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Perturbations of bi-continuous semigroups

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Abstract. The notion of bi-continuous semigroups has recently been introduced to handle semigroups on Banach spaces that are only strongly continuous for a topology coarser than the norm-topology. In this paper, as a continuation of the systematic treatment of such semigroups started in [20–22], we provide a bounded perturbation theorem, which turns out to be quite general in view of various examples.

1. Introduction. While the theory of C_0-semigroups is well understood and has found a wide range of applications, there are important examples of semigroups of bounded linear operators on Banach spaces that are not strongly continuous on $[0,\infty)$ with respect to the norm-topology (see, e.g., [5, 20–22, 24] and also [8, 12, 18, 19, 23]). To deal with such semigroups the notion of *bi-continuous semigroups* has been introduced recently by F. Kühnemund ([20–22]). Among the semigroups that fit into this setting are adjoint semigroups ([20, 24]), evolution semigroups on $C_{\mathrm{b}}(\mathbb{R})$, semigroups induced by flows ([13–15]), implemented semigroups ([2, 3]) and the Ornstein–Uhlenbeck semigroup on $C_{\mathrm{b}}(H)$ ([9, 10, 20, 25]). In [21, 22], F. Kühnemund obtained generation and approximation theorems for such semigroups (see also [1, 6]). Although perturbation results for semigroups which are not strongly continuous were investigated for example in [11, 17] and [24, Ch. 4] in a different setting, a general perturbation theory of generators of bi-continuous semigroups is still lacking. Our aim is to close this gap and to provide a bounded perturbation theorem for bi-continuous semigroups. It turns out that some additional assumptions on the perturbing operator are needed.

In Section 2 we give some examples showing that we cannot expect that a bounded perturbation theorem holds in general. In Section 3 we prove the bounded perturbation theorem for bi-continuous semigroups, and we

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also show the existence of a Dyson–Phillips expansion and a variation of parameters formula for the perturbed semigroup. In Section 4 we return to examples and collect some conditions on bounded perturbations of bi-continuous semigroups for various topologies.

Throughout this paper X denotes a Banach space which is also endowed with a locally convex, Hausdorff topology τ. On the space (X,τ) we always assume the following (see [20, Sec. 1.1]).

Assumption 1.1. (i) The norm topology is finer than τ.

(ii) The locally convex space (X,τ) is sequentially complete on τ-closed, norm-bounded sets.

(iii) The dual space $(X,\tau)'$ is norming for $(X,\|\cdot\|)$, i.e.,

$$\|x\| = \sup_{\substack{\varphi\in(X,\tau)'\\ \|\varphi\|\le 1}} |\varphi(x)|.$$

The locally convex topology τ is determined by a directed family $\mathcal{P}$ of seminorms, and for simplicity we assume that all seminorms $p\in\mathcal{P}$ satisfy $p(x)\le\|x\|$ for all $x\in X$.

We now recall the basic notions as introduced and studied in [20–22].

Definition 1.2. A set $\mathcal{B}\subseteq\mathcal{L}(X)$ of bounded linear operators is said to be *bi-equicontinuous* (for the topology τ), if for every norm-bounded τ-null sequence x_n

$$\tau\text{-}\lim_{n\to\infty} Bx_n = 0$$

uniformly for $B\in\mathcal{B}$. A family $\{T(s) : s\in\mathbb{R}_+\}$ of operators is *locally bi-equicontinuous*, if $\{T(s) : s\in[0,t]\}$ is bi-equicontinuous for all $t>0$.

Definition 1.3. A function $T:\mathbb{R}_+\to\mathcal{L}(X)$ is called a *bi-continuous semigroup*, if

(i) $T(0)=I$ and $T(s+t)=T(s)T(t)$ for all $t,s\ge 0$,

(ii) T is locally bounded, i.e., $T|_{[0,t]}$ is a norm-bounded function for some (and in this case for all) $t>0$,

(iii) $\{T(t) : t\in\mathbb{R}_+\}$ is locally bi-equicontinuous,

(iv) the maps $t\mapsto T(t)x$ are τ-continuous for all $x\in X$.

Since

$$\|x\| = \sup\{p(x) : p\in\mathcal{P}\} \quad \text{for all } x\in X$$

by Assumption 1.1, the function $\|\cdot\|$ is τ-lower semicontinuous. This has an important consequence which we state explicitly in the following proposition.

Proposition 1.4. *The unit ball in X is τ-closed.*

The following notion will become important in Section 3, when we extend a non-densely defined operator to the whole space X.

DEFINITION 1.5. Let $Y \subseteq X$ be an arbitrary subset. It is called *bi-dense*, if for all $x \in X$ there exists a norm-bounded sequence x_n which converges to x in τ. If moreover there exists a $\delta > 1$ such that for all $x \in X$ there is a sequence x_n τ-convergent to x with

$$\|x_n\| \le \delta \|x\| \quad \text{for all } n \in \mathbb{N}, \tag{1}$$

then we say that Y is *δ-bi-dense* in X for the topology τ.

The following two results show that requiring δ-bi-denseness instead of bi-denseness is not so restrictive.

REMARK 1.6. Suppose that the topology τ is metrisable. Then any bi-dense set $D \subseteq X$ which contains 0 is δ-bi-dense for arbitrary $\delta > 1$. This can be seen by means of a simple diagonal process. In fact, take an arbitrary $x \in X$ and suppose that

$$p_1 \le p_2 \le \dots \le p_n \le \dots$$

is cofinal in $\mathcal{P}$. Let $n \in \mathbb{N}$ and choose $x_n \in D$ such that $p_n(x_n - x) \le 1/n$. Then $x_n \xrightarrow{\tau} x$ and $\|x_n\| \le \delta \|x\|$ for $n \in \mathbb{N}$ sufficiently large.

PROPOSITION 1.7. *Let T be a bi-continuous semigroup, and denote its generator by $(A, D(A))$. Then $D(A)$ is δ-bi-dense in X for an appropriate $\delta > 1$.*

Proof. Take $x \in X$. Then it is shown in [20, Sec. 1.2] that

$$\tau\text{-}\lim_{n \to \infty} nR(n, A)x = x.$$

Let $M > 0$ and $\omega \in \mathbb{R}$ satisfy

$$\|T(t)\| \le Me^{\omega t} \qquad \text{for } t \ge 0. \tag{2}$$

Since $(A, D(A))$ is a Hille–Yosida operator, we have

$$\|nR(n, A)x\| \le \frac{Mn}{n - \omega} \|x\| \qquad \text{for all } t \ge 0,$$

hence taking $x_n = nR(n, A)x$ and any $\delta > M$ we see that (1) is satisfied for large $n \in \mathbb{N}$. ∎

2. Counterexamples. In this section we show that in general the perturbation of the generator of a bi-continuous semigroup by a norm-bounded operator does not generate a bi-continuous semigroup. In fact, it suffices to show that a norm-bounded operator is not necessarily the generator of a bi-continuous semigroup.

The first example deals with adjoint semigroups which are bi-continuous for the weak*-topology ([20, Sec. 3.5], [24]). Namely, we show that not all bounded operators on dual spaces are generators of weak*-bi-continuous semigroups.

EXAMPLE 2.1. Consider the Banach spaces $X := C([0,1])$ and $X' = \mathcal{M}([0,1])$, the space of all complex Borel measures on $[0,1]$. Define the following shift operator on X':

$$(B\mu)(S) = \mu((S-3/4)\cap[0,1]) \quad \text{for all } \mu\in\mathcal{M}([0,1]),\ S\subseteq[0,1] \text{ a Borel set.}$$

Then B is a norm-continuous operator on $\mathcal{M}([0,1])$ and clearly $B^2=0$. We show that B is not weak*-continuous. Indeed, consider the sequence of Dirac measures $\delta_{3/4-1/(n+1)}$. It obviously converges to $\delta_{3/4}$ in the weak*-topology $\sigma(X',X)$. Further, $B\delta_{3/4-1/(n+1)}=0$, whereas $B\delta_{3/4}=\delta_0$, showing that B is not continuous for the topology $\sigma(X',X)$. The C_0-semigroup T generated by B is given by

$$T(t) = I + tB, \qquad t\geq 0,$$

which shows that T is not bi-continuous for the weak*-topology $\sigma(X',X)$.

The following two examples are given on $C_{\mathrm{b}}(\mathbb{R})$. Both rely on the fact that there exist Baire measures on the Stone–Čech compactification $\beta\mathbb{R}$ of $\mathbb{R}$ with support disjoint from $\mathbb{R}$.

EXAMPLE 2.2. Let $X = C_{\mathrm{b}}(\mathbb{R})$ and take $x\in\beta\mathbb{R}\setminus\mathbb{R}$ and consider the linear operator $B = \mathbf{1}\otimes\delta_x$ on X. Then B is a contractive projection. Now, we show that B is not continuous with respect to the compact-open topology τ_{c}. Let f_n be a sequence of continuous functions satisfying

$$f_n|_{[-n,n]} = 1, \qquad \operatorname{supp} f_n \subseteq [-(n+1), n+1].$$

Then $f_n\to\mathbf{1}$ in the topology τ_{c}, but $Bf_n=0$ since the function f_n vanishes outside a compact set of $\mathbb{R}$, while $B\mathbf{1}=\mathbf{1}$. Hence, B is not continuous with respect to τ_{c}. As before, the C_0-semigroup T generated by B is given by the series

$$T(t) = \sum_{n=0}^{\infty}\frac{(Bt)^n}{n!} = I + \sum_{n=1}^{\infty}\frac{t^n}{n!}B = I - B + e^tB, \quad t\geq 0.$$

This implies that $T(t)$ is not sequentially τ_{c}-continuous on bounded sets unless $t=0$, therefore B is not the generator of a bi-continuous semigroup with respect to τ_{c}.

EXAMPLE 2.3. Again we work on $X = C_{\mathrm{b}}(\mathbb{R})$ and take a Banach limit, i.e., a linear functional φ with

$$\|\varphi\| = 1, \quad \varphi(\mathbf{1}) = 1, \quad \varphi(f(\cdot+r)) = \varphi(f) \quad \text{for all } f\in X \text{ and } r\in\mathbb{R}.$$

For such a φ, we define the norm-bounded linear operator

$$B = \mathbf{1}\otimes\varphi.$$

As in Example 2.2, one shows that B is not τ_{c}-continuous on norm-bounded sets, hence B is not the generator of a bi-continuous semigroup for the topology τ_{c}. Now, we show that there exists a non-trivial bi-continuous semigroup

S with generator $(A, D(A))$ for which $(A+B, D(A))$ is not the generator of a bi-continuous semigroup.

It is not hard to check that the translation group S on X is τ_c-bi-continuous (cf. [20, Sec. 3.2]). Denote its generator by $(A, D(A))$. We now show that a bounded perturbation of S need not be bi-continuous. We take the above B as the perturbing operator and make use of the translation invariance of φ, which implies that B commutes with the semigroup S. Consider the function

$$\mathbb{R} \ni t \mapsto T(t) = S(t)(I - B + e^t B) = (I - B + e^t B)S(t).$$

Then T obviously has the semigroup property and is τ_c-strongly continuous. A straightforward computation shows that for all $x \in D(A)$ the orbits $t \mapsto T(t)x$ are τ_c-differentiable with derivative $(A+B)T(t)x$. If $(A+B, D(A))$ is the generator of a bi-continuous semigroup, then this semigroup has to coincide with T. However, T is not locally bi-equicontinuous, which can be seen by taking the sequence of functions f_n as before. Therefore $(A+B, D(A))$ cannot be the generator of a bi-continuous semigroup.

3. Bounded perturbation of bi-continuous semigroups. We now turn to positive results concerning bounded perturbations of bi-continuous semigroups and prove that a bi-continuous semigroup can be perturbed by a bounded operator provided that the perturbing operator is also sequentially τ-continuous on norm-bounded sets. We make use of abstract Volterra operators as in [16, Sec. III.1] and put all the necessary properties of a bi-continuous semigroup into the Banach space on which this operator acts.

Definition 3.1. For $t_0 > 0$ consider

$$\mathcal{X}_{t_0} := \{T : [0, t_0] \to \mathcal{L}(X)\ \tau\text{-strongly continuous, norm-bounded, and } \{T(t) : t \in [0, t_0]\} \text{ bi-equicontinuous}\}.$$

It is clear that $\mathcal{X}_{t_0}$ is a linear space.

Lemma 3.2. *The space $\mathcal{X}_{t_0}$ is complete for the supremum norm*

$$\|T\|_\infty := \sup\{\|T(t)\| : t \in [0, t_0]\} \quad \textit{for } T \in \mathcal{X}_{t_0}.$$

Proof. We show that $\mathcal{X}_{t_0}$ is a closed subspace of the Banach space $B([0, t_0], \mathcal{L}(X))$ of all bounded functions from $[0, t_0]$ to the space of bounded linear operators $\mathcal{L}(X)$ endowed with the supremum norm. Let $T_n \in \mathcal{X}_{t_0}$ converge to $T \in B([0, t_0], \mathcal{L}(X))$ and take any $x \in X$. Then $T_n(\cdot)x$ converges to $T(\cdot)x$ for τ uniformly in $t \in [0, t_0]$. Indeed, for any $p \in \mathcal{P}$ we have

$$p(T_n(t)x - T(t)x) \leq \|T_n(t)x - T(t)x\| \leq \|T_n - T\|_\infty \to 0.$$

Hence, T is τ-strongly continuous. The norm-boundedness of T is trivial. In order to prove the bi-equicontinuity of the set $\{T(t) : t \in [0, t_0]\}$ we take a norm-bounded τ-null sequence x_n and $\varepsilon > 0$. Choose $m \in \mathbb{N}$ such that $\|T_m - T\|_\infty < \varepsilon/2$. Then we obtain

$$p(T(t)x_n) \le p((T(t) - T_m(t))x_n) + p(T_m(t)x_n) \le \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for n sufficiently large, by the bi-equicontinuity of $\{T_m(t) : t \in [0, t_0]\}$. ∎

LEMMA 3.3. *The Banach space $\mathcal{X}_{t_0}$ is a Banach algebra.*

Proof. Take $0 < t < t_0$ and $h_n \in \mathbb{R}$ converging to 0. For $S, T \in \mathcal{X}_{t_0}$ and $x \in X$ we write

$$\begin{aligned} &T(t + h_n)S(t + h_n)x - T(t)S(t)x \\ &\quad = T(t + h_n)S(t + h_n)x - T(t + h_n)S(t)x + T(t + h_n)S(t)x - T(t)S(t)x. \end{aligned}$$

Clearly we have

$$T(t + h_n)S(t)x - T(t)S(t)x \xrightarrow{\tau} 0.$$

From the bi-equicontinuity of $\{T(t) : t \in [0, t_0]\}$ it also follows that

$$T(t + h_n)S(t + h_n)x - T(t + h_n)S(t)x = T(t + h_n)(S(t + h_n) - S(t))x \xrightarrow{\tau} 0.$$

The left (respectively right) τ-strong continuity of $T \cdot S$ at the endpoints of $[0, t_0]$ can be proved analogously. The norm-boundedness of $t \mapsto T(t)S(t)$ is obvious. It remains to show the bi-equicontinuity of $\{T(t)S(t) : t \in [0, t_0]\}$. Suppose the contrary, i.e., there exists a norm-bounded τ-null sequence x_n, an $\varepsilon > 0$ and a seminorm $p \in \mathcal{P}$ such that for all $n \in \mathbb{N}$ there exists $t_n \in [0, t_0]$ convergent and satisfying

$$p(T(t_n)S(t_n)x_n) > \varepsilon.$$

This leads to a contradiction since $S(t_n)x_n$ is norm-bounded and τ-convergent to 0 (by the bi-equicontinuity of S), hence by the bi-equicontinuity of T we conclude that

$$p(T(t_n)S(t_n)x_n) \to 0. \ ∎$$

Take now a τ-bi-continuous semigroup T and suppose that $B \in \mathcal{L}(X)$ is also sequentially τ-continuous on norm-bounded sets. The previous two lemmas enable us to define an abstract Volterra operator V_{t_0} associated to the semigroup T on the Banach space $\mathcal{X}_{t_0}$ in the following way. For $t \in [0, t_0]$ and $x \in X$ set

$$[V_{t_0}S](t)x := \int_0^t T(t - s)BS(s)x\, ds, \quad x \in X, \tag{3}$$

for any $S \in \mathcal{X}_{t_0}$. The integral exists in the τ topology since the mapping $s \mapsto T(t - s)BS(s)$ is τ-continuous by assumption and Lemma 3.3. We now prove that $V_{t_0} \in \mathcal{L}(\mathcal{X}_{t_0})$ and compute its spectral radius $r(V_{t_0})$.

Notice that $[V_{t_0}S](t)$ (for $t \in [0, t_0]$) is independent of the particular choice of t_0.

LEMMA 3.4. *The linear operator defined in* (3) *maps* $\mathcal{X}_{t_0}$ *into itself*, *is bounded*, *and has spectral radius* $r(V_{t_0}) = 0$.

Proof. Take $S \in \mathcal{X}_{t_0}$, $t \in [0, t_0]$ and $x \in X$, and write

$$\|[V_{t_0}S](t)x\| = \sup_{\substack{\phi\in(X,\tau)'\\ \|\phi\|\le 1}} \Big|\Big\langle \int_0^t T(t-s)BS(s)x\,ds, \phi\Big\rangle\Big| \le \sup_{\substack{\phi\in(X,\tau)'\\ \|\phi\|\le 1}} \int_0^t |\langle T(t-s)BS(s)x, \phi\rangle|\,ds \le t_0\|T|_{[0,t_0]}\|_\infty \cdot \|B\| \cdot \|S\|_\infty \cdot \|x\|, \tag{4}$$

which proves that $[V_{t_0}S](t) \in \mathcal{L}(X)$.

Second, we show that $\{[V_{t_0}S](t) : t \in [0, t_0]\}$ is bi-equicontinuous. To do this, consider a norm-bounded τ-null sequence x_n and take $p \in \mathcal{P}$ and $\varepsilon > 0$. Then

$$p([V_{t_0}S](t)x_n) = p\Big(\int_0^t T(t-s)BS(s)x_n\,ds\Big) \le \int_0^t p(T(t-s)BS(s)x_n)\,ds \le t_0\varepsilon$$

for n sufficiently large since $\{T(t-s)BS(s) : s \in [0, t_0]\}$ is bi-equicontinuous by Lemma 3.3. Clearly $[V_{t_0}S](\cdot)$ is norm-bounded and τ-strongly continuous, hence $V_{t_0}S \in \mathcal{X}_{t_0}$.

The operator V_{t_0} is the restriction of the abstract Volterra operator defined on the space of all strongly continuous functions $C([0, t_0], \mathcal{L}_s(X))$, hence it is bounded and has spectral radius $r(V_{t_0}) = 0$ (see [16, III.1.5]). In fact,

$$\|V_{t_0}^n S\| \le \frac{(t_0 \cdot \|T|_{[0,t_0]}\|_\infty \cdot \|B\|)^n}{n!}\|S\|_\infty \quad \text{for } n \in \mathbb{N} \text{ and } S \in \mathcal{X}_{t_0}. \ \blacksquare \tag{5}$$

The previous lemma plays a key role in the following since it implies $1 \in \varrho(V_{t_0})$. Furthermore, the resolvent of V_{t_0} at 1 is given by

$$R(1, V_{t_0}) = \sum_{n=0}^{\infty} V_{t_0}^n,$$

the series converging in the operator norm of $\mathcal{L}(\mathcal{X}_{t_0})$. We are now ready to prove our main theorem.

THEOREM 3.5. *Let* T *be a bi-continuous semigroup with generator* $(A, D(A))$ *and suppose that* $B \in \mathcal{L}(X)$ *is sequentially* τ*-continuous on norm-bounded sets. Then* $(A + B, D(A))$ *is also the generator of a bi-continuous* *semigroup S. Moreover S is given by the Dyson–Phillips series*

$$S(t) = \sum_{n=0}^{\infty} T_n(t), \quad t \geq 0, \tag{6}$$

which is uniformly norm-convergent on compact intervals. Here,

$$T_0(t) := T(t), \qquad T_n(t) := \int_0^t T(t-s)BT_{n-1}(s)\,ds \quad \text{for } n > 0,$$

where the integral is understood in the τ-strong topology.

Proof. Let $t_0 > 0$ arbitrary, and define the abstract Volterra operator V_{t_0} on the space $\mathcal{X}_{t_0}$ as above. Further set

$$S_{t_0}(t) := \delta_t(R(1, V_{t_0})T|_{[0,t_0]}) \tag{7}$$

for each $t \in [0, t_0]$ (δ_t is the Dirac measure at t). In the following we will write T instead of the restriction $T|_{[0,t_0]}$. From the definition of V_{t_0}, it is immediate that

$$S_{t_0}(t)x = S_{t_0'}(t)x \quad \text{for all } t \leq t_0' \leq t_0 \text{ and } x \in X.$$

This enables us to define

$$S(t) := \begin{cases} I & \text{for } t = 0, \\ S_t(t) & \text{for } t > 0. \end{cases}$$

First, we show that S is a semigroup. As a first step, we prove that

$$S_{t_0}(t+s) = S_{t_0}(t)S_{t_0}(s)$$

whenever $0 \leq s, t \leq s + t \leq t_0$. Indeed, for such t and s, by definition we have

$$S_{t_0}(t)S_{t_0}(s) = \sum_{n=0}^{\infty} [V_{t_0}^n T](t) \cdot \sum_{n=0}^{\infty} [V_{t_0}^n T](s).$$

Since the series converges in the operator norm, the Cauchy product yields

$$S_{t_0}(t)S_{t_0}(s) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} [V_{t_0}^k T](t)[V_{t_0}^{n-k} T](s).$$

Therefore it remains to show that

$$\sum_{k=0}^{n} [V_{t_0}^k T](t)[V_{t_0}^{n-k} T](s) = [V_{t_0}^n T](t+s),$$

which obviously holds for $n = 0$ and then can be proved by induction.

Since $S|_{[0,t_0]} = S_{t_0} \in \mathcal{X}_{t_0}$, we see immediately that S is a bi-continuous semigroup and from the definition it is straightforward that $T_n(t) = (V_t^n T|_{[0,t]})(t)$ and hence (6) is satisfied. The uniform convergence on compact intervals follows from the continuity of δ_t.

We claim that the generator $(C, D(C))$ of S is $(A+B, D(A))$. To prove this, we first remark that $(A, D(A))$ is a Hille–Yosida operator (cf. [20, Sec. 1.5] and [21]) and therefore its bounded perturbation $(A+B, D(A))$ is also a Hille–Yosida operator (see, e.g., [4, Theorem 3.5.5]), in particular its resolvent set is not empty. Now, take $x \in X$ arbitrary. Then

$$\frac{[V_{t_0}T](h)x}{h} = \frac{1}{h}\int_0^h T(h-s)BT(s)x\,ds,$$

from which we deduce that

$$\frac{[V_{t_0}T](h)x}{h} \xrightarrow{\tau} Bx. \tag{8}$$

Indeed, by the τ-continuity of the orbits $s \mapsto S(s)Bx$ one can choose $0 < \delta < t_0$ for a given $\varepsilon > 0$ such that

$$p(S(h-s)Bx - Bx) \le \varepsilon/2$$

whenever $h \in [0, \delta]$ and $s \in [0, h]$. Further, by taking δ possibly smaller, we see by the bi-equicontinuity of $\{S(s) : s \in [0, t_0]\}$ and the τ-continuity of $s \mapsto BT(s)x$ that

$$\begin{aligned} p\biggl(\frac{1}{h}\int_0^h & S(h-s)BT(s)x\,ds - Bx\biggr) \\ &\le \frac{1}{h}\int_0^h p(S(h-s)Bx - Bx)\,ds + \frac{1}{h}\int_0^h p(S(h-s)(BT(s)x - Bx))\,ds \\ &\le \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

For $x \in D(A)$ we have

$$\frac{S(h)x - x}{h} = \frac{T(h)x - x}{h} + \frac{[V_{t_0}T](h)x}{h} + \sum_{n=2}^{\infty} \frac{[V_{t_0}^n T](h)x}{h}. \tag{9}$$

Using (5) we see that

$$\biggl\| \sum_{n=2}^{\infty} \frac{[V_{t_0}^n T](h)x}{h} \biggr\| \le h \sum_{n=2}^{\infty} \frac{h^{n-2}(\|T\|_\infty \cdot \|B\|)^n}{n!}\, \|T\|_\infty \cdot \|x\| \le hC\|x\| \tag{10}$$

for all $x \in X$ and for some constant $C > 0$, which shows that the third term in (9) converges to zero in the topology τ as $h \to 0$. Putting (8) and (9) together we obtain

$$\frac{S(h)x - x}{h} \xrightarrow{\tau} Ax + Bx,$$

proving that $A + B \subseteq C$. This together with the above remark on the non-empty resolvent set of $A + B$ implies that $A + B = C$. ∎

COROLLARY 3.6. *If T is a bi-continuous semigroup with generator $(A, D(A))$ and $B \in \mathcal{L}(X)$ is sequentially τ-continuous on norm-bounded sets, then the semigroup S generated by $(A+B, D(A))$ satisfies the integral equation*

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\,ds, \tag{IE}$$

$$T(t)x = S(t)x - \int_0^t S(s)BT(t-s)x\,ds \tag{IE$'$}$$

for all $x \in X$ and $t \geq 0$, where the integral is understood in the τ-Riemannian sense. As a consequence

$$\|T(t) - S(t)\| \leq tM$$

for all $t \in [0, t_0]$ and some constant $M > 0$.

Proof. The equation (IE) is just a reformulation of (7). The integral equation (IE$'$) can be deduced from (IE) by applying the bounded perturbation theorem with the operator $-B$. For the third assertion, let $t_0 \geq 0$ be arbitrary and set

$$M := \|B\| \cdot \sup\{\|T(t)\| : t \in [0, t_0]\} \cdot \sup\{\|S(t)\| : t \in [0, t_0]\}.$$

Then the desired inequality follows immediately from (IE). ∎

COROLLARY 3.7. *Let T be a bi-continuous semigroup with generator $(A, D(A))$ and $B \in \mathcal{L}(X)$ an operator which is sequentially τ-continuous on norm-bounded sets. Denote by S the semigroup generated by $(A+B, D(A))$. Then the operators*

$$U(0) := 0 \quad \text{and} \quad U(t) := \frac{S(t) - T(t)}{t} \quad \text{for } t > 0$$

form a locally bi-equicontinuous family.

Proof. Let $\varepsilon > 0$ and $x_n \xrightarrow{\tau} 0$ be a norm-bounded sequence. We make use of (IE) and write, for $p \in \mathcal{P}$,

$$p(U(t)x_n) = \frac{1}{t}\,p(S(t)x_n - T(t)x_n) \leq \frac{1}{t}\int_0^t p(T(t-s)BS(s)x_n)\,ds \leq \frac{1}{t} \cdot t\varepsilon = \varepsilon,$$

the last inequality being true for sufficiently large $n \in \mathbb{N}$ since the family of operators $\{T(t-s)BS(s) : s \in [0, t]\}$ is bi-equicontinuous by Lemma 3.3. ∎

We show that these two corollaries together characterise bounded perturbations of a given bi-continuous semigroup T (cf. [16, III, Corollary 3.12]).

THEOREM 3.8. *Let T and S be bi-continuous semigroups with generators $(A, D(A))$ and $(C, D(C))$ respectively, and suppose that*

(i) *there exists* $M > 0$ *such that*

$$\|T(t) - S(t)\| \leq tM$$

for all $t \in [0, 1]$,

(ii) *the family*

$$U(0) := 0, \qquad U(t) := \frac{S(t) - T(t)}{t} \qquad \text{for } t > 0$$

is locally bi-equicontinuous.

If $D(A) \cap D(C)$ *is* δ*-bi-dense for some* $\delta > 0$, *then there exists a bounded operator* B *which is sequentially* τ*-continuous on norm-bounded sets such that* $C = A + B$.

Proof. Let $x \in D(A) \cap D(C)$. Then

$$U(1/n)x = n \cdot (S(1/n)x - x) + n \cdot (x - T(1/n)) \xrightarrow{\tau} Cx - Ax \tag{11}$$

as $n \to \infty$. We define

$$Bx := \tau\text{-}\lim_{n\to\infty} U(1/n)x$$

for $x \in D(A) \cap D(C)$. Now, let $x \in X$ be arbitrary, and take a norm-bounded sequence $x_n \in D(A) \cap D(C)$ which is τ-convergent to x and satisfies (1). Then, for arbitrary $p \in \mathcal{P}$ we conclude by (ii) that

$$\begin{aligned} p(B(x_m - x_k)) &\leq \overline{\lim_{n\to\infty}}\, p(U(1/n)(x_m - x_k)) \\ &\leq \overline{\lim_{n\to\infty}}\, p(U(1/n)(x_m - x)) + \overline{\lim_{n\to\infty}}\, p(U(1/n)(x - x_k)) \leq 2\varepsilon \end{aligned} \tag{12}$$

if $m, k \in \mathbb{N}$ are sufficiently large. Therefore, Bx_m is a norm-bounded, τ-Cauchy sequence, hence it is convergent. So we can extend B to the whole space X by

$$Bx := \tau\text{-}\lim_{m\to\infty} Bx_m.$$

We claim that B is a bounded operator. Indeed, it follows from (i) and Proposition 1.4 that

$$\|Bx\| \leq \overline{\lim_{m\to\infty}}\, \|Bx_m\| \leq \overline{\lim_{m\to\infty}}\, \overline{\lim_{n\to\infty}}\, \|U(1/n)x_m\| \leq \overline{\lim_{m\to\infty}}\, M\|x_m\| \leq M\delta\|x\|$$

for all $x \in X$, proving $B \in \mathcal{L}(X)$. From (12) it is straightforward that B is sequentially τ-continuous on norm-bounded sets.

We conclude the proof by showing that $A + B = C$. To this end, let $\overline{S}$ be the bi-continuous semigroup generated by $(A + B, D(A))$ and consider $x \in D(A) \cap D(C)$. Then it is straightforward from (11) that

$$\frac{d}{dt}(\overline{S}(t)x - S(t)x) = 0$$

for all $t \geq 0$, taking the derivative in the τ topology. Also

$$\overline{S}(0)x - S(0)x = 0,$$

therefore

$$(\overline{S}(t) - S(t))x = 0 \quad \text{for all } x \in D(A) \cap D(C).$$

Since $D(A) \cap D(C)$ is bi-dense in X, we obtain $\overline{S} = S$, hence $A + B = C$. ∎

4. Examples. It is shown in [20, Sec. 3.5] that adjoint semigroups are bi-continuous with respect to the weak*-topology. Therefore, we consider bounded perturbations of bi-continuous semigroups with respect to various topologies on the dual space X'. First, we remark that if T is a weak*-continuous semigroup on X' with generator $(A, D(A))$ and X is invariant under the operators $T(t)'$, then T is the dual of a C_0-semigroup S (since $T'|_X$ is weakly, hence strongly continuous). From the bounded perturbation theorem for C_0-semigroups it follows that if $B \in \mathcal{L}(X)$, then $(A + B', D(A))$ is the generator of a bi-continuous semigroup with respect to the weak*-topology $\sigma(X', X)$. We now show that weak*-bi-continuous semigroups are, in certain cases, adjoint semigroups of C_0-semigroups. To do this, we need the following lemma.

LEMMA 4.1. *Let $T \in \mathcal{L}(X')$ and suppose that T is weak*-continuous on norm-bounded sets. Then there exists $S \in \mathcal{L}(X)$ such that $S' = T$.*

Proof. It is enough to show that $T' \in \mathcal{L}(X'')$ leaves X invariant. This follows if we prove that T has a weak*-adjoint

$$T'^{w^*} : X \to X$$

which, in this case, coincides with $T'|_X$. For $x \in X$ consider the linear form

$$\varphi_x(\cdot) := \langle T\cdot, x\rangle.$$

Then the restriction $\varphi_x|_B$ to the unit ball B in X' is weak*-continuous, so $\ker \varphi_x \cap B$ is weak*-closed. From the Krein–Šmulian Theorem (see [26, Ch. IV, Sec. 6]) it follows that $\ker \varphi_x$ is weak*-closed, hence φ_x is weak*-continuous. Therefore T has a weak*-adjoint T'^{w^*} given by $T'^{w^*}x = \varphi_x$. ∎

An immediate consequence of this lemma and the previous remarks is the characterisation of certain weak*-bi-continuous semigroups.

THEOREM 4.2. *If T is a weak*-bi-continuous semigroup in X' and $T(t)$ is weak*-continuous on norm-bounded sets for $t \geq 0$, then there exists a C_0-semigroup S in X with $S' = T$.*

The assumptions of the previous theorem are satisfied, for example, when T is a weak*-bi-continuous semigroup on X' and X is separable, because in this case the unit ball $B(0,1) \subseteq X'$ is metrisable for the weak*-topology. Thus the following proposition becomes trivial in view of the bounded perturbation theorem for C_0-semigroups. Nevertheless, the proposition is valid for all weak*-bi-continuous semigroups.

PROPOSITION 4.3. *Let T be a bi-continuous semigroup on X' with respect to the weak*-topology with generator $(A, D(A))$. Then for every $B \in \mathcal{L}(X)$, $(A + B', D(A))$ is the generator of a bi-continuous semigroup on X' with respect to the weak*-topology $\sigma(X', X)$.*

Proof. In view of Theorem 3.5, it suffices to show B' is $\sigma(X', X)$-continuous (on norm-bounded sets) for any operator $B \in \mathcal{L}(X)$. To this end, consider the $\sigma(X', X)$ open neighbourhood U of 0 in X' determined by the seminorms $p_i(\cdot) = |\langle x_i, \cdot\rangle|$, $x_i \in X$, $i = 1, \ldots, n$ and the positive real number ε,

$$U = \{x' : x' \in X',\, p_i(x') < \varepsilon,\, i = 1, \ldots, n\}.$$

Then we have to find a $\sigma(X', X)$-neighbourhood of 0 for which $B'V \subseteq U$. However, it is obvious that the $\sigma(X', X)$-neighbourhood V determined by the seminorms $q_i(\cdot) = |\langle Bx_i, \cdot\rangle|$ and $\varepsilon > 0$ fulfills this requirement. ∎

Next, we prove a similar result for the Mackey topology $\mu(X', X)$ (see [26, Ch. IV, Sec. 3]), i.e., the topology determined by the family of seminorms

$$\{\sup_{x\in K} |\langle x, x'\rangle| : K \subseteq X \text{ is weakly compact, absolutely convex}\}.$$

PROPOSITION 4.4. *Let T be a bi-continuous semigroup on X' with respect to the Mackey topology with generator $(A, D(A))$ and $B \in \mathcal{L}(X)$. Then $(A+B', D(A))$ is the generator of a bi-continuous semigroup on X' with respect to the Mackey topology.*

Proof. We show first that an operator B' satisfying the above assumptions is also continuous for the Mackey topology. In Proposition 4.3 it is shown that B is weakly continuous. Next, we suppose that $K \subseteq X$ is a weakly compact, absolutely convex set. Then $BK \subseteq X$ is also weakly compact and absolutely convex. For a given $\mu(X', X)$-neighbourhood U of 0,

$$U := \{x' : x' \in X',\, p_i(x') < \varepsilon,\, i = 1, \ldots, n\},$$

with

$$p_i(x') = \sup\{|\langle x, x'\rangle| : x \in K_i\} \quad \text{and} \quad \varepsilon > 0,$$

we have to find a $\mu(X', X)$-neighbourhood $V \subseteq X'$ of 0 such that $B'V \subseteq U$. But the neighbourhood V determined by the seminorms

$$q_i(x') = \sup\{|\langle x, x'\rangle| : x \in BK_i\}$$

and $\varepsilon > 0$ fulfills this requirement. We have seen that the assumptions of Theorem 3.5 are satisfied, therefore our statement is proved. ∎

Finally, we give some examples of τ_c-continuous operators.

REMARK 4.5. Let (Ω, τ) be a locally compact, Hausdorff topological space, and suppose that $\Phi : \Omega \to \Omega$ is continuous. Then the linear operator B on $C_b(\Omega)$ defined as

$$(Bf)(x) = f(\Phi(x)) \quad \text{for } x \in \Omega,$$

is continuous on $C_b(\Omega)$ with respect to the compact-open topology τ_c. Hence Theorem 3.5 can be applied to semigroups induced by jointly continuous flows (see [13–15]) and for these "local" perturbations.

REMARK 4.6. For a given function $f \in C_b(\mathbb{R})$ the multiplication operator

$$V_f : C_b(\mathbb{R}) \to C_b(\mathbb{R}), \qquad V_f(g) := f \cdot g,$$

is norm-bounded and τ_c-continuous on norm-bounded sets.

As an application, we consider the heat equation in $C_b(\mathbb{R})$ with bounded, continuous potential $V = V_f$:

$$\text{(HE)} \qquad \begin{cases} u'(t) = \Delta u(t) + V u(t) & \text{for all } t \geq 0, \\ u(0) = u_0, \quad u_0 \in C_b(\mathbb{R}). \end{cases}$$

Let

$$D(\Delta) := \{f : f \in C_b(\mathbb{R}),\ f'' \text{ exists}, f'' \in C_b(\mathbb{R})\}.$$

Then it is easy to see that $(\Delta, D(\Delta))$ is the generator of a bi-continuous semigroup. Indeed, it generates the Gaussian semigroup

$$[P(t)f](x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-|y-x|^2/4t} f(y)\, dy \qquad \text{for } f \in C_b(\mathbb{R}).$$

In order to solve the abstract Cauchy problem (cf. [7]) corresponding to the equation (HE) it suffices to show that $(\Delta + V, D(\Delta))$ is the generator of a bi-continuous semigroup. This is, however, a straightforward consequence of Theorem 3.5 and Remark 4.6 above.

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