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An inequality between the James and James type constants in Banach spaces

by

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Abstract. We consider the James and Schäffer type constants recently introduced by Takahashi. We prove an equality between James (resp. Schäffer) type constants and the modulus of convexity (resp. smoothness). By using these equalities, we obtain some estimates for the new constants in terms of the James constant. As a result, we improve an inequality between the Zbăganu and James constants.

1. Introduction. Recently, the problem of finding the relation between the James constant $J(X)$ and the von Neumann–Jordan constant $C_{\mathrm{NJ}}(X)$ has been investigated by several authors. This problem was originally studied by Kato, Maligranda and Takahashi [16] who proved

$$C_{\mathrm{NJ}}(X) \le \frac{[J(X)]^2}{1+[J(X)-1]^2}. \tag{1.1}$$

Since then much effort has gone into improving this inequality; see e.g., [2, 17–22, 24, 25]. There are several methods for estimating $C_{\mathrm{NJ}}(X)$. Among them we mention one due to Alonso, Martín and Papini [2], which mainly relies on the inequality

$$C'_{\mathrm{NJ}}(X) \le J(X), \tag{1.2}$$

where $C'_{\mathrm{NJ}}(X)$ is a constant introduced by Gao [12]. In [21] Wang improved (1.2) as

$$C'_{\mathrm{NJ}}(X) \le 1+\frac{4(J(X)-1)^2}{J^2(X)}. \tag{1.3}$$

In addition, Wang and Pang [22] obtained a similar inequality:

$$A_2(X) \le 1+\sqrt{J(X)-1}, \tag{1.4}$$

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where $A_2(X)$ is a constant introduced by Baronti, Casini and Papini [6]. Later, inequality (1.4) was improved independently by Takahashi and Kato [20] and Wang [21] as

$$A_2(X) \le \frac{3J(X)-2}{J(X)}. \tag{1.5}$$

This inequality enabled them to improve (1.1) as

$$C_{\mathrm{NJ}}(X) \le J(X),$$

which was also obtained by Yang and Li [25].

It is readily seen that inequalities (1.2)–(1.5) play an important role in estimating the von Neumann–Jordan constant. We note that both $A_2(X)$ and $C'_{\mathrm{NJ}}(X)$ fall into the class of James type constants, recently introduced by Takahashi [19]. The aim of this paper is to present a general method for estimating James type constants. Moreover we get an inequality between the James and Schäffer type constants from an equality for the modulus of smoothness. As an application, we strengthen an inequality between the Zbăganu and James constants.

2. Preliminaries and notation. Throughout this paper, B_X and S_X respectively denote the unit ball and the unit sphere of a Banach space X. The *von Neumann–Jordan constant,* introduced by Clarkson [9], is defined as

$$C_{\mathrm{NJ}}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2+\|y\|^2)} : x \in S_X,\, y \in B_X \right\}.$$

A Banach space X is called *uniformly nonsquare* ((UNS) for short), in the sense of James, if there exists a positive number $\delta < 2$ such that

$$\min(\|x-y\|, \|x+y\|) \le \delta$$

for all $x, y \in S_X$. The *James constant*

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in S_X\}$$

is introduced to characterize this concept: obviously X is (UNS) in the sense of James if and only if $J(X) < 2$. Now let us turn to another definition of uniform nonsquareness. A Banach space X is called *uniformly nonsquare* in the sense of Schäffer if there exists a $\lambda > 1$ such that

$$\max(\|x-y\|, \|x+y\|) \ge \lambda$$

for all $x, y \in S_X$. The *Schäffer constant,* defined by

$$S(X) = \inf\{\max(\|x-y\|, \|x+y\|) : x, y \in S_X\},$$

is introduced to characterize this concept: obviously X is uniformly nonsquare in the sense of Schäffer if and only if $S(X) > 1$. From the equality

$J(X)S(X) = 2$ (see [16]), we know that these two definitions of (UNS) are equivalent. For more information on these constants, we refer to [13, 14, 16].

Recall the generalized mean defined by

$$\mathcal{M}_t(a,b) := \left(\frac{a^t + b^t}{2}\right)^{1/t},$$

where a and b are two positive real numbers. It is well known that $\mathcal{M}_t(a,b)$ is nondecreasing and

$$\mathcal{M}_{-\infty}(a,b) := \lim_{t\to-\infty} \mathcal{M}_t(a,b) = \min(a,b),$$
$$\mathcal{M}_{+\infty}(a,b) := \lim_{t\to+\infty} \mathcal{M}_t(a,b) = \max(a,b),$$

and $\mathcal{M}_0(a,b) = \lim_{t\to 0} \mathcal{M}_t(a,b) = \sqrt{ab}$. By using the generalized mean, we now recall two classes of geometric constants, which include the James and Schäffer constants, respectively.

Definition 2.1 (Takahashi [19]). (1) For $\tau \geq 0$ and $t \in [-\infty, +\infty)$, a *James type constant* is defined by

$$J_{X,t}(\tau) = \sup\{\mathcal{M}_t(\|x - \tau y\|, \|x + \tau y\|) : x, y \in S_X\}.$$

(2) For $\tau \geq 0$ and $1 < t \leq +\infty$, a *Schäffer type constant* is defined by

$$S_{X,t}(\tau) = \inf\{\mathcal{M}_t(\|x - \tau y\|, \|x + \tau y\|) : x, y \in S_X\}.$$

Obviously $J_{X,t}(\tau)$ includes some known constants, such as Alonso–Llorens-Fuster's constant $T(X)$ [1], Baronti–Casini–Papini's constant $A_2(X)$ [6], Gao's constant $C'_{\mathrm{NJ}}(X)$ [12] and Yang–Wang's modulus $\gamma_X(t)$ [26]. Also $S_{X,t}(\tau)$ is an extension of $S(X)$, including Gao's constant $f(X)$ [12] as a special case.

Notation. $J_t := J_{X,t}(1), J := J(X), S_t := S_{X,t}(1)$ and $S := S(X)$.

3. Main results

3.1. James type constants. The *modulus of convexity* $\delta_X : [0,2] \to [0,1]$ is defined as

$$\delta_X(\epsilon) = \inf\{1 - \|x+y\|/2 : x, y \in S_X,\ \|x - y\| \geq \epsilon\}.$$

Obviously, δ_X is nondecreasing on $[0,2]$; moreover, the function $\delta_X(\epsilon)/\epsilon$ is also nondecreasing on $(0,2]$ (see [11]). The equality

$$J = \sup_{0\leq\epsilon\leq 2} \{\epsilon : \delta_X(\epsilon) \leq 1 - \epsilon/2\}$$

holds for any nontrivial space, while the equality

$$1 - \delta_X(J) = J/2 \tag{3.1}$$

holds whenever X is uniformly nonsquare (see [8]).

We recall several known results on the modulus of convexity:

$$A_2(X) = 1 + \sup_{\sqrt{2}\le\epsilon\le 2} \{\epsilon/2 - \delta_X(\epsilon)\}$$

by Baronti, Casini and Papini [6, Proposition 2.4];

$$T(X) = \sup_{0\le\epsilon\le 2} \sqrt{2\epsilon(1-\delta_X(\epsilon))}$$

by Alonso and Llorens-Fuster [1, Theorem 11]; and

$$C'_{\mathrm{NJ}}(X) = \sup_{0\le\epsilon\le 2} \{\epsilon^2/4 + (1-\delta_X(\epsilon))^2\}$$

by Alonso, Martín and Papini [2, Proposition 4]. We generalize the formulas above as follows.

Theorem 3.1. *Let $t \in \mathbb{R}$. Then for any Banach space X,*

$$J_t = \sup\{\mathcal{M}_t(\epsilon, 2(1-\delta_X(\epsilon))) : J \le \epsilon \le 2\}. \tag{3.2}$$

Proof. Let $\epsilon \in [0,2]$ and let $\eta > 0$ be sufficiently small. Then there exist $x, y \in S_X$ with $\|x-y\| = \epsilon$ and $\|x+y\| \ge 2(1-\delta_X(\epsilon)) - \eta$. Therefore

$$\mathcal{M}_t(\epsilon, 2(1-\delta_X(\epsilon)) - \eta) \le \mathcal{M}_t(\|x-y\|, \|x+y\|) \le J_t.$$

Since ϵ is arbitrary, by letting $\eta \to 0$ we get

$$J_t \ge \sup_{0\le\epsilon\le 2} \mathcal{M}_t(\epsilon, 2(1-\delta_X(\epsilon))) \ge \sup_{J\le\epsilon\le 2} \mathcal{M}_t(\epsilon, 2(1-\delta_X(\epsilon))).$$

To show the opposite inequality let $x, y \in S_X$. If $\max(\|x-y\|, \|x+y\|) \ge J$, then we let $\|x-y\| = \epsilon$ and assume without loss of generality that $\|x-y\| = \max(\|x-y\|, \|x+y\|)$. It follows that $\epsilon \ge J$, $\|x+y\| \le 2(1-\delta_X(\epsilon))$ and

$$\begin{aligned}\mathcal{M}_t(\|x-y\|, \|x+y\|) &\le \mathcal{M}_t(\epsilon, 2(1-\delta_X(\epsilon))) \\ &\le \sup_{J\le\epsilon\le 2} \mathcal{M}_t(\epsilon, 2(1-\delta_X(\epsilon))).\end{aligned}$$

Otherwise, if $\max(\|x-y\|, \|x+y\|) \le J$, then

$$\begin{aligned}\mathcal{M}_t(\|x-y\|, \|x+y\|) &\le J = \mathcal{M}_t(J, 2(1-\delta_X(J))) \\ &\le \sup_{J\le\epsilon\le 2} \mathcal{M}_t(\epsilon, 2(1-\delta_X(\epsilon))).\end{aligned}$$

Altogether,

$$J_t \le \sup_{J\le\epsilon\le 2} \mathcal{M}_t(\epsilon, 2(1-\delta_X(\epsilon))),$$

which completes the proof. ■

Remark 3.2. To compute J_t, it suffices to consider the function $\mathcal{M}_t(\epsilon, 2(1-\delta_X(\epsilon)))$ on $[J,2]$ instead of $[0,2]$. So our result generalizes and improves some known results.

EXAMPLE 3.3 (ℓ_∞-ℓ_1 space). Let X be $\mathbb{R}^2$ with the norm

$$\|x\| = \|(x_1, x_2)\| = \begin{cases} \max(|x_1|, |x_2|) & \text{if } x_1x_2 \geq 0, \\ |x_1| + |x_2| & \text{if } x_1x_2 \leq 0. \end{cases}$$

Since $\delta_X(\epsilon) = \max(0, (\epsilon - 1)/2)$ and $J = 3/2$ (see [15, 16]), we get

$$J_t = \max_{3/2 \leq \epsilon \leq 2} \left(\frac{\epsilon^t + (3-\epsilon)^t}{2} \right)^{1/t}.$$

It is easy to see that $J_t = 3/2$ for $t \leq 1$ and

$$J_t = \left(\frac{1 + 2^t}{2} \right)^{1/t} \quad \text{for } t \geq 1.$$

EXAMPLE 3.4 (ℓ_2-ℓ_1 space). Let X be $\mathbb{R}^2$ with the norm

$$\|x\| = \|(x_1, x_2)\| = \begin{cases} (|x_1|^2 + |x_2|^2)^{1/2} & \text{if } x_1x_2 \geq 0, \\ |x_1| + |x_2| & \text{if } x_1x_2 \leq 0. \end{cases}$$

Since $J = \sqrt{8/3}$ and

$$\delta_X(\epsilon) = 1 - \sqrt{1 - \epsilon^2/8} \quad \text{for } \sqrt{8/3} \leq \epsilon \leq 2$$

(see [15, 16]), we get

$$J_t = \max_{\sqrt{8/3} \leq \epsilon \leq 2} \left(\frac{\epsilon^t + (4 - \epsilon^2/2)^{t/2}}{2} \right)^{1/t}.$$

A simple calculation shows that

$$J_t = \sqrt{2} \left(\frac{1 + 2^{t/2}}{2} \right)^{1/t} \quad \text{for } t \geq 0,$$

and

$$J_t = 2 \left(\frac{1 + 2^{t/(t-2)}}{2} \right)^{1/t - 1/2} \quad \text{for } t \leq 0.$$

We now extend and improve inequalities (1.2)–(1.5) to the following form.

THEOREM 3.5. *Let $t \geq 0$. Then for any Banach space X,*

$$J_t \leq \frac{2\mathcal{M}_t(J, 2(J-1))}{J}. \tag{3.3}$$

Proof. Since in the case $J = 2$ the inequality is obvious, we assume $J < 2$, that is, X is (UNS). By the monotonicity of $\delta_X(\epsilon)/\epsilon$ and (3.1), we have

$$\delta_X(\epsilon) \geq \frac{\delta_X(J)}{J}\epsilon = \frac{(2-J)\epsilon}{2J} = \frac{(S-1)\epsilon}{2}$$

for any $\epsilon \geq J$. This together with Theorem 3.1 yields

$$\begin{aligned} J_t &= \sup_{J\leq\epsilon\leq 2} \mathcal{M}_t(\epsilon, 2(1-\delta_X(\epsilon))) \\ &\leq \sup_{J\leq\epsilon\leq 2} \mathcal{M}_t(\epsilon, 2-(S-1)\epsilon) = 2\mathcal{M}_t(1, 2-S) \\ &= \frac{2\mathcal{M}_t(J, 2(J-1))}{J}. \ \blacksquare \end{aligned}$$

REMARK 3.6. (1) When X is (UNS), by a simple calculation, we can extend the above result from $[0, +\infty)$ to $[t_0, +\infty)$, where $t_0 = 1 - 1/\log_{S-1}(2-S)$. The upper bound of J_t is $\mathcal{M}_t(\epsilon_t, 2-(S-1)\epsilon_t)$ with $\epsilon_t = 2/(S-1+(S-1)^{1/(1-t)})$ whenever $t < t_0$.

(2) It is readily seen that inequalities (1.2)–(1.5) are all included in (3.3), which shows that our result generalizes some known results.

COROLLARY 3.7. *X is (UNS) $\Leftrightarrow J_t < 2$ for some $t \in \mathbb{R}$.*

Proof. By the definition of J_t, it is not hard to check that $J_t \leq 2$ for all $t \in \mathbb{R}$. Thus our assertion is equivalent to $J = 2 \Leftrightarrow J_t = 2$ for all $t \in \mathbb{R}$. That $J = 2 \Rightarrow J_t = 2$ for all $t \in \mathbb{R}$ follows from $J_t \geq J$. To see the converse, let $t \geq 0$ be fixed. By assumption $J_t = 2$. It follows from (3.3) that

$$J = J(J_t/2) \leq \mathcal{M}_t(J, 2(J-1)) \leq J,$$

which implies $\mathcal{M}_t(J, 2(J-1)) = J$ and thus $J = 2$. $\blacksquare$

3.2. Schäffer type constant. The *modulus of smoothness* $\varrho_X : [0,2] \to [0,1]$ is defined as

$$\varrho_X(\epsilon) = \sup\{1 - \|x+y\|/2 : x, y \in S_X, \|x-y\| \leq \epsilon\}.$$

It is known that ϱ_X is continuous and convex on $[0,2]$; moreover, the function $\varrho_X(\epsilon)/\epsilon$ is also nondecreasing on $(0,2]$. In addition

$$1 - \varrho_X(S) = S/2$$

for any Banach space X (see e.g. [3, 4, 7, 5]). By using related properties, we state a relation between this modulus and Schäffer type constants.

THEOREM 3.8. *Let $t > 1$. Then for any Banach space X,*

$$S_t = \min\{\mathcal{M}_t(\epsilon, 2(1-\varrho_X(\epsilon))) : 0 \leq \epsilon \leq S\}.$$

Proof. Let $\epsilon \in [0,2]$ be fixed. For sufficiently small $\eta > 0$, there exist $x, y \in S_X$ with $\|x-y\| = \epsilon$ so that $\|x+y\| \leq 2(1-\varrho_X(\epsilon)) + \eta$. Hence

$$\mathcal{M}_t(\epsilon, 2(1-\varrho_X(\epsilon)) + \eta) \geq \mathcal{M}_t(\|x-y\|, \|x+y\|) \geq S_t.$$

Since ϵ is arbitrary, by letting $\eta \to 0$ we obtain

$$S_t \leq \inf_{0\leq\epsilon\leq 2} \mathcal{M}_t(\epsilon, 2(1-\varrho_X(\epsilon))) \leq \inf_{0\leq\epsilon\leq S} \mathcal{M}_t(\epsilon, 2(1-\varrho_X(\epsilon))).$$

To see the opposite inequality, let $x, y \in S_X$. If $\min(\|x-y\|, \|x+y\|) \le S$, then we assume without loss of generality that $\|x-y\| = \min(\|x-y\|, \|x+y\|)$ and let $\|x-y\| = \epsilon$. Then $\epsilon \le S$, $\|x+y\| \ge 2(1-\varrho_X(\epsilon))$ and

$$\mathcal{M}_t(\|x-y\|, \|x+y\|) \ge \mathcal{M}_t(\epsilon, 2(1-\varrho_X(\epsilon))) \ge \inf_{0\le\epsilon\le S} \mathcal{M}_t(\epsilon, 2(1-\varrho_X(\epsilon))).$$

Otherwise, if $\min(\|x-y\|, \|x+y\|) \ge S$, then

$$\begin{aligned}\mathcal{M}_t(\|x-y\|, \|x+y\|) &\ge S = \mathcal{M}_t(S, 2(1-\varrho_X(S))) \\ &\ge \inf_{0\le\epsilon\le S} \mathcal{M}_t(\epsilon, 2(1-\varrho_X(\epsilon))).\end{aligned}$$

Altogether

$$S_t \ge \inf_{0\le\epsilon\le S} \mathcal{M}_t(\epsilon, 2(1-\varrho_X(\epsilon))).$$

Thus the result follows from the continuity of $\varrho_X(\epsilon)$. ∎

EXAMPLE 3.9. Let X be the ℓ_∞-ℓ_1 space defined in Example 3.3. From $\varrho_X(\epsilon) = \max(\epsilon/4, \epsilon-1)$ (see [7]), it follows that

$$S_t = \min_{0\le\epsilon\le 4/3} \left(\frac{\epsilon^t + (2-\epsilon/2)^t}{2}\right)^{1/t} = 2\left(\frac{2}{1+2^{t'}}\right)^{1/t'},$$

where $t' > 1$, $1/t + 1/t' = 1$.

EXAMPLE 3.10 (ℓ_2-ℓ_∞ space). Let X be $\mathbb{R}^2$ with the norm

$$\|x\| = \|(x_1, x_2)\| = \begin{cases} (|x_1|^2 + |x_2|^2)^{1/2} & \text{if } x_1x_2 \ge 0, \\ \max(|x_1|, |x_2|) & \text{if } x_1x_2 \le 0. \end{cases}$$

Since $J = 1 + 1/\sqrt{2}$, we have $S = 2(2-\sqrt{2})$. It is proved in [10] that

$$\varrho_X(\epsilon) = \max\left(\frac{\epsilon}{2\sqrt{2}}, \frac{\epsilon}{\sqrt{2}} + 1 - \sqrt{2}\right),$$

which implies that

$$S_t = \min_{0\le\epsilon\le 2(2-\sqrt{2})} \left(\frac{\epsilon^t + (2-\epsilon/\sqrt{2})^t}{2}\right)^{1/t} = \sqrt{2}\left(\frac{2}{1+2^{t'/2}}\right)^{1/t'},$$

where $t' > 1$, $1/t + 1/t' = 1$.

It is readily seen that $S_t \le S$ for every $t > 1$. By the formula $JS = 2$, we see that the upper bound of S_t is $2/J$. We now consider its lower bound.

THEOREM 3.11. *For any Banach space X,*

$$S_t \ge 1/\mathcal{M}_{t'}(1, J-1), \tag{3.4}$$

where $t, t' > 1$ with $1/t + 1/t' = 1$.

Proof. Since $\varrho_X(\epsilon)/\epsilon$ is nondecreasing and $2(1-\varrho_X(S)) = S$, it follows that

$$\varrho_X(\epsilon) \le \frac{\varrho_X(S)}{S}\epsilon = \frac{(2-S)\epsilon}{2S} = \frac{(J-1)\epsilon}{2}$$

for all $\epsilon \in [0, S]$. Using Theorem 3.8 now yields

$$S_t = \min_{0\le\epsilon\le S} \mathcal{M}_t(\epsilon, 2(1-\varrho_X(\epsilon))) \ge \min_{0\le\epsilon\le S} \mathcal{M}_t(\epsilon, 2-(J-1)\epsilon).$$

Since $\mathcal{M}_t(\epsilon, 2-(J-1)\epsilon)$ attains its minimum at

$$\epsilon_t := \frac{2}{(J-1)+(J-1)^{-1/(t-1)}} \le \frac{2}{(J-1)+1} = S,$$

we infer that

$$S_t \ge \mathcal{M}_t(\epsilon_t, 2-(J-1)\epsilon_t) = 1/\mathcal{M}_{t'}(1, J-1). \ \blacksquare$$

REMARK 3.12. (1) Example 3.9 shows that inequality (3.4) is sharp even for a uniformly nonsquare space.

(2) Theorem 3.11 is an improvement of [23, Theorem 2]. In fact, for $t = 2$, inequality (3.4) is reduced to

$$f(X) \ge \frac{4}{1+(J-1)^2},$$

where $f(X) = 2(S_2)^2$ is a constant introduced in [12].

COROLLARY 3.13. *X is (UNS) $\Leftrightarrow S_t > 1$ for some $t > 1$.*

Proof. By the definition of S_t, we know that $S_t \ge 1$ for all $t > 1$. Thus our assertion is equivalent to $J = 2 \Leftrightarrow S_t = 1$ for all $t > 1$. That $J = 2 \Rightarrow S_t = 1$ for all $t > 1$ follows from the estimate $S_t \le 2/J$. To see the converse, we deduce from (3.4) that

$$1 \ge \mathcal{M}_{t'}(1, J-1) \ge 1/S_t = 1,$$

which implies $\mathcal{M}_{t'}(1, J-1) = 1$ and thus $J = 2$. $\blacksquare$

3.3. Zbăganu constant. The *Zbăganu constant,* introduced in [27], is defined as

$$C_{\mathrm{Z}}(X) = \sup\left\{\frac{\|x+y\|\,\|x-y\|}{\|x\|^2+\|y\|^2} : x \in S_X,\, y \in B_X\right\}.$$

Recently Takahashi studied the relation between the Zbăganu and James constants and proved the following inequality (see [19, Theorem 18]):

$$C_{\mathrm{Z}}(X) \le \frac{J+\sqrt{J^2+(2-J)^2}}{2}. \tag{3.5}$$

Applying the previous result, we can improve (3.5) to the following form.

THEOREM 3.14. *For any Banach space X,*

$$C_{\mathrm{Z}}(X) \le \frac{2(J-1)+\sqrt{4(J-1)^2+(2-J)^2}}{J}.$$

Proof. Observe first that $\|x \pm \tau y\| \le \tau\|x \pm y\| + (1-\tau)$ for every $0 \le \tau \le 1$ and $x \in S_X$. This together with Theorem 3.5 leads to

$$\begin{aligned}\frac{J_{X,0}^2(\tau)}{1+\tau^2} &= \sup\left\{\frac{\|x+\tau y\|\,\|x-\tau y\|}{1+\tau^2} : x, y \in S_X\right\}\\ &\le \frac{J_{X,0}^2(1)\tau^2 + 2J_{X,1}(1)\tau(1-\tau) + (1-\tau)^2}{1+\tau^2}\\ &\le \frac{(3J-4)\tau^2 + 4(J-1)\tau + J}{J(1+\tau^2)}.\end{aligned}$$

It follows from a simple computation that

$$\begin{aligned}C_{\mathrm{Z}}(X) &= \sup\left\{\frac{J_{X,0}^2(\tau)}{1+\tau^2} : 0 \le \tau \le 1\right\}\\ &\le \max_{0\le\tau\le1} \frac{(3J-4)\tau^2 + 4(J-1)\tau + J}{J(1+\tau^2)}\\ &= \frac{2(J-1)+\sqrt{4(J-1)^2+(2-J)^2}}{J}. \quad \blacksquare\end{aligned}$$

REMARK 3.15. Since $\sqrt{2} \le J \le 2$, one gets

$$\frac{2(J-1)+\sqrt{4(J-1)^2+(2-J)^2}}{J} \le J \le \frac{J+\sqrt{J^2+(2-J)^2}}{2}.$$

Indeed, the right-hand inequality is obvious and the left-hand inequality follows from the following chain:

$$\begin{aligned}2(J-1)+\sqrt{4(J-1)^2+(2-J)^2} \le J^2\\ &\Leftrightarrow 4(J-1)^2+(2-J)^2 \le (1+(J-1)^2)^2\\ &\Leftrightarrow (2-J)^2 \le (1-(J-1)^2)^2\\ &\Leftrightarrow (J-1)^2 \le J-1.\end{aligned}$$

Thus our result significantly improves (3.5).

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