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Operator equations and subscalarity

by

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Abstract. We consider the system of operator equations $ABA = A^2$ and $BAB = B^2$. Let (A, B) be a solution to this system. We give several connections among the operators A, B, AB, and BA. We first prove that A is subscalar of finite order if and only if B is, which is equivalent to the subscalarity of AB or BA with finite order. As a corollary, if A is subscalar and its spectrum has nonempty interior, then B has a nontrivial invariant subspace. We also provide examples of subscalar operator matrices. Moreover, we deal with algebraicity, power boundedness, and quasitriangularity, using some power properties obtained from the operator equations.

1. Introduction. Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_{\mathrm{p}}(T)$, $\sigma_{\mathrm{ap}}(T)$, $\sigma_{\mathrm{e}}(T)$, $\sigma_{\mathrm{le}}(T)$, and $\sigma_{\mathrm{re}}(T)$ for the spectrum, point spectrum, approximate point spectrum, essential spectrum, left essential spectrum, and right essential spectrum of T, respectively.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(TT^*)^p \le (T^*T)^p$, where $0 < p < \infty$. When $p = 1$ (respectively, $p = 1/2$), T is called *hyponormal* (resp. *semi-hyponormal*). By Löwner's inequality, if $0 < q < p$, then every p-hyponormal operator is q-hyponormal.

An operator $S \in \mathcal{L}(\mathcal{H})$ is called *scalar* of order m $(0 \le m \le \infty)$ if it has a spectral distribution of order m, i.e., if there is a continuous unital homomorphism of topological algebras

$$\Phi : C_0^m(\mathbb{C}) \to \mathcal{L}(\mathcal{H})$$

such that $\Phi(z) = S$, where as usual z stands for the identical function on $\mathbb{C}$, and $C_0^m(\mathbb{C})$ for the Fréchet space of all continuously differentiable functions of order m with compact support. An operator is *subscalar* of order m if it is similar to the restriction of a scalar operator of order m to an invariant subspace. M. Putinar [14] showed that every hyponormal operator is subscalar of order 2, and S. Brown [2] used this result to prove that hyponormal

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operators with thick spectra have nontrivial invariant subspaces. Similarly, a partial solution of the invariant subspace problem with respect to subscalar operators is due to J. Eschmeier [6].

I. Vidav [16] showed that A and B are self-adjoint operators satisfying $ABA = A^2$ and $BAB = B^2$ if and only if $FF^* = A$ and $F^*F = B$ for some idempotent operator F. We now consider the system of operator equations

$$ABA = A^2 \quad \text{and} \quad BAB = B^2 \tag{1.1}$$

without self-adjointness.

C. Schmoeger [15] provided a method to get solutions of (1.1): if $A := PQ$ and $B := QP$ where P and Q are idempotent operators, then (A, B) is a solution of (1.1). Using this construction, we give the following examples. Set

$$P = \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1/2 & b \\ c & 1/2 \end{pmatrix}$$

where $a, b, c \in \mathbb{C}$ and $bc = 1/4$. Then $P^2 = P$ and $Q^2 = Q$. Hence

$$A := PQ = \begin{pmatrix} 0 & 0 \\ a/2 + c & ab + 1/2 \end{pmatrix} \quad \text{and} \quad B := QP = \begin{pmatrix} ab & b \\ a/2 & 1/2 \end{pmatrix}$$

satisfy (1.1). But, in general, neither A nor B are self-adjoint. In this case, if we assume that $a = 2\overline{b}$ and $a \neq -2c$, then it is easy to see that B is normal, but A is not.

We give another example. Let

$$P = \begin{pmatrix} I & 0 \\ -S & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} T & S^{-1} \\ S & I - T \end{pmatrix}$$

where $T \in \mathcal{L}(\mathcal{H})$ is an algebraic operator such that $T^2 - T + I = 0$ and $S \in \mathcal{L}(\mathcal{H})$ is an invertible operator commuting with T. Define

$$A := PQ = \begin{pmatrix} T & S^{-1} \\ -ST & -I \end{pmatrix} \quad \text{and} \quad B := QP = \begin{pmatrix} T^2 & 0 \\ ST & 0 \end{pmatrix}.$$

Since both P and Q are idempotent operators, (A, B) is a solution to (1.1). It seems to be much easier to characterize B than A. Hence, in order to obtain some properties of A which are preserved through the operator equations (1.1), it suffices to deal with B.

In this paper, we consider the operator equations (1.1). Let (A, B) be a solution to (1.1). We give several connections among the operators A, B, AB, and BA. We first prove that A is subscalar of finite order if and only if B is, which is equivalent to the subscalarity of AB or BA with finite order. As a corollary, if A is subscalar and its spectrum has nonempty interior, then B has a nontrivial invariant subspace. We also provide examples of subscalar operator matrices. Moreover, we deal with algebraicity, power boundedness, quasitriangularity, using some power properties obtained from the operator equations.

2. Preliminaries. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single-valued extension property* (or SVEP) if for every open subset G of $\mathbb{C}$, the only analytic function $f : G \to \mathcal{H}$ such that $(T-z)f(z) \equiv 0$ on G is the zero function on G. For an operator $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, the *local resolvent set* $\rho_T(x)$ of T at x is the union of all open subsets G of $\mathbb{C}$ such that there exists an analytic function $f : G \to \mathcal{H}$ with $(T-z)f(z) \equiv x$ on G; hence, every local resolvent set is open in $\mathbb{C}$. We denote the *local spectrum* of T at $x \in \mathcal{H}$ by

$$\sigma_T(x) = \mathbb{C} \setminus \rho_T(x),$$

and the *local spectral subspace* for T is defined by

$$\mathcal{H}_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$$

where F is a subset of $\mathbb{C}$. From [12, Proposition 1.2.16], we know that $T \in \mathcal{L}(\mathcal{H})$ has the single-valued extension property if and only if $\mathcal{H}_T(\emptyset) = \{0\}$. Moreover, if T has the single-valued extension property, then $\sigma(T) = \bigcup\{\sigma_T(x) : x \in \mathcal{H}\}$ by [12, Proposition 1.3.2].

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $\{f_n\}$ of $\mathcal{H}$-valued analytic functions on G such that $(T-z)f_n(z)$ converges to 0 in norm, uniformly on compact subsets of G, $f_n(z)$ also converges to 0 in norm, uniformly on compact subsets of G. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Dunford's property* (C) if $\mathcal{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$. We say that $T \in \mathcal{L}(\mathcal{H})$ is *decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that

$$\mathcal{H} = \mathcal{X} + \mathcal{Y}, \quad \sigma(T|_{\mathcal{X}}) \subset \overline{U}, \quad \sigma(T|_{\mathcal{Y}}) \subset \overline{V}.$$

It is well known from [12, Theorem 1.2.7, Proposition 1.2.19] that

Decomposable $\Rightarrow$ Property (β) $\Rightarrow$ Dunford's property (C) $\Rightarrow$ SVEP.

Let z be the coordinate in $\mathbb{C}$, and let $d\mu(z)$, or simply $d\mu$, denote the planar Lebesgue measure. Let U be a bounded open subset of $\mathbb{C}$. We shall usually denote by $L^2(U, \mathcal{H})$ the Hilbert space of measurable functions $f : U \to \mathcal{H}$ such that

$$\|f\|_{2,U} = \Big(\int_U \|f(z)\|^2 \, d\mu\Big)^{1/2} < \infty.$$

We define the *Bergman space for* U by $A^2(U, \mathcal{H}) = L^2(U, \mathcal{H}) \cap \mathcal{O}(U, \mathcal{H})$ where $\mathcal{O}(U, \mathcal{H})$ is the Fréchet space of analytic $\mathcal{H}$-valued functions on U. Then $A^2(U, \mathcal{H})$ is a closed subspace and $A^2(U, \mathcal{H}) = \{f \in L^2(U, \mathcal{H}) : \bar{\partial} f = 0\}$

where $\bar{\partial}$ represents the partial derivative, in the sense of distributions, with respect to $\overline{z}$.

Now, we introduce a special Sobolev type space. Let U be a bounded open subset of $\mathbb{C}$. For a fixed non-negative integer m, the vector-valued Sobolev space $W^m(U,\mathcal{H})$ is the space of functions $f \in L^2(U,\mathcal{H})$ whose derivatives $\bar{\partial} f, \bar{\partial}^2 f, \ldots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U,\mathcal{H})$. Endowed with the norm

$$\|f\|^2_{W^m} = \sum_{i=0}^{m} \|\bar{\partial}^i f\|^2_{2,U},$$

$W^m(U,\mathcal{H})$ becomes a Hilbert space, continuously contained in $L^2(U,\mathcal{H})$. The linear operator M of multiplication by z on $W^m(U,\mathcal{H})$ is continuous and it has a spectral distribution Φ_M of order m defined by $\Phi_M(\varphi)f = \varphi f$ for $\varphi \in C_0^m(\mathbb{C})$ and $f \in W^m(U,\mathcal{H})$. Hence M is a scalar operator of order m.

3. Main results. In this section, we study several properties preserved through the operator equations $ABA = A^2$ and $BAB = B^2$. If (A,B) is a solution to these operator equations, then $\sigma(A) = \sigma(B) = \sigma(AB) = \sigma(BA)$ by [15, Proposition 2.6, Corollary 2.7]; set

$$\sigma := \sigma(A) = \sigma(B) = \sigma(AB) = \sigma(BA)$$

throughout this section. We start with the following lemmas on subscalarity, obtained by applying Putinar's method [14].

Lemma 3.1. *Let $A, B \in \mathcal{L}(\mathcal{H})$ satisfy $ABA = A^2$ and $BAB = B^2$. Suppose that A is subscalar of finite order, and set $T = B$ or $T = BA$. For an open disk D in $\mathbb{C}$ containing σ and a positive integer m, define the map $V_m : \mathcal{H} \to H(D)$ by*

$$V_m h = \widetilde{1 \otimes h} \equiv 1 \otimes h + \overline{(T-z)W^m(D,\mathcal{H})},$$

where $H(D) = W^m(D,\mathcal{H})/\overline{(T-z)W^m(D,\mathcal{H})}$ and $1 \otimes h$ denotes the constant function sending $z \in D$ to h. Then V_m is one-to-one and has closed range for some positive integer m.

Proof. Let D_1 be an open disk in $\mathbb{C}$ containing σ with $\overline{D_1} \subset D$. From [7, Proposition 3.1], there exist a constant $C > 0$ and a positive integer k such that

$$\|f\|_{2,D_1} \le C \sum_{i=0}^{k} \|(A-z)\bar{\partial}^i f\|_{2,D} \tag{3.1}$$

for $f \in C^\infty(\mathbb{C},\mathcal{H})$, where $C^\infty(\mathbb{C},\mathcal{H})$ denotes the Fréchet space of all infinitely differentiable $\mathcal{H}$-valued functions on $\mathbb{C}$.

(i) For $T = B$, we will show that V_{k+4} is one-to-one and has closed range, i.e., is bounded below. For this, it is enough to prove that if $\{h_n\} \subset \mathcal{H}$ and

$\{f_n\} \subset W^{k+4}(D,\mathcal{H})$ are sequences such that

$$\lim_{n\to\infty} \|(B-z)f_n + 1\otimes h_n\|_{W^{k+4}} = 0, \tag{3.2}$$

then $\lim_{n\to\infty}\|h_n\| = 0$. Choose an open disk D_2 in $\mathbb{C}$ containing σ such that $\overline{D_2} \subset D_1$. By the definition of the norm of Sobolev space, (3.2) implies that

$$\lim_{n\to\infty} \|(B-z)\bar{\partial}^i f_n\|_{2,D} = 0 \quad \text{for } i = 1,\ldots,k+4. \tag{3.3}$$

Multiplying both sides of (3.3) by B, we have

$$\lim_{n\to\infty} \|(B^2-zB)\bar{\partial}^i f_n\|_{2,D} = 0 \quad \text{for } i = 1,\ldots,k+4. \tag{3.4}$$

Since $BAB = B^2$, we get

$$\lim_{n\to\infty} \|(BA-z)B\bar{\partial}^i f_n\|_{2,D} = 0 \quad \text{for } i = 1,\ldots,k+4,$$

and so

$$\lim_{n\to\infty} \|(ABA-zA)B\bar{\partial}^i f_n\|_{2,D} = 0 \quad \text{for } i = 1,\ldots,k+4.$$

Since $ABA = A^2$, we infer that

$$\lim_{n\to\infty} \|(A-z)AB\bar{\partial}^i f_n\|_{2,D} = 0 \quad \text{for } i = 1,\ldots,k+4. \tag{3.5}$$

Since $C^\infty(\bar{D},\mathcal{H})$ is dense in $W^k(D,\mathcal{H})$ (see [14, p. 388]), equation (3.1) holds for $f \in W^k(D,\mathcal{H})$. Hence (3.5) yields

$$\lim_{n\to\infty} \|AB\bar{\partial}^i f_n\|_{2,D_1} = 0 \quad \text{for } i = 1,2,3,4.$$

Since $BAB = B^2$, it follows from (3.3) that

$$\lim_{n\to\infty} \|z^2\bar{\partial}^i f_n\|_{2,D_1} = 0 \quad \text{for } i = 1,2,3,4. \tag{3.6}$$

Due to the hyponormality of the zero operator and [14, Corollary 2.2], we have

$$\lim_{n\to\infty} \|(I-P)z\bar{\partial}^i f_n\|_{2,D_1} = 0 \quad \text{for } i = 0,1,2,$$

where P denotes the orthogonal projection of $L^2(D_1,\mathcal{H})$ onto $A^2(D_1,\mathcal{H})$. This yields

$$\lim_{n\to\infty} \|zP(z\bar{\partial}^i f_n)\|_{2,D_1} = 0 \quad \text{for } i = 1,2.$$

Since the zero operator has Bishop's property (β), we infer from [14, Lemma 1.1] that

$$\lim_{n\to\infty} \|P(z\bar{\partial}^i f_n)\|_{2,D_2} = 0 \quad \text{for } i = 1,2.$$

Thus

$$\lim_{n\to\infty} \|z\bar{\partial}^i f_n\|_{2,D_2} = 0 \quad \text{for } i = 1,2. \tag{3.7}$$

Hence, again applying [14, Corollary 2.2], we have

$$\lim_{n\to\infty} \|(I-P)f_n\|_{2,D_2} = 0. \tag{3.8}$$

Combining (3.2) and (3.8), we have

$$\lim_{n\to\infty} \|(B-z)Pf_n + 1\otimes h_n\|_{2,D_2} = 0. \tag{3.9}$$

Let Γ be a closed curve in D_2 surrounding σ. Since

$$\lim_{n\to\infty} \|(B-z)Pf_n(z) + h_n\| = 0$$

uniformly on compact subsets of D_2 (see [14, Lemma 1.1]), applying the Riesz–Dunford functional calculus, we obtain

$$\lim_{n\to\infty} \left\| \frac{1}{2\pi i} \int_\Gamma Pf_n(z)\,dz + h_n \right\| = 0.$$

But $\frac{1}{2\pi i}\int_\Gamma Pf_n(z)\,dz = 0$ by Cauchy's theorem, and so $\lim_{n\to\infty}\|h_n\| = 0$.

(ii) If $T = BA$, consider arbitrary sequences $\{h_n\} \subset \mathcal{H}$ and $\{f_n\} \subset W^{k+2}(D,\mathcal{H})$ such that

$$\lim_{n\to\infty} \|(BA-z)f_n + 1\otimes h_n\|_{W^{k+2}} = 0.$$

Then

$$\lim_{n\to\infty} \|(BA-z)\bar{\partial}^i f_n\|_{2,D} = 0 \quad \text{for } i = 1,\ldots,k+2. \tag{3.10}$$

Since $ABA = A^2$, we have

$$\lim_{n\to\infty} \|(A-z)A\bar{\partial}^i f_n\|_{2,D} = 0 \quad \text{for } i = 1,\ldots,k+2.$$

We deduce from (3.1) that

$$\lim_{n\to\infty} \|A\bar{\partial}^i f_n\|_{2,D_1} = 0 \quad \text{for } i = 1,2.$$

This yields

$$\lim_{n\to\infty} \|BA\bar{\partial}^i f_n\|_{2,D_1} = 0 \quad \text{for } i = 1,2. \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\lim_{n\to\infty} \|z\bar{\partial}^i f_n\|_{2,D_1} = 0 \quad \text{for } i = 1,2.$$

Hence $\lim_{n\to\infty}\|h_n\| = 0$ by the procedure after equation (3.7) in (i), and so V_{k+2} is bounded below, which completes the proof. ∎

Similarly, we can state the following lemma.

Lemma 3.2. *Let $A, B \in \mathcal{L}(\mathcal{H})$ satisfy $ABA = A^2$ and $BAB = B^2$. Suppose that AB is subscalar of finite order, and set $T = BA$ or $T = A$. For an open disk D in $\mathbb{C}$ containing σ and a positive integer m, define the map $V_m : \mathcal{H} \to H(D)$ as in Lemma 3.1. Then V_m is one-to-one and has closed range for some positive integer m.*

Proof. Assume that D_1 is an open disk containing σ such that $\overline{D_1} \subset D$. From [7], we can find a constant $C > 0$ and a positive integer ℓ such that

$$\|f\|_{2,D_1} \le C \sum_{i=0}^{\ell} \|(AB - z)\bar{\partial}^i f\|_{2,D} \quad \text{for } f \in C^\infty(\mathbb{C}, \mathcal{H}). \tag{3.12}$$

(i) Consider the case of $T = BA$. Let $\{h_n\} \subset \mathcal{H}$ and $\{f_n\} \subset W^{\ell+2}(D, \mathcal{H})$ be sequences such that

$$\lim_{n\to\infty} \|(BA - z)f_n + 1 \otimes h_n\|_{W^{\ell+2}} = 0.$$

By the definition of the norm on $W^{\ell+2}(D, \mathcal{H})$, we have

$$\lim_{n\to\infty} \|(BA - z)\bar{\partial}^i f_n\|_{2,D} = 0 \quad \text{for } i = 1, \dots, \ell + 2. \tag{3.13}$$

Multiplying (3.13) by A, we obtain

$$\lim_{n\to\infty} \|(AB - z)A\bar{\partial}^i f_n\|_{2,D} = 0 \quad \text{for } i = 1, \dots, \ell + 2. \tag{3.14}$$

Then it follows from (3.12) that

$$\lim_{n\to\infty} \|A\bar{\partial}^i f_n\|_{2,D_1} = 0 \quad \text{for } i = 1, 2. \tag{3.15}$$

Combining (3.15) with (3.13), we get

$$\lim_{n\to\infty} \|z\bar{\partial}^i f_n\|_{2,D_1} = 0 \quad \text{for } i = 1, 2.$$

Hence, as in the proof of Lemma 3.1, we can show that $V_{\ell+2}$ is one-to-one and has closed range.

(ii) Set $T = A$, and let $\{h_n\} \subset \mathcal{H}$ and $\{f_n\} \subset W^{\ell+2}(D, \mathcal{H})$ be sequences such that

$$\lim_{n\to\infty} \|(A - z)f_n + 1 \otimes h_n\|_{W^{\ell+2}} = 0.$$

Since

$$\lim_{n\to\infty} \|(A - z)\bar{\partial}^i f_n\|_{2,D} = 0 \quad \text{for } i = 1, \dots, \ell + 2 \tag{3.16}$$

and $ABA = A^2$, multiplying (3.16) by A, we have

$$\lim_{n\to\infty} \|(AB - z)A\bar{\partial}^i f_n\|_{2,D} = 0 \quad \text{for } i = 1, \dots, \ell + 2.$$

By the same process as in (3.14), the map $V_{\ell+2}$ is one-to-one and has closed range. ∎

LEMMA 3.3. *Let $T \in \mathcal{L}(\mathcal{H})$. For an open disk D in $\mathbb{C}$ containing $\sigma(T)$ and a positive integer m, define $V_m : \mathcal{H} \to H(D)$ as in Lemmas* 3.1 *and* 3.2. *If V_m is one-to-one and has closed range for some positive integer m, then T is subscalar of order m.*

Proof. We write $\widetilde{f}$ for the class of a vector $f \in W^m(D,\mathcal{H})$ on the quotient space $H(D) := W^m(D,\mathcal{H})/\overline{(T-z)W^m(D,\mathcal{H})}$, that is, $\widetilde{f} = f + \overline{(T-z)W^m(D,\mathcal{H})}$. Similarly, $\widetilde{S}$ stands for the operator, induced by an operator $S \in \mathcal{L}(\mathcal{H})$, on $H(D)$ given by

$$\widetilde{S}\widetilde{f} = Sf + \overline{(T-z)W^m(D,\mathcal{H})}.$$

Let M be the operator of multiplication by z on $W^m(D,\mathcal{H})$. As noted at the end of Section 2, M is a scalar operator of order m with spectral distribution $\Phi_M : C_0^m(\mathbb{C}) \to \mathcal{L}(W^m(D,\mathcal{H}))$ defined by $\Phi_M(\varphi)f = \varphi f$ for $\varphi \in C_0^m(\mathbb{C})$ and $f \in W^m(D,\mathcal{H})$. Since the range of $T-z$ is invariant under M, the operator $\widetilde{M}$ is well-defined. Moreover, the spectral distribution Φ_M of M commutes with $T-z$, and so $\widetilde{M}$ is also scalar of order m with spectral distribution $\widetilde{\Phi}_M$. Since

$$V_mTh = \widetilde{1\otimes Th} = \widetilde{z\otimes h} = \widetilde{M}(\widetilde{1\otimes h}) = \widetilde{M}V_mh$$

for all $h \in \mathcal{H}$, we have $V_mT = \widetilde{M}V_m$. In particular, $\operatorname{ran}(V_m)$ is an invariant subspace for $\widetilde{M}$. Since T is similar to the restriction $\widetilde{M}|_{\operatorname{ran}(V_m)}$ and $\widetilde{M}$ is scalar of order m, we conclude that T is subscalar of order m. ■

As an application of Lemmas 3.1–3.3, we get the following theorem.

Theorem 3.4. *Let $A, B \in \mathcal{L}(\mathcal{H})$ satisfy $ABA = A^2$ and $BAB = B^2$. Then the following statements are equivalent:*

(i) *A is subscalar of finite order.*
(ii) *B is subscalar of finite order.*
(iii) *AB is subscalar of finite order.*
(iv) *BA is subscalar of finite order.*

Proof. If A is subscalar of finite order, then it follows from Lemmas 3.1 and 3.3 that B and BA are subscalar of finite order. Hence we have (i)⇒(ii) and (i)⇒(iv). Similarly, combining Lemma 3.3 with Lemma 3.2, we find that (iii)⇒(iv) and (iii)⇒(i). By symmetry, the implications (ii)⇒(iii) and (vi)⇒(iii) also hold. ■

Remark 3.5. Let A and B be operators in $\mathcal{L}(\mathcal{H})$ satisfying the operator equations $ABA = A^2$ and $BAB = B^2$. Since the pair (A^*, B^*) is also a solution to these equations, the following statements are obviously equivalent from Theorem 3.4:

(i) A^* is subscalar of finite order.
(ii) B^* is subscalar of finite order.
(iii) A^*B^* is subscalar of finite order.
(iv) B^*A^* is subscalar of finite order.

In addition, the equivalence statements hold when we replace subscalarity of finite order with Bishop's property (β). Indeed, if A has Bishop's property (β), assume that $\{f_n\}$ is a sequence of analytic functions on an open set G in $\mathbb{C}$ such that $\lim_{n\to\infty}\|(B-z)f_n(z)\|=0$ uniformly on compact subsets of G. Applying the procedure from (3.3) to (3.5) in the proof of Lemma 3.1, we find that $\lim_{n\to\infty}\|(A-z)ABf_n(z)\|=0$ uniformly on compact subsets of G. Since A has Bishop's property (β), it follows that $\lim_{n\to\infty}\|ABf_n(z)\|=0$ uniformly on compact subsets of G. Since $BAB = B^2$ and $\lim_{n\to\infty}\|(B-z)f_n(z)\|=0$ uniformly on compact subsets of G, we have $\lim_{n\to\infty}\|z^2f_n(z)\|=0$ uniformly on compact subsets of G, which implies that $\lim_{n\to\infty}\|f_n(z)\|=0$ uniformly on compact subsets of G since every zero operator has Bishop's property (β). Hence, if A has Bishop's property (β), then so does B, and vice versa. Similarly, the remaining equivalence statements are obtained by applying the methods in Lemmas 3.1 and 3.2 (see [5, Theorem 2.9] for more details).

Recall that an operator $T\in\mathcal{L}(\mathcal{H})$ is said to be a *class A* operator if the absolute value condition $|T^2|\geq|T|^2$ holds where $|T|:=(T^*T)^{1/2}$.

COROLLARY 3.6. *Let $A,B\in\mathcal{L}(\mathcal{H})$ satisfy $ABA=A^2$ and $BAB=B^2$. If A is a class A operator, then B is subscalar of finite order.*

Proof. Since every class A operator is subscalar of finite order by [9, Theorem 3.2], the assertion follows from Theorem 3.4. ∎

COROLLARY 3.7. *Let $A,B\in\mathcal{L}(\mathcal{H})$ satisfy $ABA=A^2$ and $BAB=B^2$. If A is subscalar of finite order and $\sigma(A)$ has nonempty interior, then B, AB, and BA each have nontrivial invariant subspaces.*

Proof. Since $\sigma(A)=\sigma(B)=\sigma(AB)=\sigma(BA)$ from [15, Proposition 2.6, Corollary 2.7], the conclusion follows from Theorem 3.4 and [6]. ∎

For an operator $T\in\mathcal{L}(\mathcal{H})$, a vector $x\in\mathcal{H}$ is said to be *cyclic* if the linear span of the orbit $O(x,T):=\{T^nx : n=0,1,2,\ldots\}$ is dense in $\mathcal{H}$, i.e., $\bigvee O(x,T)=\mathcal{H}$. If there is a cyclic vector x for T, then we say that T is a *cyclic* operator. We denote the *local spectral radius* of T at x by $r_T(x):=\limsup_{n\to\infty}\|T^nx\|^{1/n}$, while $r(T):=\sup\{|\lambda| : \lambda\in\sigma(T)\}$ is the *spectral radius* of T. For an operator $T\in\mathcal{L}(\mathcal{H})$, a T-invariant subspace $\mathcal{M}$ is said to be a *spectral maximal space* of T if $\mathcal{M}$ contains any T-invariant subspace $\mathcal{N}$ satisfying $\sigma(T|_{\mathcal{N}})\subset\sigma(T|_{\mathcal{M}})$.

COROLLARY 3.8. *Let $A,B\in\mathcal{L}(\mathcal{H})$ satisfy $ABA=A^2$ and $BAB=B^2$, and suppose A is subscalar of finite order. Then:*

(i) *B, AB, and BA have Bishop's property (β), Dunford's property (C), and the single-valued extension property.*
(ii) *$\sigma_B(x)=\sigma(B)$ and $r_B(x)=r(B)$ whenever $x\in\mathcal{H}$ is any cyclic vector for B. Moreover, $r_B(x)=\lim_{n\to\infty}\|B^nx\|^{1/n}$ for all $x\in\mathcal{H}$.*

(iii) *For each closed subset F of $\mathbb{C}$, $\mathcal{H}_B(F)$ is a spectral maximal space of B, $\sigma(B|_{\mathcal{H}_B(F)}) \subset \sigma(B) \cap F$, and $\sigma_B(x) = \sigma_{B|_{\mathcal{H}_B(F)}}(x)$ for all $x \in \mathcal{H}_B(F)$.*

(iv) *If $f : D \to \mathbb{C}$ is an analytic function on an open neighborhood D of $\sigma(A)$, then $\sigma_{f(B)}(x) = f(\sigma_B(x))$.*

Proof. (i) It suffices to prove that B, AB, and BA have Bishop's property (β). Since Bishop's property (β) is transmitted from an operator to its restrictions to closed invariant subspaces, we are reduced by Theorem 3.4 to the case of a scalar operator. Since every scalar operator has Bishop's property (β) (see [14, p. 394, Remarks 4]), it follows that B, AB, and BA have Bishop's property (β).

(ii) By (i), B has Bishop's property (β), and the results follow from [12, p. 238].

(iii) This follows from [12, Proposition 1.2.20].

(iv) Since B has the single-valued extension property from (i) and $\sigma(A) = \sigma(B)$, the assertion follows from [12, Theorem 3.3.8]. ∎

Corollary 3.9. *If $A, B \in \mathcal{L}(\mathcal{H})$ satisfy $ABA = A^2$ and $BAB = B^2$, then the following statements are equivalent:*

(i) *A is decomposable.*
(ii) *B is decomposable.*
(iii) *AB is decomposable.*
(iv) *BA is decomposable.*

In particular, if A is decomposable, then $\mathcal{H}_B(F)$ is a spectral maximal space of B for any closed subset F of $\mathbb{C}$.

Proof. This follows immediately from Remark 3.5 and [3, Chapter 2, Proposition 3.8]. ∎

It is not easy to show that a given 2×2 operator matrix is subscalar. But if we use the operator equations, we have some advantages. In the following theorem, we prove that some 2×2 operator matrices have scalar extensions.

Theorem 3.10.

(i) *If $T \in \mathcal{L}(\mathcal{H})$ is an operator such that $T^2 - \gamma T + \delta I$ is hyponormal for some $\gamma, \delta \in \mathbb{C} \setminus \{0\}$, then*

$$\begin{pmatrix} \gamma T^2 & \gamma^2 T \\ (\delta I - \gamma T)T & \gamma(\delta I - \gamma T) \end{pmatrix}$$

is subscalar of finite order.

(ii) *If $S, T, R \in \mathcal{L}(\mathcal{H})$ satisfy that RS is hyponormal and $ST = TS = \frac{1}{4}I$, then*

$$\begin{pmatrix} SR & S \\ \frac{1}{2}R & \frac{1}{2}I \end{pmatrix}$$

is subscalar of finite order.

(iii) *Suppose that $S, T, R \in \mathcal{L}(\mathcal{H})$ are mutually commuting operators such that $T^2 + RS = T$. If T is hyponormal, then*

$$\begin{pmatrix} T & S \\ -RT & -RS \end{pmatrix}$$

is subscalar of finite order.

Proof. (i) Let

$$P = \begin{pmatrix} \frac{\gamma}{\delta}T & \frac{\gamma}{\delta}T \\ I - \frac{\gamma}{\delta}T & I - \frac{\gamma}{\delta}T \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 \\ \frac{1}{\gamma}T & I \end{pmatrix}.$$

Define

$$A := PQ = \frac{1}{\gamma\delta}\begin{pmatrix} \gamma T^2 & \gamma^2 T \\ (\delta I - \gamma T)T & \gamma(\delta I - \gamma T) \end{pmatrix},$$

$$B := QP = \frac{1}{\delta}\begin{pmatrix} 0 & 0 \\ T^2 - \gamma T + \delta I & T^2 - \gamma T + \delta I \end{pmatrix}.$$

Since $P^2 = P$ and $Q^2 = Q$, we have $ABA = A^2$ and $BAB = B^2$. For the unitary operator $U = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, we have

$$U^*BU = \begin{pmatrix} T^2 - \gamma T + \delta I & T^2 - \gamma T + \delta I \\ 0 & 0 \end{pmatrix}.$$

Since $T^2 - \gamma T + \delta I$ is hyponormal, U^*BU is subscalar of finite order from [10, Theorem 4.5], and so is B. Thus A is also subscalar of finite order by Theorem 3.4.

(ii) Set

$$P = \begin{pmatrix} \frac{1}{2}I & S \\ T & \frac{1}{2}I \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 \\ R & I \end{pmatrix}.$$

Define

$$A := PQ = \begin{pmatrix} SR & S \\ \frac{1}{2}R & \frac{1}{2}I \end{pmatrix} \quad \text{and} \quad B := QP = \begin{pmatrix} 0 & 0 \\ \frac{1}{2}R + T & RS + \frac{1}{2}I \end{pmatrix}.$$

Since $P^2 = P$ and $Q^2 = Q$, we see that $ABA = A^2$ and $BAB = B^2$. Furthermore, since $RS + \frac{1}{2}I$ is hyponormal, B is subscalar of finite order by [10, Theorem 4.5]. Therefore, Theorem 3.4 shows that A is subscalar of finite order.

(iii) Set

$$P = \begin{pmatrix} I & 0 \\ -R & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} T & S \\ R & I-T \end{pmatrix}.$$

By the hypotheses, $P^2 = P$ and $Q^2 = Q$. Define

$$A := PQ = \begin{pmatrix} T & S \\ -RT & -RS \end{pmatrix} \quad \text{and} \quad B := QP = \begin{pmatrix} T^2 & 0 \\ TR & 0 \end{pmatrix}.$$

Since T is hyponormal, T^2 is subscalar of finite order from [14, Proposition 2.5(d), Remarks 2], and so is B. Thus, we conclude from Theorem 3.4 that A is subscalar of finite order. ■

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *algebraic* if there exists a nonzero polynomial p such that $p(T) = 0$.

THEOREM 3.11. *If $A, B \in \mathcal{L}(\mathcal{H})$ satisfy $ABA = A^2$ and $BAB = B^2$, then the following statements are equivalent:*

(i) *A is algebraic.*
(ii) *B is algebraic.*
(iii) *AB is algebraic.*
(iv) *BA is algebraic.*

Proof. Note that

$$\begin{cases} B\Big(\sum_{j=0}^{n} a_j A^j\Big)B = a_0 B^2 + \sum_{j=1}^{n} a_j BA^j B, \\ A\Big(\sum_{j=0}^{n} a_j B^j\Big)A = a_0 A^2 + \sum_{j=1}^{n} a_j AB^j A, \end{cases}$$

where $a_0, a_1, \ldots, a_n \in \mathbb{C}$ and n is any positive integer. Thus, we can obtain the equivalence (i)⇔(ii) if the following identities hold:

$$BA^j B = B^{j+1} \quad \text{and} \quad AB^j A = A^{j+1} \tag{3.17}$$

for any positive integer j. If $j = 1$, then (3.17) holds clearly. If (3.17) holds for some positive integer j, we get

$$BA^{j+1}B = B(AB^jA)B = (BAB)B^{j-1}AB = B^{j+1}AB = B^j(BAB) = B^{j+2}.$$

Similarly, $AB^{j+1}A = A^{j+2}$. Thus the equalities in (3.17) are true for all positive integers j, and so (i)⇔(ii).

We next prove that

$$(BA)^j = B^2A^{j-1} \quad \text{and} \quad (AB)^j = A^2B^{j-1} \tag{3.18}$$

for every positive integer $j \geq 2$. We first note that $(BA)^2 = (BAB)A = B^2A$. If $(BA)^j = B^2A^{j-1}$ for some positive integer $j \geq 2$, then

$$(BA)^{j+1} = B^2A^{j-1}BA = B^2A^{j-2}(ABA) = B^2A^j.$$

By induction on j, the identity $(BA)^j = B^2A^{j-1}$ is true for every integer $j \geq 2$. We obtain the identity $(AB)^j = A^2B^{j-1}$ for each integer $j \geq 2$, using a similar method. If $p(A) = 0$ for some nonzero polynomial $p(z) = \sum_{j=0}^n a_j z^j$, then it follows from (3.18) that

$$0 = B^2p(A)A = \sum_{j=0}^n a_j B^2 A^{j+1} = \sum_{j=0}^n a_j (BA)^{j+2},$$

which shows that BA is algebraic. Hence (i)⇒(iv). Applying the second equality in (3.18), we have (ii)⇒(iii).

If AB is algebraic, then $p(AB) = 0$ where $p(z) = \sum_{j=0}^n a_j z^j$ is some nonzero polynomial. Combining (3.17) and (3.18), we obtain

$$(AB)^jA = A^{j+1} \quad \text{and} \quad (BA)^jB = B^{j+1} \tag{3.19}$$

for any integer $j \geq 0$. The first identity of (3.19) yields

$$0 = p(AB)A = \sum_{j=0}^n a_j (AB)^j A = \sum_{j=0}^n a_j A^{j+1},$$

which means that A is algebraic. Therefore (iii)⇒(i). By symmetry, we obtain (iv)⇒(ii). ∎

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *power bounded* if there exists a constant $M > 0$ such that $\|T^n\| \leq M$ for every positive integer n.

THEOREM 3.12. *If $A, B \in \mathcal{L}(\mathcal{H})$ satisfy $ABA = A^2$ and $BAB = B^2$, then the following statements are equivalent:*

(i) *A is power bounded.*
(ii) *B is power bounded.*
(iii) *AB is power bounded.*
(iv) *BA is power bounded.*

Proof. We see from (3.17) that

$$\|B^n\| \leq \|B\|^2\|A^{n-1}\|$$

for all integers $n \geq 2$, which ensures that (i)⇒(ii). Applying (3.18), we get

$$\|(BA)^n\| \leq \|B\|^2\|A^{n-1}\|$$

for all $n \geq 2$, and so (i)⇒(iv). Furthermore, (3.19) implies that

$$\|A^n\| \leq \|A\|\|(AB)^{n-1}\|$$

for all $n \geq 1$. Therefore (iii)⇒(i). By symmetry, we have the implications (ii)⇒(i), (ii)⇒(iii), and (iv)⇒(ii). ∎

Corollary 3.13. *Let $A, B \in \mathcal{L}(\mathcal{H})$ satisfy $ABA = A^2$ and $BAB = B^2$. If A is power bounded, then so is $B \otimes U$ for every contraction $U \in \mathcal{L}(\mathcal{H})$.*

Proof. Since $\|(B\otimes U)^n\| = \|B^n\|\,\|U^n\| \leq \|B^n\|$ for all n and B is power bounded from Theorem 3.12, the proof is complete. ∎

An operator T in $\mathcal{L}(\mathcal{H})$ is called *quasitriangular* if it can be written as a sum $T = T_0 + K$, where T_0 is a triangular operator (i.e., there exists an orthonormal basis for $\mathcal{H}$ with respect to which the matrix for T_0 is upper triangular) and K is a compact operator in $\mathcal{L}(\mathcal{H})$. We say that T is *biquasitriangular* if both T and T^* are quasitriangular (see [13] for more details). For $T \in \mathcal{L}(\mathcal{H})$, a *hole* in $\sigma_{\mathrm{e}}(T)$ is a bounded component of $\mathbb{C}\setminus\sigma_{\mathrm{e}}(T)$. A *pseudohole* in $\sigma_{\mathrm{e}}(T)$ is a component of $\sigma_{\mathrm{e}}(T) \setminus \sigma_{\mathrm{le}}(T)$ or $\sigma_{\mathrm{e}}(T) \setminus \sigma_{\mathrm{re}}(T)$. The *spectral picture* of $T \in \mathcal{L}(\mathcal{H})$ (notation: $\mathrm{SP}(T)$) is the structure consisting of the set $\sigma_{\mathrm{e}}(T)$, the collection of holes and pseudoholes in $\sigma_{\mathrm{e}}(T)$, and the indices associated with these holes and pseudoholes (see [13] for more details). According to Apostol–Foiaş–Voiculescu (see [13, Theorem 1.31]), an operator $T \in \mathcal{L}(\mathcal{H})$ is quasitriangular if and only if $\mathrm{SP}(T)$ contains no hole or pseudohole with a negative index. Using this result, we show the following theorem.

Theorem 3.14. *If $A, B \in \mathcal{L}(\mathcal{H})$ satisfy $ABA = A^2$ and $BAB = B^2$, then the following statements are equivalent:*

(i) *A is quasitriangular.*
(ii) *B is quasitriangular.*
(iii) *AB is quasitriangular.*
(iv) *BA is quasitriangular.*

Proof. From [5, Theorem 2.5], we know that

$$\begin{cases} \sigma_{\mathrm{e}}(A) = \sigma_{\mathrm{e}}(B) = \sigma_{\mathrm{e}}(AB) = \sigma_{\mathrm{e}}(BA), \\ \sigma_{\mathrm{le}}(A) = \sigma_{\mathrm{le}}(B) = \sigma_{\mathrm{le}}(AB) = \sigma_{\mathrm{le}}(BA), \\ \sigma_{\mathrm{re}}(A) = \sigma_{\mathrm{re}}(B) = \sigma_{\mathrm{re}}(AB) = \sigma_{\mathrm{re}}(BA). \end{cases} \tag{3.20}$$

Moreover, if λ belongs to a hole or pseudohole in $\sigma_{\mathrm{e}}(A)$, then all the indices of $A-\lambda$, $B-\lambda$, $AB-\lambda$, and $BA-\lambda$ are equal by [15, Corollary 2.12]. Thus, all of A, B, AB, and BA have the same spectral picture, which completes the proof by [13, Theorem 1.31]. ∎

Corollary 3.15. *Let $A, B \in \mathcal{L}(\mathcal{H})$ satisfy $ABA = A^2$ and $BAB = B^2$. Then A is biquasitriangular if and only if B is biquasitriangular. In particular, if $\sigma(A)$ is countable, then B is biquasitriangular. Moreover, if A is not biquasitriangular, then B has a nontrivial hyperinvariant subspace.*

Proof. Since $A^*B^*A^* = A^{*2}$ and $B^*A^*B^* = B^{*2}$, the first assertion holds by Theorem 3.14. If $\sigma(A)$ is countable, then A is biquasitriangular from [8, p. 862, Corollary], and so is B by the first statement. Moreover, if A is not biquasitriangular, then neither is B. Hence B has a nontrivial hyperinvariant subspace by [1]. ∎

Finally, we consider local spectra of operators A and B such that $ABA = A^2$ and $BAB = B^2$.

PROPOSITION 3.16. *If $A, B \in \mathcal{L}(\mathcal{H})$ satisfy $ABA = A^2$ and $BAB = B^2$, then:*

(i) *$\sigma_A(ABx) \subset \sigma_B(x)$ and $\sigma_B(BAx) \subset \sigma_A(x)$ for all $x \in \mathcal{H}$.*
(ii) *$AB(\mathcal{H}_B(F)) \subset \mathcal{H}_A(F)$ and $BA(\mathcal{H}_A(F)) \subset \mathcal{H}_B(F)$ for any closed set F in $\mathbb{C}$.*
(iii) *$A\mathcal{H}_A(F) \subset \mathcal{H}_{AB}(F)$ and $B\mathcal{H}_B(F) \subset \mathcal{H}_{BA}(F)$ for any closed set F in $\mathbb{C}$.*

Proof. (i) It suffices to show the first inclusion. Let $x \in \mathcal{H}$ and $\lambda \in \rho_B(x)$. We can choose a neighborhood D of λ and an analytic function $f : D \to \mathcal{H}$ such that $(B - z)f(z) = x$ for all $z \in D$. Since

$$(A - z)ABf(z) = (AB - z)ABf(z) = AB(B - z)f(z) = ABx$$

for all $z \in D$, we infer that $\lambda \in \rho_A(ABx)$, and so $\rho_B(x) \subset \rho_A(ABx)$, i.e. $\sigma_B(x) \supset \sigma_A(ABx)$.

(ii) Let F be any closed set in $\mathbb{C}$. If $x \in \mathcal{H}_B(F)$, then it follows from (i) that $\sigma_A(ABx) \subset \sigma_B(x) \subset F$. Thus $ABx \in \mathcal{H}_A(F)$, and so $AB(\mathcal{H}_B(F)) \subset \mathcal{H}_A(F)$. By symmetry, $BA(\mathcal{H}_A(F)) \subset \mathcal{H}_B(F)$.

(iii) To obtain the inclusion $A\mathcal{H}_A(F) \subset \mathcal{H}_{AB}(F)$, it is enough to prove that $\sigma_{AB}(Ax) \subset \sigma_A(x)$, or equivalently $\rho_A(x) \subset \rho_{AB}(Ax)$ for all $x \in \mathcal{H}$; indeed, whenever $x \in \mathcal{H}_A(F)$, we have $\sigma_{AB}(Ax) \subset \sigma_A(x) \subset F$, i.e., $Ax \in \mathcal{H}_{AB}(F)$. Let $x \in \mathcal{H}$. If $\lambda \in \rho_A(x)$, then there exists an $\mathcal{H}$-valued analytic function f on some neighborhood D of λ such that $(A - z)f(z) = x$ for all $z \in D$. Since $A^2 = ABA$, we have

$$(AB - z)Af(z) = A(A - z)f(z) = Ax$$

for all $z \in D$, which means that $\lambda \in \rho_{AB}(Ax)$. Therefore $\rho_A(x) \subset \rho_{AB}(Ax)$. We obtain the second inclusion in (iii) by symmetry. ∎

COROLLARY 3.17. *Let $A, B \in \mathcal{L}(\mathcal{H})$ satisfy $ABA = A^2$ and $BAB = B^2$. If F is any closed subset of $\mathbb{C}$ with $0 \in F$, then $\mathcal{H}_A(F)$ is closed if and only if $\mathcal{H}_B(F)$ is closed.*

Proof. Let F be any closed subset of $\mathbb{C}$ with $0 \in F$, and suppose that $\mathcal{H}_A(F)$ is closed. Let $\{x_n\}$ be a sequence in $\mathcal{H}_B(F)$ such that $x_n \to x$ in norm for some $x \in \mathcal{H}$. From Proposition 3.16, we have

$$ABx_n \in AB(\mathcal{H}_B(F)) \subset \mathcal{H}_A(F)$$

for all n. Since $\mathcal{H}_A(F)$ is closed, we see that $ABx = \lim_{n\to\infty} ABx_n \in \mathcal{H}_A(F)$. Since $BA^2B = B^3$ by (3.17), it follows from Proposition 3.16 that

$$B^3x = BA^2Bx \in BA(\mathcal{H}_A(F)) \subset \mathcal{H}_B(F). \tag{3.21}$$

We note that if $T \in \mathcal{L}(\mathcal{H})$ and $(T-\lambda)x \in \mathcal{H}_T(F)$ for some $\lambda \in F$, then $x \in \mathcal{H}_T(F)$ (see [12, Proposition 1.2.16]). Since $(B-0)B^2x \in \mathcal{H}_B(F)$ from (3.21) and $0 \in F$, we have $B^2x \in \mathcal{H}_B(F)$. Repeating this process, we conclude that $x \in \mathcal{H}_B(F)$. Hence $\mathcal{H}_B(F)$ is closed. The converse statement holds by symmetry. ∎

For $A, B \in \mathcal{L}(\mathcal{H})$, define the *commutator* $C(A,B) : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ by $C(A,B)T = AT - TB$. It is easy to see that $C(A,B)$ is a bounded linear operator and $C^n(A,B) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}A^{n-k}TB^k$ for each positive integer n, where $C^n(A,B)$, usually called a *higher order commutator*, stands for the nth power of $C(A,B)$ (see [3] and [12] for more details). As an application of Proposition 3.16, we give some local spectral properties of the commutators $C(A,B)$ and $C(B,A)$ when (A,B) is a solution of the system $ABA = A^2$ and $BAB = B^2$.

Corollary 3.18. *Let $A, B \in \mathcal{L}(\mathcal{H})$ satisfy $ABA = A^2$ and $BAB = B^2$. If A is decomposable, then $C(A,B)$ has the single-valued extension property and* $\lim_{n\to\infty} \|C^n(A,B)(AB)\|^{1/n} = 0$. *Moreover, $\sigma_{C(A,B)}(AB) = \{0\}$ if $AB \neq 0$, and $\sigma_{C(A,B)}(AB) = \emptyset$ if AB is the zero operator on $\mathcal{H}$.*

Proof. If A is decomposable, then B is also decomposable from Corollary 3.9. Hence $C(A,B)$ has the single-valued extension property by [12, Proposition 3.4.6] and we deduce that $\lim_{n\to\infty} \|C^n(A,B)(AB)\|^{1/n} = 0$ by combining Proposition 3.16 with [3, p. 48, Theorem 3.3]. Since $C(A,B)$ has the single-valued extension property, we know from [12, Proposition 1.2.16] that the only operator $T \in \mathcal{L}(\mathcal{H})$ such that $\sigma_{C(A,B)}(T) = \emptyset$ is $T \equiv 0$ on $\mathcal{H}$, and thus the results on the local spectral radii follow from [12, Proposition 3.3.13]. ∎

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