}$, $\ell_p(\ell_2)$.

THEOREM 3.5. *The following quasi-Banach spaces have a unique unconditional basis up to permutation*: $\ell_p \oplus \ell_2$, $\ell_p(\ell_2^n)_{n=1}^{\infty}$, $\ell_2 \oplus \ell_p(\ell_2^n)_{n=1}^{\infty}$, $\ell_p(\ell_2)$.

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