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Numerical index with respect to an operator

by

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Abstract. We introduce new concepts of numerical range and numerical radius of one operator with respect to another one, which generalize in a natural way the known concepts of numerical range and numerical radius. We study basic properties of these new concepts and present some examples.

1. Introduction. In our paper the letters X and Y stand for Banach spaces over a field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). The space of all bounded linear operators from X into Y is denoted by $L(X,Y)$, shortened to $L(X)$ if $X = Y$. The dual space to X is denoted by X^*. For a Banach space X, we denote by B_X and S_X the corresponding unit ball and unit sphere.

Let $T \in L(X)$. The *numerical range* of T, $V(T)$, the *numerical radius* of T, $\nu(T)$, and the *numerical index* of the space X, $n(X)$, are defined as follows:

$$\begin{aligned} V(T) &:= \{x^*(Tx) : x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\}, \\ \nu(T) &:= \sup\{|\lambda| : \lambda \in V(T)\}, \\ n(X) &:= \inf\{\nu(T) : T \in L(X),\ \|T\| = 1\}. \end{aligned}$$

It is well-known (see [3, 5]) that

$$\sup \operatorname{Re} V(T) = \lim_{\alpha \downarrow 0} \frac{\|\mathrm{Id} + \alpha T\| - 1}{\alpha}$$

and

$$\nu(T) = \max_{\omega \in \mathbb{T}} \lim_{\alpha \downarrow 0} \frac{\|\mathrm{Id} + \alpha\omega T\| - 1}{\alpha}$$

where $\mathbb{T}$ stands for the unit sphere of the base field K. Also we have

$$(\nu(T) = \|T\|) \ \Leftrightarrow\ \Big(\max_{\omega \in \mathbb{T}} \|\mathrm{Id} + \omega T\| = 1 + \|T\|\Big) \tag{1.1}$$

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(see [7, Lemma 2.3]). Therefore,

$$(n(X)=1) \Leftrightarrow \Big(\forall T \in L(X)\ \max_{\omega\in\mathbb{T}} \|\mathrm{Id}+\omega T\| = 1+\|T\|\Big). \tag{1.2}$$

In this note we generalize the concepts of numerical range, numerical radius and numerical index, considering instead of the identity operator some other operator $G \in L(X,Y)$ in such a way that relation (1.1) (and hence (1.2)) extends to these new general notions. The main difficulty that we had to overcome is that $L(X,Y)$ is not a Banach algebra, so we cannot use the technique of Banach algebras [3] in the study of numerical radius. Hence, the proof of the generalized version of (1.1) is different from the original one, and is more geometric in nature. We also present some examples.

2. Definitions. Before we present the generalized definitions, we recall the famous Bishop–Phelps–Bollobás theorem.

Theorem 2.1 (Bishop–Phelps–Bollobás). *Let X be a real Banach space and $0<\varepsilon<1/2$. Suppose $x \in S_X$ and $x^* \in S_{X^*}$ satisfy $x^*(x) \ge 1-\varepsilon^2/2$. Then there exists $(y,y^*) \in S_X \times S_{X^*}$ such that $y^*(y)=1$, $\|x-y\| < \varepsilon+\varepsilon^2$ and $\|x^*-y^*\| \le \varepsilon$.*

Proof. See [2]. ∎

Note that:

(a) Theorem 2.1 is applicable to real parts of complex functionals.

(b) The norms of a complex functional and of its real part are the same.

Taking into account the above comments we can reformulate the above theorem for arbitrary (real or complex) spaces in the following form:

Theorem 2.2. *Let X be a Banach space and $0<\varepsilon<1/2$. Suppose $x \in S_X$ and $x^* \in S_{X^*}$ satisfy $\operatorname{Re} x^*(x) \ge 1-\varepsilon^2/2$. Then there exists $(y,y^*) \in S_X \times S_{X^*}$ such that $y^*(y)=1$, $\|x-y\| < \varepsilon+\varepsilon^2$ and $\|x^*-y^*\| \le \varepsilon$.*

Observe also that according to [4, Theorem 2.1], the summand ε^2 in the estimate for $\|x-y\|$ above can be omitted, and that in fact the result remains valid for $0<\varepsilon<2$ and for $x \in B_X$ and $x^* \in B_{X^*}$.

Definition 2.3. For $T, G \in L(X,Y)$ with $\|G\|=1$ we define the *numerical range of T with respect to G*, $V_G(T)$, as follows:

$$V_G(T) := \bigcap_{\varepsilon>0} \overline{\{x^*(Tx) : x \in S_X,\ x^* \in S_{Y^*},\ \operatorname{Re} x^*(Gx) > 1-\varepsilon\}}.$$

Note that $V_G(T)$ is a closed set, but $V(T)$ may not be closed. Nevertheless, the following lemma shows that for $G=\mathrm{Id}: X \to X$ the new definition almost agrees with the classical one.

LEMMA 2.4. $V_{\mathrm{Id}}(T) = \overline{V(T)}$.

Proof. We have

$$V_{\mathrm{Id}}(T) = \bigcap_{\varepsilon>0} \overline{\{x^*(Tx) : x \in S_X,\ x^* \in S_{X^*},\ \operatorname{Re} x^*(x) > 1-\varepsilon\}}.$$

If $\lambda \in V(T)$ then there are $x \in S_X$ and $x^* \in S_{X^*}$ such that $x^*(x) = 1$ and $\lambda = x^*(Tx)$. Since $\operatorname{Re} x^*(x) = x^*(x) = 1$, we have $\lambda \in \{x^*(Tx) : x \in S_X, x^* \in S_{X^*}, \operatorname{Re} x^*(x) > 1-\varepsilon\}$ for all $\varepsilon > 0$, and hence

$$\lambda \in \bigcap_{\varepsilon>0} \overline{\{x^*(Tx) : x \in S_X,\ x^* \in S_{X^*},\ \operatorname{Re} x^*(x) > 1-\varepsilon\}}.$$

This means that $V(T) \subseteq V_{\mathrm{Id}}(T)$, so $\overline{V(T)} \subseteq \overline{V_{\mathrm{Id}}(T)} = V_{\mathrm{Id}}(T)$.

For the inverse inclusion consider an arbitrary $\mu \in V_{\mathrm{Id}}(T)$. Then for all $\delta > 0$,

$$\mu \in \overline{\{x^*(Tx) : x \in S_X,\ x^* \in S_{X^*},\ \operatorname{Re} x^*(x) > 1-\delta\}},$$

which means that for all $\varepsilon, \delta > 0$, there are $x \in S_X$ and $x^* \in S_{X^*}$ such that $\operatorname{Re} x^*(x) > 1-\delta$ and $|\mu - x^*(Tx)| < \varepsilon$. If in particular $0 < \varepsilon < 1/2$ and $\delta = \varepsilon^2/2$ then we get $x \in S_X$ and $x^* \in S_{X^*}$ such that $\operatorname{Re} x^*(x) > 1 - \varepsilon^2/2$ and $|\mu - x^*(Tx)| < \varepsilon$. Theorem 2.2 implies that there exist $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|x^* - y^*\| < \varepsilon$ and $\|x - y\| < \varepsilon + \varepsilon^2$. Obviously, $y^*(Ty) \in V(T)$. Now, we have

$$|y^*(Ty) - \mu| \le |y^*(Ty) - x^*(Tx)| + |x^*(Tx) - \mu| \le |y^*(Ty) - x^*(Tx)| + \varepsilon.$$

Let us estimate $|y^*(Ty) - x^*(Tx)|$:

$$\begin{aligned} |y^*(Ty) - x^*(Tx)| &= |y^*(Ty) - x^*(Ty) + x^*(Ty) - x^*(Tx)| \\ &\le |(y^* - x^*)(Ty)| + |x^*(T(y-x))| \\ &\le \|x^* - y^*\|\,\|T\|\,\|y\| + \|x^*\|\,\|T\|\,\|y - x\|. \end{aligned}$$

Therefore $|y^*(Ty) - \mu| \le \|T\|(2\varepsilon + \varepsilon^2) + \varepsilon$. Hence $V_{\mathrm{Id}}(T) \subseteq \overline{V(T)}$. ∎

DEFINITION 2.5. Suppose $T, G \in L(X,Y)$ and $\|G\| = 1$. We define the *numerical radius of T with respect to G*, $\nu_G(T)$, as follows:

$$\nu_G(T) := \max\{|t| : t \in V_G(T)\}.$$

Again, if $G = \mathrm{Id}$ then $\nu_{\mathrm{Id}}(T) = \max\{|t| : t \in V_{\mathrm{Id}}(T)\}$. Since $V_{\mathrm{Id}}(T) = \overline{V(T)}$, we have

$$\nu_{\mathrm{Id}}(T) = \max\{|t| : t \in \overline{V(T)}\} = \max\{|t| : t \in V(T)\} = \nu(T).$$

Finally, we extend the definition of numerical index:

DEFINITION 2.6. Suppose $T, G \in L(X,Y)$ and $\|G\| = 1$. We define $n_G(X,Y)$, the *numerical index of the pair (X,Y) with respect to G*, by

$$n_G(X,Y) = \inf\{\nu_G(T) : T \in L(X,Y),\ \|T\| = 1\}.$$

In the case of $X = Y$ we simplify the notation to $n_G(X)$. Evidently,

$$\begin{aligned}(2.1)\qquad n_{\mathrm{Id}}(X) &= \inf\{\nu_{\mathrm{Id}}(T) : T \in L(X),\ \|T\| = 1\}\\ &= \inf\{\nu(T) : T \in L(X),\ \|T\| = 1\} = n(X).\end{aligned}$$

We conclude this section with the following remark.

Remark 2.7. Obviously, by the definition $V_G(T)$ is a closed subset of $\mathbb{C}$. Since $|x^*(Tx)| \le \|x^*\|\,\|T\|\,\|x\| = \|T\|$ for all $x^* \in S_{Y^*}$ and $x \in S_X$, $V_G(T)$ is a compact subset of $\mathbb{C}$. Also, we have $\nu_G(T) \le \|T\|$.

3. Main results. Now, we are able to generalize relations (1.1) and (1.2). Observe that our proofs in the general case are even simpler than the original ones.

Lemma 3.1. *If* $\nu_G(T) = \|T\|$ *then* $\max_{\omega\in\mathbb{T}} \|G + \omega T\| = 1 + \|T\|$.

Proof. If $\nu_G(T) = \|T\|$, then the definition of $\nu_G(T)$ implies there is a $t = re^{i\theta} \in V_G(T)$ such that $r = |t| = \|T\|$. The inclusion $t \in V_G(T)$ means that for all $\varepsilon > 0$ we have

$$t \in \overline{\{x^*(Tx) : x \in S_X,\ x^* \in S_{Y^*},\ \operatorname{Re} x^*(Gx) > 1 - \varepsilon\}}.$$

Hence, for every $\varepsilon > 0$ there is a pair $(x, x^*) \in S_X \times S_{Y^*}$ such that $\operatorname{Re} x^*(Gx) > 1 - \varepsilon$ and $|x^*(Tx) - t| < \varepsilon$. Then

$$(3.1)\qquad |\operatorname{Re} e^{-i\theta}x^*(T(x)) - r| \le |e^{-i\theta}x^*(T(x)) - r| = |x^*(Tx) - t| < \varepsilon.$$

Now,

$$\begin{aligned}\max_{\omega\in\mathbb{T}} \|G + \omega T\| &\ge \|G + e^{-i\theta}T\| \ge |x^*((G + e^{-i\theta}T)(x))|\\ &\ge \operatorname{Re} x^*((G + e^{-i\theta}T)(x)) = \operatorname{Re} x^*(G(x)) + \operatorname{Re} e^{-i\theta}x^*(T(x)),\end{aligned}$$

which together with (3.1) means that

$$\max_{\omega\in\mathbb{T}} \|G + \omega T\| \ge \operatorname{Re} x^*(G(x)) + r - \varepsilon \ge 1 - \varepsilon + r - \varepsilon = 1 + \|T\| - 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\max_{\omega\in\mathbb{T}} \|G + \omega T\| \ge 1 + \|T\|$. By the triangle inequality,

$$\max_{\omega\in\mathbb{T}} \|G + \omega T\| \le 1 + \|T\|,$$

so $\max_{\omega\in\mathbb{T}} \|G + \omega T\| = 1 + \|T\|$. ∎

The next lemma is the converse of the previous one.

Lemma 3.2. *If* $\max_{\omega\in\mathbb{T}} \|G + \omega T\| = 1 + \|T\|$ *then* $\nu_G(T) = \|T\|$.

Proof. Since $\max_{\omega\in\mathbb{T}} \|G + \omega T\| = 1 + \|T\|$, there is an $\omega \in \mathbb{T}$ such that $\|G + \omega T\| = 1 + \|T\|$. For this ω we have

$$1 + \|T\| = \|G + \omega T\| = \sup\{\|(G + \omega T)x\| : x \in S_X\},$$

so there is a sequence $\{x_n\} \subseteq S_X$ such that

$$\|(G+\omega T)x_n\| > 1 + \|T\| - \frac{1}{n}, \quad n = 1, 2, \ldots .$$

Let us select supporting functionals x_n^* at points $(G+\omega T)x_n$, i.e. $x_n^* \in S_{Y^*}$ such that $x_n^*((G+\omega T)x_n) = \operatorname{Re} x_n^*((G+\omega T)x_n) = \|(G+\omega T)x_n\|$. Then

$$1 + \|T\| - \frac{1}{n} < \operatorname{Re} x_n^*(G(x_n)) + \operatorname{Re} \omega x_n^*(T(x_n)) \le 1 + \|T\|$$

Thus, since evidently

$$\operatorname{Re} x_n^*(G(x_n)) \le 1 \quad \text{and} \quad \operatorname{Re} \omega x_n^*(T(x_n)) \le \|T\|,$$

we have

$$1 - \frac{1}{n} < \operatorname{Re} x_n^*(G(x_n)) \le 1$$

and

$$\|T\| - \frac{1}{n} < \operatorname{Re} \omega x_n^*(T(x_n)) \le \|T\|.$$

Consider $D := \{z \in \mathbb{C} : \|T\| - 1/n < \operatorname{Re} z \text{ and } |z| \le \|T\|\}$. Obviously, D is the intersection of the disc of radius $\|T\|$ centered at 0 with the half-plane $\operatorname{Re} z > \|T\| - 1/n$. The maximal distance to the point $(\|T\|, 0)$ from the points of D is $((1/n)^2 + (2\|T\|/n - 1/n^2))^{1/2}$, which tends to zero as $n \to \infty$. Clearly, $\omega x_n^*(T(x_n)) \in D$. Thus the above argument implies that $\omega x_n^*(T(x_n)) \to \|T\|$. Therefore,

$$\operatorname{Re} x_n^*(G(x_n)) \to 1, \qquad x_n^*(T(x_n)) \to \omega^{-1}\|T\|.$$

So, we have demonstrated that for all $\varepsilon, \delta > 0$, there exists $(x, x^*) \in S_X \times S_{Y^*}$ such that $\operatorname{Re} x^*(G(x)) > 1 - \varepsilon$ and $|x^*(Tx) - \omega^{-1}\|T\|\,| < \delta$. This means that for all $\varepsilon > 0$,

$$\omega^{-1}\|T\| \in \overline{\{x^*(Tx) : x \in S_X,\ x^* \in S_{Y^*},\ \operatorname{Re} x^*(Gx) > 1 - \varepsilon\}}.$$

According to the definition, this means that $\omega^{-1}\|T\| \in V_G(T)$ and consequently $\|T\| = \nu_G(T)$. ∎

These two lemmas evidently imply the following theorem:

Theorem 3.3. *Let $G \in L(X,Y)$ and $\|G\| = 1$. Then $n_G(X,Y) = 1$ if and only if for every $T \in L(X,Y)$,*

$$\max_{\omega \in \mathbb{T}} \|G + \omega T\| = 1 + \|T\|.$$

4. 'Spears" and examples of operators G with $n_G(X,Y) = 1$.

According to (2.1), $n_{\mathrm{Id}}(X) = n(X)$, so evident examples of operators G with $n_G(X,Y) = 1$ are provided by $G = \mathrm{Id} : X \to X$, where X is a space with numerical index one. In this section we present other examples, very different from the identity. In particular $n_G(X,Y)$ can be equal to 1 for some operators that are neither surjective nor injective.

Definition 4.1. An element $z \in X$ is called a *spear* if for every $x \in X$ there is a modulus one scalar t for which $\|z+tx\| = 1+\|x\|$. In such a case X is said to be *spear-containing*.

Note that substituting $x = 0$ into the above definition we find that for every spear $z \in X$ necessarily $\|z\| = 1$.

According to Theorem 3.3, $G \in S_{L(X,Y)}$ is a spear if and only if $n_G(X,Y) = 1$, so in this case the space $L(X,Y)$ is spear-containing. Let us list some other, simpler examples of spears.

- ℓ_1 is spear-containing. The only examples of spears in ℓ_1 are vectors of the form θe_n, where $|\theta| = 1$ and e_n is the nth member of the canonical basis, i.e. all coordinates of e_n are zeros except the nth that equals 1.
- $L_1(\Omega, \Sigma, \mu)$ is spear-containing if and only if the measure μ has atoms. In that case all spears in $L_1(\Omega, \Sigma, \mu)$ are of the form θf, where $|\theta| = 1$ and f is the characteristic function of some atom.
- $C(K)$ is spear-containing. The spears are functions f such that $|f(t)| = 1$ at all points $t \in K$. For the same reason $L_\infty(\Omega, \Sigma, \mu)$ is spear-containing.

Theorem 4.2. *Let $G \in L(\ell_1, Y)$, $\|G\| = 1$ and let $(e_n) \subset \ell_1$ be the canonical basis. Then $n_G(\ell_1, Y) = 1$ if and only if $G(e_n)$ is a spear in Y for all $n \in \mathbb{N}$.*

Proof. Recall that for any bounded operator $U : \ell_1 \to Y$ we have $\|U\| = \sup_{n\in\mathbb{N}} \|Ue_n\|$ (see, for example, [6, §6.4.3, Exercise 4]). Consequently, for every $T \in L(\ell_1, Y)$,

$$\max_{\omega\in\mathbb{T}} \|G + \omega T\| = \max_{\omega\in\mathbb{T}} \sup_{n\in\mathbb{N}} \|Ge_n + \omega Te_n\| = \sup_{n\in\mathbb{N}} \max_{\omega\in\mathbb{T}} \|Ge_n + \omega Te_n\|. \tag{4.1}$$

If all the $G(e_n)$ are spears in Y, this leads to the equality

$$\max_{\omega\in\mathbb{T}} \|G + \omega T\| = \sup_{n\in\mathbb{N}} \max_{\omega\in\mathbb{T}} (1 + \|Te_n\|) = 1 + \|T\|,$$

which, thanks to Theorem 3.3, means that $n_G(\ell_1, Y) = 1$.

Now assume that $G(e_m)$ is not a spear for some $m \in \mathbb{N}$. Then there is a $y \in Y$ such that $\max_{\omega\in\mathbb{T}} \|Ge_m + \omega y\| < 1 + \|y\|$. Consider the operator $T \in L(\ell_1, Y)$ defined by

$$T(x_1, x_2, \ldots) = x_m y.$$

Then $\|T\| = \|y\|$ and according to (4.1),

$$\max_{\omega\in\mathbb{T}} \|G + \omega T\| = \sup_{n\in\mathbb{N}} \max_{\omega\in\mathbb{T}} \|Ge_n + \omega Te_n\| < 1 + \|y\| = 1 + \|T\|,$$

so G does not satisfy the condition of Theorem 3.3. ∎

Finally we remark that the last theorem enables us to construct a lot of operators with $n_G(\ell_1, Y) = 1$ that have properties very different from the identity operator. For example, the operator $G \in L(\ell_1, \ell_1)$ defined by

$$G(x_1, x_2, \ldots) = \Big(\sum_{n\in\mathbb{N}} x_n\Big) e_1$$

has $n_G(\ell_1, \ell_1) = 1$ but is neither surjective nor injective.

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