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# Riesz sequences and arithmetic progressions

by

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**Abstract.** Given a set $\mathcal{S}$ of positive measure on the circle and a set $\Lambda$ of integers, one can ask whether $E(\Lambda) := \{e^{i\lambda t}\}_{\lambda\in\Lambda}$ is a Riesz sequence in $L^2(\mathcal{S})$.

We consider this question in connection with some arithmetic properties of the set $\Lambda$. Improving a result of Bownik and Speegle (2006), we construct a set $\mathcal{S}$ such that $E(\Lambda)$ is never a Riesz sequence if $\Lambda$ contains an arithmetic progression of length $N$ and step $\ell = O(N^{1-\varepsilon})$ with $N$ arbitrarily large. On the other hand, we prove that every set $\mathcal{S}$ admits a Riesz sequence $E(\Lambda)$ such that $\Lambda$ does contain arithmetic progressions of length $N$ and step $\ell = O(N)$ with $N$ arbitrarily large.

**1. Introduction.** We use the following notation:

- $\Lambda$ — a set of integers.
- $\mathcal{S}$ — a set of positive measure on the circle $\mathbb{T}$.
- $|\mathcal{S}|$ — the Lebesgue measure of $\mathcal{S}$.

For $A, B \subset \mathbb{R}$ and $x \in \mathbb{R}$ we let

$$A + B := \{\alpha + \beta \mid \alpha \in A,\ \beta \in B\}, \qquad x \cdot A := \{x \cdot \alpha \mid \alpha \in A\}.$$

A sequence $\{\varphi_i\}_{i\in I}$ of elements in a Hilbert space $\mathcal{H}$ is called a *Riesz sequence* (RS) if there are positive constants $c$, $C$ such that

$$c\sum_{i\in I}|a_i|^2 \le \Big\|\sum_{i\in I} a_i\varphi_i\Big\|^2 \le C\sum_{i\in I}|a_i|^2$$

for every finite sequence $\{a_i\}_{i\in I}$ of scalars.

Given $\Lambda \subset \mathbb{Z}$ (referred to as a *set of frequencies*) we denote

$$E(\Lambda) := \{e^{i\lambda t}\}_{\lambda\in\Lambda}.$$

The following result is classical (see [9, p. 203, Lemma 6.5]):

- *If $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{Z}$ is lacunary in the sense of Hadamard, i.e.*

$$\frac{\lambda_{n+1}}{\lambda_n} \ge q > 1, \qquad n \in \mathbb{N},$$

*then $E(\Lambda)$ forms a RS in $L^2(\mathcal{S})$ for every $\mathcal{S} \subset \mathbb{T}$ with $|\mathcal{S}| > 0$.*

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The following generalization is due to I. M. Mikheev [7, Thm. 7]:

- *If $E(\Lambda)$ is an $S_p$-system for some $p > 2$, i.e.*

$$\Big\| \sum_{\lambda\in\Lambda} a_\lambda e^{i\lambda t} \Big\|_{L^p(\mathbb{T})} \le C \Big\| \sum_{\lambda\in\Lambda} a_\lambda e^{i\lambda t} \Big\|_{L^2(\mathbb{T})}$$

  *with some $C > 0$ for every finite sequence $\{a_\lambda\}_{\lambda\in\Lambda}$ of scalars, then it forms a RS in $L^2(\mathcal{S})$ for every $\mathcal{S} \subset \mathbb{T}$ with $|\mathcal{S}| > 0$.*

J. Bourgain and L. Tzafriri proved the following result as a consequence of their "restricted invertibility theorem" [2, Thm. 2.2]:

- *Given $\mathcal{S} \subset \mathbb{T}$, there is a set $\Lambda$ of integers with positive asymptotic density*

$$\operatorname{dens} \Lambda := \lim_{N\to\infty} \frac{\#\{\Lambda \cap [-N, N]\}}{2N} > C|\mathcal{S}|$$

  *such that $E(\Lambda)$ is a RS in $L^2(\mathcal{S})$.*

(Here and below, $C$ denotes positive absolute constants, which might be different from one another.)

W. Lawton [5, Cor. 2.1], *assuming the Feichtinger conjecture for exponentials*, proved the following proposition:

(L) *For every $\mathcal{S}$ there is a set of frequencies $\Lambda \subset \mathbb{Z}$ which is syndetic, that is, $\Lambda + \{0, \dots, n-1\} = \mathbb{Z}$ for some $n \in \mathbb{N}$, and such that $E(\Lambda)$ is a RS in $L^2(\mathcal{S})$.*

Recall that the Feichtinger conjecture says that every bounded frame in a Hilbert space can be decomposed into a finite family of RSs. This claim turned out to be equivalent to the Kadison–Singer conjecture (see [4]). The latter conjecture has recently been proved by A. Marcus, D. Spielman and N. Srivastava [6], so proposition (L) holds unconditionally.

Notice that in some results above, the system $E(\Lambda)$ serves as a RS for all sets $\mathcal{S}$; however, the set of frequencies $\Lambda$ is then quite sparse. In others, $\Lambda$ is rather dense but it works for an $\mathcal{S}$ given in advance.

One could wonder whether one can somehow combine the density and "universality" properties. It turns out this is indeed possible. In [8], a sequence $\Lambda \subset \mathbb{R}$ has been constructed such that $E(\Lambda)$ forms a RS in $L^2(\mathcal{S})$ for any open set $\mathcal{S}$ of a given measure, and the set of frequencies has optimal density, proportional to $|\mathcal{S}|$. This is not true for nowhere dense sets $\mathcal{S}$.

**2. Results.** In this paper we consider sets of frequencies $\Lambda$ which contain arbitrarily long arithmetic progressions. Below we denote by $N$ the length of a progression, and by $\ell$ its step. Given $\Lambda$ which contains arbitrarily long arithmetic progressions there exists a set $\mathcal{S} \subset \mathbb{T}$ of positive measure such that $E(\Lambda)$ is not a RS in $L^2(\mathcal{S})$ (see [7]).

In the case where $\ell$ grows slowly with respect to $N$, one can define $\mathcal{S}$ independent of $\Lambda$.

A quantitative version of such a result was proved in [3]:

- *There exists a set $\mathcal{S}$ such that $E(\Lambda)$ is not a RS in $L^2(\mathcal{S})$ whenewer $\Lambda$ contains arithmetic progressions of length $N_j$ and step*

$$\ell_j = o(N_j^{1/2} \log^{-3} N_j) \quad (N_1 < N_2 < \cdots).$$

The proof is based on some estimates of the discrepancy of sequences of the form $\{\alpha k\}_{k\in\mathbb{N}}$ on the circle.

Using a different approach we prove a stronger result:

THEOREM 1. *There exists a set $\mathcal{S} \subset \mathbb{T}$ such that if a set $\Lambda \subset \mathbb{Z}$ contains arithmetic progressions of length $N$ $(= N_1 < N_2 < \cdots)$ and step $\ell = O(N^\alpha)$, $\alpha < 1$, then $E(\Lambda)$ is not a RS in $L^2(\mathcal{S})$.*

Here one can construct $\mathcal{S}$ not depending on $\alpha$ and with arbitrarily small measure of the complement.

The next theorem shows that the result is sharp.

THEOREM 2. *Given a set $\mathcal{S} \subset \mathbb{T}$ of positive measure, there is a set $\Lambda \subset \mathbb{Z}$ such that:*

(i) *For infinitely many $N$'s $\Lambda$ contains an arithmetic progression of length $N$ and step $\ell = O(N)$.*
(ii) *$E(\Lambda)$ forms a RS in $L^2(\mathcal{S})$.*

Slightly increasing the bound for $\ell$, one can get a version of Theorem 2 which admits a progression of any length:

THEOREM 3. *Given $\mathcal{S}$ one can find $\Lambda$ with property* (ii) *above and such that*

(i′) *For every $\alpha > 1$ and for every $N \in \mathbb{N}$ the set $\Lambda$ contains an arithmetic progression of length $N$ and step $\ell < C(\alpha)N^\alpha$.*

## 3. Proof of Theorem 1

*Proof.* Fix $\varepsilon > 0$. Take a decreasing sequence $\{\delta(\ell)\}_{\ell\in\mathbb{N}}$ of positive numbers such that

(a) $\sum_{\ell\in\mathbb{N}} \delta(\ell) < \varepsilon/2$,
(b) $\delta(\ell) \cdot \ell^{1/\alpha} \to \infty$ as $\ell \to \infty$ for all $\alpha \in (0,1)$,

For every $\ell \in \mathbb{N}$ set $I_\ell = (-\delta(\ell), \delta(\ell))$ and let $\tilde{I}_\ell$ be the $2\pi$-periodic extension of $I_\ell$, i.e.

$$\tilde{I}_\ell = \bigcup_{k\in\mathbb{Z}} (I_\ell + 2\pi k).$$

We define

$$(1)\qquad I_{[\ell]} = \left(\frac{1}{\ell}\cdot \tilde{I}_\ell\right) \cap [-\pi,\pi] \quad\text{and}\quad \mathcal{S} = \mathbb{T} \setminus \bigcup_{\ell\in\mathbb{N}} I_{[\ell]} = \Big(\bigcup_{\ell\in\mathbb{N}} I_{[\ell]}\Big)^c,$$

whence

$$|\mathcal{S}| \ge 1 - \sum_{\ell=1}^{\infty} |I_{[\ell]}| = 1 - \sum_{\ell=1}^{\infty} 2\delta(\ell) > 1-\varepsilon.$$

Fix $\alpha < 1$ and let $\Lambda \subset \mathbb{Z}$ be such that one can find arbitrarily large $N \in \mathbb{N}$ for which

$$\{M+\ell,\dots,M+N\cdot\ell\} \subset \Lambda,$$

with some $M = M(N) \in \mathbb{Z}$, $\ell = \ell(N) \in \mathbb{N}$ and

$$(2)\qquad \ell < C(\alpha)N^\alpha.$$

Recall that by (1) we have $t \in I_{[\ell]}$ if and only if $t\ell \in \tilde{I}_\ell \cap [-\pi\ell,\pi\ell]$. Since $\mathcal{S}$ lies inside the complement of $I_{[\ell]}$, we get

$$\int_{\mathcal{S}} \Big|\sum_{k=1}^{N} c(k)e^{i(M+k\ell)t}\Big|^2 \frac{dt}{2\pi} \le \int_{I_{[\ell]}^c} \Big|\sum_{k=1}^{N} c(k)e^{i(M+k\ell)t}\Big|^2 \frac{dt}{2\pi}$$
$$= \int_{[-\pi\ell,\pi\ell]\setminus \tilde{I}_\ell} \Big|\sum_{k=1}^{N} c(k)e^{ik\tau}\Big|^2 \frac{d\tau}{2\pi\ell} = \int_{I_\ell^c} \Big|\sum_{k=1}^{N} c(k)e^{ik\tau}\Big|^2 \frac{d\tau}{2\pi}.$$

To complete the proof, it is enough to show that $\|\sum_{k=1}^N c(k)e^{ik\tau}\|_{L^2(I_\ell^c)}$ can be made arbitrarily small while keeping $\sum_{k=1}^N |c(k)|^2$ bounded away from zero. This observation allows us to reformulate the problem as a norm concentration problem for trigonometric polynomials of degree $N$ on the interval $I_\ell$.

Let

$$P_N(t) = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{ikt},$$

so $\|P_N\|_{L^2(\mathbb{T})} = 1$. Moreover, for every $t \in \mathbb{T}$ we have $|P_N(t)| \le \frac{1}{\sqrt{N}\sin\frac{t}{2}}$, hence

$$\int_{I_\ell^c} |P_N(t)|^2 \frac{dt}{2\pi} \le \frac{1}{N}\int_{\delta(\ell)}^{\pi} \frac{dt}{\sin^2\frac{t}{2}} < \frac{C}{N}\int_{\delta(\ell)}^{\pi} \frac{dt}{t^2} < \frac{C}{\delta(\ell)N} < \frac{C(\alpha)}{\delta(\ell)\ell^{1/\alpha}},$$

where the last inequality holds for every $N$ for which (2) holds. Using condition (b) we see that indeed the last term can be made arbitrarily small, and so $E(\Lambda)$ is not a RS in $L^2(\mathcal{S})$. ∎

**4. Proof of Theorem 2.** For $n \in \mathbb{N}$ we define

$$B_n := \{n, 2n, \dots, n^2\}.$$

LEMMA 4. *Let $\mathcal{P}$ be the set of all prime numbers. Then the blocks $\{B_p\}_{p\in\mathcal{P}}$ are pairwise disjoint.*

*Proof.* Let $p < q$ be prime numbers. Notice that $a \in B_p \cap B_q$ if and only if there exist $1 \le m \le p$ and $1 \le k \le q$ such that

$$a = mp = kq,$$

which is possible only if $q$ divides $m$. But since $m < q$ this cannot happen and so such an $a$ does not exist. ∎

LEMMA 5. *Let $\{a(n)\}_{n\in\mathbb{N}}$ be a sequence of non-negative numbers such that $\sum_{n=1}^{\infty} a(n) \le 1$. Then for every $\varepsilon > 0$ there exist infinitely many $n \in \mathbb{N}$ such that*

$$\sum_{\ell=1}^{n} a(\ell n) < \frac{\varepsilon}{n}.$$

*Proof.* By Lemma 4 we may write

$$\sum_{n=1}^{\infty} a(n) \ge \sum_{p\in\mathcal{P}} \sum_{\ell=1}^{p} a(\ell p).$$

Assuming the contrary for some $\varepsilon$, i.e. for all but finitely many $p \in \mathcal{P}$ we have $\sum_{\ell=1}^{p} a(\ell p) \ge \varepsilon/p$, we get a contradiction to the well-known fact that $\sum_{p\in\mathcal{P}} 1/p = \infty$. ∎

COROLLARY 6. *Let $\{a(n)\}_{n\in\mathbb{N}}$ be as in Lemma 5. Then for every $\varepsilon > 0$ there exist infinitely many $n \in \mathbb{N}$ such that*

$$\sum_{\substack{\lambda,\mu\in B_n\\ \mu<\lambda}} a(\lambda - \mu) < \varepsilon. \tag{3}$$

*Proof.* Every $\mu < \lambda$ from $B_n$ must take the form

$$\lambda = kn, \quad \mu = k'n, \quad 1 \le k' < k \le n,$$

hence $\lambda - \mu = \ell n$ for some $\ell \in \{1, \dots, n-1\}$. From Lemma 5 we get, for infinitely many $n \in \mathbb{N}$,

$$\sum_{\substack{\lambda,\mu\in B_n\\ \mu<\lambda}} a(\lambda - \mu) = \sum_{\ell=1}^{n} (n-\ell) a(\ell n) \le n \sum_{\ell=1}^{n} a(\ell n) < \varepsilon. \ \blacksquare$$

Given a sequence $B \subset \mathbb{R}$, we say that a positive number $\gamma$ is a *lower Riesz bound* (in $L^2(\mathcal{S})$) for the sequence $E(B)$ if

$$\Big\| \sum_{\lambda\in B} c(\lambda) e^{i\lambda t} \Big\|^2_{L^2(\mathcal{S})} \ge \gamma \sum_{\lambda\in B} |c(\lambda)|^2$$

for every finite sequence $\{c(\lambda)\}_{\lambda\in B}$ of scalars.

LEMMA 7. *Given $\mathcal{S} \subset \mathbb{T}$ of positive measure, there exists a constant $\gamma = \gamma(\mathcal{S}) > 0$ which is a lower Riesz bound (in $L^2(\mathcal{S})$) for $E(B_n)$ for infinitely many $n \in \mathbb{N}$.*

*Proof.* Let $\mathcal{S} \subset \mathbb{T}$ with $|\mathcal{S}| > 0$. Applying Corollary 6 to the sequence $\{a(n)\}_{n\in\mathbb{N}} := \{|\widehat{\mathbb{1}_\mathcal{S}}(n)|^2\}_{n\in\mathbb{N}}$ (where $\mathbb{1}_\mathcal{S}$ is the indicator function of $\mathcal{S}$), we get for every $\varepsilon > 0$ infinitely many $n \in \mathbb{N}$ for which (3) holds. We write

$$\int_\mathcal{S} \Big| \sum_{\lambda\in B_n} c(\lambda)e^{i\lambda t}\Big|^2 \frac{dt}{2\pi} = \int_\mathcal{S} \Big( \sum_{\lambda\in B_n} |c(\lambda)|^2 + \sum_{\substack{\lambda,\mu\in B_n\\ \lambda\neq\mu}} c(\lambda)\,\overline{c(\mu)}\, e^{i(\lambda-\mu)t}\Big) \frac{dt}{2\pi}$$
$$= |\mathcal{S}| \sum_{\lambda\in B_n} |c(\lambda)|^2 + \sum_{\substack{\lambda,\mu\in B_n\\ \lambda\neq\mu}} c(\lambda)\,\overline{c(\mu)}\,\widehat{\mathbb{1}_\mathcal{S}}(\mu-\lambda).$$

By the Cauchy–Schwarz inequality,

$$\Big| \sum_{\substack{\lambda,\mu\in B_n\\ \lambda\neq\mu}} c(\lambda)\,\overline{c(\mu)}\,\widehat{\mathbb{1}_\mathcal{S}}(\mu-\lambda)\Big|$$
$$\le \Big( \sum_{\lambda,\mu\in B_n} |c(\lambda)\,\overline{c(\mu)}|^2\Big)^{1/2} \Big( \sum_{\substack{\lambda,\mu\in B_n\\ \lambda\neq\mu}} |\widehat{\mathbb{1}_\mathcal{S}}(\mu-\lambda)|^2\Big)^{1/2}$$
$$= \sum_{\lambda\in B_n} |c(\lambda)|^2 \Big( \sum_{\substack{\lambda,\mu\in B_n\\ \lambda\neq\mu}} |\widehat{\mathbb{1}_\mathcal{S}}(\mu-\lambda)|^2\Big)^{1/2}.$$

By (3) we get

$$\sum_{\substack{\lambda,\mu\in B_n\\ \lambda\neq\mu}} |\widehat{\mathbb{1}_\mathcal{S}}(\mu-\lambda)|^2 = 2 \sum_{\substack{\lambda,\mu\in B_n\\ \mu<\lambda}} |\widehat{\mathbb{1}_\mathcal{S}}(\mu-\lambda)|^2 < 2\varepsilon,$$

hence

$$\int_\mathcal{S} \Big| \sum_{\lambda\in B_n} c(\lambda)e^{i\lambda t}\Big|^2 \frac{dt}{2\pi} \ge (|\mathcal{S}| - (2\varepsilon)^{1/2}) \sum_{\lambda\in B_n} |c(\lambda)|^2 \ge \frac{|\mathcal{S}|}{2} \sum_{\lambda\in B_n} |c(\lambda)|^2.$$

Fixing some $\varepsilon < |\mathcal{S}|^2/8$, we see that the last inequality holds for infinitely many $n \in \mathbb{N}$. ■

The next lemma shows how to combine blocks which correspond to different progressions.

LEMMA 8. *Let $\gamma > 0$, $\mathcal{S} \subset \mathbb{T}$ with $|\mathcal{S}| > 0$, and $A_1, A_2 \subset \mathbb{N}$ finite subsets such that $\gamma$ is a lower Riesz bound (in $L^2(\mathcal{S})$) for $E(A_j)$, $j = 1, 2$. Then for any $0 < \gamma' < \gamma$ there exists $M \in \mathbb{Z}$ such that the system $E(A_1 \cup (M + A_2))$ has $\gamma'$ as a lower Riesz bound.*

*Proof.* Denote $P_j(t) = \sum_{\lambda\in A_j} c_j(\lambda)e^{i\lambda t}$, $j = 1, 2$. Notice that for sufficiently large $M = M(\mathcal{S})$, the polynomials $P_1$ and $e^{iMt}P_2$ are "almost orthogonal" on $\mathcal{S}$, meaning

$$\int_{\mathcal{S}} |P_1(t) + e^{iMt} \cdot P_2(t)|^2 \frac{dt}{2\pi} = \|P_1\|^2_{L^2(\mathcal{S})} + \|P_2\|^2_{L^2(\mathcal{S})} + o(1),$$

where the last term is uniform with respect to all polynomials having $\|P\|_{L^2(\mathbb{T})} = 1$. ∎

Now we are ready to finish the proof of Theorem 2. Given $\mathcal{S}$ take $\gamma$ from Lemma 7 and denote by $\mathcal{N}$ the set of all natural numbers $n$ for which $\gamma$ is a lower Riesz bound (in $L^2(\mathcal{S})$) for $E(B_n)$. Define

$$\Lambda = \bigcup_{n\in\mathcal{N}} (M_n + B_n).$$

By Lemma 8 we can define subsequently, for every $n \in \mathcal{N}$, an integer $M_n$ such that for any partial union

$$\Lambda(N) = \bigcup_{\substack{n\in\mathcal{N}\\ n<N}} (M_n + B_n), \qquad N \in \mathcal{N},$$

the corresponding exponential system $E(\Lambda(N))$ has lower Riesz bound $\frac{\gamma}{2} \cdot \left(1 + \frac{1}{N}\right)$, so we conclude that $E(\Lambda)$ is a RS in $L^2(\mathcal{S})$.

**5. Proof of Theorem 3.** In order to obtain $\Lambda$ which satisfies property (i′) we will need the following result.

THEOREM A ([1, Thm. 13.12]). *Let $d(n)$ denote the number of divisors of an integer $n$. Then $d(n) = o(n^\varepsilon)$ for every $\varepsilon > 0$.*

The next lemma will be used to control the contribution of blocks when they are not disjoint.

LEMMA 9. *Let $\{a(n)\}_{n\in\mathbb{N}}$ be a sequence of non-negative numbers such that $\sum_{n=1}^{\infty} a(n) \le 1$. Then for every $\alpha > 1$ there exist $\varepsilon(\alpha) > 0$ and $\nu(\alpha) \in \mathbb{N}$ such that for every $N \ge \nu(\alpha)$ one can find an integer $\ell_{\alpha,N} < N^\alpha$ satisfying*

$$\sum_{n=1}^{N} a(n\ell_{\alpha,N}) < \frac{1}{N^{1+\varepsilon(\alpha)}}. \tag{4}$$

*Proof.* Fix $\alpha > 1$ and apply Theorem A with $\varepsilon$ small enough, depending on $\alpha$, to be chosen later. We get the inequality $d(k) < k^\varepsilon$ for every $k \ge \nu(\alpha)$. Fix $N \ge \nu(\alpha)$, and notice that for every $L \in \mathbb{N}$,

$$\sum_{\ell=1}^{L}\sum_{n=1}^{N} a(n\ell) \le \sum_{k=1}^{LN} d(k)a(k) < (LN)^\varepsilon.$$

It follows that there exists an integer $0 < \ell < L$ such that

$$\sum_{n=1}^{N} a(n\ell) < \frac{(LN)^{\varepsilon}}{L} = \frac{N^{\varepsilon}}{L^{1-\varepsilon}}.$$

In order to get (4) we require

$$\frac{N^{\varepsilon}}{L^{1-\varepsilon}} < \frac{1}{N^{1+\varepsilon}},$$

which yields

$$N^{\frac{1+2\varepsilon}{1-\varepsilon}} < L.$$

Therefore, choosing $\varepsilon = \varepsilon(\alpha)$ sufficiently small we see that $L$ may be chosen to be smaller than $N^{\alpha}$. ∎

Setting

$$B_{\alpha,N} := \{\ell_{\alpha,N},\, 2\ell_{\alpha,N}, \ldots, N\ell_{\alpha,N}\},$$

we get

Corollary 10. *Let $\{a(n)\}_{n\in\mathbb{N}}$ be as in Lemma 9. For every $\alpha > 1$ and $N \geq \nu(\alpha)$,*

$$\sum_{\substack{\lambda,\mu\in B_{\alpha,N}\\ \mu<\lambda}} a(\lambda-\mu) < \frac{1}{N^{\varepsilon(\alpha)}}. \tag{5}$$

The proof is identical to that of Corollary 6.

We now combine our estimates.

Lemma 11. *Given $\mathcal{S} \subset \mathbb{T}$ of positive measure, there exists a constant $\gamma = \gamma(\mathcal{S}) > 0$ such that for every $\alpha > 1$ there exists $N(\alpha) \in \mathbb{N}$ for which the following holds: For every integer $N \geq N(\alpha)$ one can find $\ell_{\alpha,N} \in \mathbb{N}$ with $\ell_{\alpha,N} < N^{\alpha}$ such that $\gamma$ is a lower Riesz bound (in $L^2(\mathcal{S})$) for $E(B_{\alpha,N})$.*

*Proof.* Let $\mathcal{S} \subset \mathbb{T}$ with $|\mathcal{S}| > 0$. We fix $\alpha > 1$ and apply Corollary 10 to the sequence $\{a(n)\}_{n\in\mathbb{N}} := \{|\widehat{\mathbb{1}_{\mathcal{S}}}(n)|^2\}_{n\in\mathbb{N}}$; we get $\varepsilon(\alpha)$ and for every $N \geq \nu(\alpha)$ a positive integer $\ell_{\alpha,N} < N^{\alpha}$ satisfying (5). Proceeding as in the proof of Lemma 7, we get

$$\int_{\mathcal{S}} \Big|\sum_{\lambda\in B_{\alpha,N}} c(\lambda)e^{i\lambda t}\Big|^2\,dt \geq \Big(|\mathcal{S}| - \frac{C}{N^{\varepsilon(\alpha)/2}}\Big)\sum_{\lambda\in B_{\alpha,N}} |c(\lambda)|^2 \geq \frac{|\mathcal{S}|}{2}\sum_{\lambda\in B_{\alpha,N}} |c(\lambda)|^2,$$

where the last inequality holds for all $N \geq N(\alpha)$. ∎

For the last step of the proof we will use a diagonal process. Given $\mathcal{S}$, find $\gamma$ using Lemma 11. This provides, for every $\alpha > 1$ and every $N \geq N(\alpha)$, a block $B_{\alpha,N}$ such that $\gamma$ is a lower Riesz bound (in $L^2(\mathcal{S})$) for $E(B_{\alpha,N})$.

Let $\alpha_k \to 1$ be a decreasing sequence. Define

$$\Lambda = \bigcup_{k\in\mathbb{N}} \bigcup_{N=N(\alpha_k)}^{N(\alpha_{k+1})-1} (M_N + B_{\alpha_k,N}).$$

Again, by Lemma 8, we can make sure any partial union has lower Riesz bound not smaller than $\gamma/2$, and so $E(\Lambda)$ is a RS in $L^2(\mathcal{S})$.

It follows directly from the construction that for every $N \in \mathbb{N}$, $\Lambda$ contains an arithmetic progression of length $N$ and step $\ell < C(\alpha)N^\alpha$, for any $\alpha > 1$, as required.

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### References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, Berlin, 1976.

[2] J. Bourgain and L. Tzafriri, *Invertibility of "large" submatrices with applications to the geometry of Banach spaces and harmonic analysis*, Israel J. Math. 57 (1987), 137–224.

[3] M. Bownik and D. Speegle, *The Feichtinger conjecture for wavelet frames, Gabor frames and frames of translates*, Canad. J. Math. 58 (2006), 1121–1143.

[4] P. G. Casazza, M. Fickus, J. C. Tremain and E. Weber, *The Kadison–Singer problem in mathematics and engineering: a detailed account*, In: Operator Theory, Operator Algebras, and Applications, Contemp. Math. 414, Amer. Math. Soc., Providence, RI, 2006, 299–355.

[5] W. Lawton, *Minimal sequences and the Kadison–Singer problem*, Bull. Malaysian Math. Sci. Soc. 33 (2010), 169–176.

[6] A. Marcus, D. A. Spielman and N. Srivastava, *Interlacing families II: Mixed characteristic polynomials and the Kadison–Singer problem*, arXiv:1306.3969 (2013).

[7] I. M. Miheev, *On lacunary series*, Math. USSR-Sb. 27 (1975), no. 4, 481–502.

[8] A. Olevskiĭ and A. Ulanovskii, *Universal sampling and interpolation of band-limited signals*, Geom. Funct. Anal. 18 (2008), 1029–1052.

[9] A. Zygmund, *Trigonometric Series. Vol. I*, 2nd ed., Cambridge Univ. Press, New York, 1959.

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