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On the (C, α) Cesàro bounded operators

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Abstract. For a given linear operator T in a complex Banach space X and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$, we define the nth Cesàro mean of order α of the powers of T by $M_n^\alpha = (A_n^\alpha)^{-1} \sum_{k=0}^n A_{n-k}^{\alpha-1} T^k$. For $\alpha = 1$, we find $M_n^1 = (n+1)^{-1} \sum_{k=0}^n T^k$, the usual Cesàro mean. We give necessary and sufficient conditions for a (C, α) bounded operator to be (C, α) strongly (weakly) ergodic.

Introduction. Let T be a bounded linear operator in a Banach space X and let $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. We say that T is *power bounded* if $\sup_n \|T^n\| < \infty$, and (C, α) *Cesàro bounded* (or (C, α) *bounded*) if

$$\sup_n \|M_n^\alpha\| = \sup_n \left\| \frac{1}{A_n^\alpha} \sum_{k=0}^n A_{n-k}^{\alpha-1} T^k \right\| < \infty.$$

In particular for $\alpha = 1$, a $(C, 1)$ Cesàro bounded operator is said to be *Cesàro bounded* (see [D], [M-S], [G-H], [D-L]). One can show that power bounded operators are (C, α) bounded (this is obvious for $\alpha > 0$). However (C, α) bounded operators need not be power bounded (see for example [D], [T-Z], and Remark and Example in the present paper).

We shall say that a (C, α) Cesàro bounded operator T is (C, α) *strongly* (resp. *weakly*) *ergodic* if there exists a bounded linear operator E on X such that $M_n^\alpha x$ converges to Ex in X for every $x \in X$ (resp. $x^* M_n^\alpha x \to x^* E x$ for all $x \in X$ and $x^* \in X^*$).

Einar Hille, applying abelian and tauberian theorems, proved the strong ergodic theorem [H, Theorem 7] which says that if a bounded linear operator T in a Banach space X is (C, α) strongly ergodic for some real number $\alpha > 0$ then T must be strongly Abel ergodic, that is, $(\lambda - 1) R(\lambda, T)$ converges in the strong operator topology as $\lambda \to 1^+$ and $(T^n/n^\alpha)x$ converges to 0 in X for all $x \in X$; he also proved that the converse holds if T is power bounded. So the power boundedness of T seems to be necessary together with strong Abel ergodicity to have the (C, α) strong ergodicity of T.

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Recently Yoshimoto [Y] gave sufficient conditions (more restrictive) for a (C,α) bounded operator T in a Banach space X to be (C,α) strongly ergodic (T need not be power bounded). Also Derriennic [D] gave sufficient conditions for a Cesàro bounded operator to be $(C,1)$ strongly ergodic but only in a reflexive Banach space.

In this paper, we consider a more general situation. For any complex number α with $\Re(\alpha) > 0$ we define the Cesàro averages of a bounded linear operator T of order α and we give necessary and sufficient conditions for a (C,α) bounded operator in a complex Banach space X to be strongly (or weakly) ergodic.

Section 1 gives some preliminaries in order to make this paper as self-contained as possible. Section 2 presents our main results. Section 3 contains an example and some propositions.

1. Preliminaries. Let X be a complex Banach space and let $B(X)$ denote the Banach algebra of all bounded linear operators from X to itself. For $T \in B(X)$, the *resolvent set* of T, denoted by $\varrho(T)$, is the set of $\lambda \in \mathbb{C}$ for which $(\lambda I - T)^{-1}$ exists as an operator in $B(X)$ with domain X. The *spectrum* of T is the complement of $\varrho(T)$ and it is denoted by $\sigma(T)$. The resolvent set $\varrho(T)$ is an open subset of $\mathbb{C}$ and $\sigma(T)$ is a nonempty compact subset of $\mathbb{C}$. So the spectral radius $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ is well defined; in fact $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$. The function $R(\lambda, T) : \lambda \in \varrho(T) \mapsto (\lambda I - T)^{-1}$ is called the *resolvent* of T. It is well known that $R(\lambda, T)$ is analytic in $\varrho(T)$ and if $T \in B(X)$ and $|\lambda| > r(T)$, then $\lambda \in \varrho(T)$ and $R(\lambda, T) = \sum_{n=0}^{\infty} T^n/\lambda^{n+1}$. The series converges in the uniform operator topology.

For any complex number α, we define

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} \quad \text{for } n \ge 1, \qquad A_0^\alpha = 1.$$

These coefficients appear in the formula

$$\frac{1}{(1-t)^{\alpha+1}} = \sum_{n=0}^{\infty} A_n^\alpha t^n, \qquad |t| < 1.$$

As in the real case, the equality

$$A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}$$

remains valid for each complex number α and for all integer $n = 0, 1, 2, \dots$ Obviously for $\alpha \in \{-1, -2, \dots\}$, $A_n^\alpha = 0$ for every integer $n \ge -\alpha$.

Let $\alpha \in \mathbb{C} \setminus \{-1, -2, \ldots\}$ and put

$$M_n^\alpha = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} T^k \qquad \text{for } n = 0, 1, 2, \ldots$$

In the following, we denote the kernel and range of a bounded linear operator T by $N(T)$ and $R(T)$, respectively. By a *projection* of a Banach space X, we mean an element P of $B(X)$ satisfying $P^2 = P$. We recall that if P is a projection of X, then $R(P)$ is a closed subspace of X and in addition $X = R(P) \oplus N(P)$. Conversely, for every direct-sum decomposition $X = Y \oplus Z$ where Y and Z are closed subspaces of X there exists a unique projection P of X such that $R(P) = Y$ and $N(P) = Z$; we call P the *projection of* X *onto* Y *along* Z.

We conclude this section with an interesting result which shows a connection between the decomposition of a Banach space X and the strong Abel summability of a bounded linear operator $T \in B(X)$:

LEMMA 1.1 ([H-P, Theorem 18.8.1]). *Let X be a Banach space and $T \in B(X)$. If there exists a sequence $(\lambda_n) \subset \varrho(T)$ and $E \in B(X)$ such that*

(1) $\lambda_n \to 1$ *as* $n \to \infty$,

(2) $\|(\lambda_n - 1)R(\lambda_n, T)x - Ex\| \to 0$ *as* $n \to \infty$ *for all* $x \in X$,

then $X = \overline{R(I-T)} \oplus N(I-T)$ and E is the projection of X onto $N(I-T)$ along $\overline{R(I-T)}$.

2. Our main results

THEOREM 2.1. *Let α be a complex number with $\Re(\alpha) > 0$, T a (C,α) bounded operator on a complex Banach space X, and $E \in B(X)$. Then*

(1) $\operatorname*{so-lim}_{n\to\infty} M_n^\alpha = E$

if and only if

(2) $\operatorname*{so-lim}_{\lambda\to 1^+} (\lambda - 1)R(\lambda, T) = E$

and

(3) $\operatorname*{so-lim}_{n\to\infty} \dfrac{T^n}{n^\alpha} = 0.$

We shall first prove that if (2) and (3) are satisfied then (1) holds. For this we need some auxiliary results.

DEFINITION 2.2. Let X be a Banach space and $T \in B(X)$. For a complex number α with $\Re(\alpha) > 0$ and an integer $l \geq 1$, we say that T satisfies *condition* $S(l, \alpha)$ if $\|(I-T)^l M_n^\alpha(T)x\| \to 0$ as $n \to \infty$ for all $x \in X$.

When $\alpha = 1$ we recover Definition 2 of [L-M].

We shall see that (2)&(3)⇒(1) comes immediately from the following crucial result:

Proposition 2.3. *Fix* $\alpha \in \mathbb{C}$ *with* $\Re(\alpha) > 0$ *and let* T *be a* (C,α) *bounded operator in a complex Banach space* X *satisfying the following two conditions*:

(1) $(\lambda-1)R(\lambda,T)x \to Ex$ *as* $\lambda \to 1^+$ *for all* $x \in X$ *and some* $E \in B(X)$.
(2) T *satisfies condition* $S(l,\alpha)$ *for some integer* $l \geq 1$.

Then T *is* (C,α) *strongly ergodic.*

Proof. By Lemma 1.1, (1) yields the decomposition $X = \overline{R(I-T)} \oplus N(I-T)$ and E is the projection of X onto $N(I-T)$ along $\overline{R(I-T)}$. It is not hard to check that $\overline{R(I-T)^n} = \overline{R(I-T)}$ for all $n \geq 1$.

Now let $x \in X$. We have $x = (I-E)x + Ex$ where $(I-E)x \in \overline{R(I-T)^l}$ and $Ex \in N(I-T)$. Thus $M_n^\alpha x - Ex = M_n^\alpha (I-E)x$. Since $\sup_n \|M_n^\alpha x\| < \infty$ for all $x \in \overline{R(I-T)} = \overline{R(I-T)^l}$ and $(I-T)^l M_n^\alpha x \to 0$ as $n \to \infty$ for all $x \in X$, the desired result follows.

Proposition 2.4. *Let* $\alpha \in \mathbb{C}$ *with* $\Re(\alpha) > 0$ *and* $T \in B(X)$ *be such that* $(T^n/n^\alpha)x \to 0$ *as* $n \to \infty$. *Then* $(T-I)^l M_n^\alpha x \to 0$ *as* $n \to \infty$ *for some integer* $l \geq 1$.

Proof. First if $\alpha \in \{1,2,\dots\}$ then

$$(T-I)^\alpha M_n^\alpha = \frac{\alpha!}{(n+1)(n+2)\dots(n+\alpha)}\, T^{\alpha+n} - P_{\alpha-1}^n(T-I),$$

where

$$P_{\alpha-1}^n(T-I) = \frac{\alpha}{n+1}\,(T-I)^{\alpha-1} + \dots + \frac{\alpha!}{(n+1)(n+2)\dots(n+\alpha)}\, I$$

(see [E, Lemma 2.3]). It is clear that $P_{\alpha-l}^n(T-I)x \to 0$ as $n \to \infty$ and by assumption $(T^n/n^\alpha)x \to 0$, thus $(T-I)^\alpha M_n^\alpha x \to 0$.

If α is not an integer, then $\alpha - l$ is not either for any integer $l \geq 1$, $M_{n+l}^{\alpha-l}$ is well defined, and we have

$$(T-I)^l M_n^\alpha = \frac{\alpha(\alpha-1)\dots(\alpha-l+1)}{(n+1)(n+2)\dots(n+l)}\, M_{n+l}^{\alpha-l} - P_{l-1}^n(T-I).$$

We now prove that for an appropriate integer $l \geq 1$, $(T-I)^l M_n^\alpha x \to 0$ as $n \to \infty$. Since $P_{l-1}^n(T-I)x \to 0$ as $n \to \infty$, it follows that $(T-I)^l M_n^\alpha x \to 0$ as $n \to \infty$ if and only if

$$\frac{\alpha(\alpha-1)\dots(\alpha-l+1)}{(n+1)(n+2)\dots(n+l)} M_{n+l}^{\alpha-l} x \to 0 \quad \text{as } n \to \infty.$$

Set $\alpha = a + ib$, $a, b \in \mathbb{R}$ with $a > 0$ (we may suppose that a is not an integer), and let $l = [a] + 1$. Then there is a unique $\beta \in [0,1[$ such that $a = [a] + \beta$.

So

$$\frac{\alpha(\alpha-1)\dots(\alpha-l+1)}{(n+1)(n+2)\dots(n+l)}\,M_{n+l}^{\alpha-l}\underset{n\to\infty}{\longrightarrow}0\quad\text{if and only if}$$

$$\frac{1}{q_{n,\alpha}n^{[a]}}\,\frac{\beta+ib}{n+1}\,M_{n+1}^{\beta-1+ib}\underset{n\to\infty}{\longrightarrow}0$$

where $q_{n,\alpha}\to 1$ as $n\to\infty$. Now we use the equality

$$\frac{\alpha+n+1}{n+1}\,M_{n+1}^{\alpha}-M_n^{\alpha}=\frac{\alpha}{n+1}M_{n+1}^{\alpha-1}$$

(see [E, Lemma 2.3]), which remains valid for all complex numbers α with $\Re(\alpha)>0$, so

$$\begin{aligned}\frac{\beta+ib}{n+1}\,M_{n+1}^{\beta-1+ib}&=\frac{\beta+ib+n+1}{n+1}\,M_{n+1}^{\beta+ib}-M_n^{\beta+ib}\\&=\frac{1}{A_n^{\beta+ib}}\Big\{T^{n+1}+\sum_{k=0}^{n}A_{n+1-k}^{\beta-2+ib}T^k\Big\}\\&=\frac{1}{A_n^{\beta+ib}}\Big\{T^{n+1}+\sum_{k=0}^{n}\frac{\beta-1+ib}{n+1-k}\,A_{n-k}^{\beta-1+ib}T^k\Big\}.\end{aligned}$$

Since $A_n^{\beta+ib}$ is equivalent to $n^{\beta+ib}/\Gamma(\beta+ib+1)$ as $n\to\infty$ (see [B-G, p. 502]), and by assumption $T^n/n^{\alpha}x\to 0$ as $n\to\infty$, we have

$$\frac{T^{n+1}}{q_{n,\alpha}n^{[a]}A_n^{\beta+ib}}\,x\to 0\qquad\text{as } n\to\infty.$$

We will be done if we prove that

$$\frac{1}{q_{n,\alpha}n^{[a]}}\,\frac{1}{A_n^{\beta+ib}}\sum_{k=0}^{n}\frac{\beta-1+ib}{n+1-k}\,A_{n-k}^{\beta-1+ib}T^kx\to 0\qquad\text{as } n\to\infty.$$

For this, put

$$\begin{aligned}I_1^n&=\frac{1}{q_{n,\alpha}n^{[a]}}\,\frac{1}{A_n^{\beta+ib}}\sum_{k=0}^{n/2}\frac{\beta-1+ib}{n+1-k}\,A_{n-k}^{\beta-1+ib}T^kx,\\I_2^n&=\frac{1}{q_{n,\alpha}n^{[a]}}\,\frac{1}{A_n^{\beta+ib}}\sum_{k=n/2+1}^{n}\frac{\beta-1+ib}{n+1-k}\,A_{n-k}^{\beta-1+ib}T^kx.\end{aligned}$$

We shall prove that both I_1^n and I_2^n converge to 0 as $n\to\infty$. We have

$$\|I_1^n\|\le\frac{1}{q_{n,\alpha}n^{[a]}}\,\frac{2|\beta-1+ib|}{n}\,\max_{k=0}^{n}\|T^kx\|\sum_{k=0}^{n/2}\left|\frac{A_{n-k}^{\beta-1+ib}}{A_n^{\beta+ib}}\right|$$

$$\le \frac{1}{q_{n,\alpha} n^{[a]}} \frac{2|\beta-1+ib|}{n} \max_{k=0}^{n} \|T^k x\| \sum_{j=0}^{n/2} \left| \frac{A_{n/2+j}^{\beta-1+ib}}{A_n^{\beta+ib}} \right|.$$

By assumption $\|(T^n/n^\alpha)x\| \to 0$ as $n \to \infty$, which yields $n^{-\alpha} \max_{k=0}^n \|T^k x\| \to 0$ as $n \to \infty$. Therefore $n^{-[a]-1} \max_{k=0}^n \|T^k x\| \to 0$ as $n \to \infty$.

We now show that the sequence $\{|A_n^{\beta+ib}|^{-1} \sum_{j=0}^{n/2} |A_{n/2+j}^{\beta-1+ib}|\}_n$ is bounded. Indeed, for any nonnegative integers k and j we have

$$A_{k+j}^{\beta-1+ib} = A_k^{\beta-1+ib} \left[\left(1 + \frac{\beta-1+ib}{k+1}\right)\left(1 + \frac{\beta-1+ib}{k+2}\right) \cdots \left(1 + \frac{\beta-1+ib}{k+j}\right)\right],$$

and in particular for $k = n/2$ and $j = 0, 1, \ldots, n/2$ we have

$$\sum_{j=0}^{n/2} A_{n/2+j}^{\beta-1+ib} = A_{n/2}^{\beta-1+ib}$$
$$\times \left[1 + \left(1 + \frac{\beta-1+ib}{n/2+1}\right) + \ldots + \left(1 + \frac{\beta-1+ib}{n/2+1}\right) \cdots \left(1 + \frac{\beta-1+ib}{n/2+n/2}\right)\right].$$

Then

$$\Big|\sum_{j=0}^{n/2} A_{n/2+j}^{\beta-1+ib}\Big| \le |A_{n/2}^{\beta-1+ib}|$$
$$\times \left[1 + \left(1 + \frac{|\beta-1+ib|}{n/2+1}\right) + \ldots + \left(1 + \frac{|\beta-1+ib|}{n/2+1}\right) \cdots \left(1 + \frac{|\beta-1+ib|}{n}\right)\right].$$

On the other hand, there exists a constant K such that

$$\left(1 + \frac{|\beta-1+ib|}{n/2+1}\right) \cdots \left(1 + \frac{|\beta-1+ib|}{n}\right) \le K.$$

Since $A_{n/2}^{\beta-1+ib}$ is equivalent to $(n/2)^{\beta-1+ib} \Gamma(\beta+ib)^{-1}$ as $n \to \infty$, it follows that

$$\Big|\sum_{j=0}^{n/2} A_{n/2+j}^{\beta-1+ib}\Big| \le \frac{n^{\beta-1}}{2^{\beta-1}|\Gamma(\beta+ib)|} (n/2+1)K.$$

From this we see that

$$\sup_n \frac{1}{|A_n^{\beta+ib}|} \sum_{j=0}^{n/2} |A_{n/2+j}^{\beta-1+ib}| < \infty.$$

Hence I_1^n converges to 0 as $n \to \infty$.

It remains to show that $I_2^n \to 0$ as $n \to \infty$. We have

$$\|I_2^n\| = \frac{1}{q_{n,\alpha} n^{[a]}} \left\| \frac{1}{A_n^{\beta+ib}} \sum_{k=0}^{n/2-1} \frac{\beta-1+ib}{k+1} A_k^{\beta-1+ib} T^{n-k} x \right\|$$

$$\leq \frac{|\beta-1+ib|}{q_{n,\alpha} n^{[a]}} \frac{\max_{k=0}^{n} \|T^k x\|}{|A_n^{\beta+ib}|} \sum_{k=0}^{\infty} \frac{|A_k^{\beta-1+ib}|}{k+1}.$$

Since $A_k^{\beta-1+ib}$ is equivalent to $k^{\beta-1+ib}/\Gamma(\beta+ib)$ as $k \to \infty$, the series converges, and for the same reason as above

$$\frac{\max_{k=0}^{n} \|T^k x\|}{n^{[a]} |A_n^{\beta+ib}|} \to 0 \quad \text{as } n \to \infty.$$

This implies that $I_2^n \to 0$ and completes the proof of Proposition 2.4.

It is well known that when α is a nonnegative real number, Hille proved in his classical paper that (1) (of Theorem 2.1) implies (3) ([H, Theorem 7]). We now prove it for $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$.

PROPOSITION 2.5. *Let $T \in B(X)$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. If $E \in B(X)$ is such that $M_n^\alpha \to E$ as $n \to \infty$ then $T^n/n^\alpha \to 0$ as $n \to \infty$.*

Proof. If $\alpha \in \{1, 2, \ldots\}$, we have

$$(T-I)M_n^\alpha = \frac{\alpha}{n+1}\,(M_{n+1}^{\alpha-1} - I).$$

Since

$$\frac{\alpha}{n+1}\, M_{n+1}^{\alpha-1} = \frac{\alpha+n+1}{n+1}\, M_{n+1}^{\alpha} - M_n^\alpha,$$

the convergence of M_n^α to E yields $(T-I)M_n^\alpha \to 0$ as $n \to \infty$. Moreover

$$(T-I)^\alpha M_n^\alpha = \frac{\alpha!}{(n+1)(n+2)\ldots(n+\alpha)}\, T^{n+\alpha} - P_{\alpha-1}^n(T-I),$$

and so the convergence of $(T-I)^\alpha M_n^\alpha$ to 0 yields $T^n/n^\alpha \to 0$ as $n \to \infty$.

If $\alpha \notin \{1, 2, \ldots\}$, set $S_n^\alpha = S_n^\alpha(T) = \sum_{k=0}^n A_{n-k}^{\alpha-1} T^k$. Then T^n can be expressed as combinations of S_n^α. In fact, we have

$$T^n = \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} S_{n-k}^\alpha(T), \qquad n \geq 0.$$

Indeed, we first have

$$\sum_{n=0}^{\infty} \bigg(\sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} A_{n-k}^{\alpha-1} \bigg) x^n$$
$$= \bigg(\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} x^k \bigg) \bigg(\sum_{k=0}^{\infty} A_k^{\alpha-1} x^k \bigg) = 1 \quad \text{ for } |x| < 1$$

because

$$(1-x)^\alpha (1-x)^{-\alpha} = \bigg(\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} x^k \bigg) \bigg(\sum_{k=0}^{\infty} A_k^{\alpha-1} x^k \bigg) = 1.$$

Thus
$$\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}A_{n-k}^{\alpha-1}=0,\qquad n\geq 1.$$
Then for $n\geq 1$ we have
$$\begin{aligned}
T^n &= \sum_{j=0}^{n}\Big[\sum_{k=0}^{n-j}(-1)^k\binom{\alpha}{k}A_{n-j-k}^{\alpha-1}\Big]T^j\\
&= \sum_{j=0}^{n}\Big[\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}A_{n-j-k}^{\alpha-1}\Big]T^j \qquad (A_{n-k-j}^{\alpha-1}=0 \text{ if } k>n-j)\\
&= \sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}\Big(\sum_{j=0}^{n-k}A_{n-k-j}^{\alpha-1}T^j\Big)\\
&= \sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}S_{n-k}^{\alpha}.
\end{aligned}$$
Since $M_n^\alpha = S_n^\alpha/A_n^\alpha$ and $\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}A_{n-k}^{\alpha-1}=0$ for $n=1,2,\dots,$ it follows that for all $n\geq 1$,
$$\begin{aligned}
T^n &= \sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}(S_{n-k}^{\alpha}-A_{n-k}^{\alpha-1}E)\\
&= \sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}A_{n-k}^{\alpha-1}\Big[\frac{A_{n-k}^{\alpha}}{A_{n-k}^{\alpha-1}}\Big\{\frac{S_{n-k}^{\alpha}}{A_{n-k}^{\alpha}}-E\Big\}+\Big(\frac{A_{n-k}^{\alpha}}{A_{n-k}^{\alpha-1}}-1\Big)E\Big]\\
&= \sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}A_{n-k}^{\alpha}\{M_{n-k}^{\alpha}-E\}\\
&\quad +\Big\{\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}A_{n-k}^{\alpha-1}\Big(\frac{\alpha+n-k}{\alpha}-1\Big)\Big\}E.
\end{aligned}$$
It is not hard to check that
$$\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}A_{n-k}^{\alpha-1}\Big(\frac{\alpha+n-k}{\alpha}-1\Big)=\sum_{k=0}^{n-1}(-1)^k\binom{\alpha}{k}A_{n-1-k}^{\alpha}=1,\qquad n\geq 1.$$
Thus $T^n/n^\alpha\to 0$ as $n\to\infty$ if and only if
$$\frac{1}{n^\alpha}\sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}A_{n-k}^{\alpha}(M_{n-k}^{\alpha}-E)\to 0\qquad \text{as } n\to\infty,$$
that is,
$$\frac{1}{n^\alpha}\sum_{k=0}^{n}(-1)^{n-k}\binom{\alpha}{n-k}A_{k}^{\alpha}(M_{k}^{\alpha}-E)\to 0\qquad \text{as } n\to\infty.$$

For a given $\varepsilon > 0$ there exists $N \geq 1$ such that $\|M_n^\alpha - E\| < \varepsilon$ for all $n > N$. Then for $n > N$,

$$\begin{aligned}
&\left\|\frac{1}{n^\alpha}\sum_{k=0}^{n}(-1)^{n-k}\binom{\alpha}{n-k}A_k^\alpha(M_k^\alpha - E)\right\| \\
&\quad \leq \frac{1}{|n^\alpha|}\max_{k=0}^{N}\|A_k^\alpha(M_k^\alpha - E)\|\sum_{k=0}^{N}\left|\binom{\alpha}{n-k}\right| \\
&\qquad + \left\|\frac{1}{n^\alpha}\sum_{k=N+1}^{n}(-1)^{n-k}\binom{\alpha}{n-k}A_k^\alpha(M_k^\alpha - E)\right\| \\
&\quad \leq \frac{1}{|n^\alpha|}\max_{k=0}^{N}\|A_k^\alpha(M_k^\alpha - E)\|\sum_{k=0}^{n}\left|\binom{\alpha}{k}\right| + \varepsilon\,\frac{1}{|n^\alpha|}\sum_{k=0}^{n}\left|\binom{\alpha}{n-k}A_k^\alpha\right|.
\end{aligned}$$

Since $|A_n^\alpha/n^\alpha| \to |\Gamma(\alpha+1)|^{-1}$ there exists a constant C so that $\sup_n |A_n^\alpha/n^\alpha| \leq C$ and also $\sup_n \max_{0\leq k\leq n} |A_k^\alpha/n^\alpha| \leq C$. We obtain

$$\begin{aligned}
&\left\|\frac{1}{n^\alpha}\sum_{k=0}^{n}(-1)^{n-k}\binom{\alpha}{n-k}A_k^\alpha(M_k^\alpha - E)\right\| \\
&\qquad\qquad \leq \left(\frac{1}{|n^\alpha|}\max_{k=0}^{N}\left\|A_k^\alpha(M_k^\alpha - E)\right\| + \varepsilon C\right)\sum_{k=0}^{n}\left|\binom{\alpha}{k}\right|.
\end{aligned}$$

The series $\sum_{k=0}^{\infty}\left|\binom{\alpha}{k}\right|$ converges because the term $\left|\binom{\alpha}{k}\right|$ is equivalent to $1/(|k^{\alpha+1}|\cdot|\Gamma(-\alpha)|)$ as $k\to\infty$ and by assumption $\Re(\alpha) > 0$. This completes the proof of Proposition 2.5.

The proof of Theorem 2.1 will be completed via the following proposition and the previous propositions.

PROPOSITION 2.6. *Let $T \in B(X)$, α a complex number with $\Re(\alpha) > 0$, and $E \in B(X)$ such that $M_n^\alpha \to E$. Then $(\lambda - 1)R(\lambda, T) \to E$ as $\lambda \to 1^+$.*

Proof. From what we have seen above, $T^n/n^\alpha \to 0$ as $n\to\infty$, and this yields $\sigma(T) \subset \overline{D(0,1)}$.

Let $\lambda > 1$. We can check that

$$(\lambda - 1)R(\lambda, T) = \left(1 - \frac{1}{\lambda}\right)^{\alpha+1}\sum_{n=0}^{\infty}A_n^\alpha M_n^\alpha(T)\left(\frac{1}{\lambda}\right)^n.$$

It follows that

$$(\lambda - 1)R(\lambda, T) - E = \left(1 - \frac{1}{\lambda}\right)^{\alpha+1}\sum_{n=0}^{\infty}A_n^\alpha(M_n^\alpha - E)\left(\frac{1}{\lambda}\right)^n$$

and the convergence of M_n^α to E as $n\to\infty$ yields the convergence of $(\lambda - 1)R(\lambda, T)$ to E as $\lambda \to 1^+$.

If we look carefully at Propositions 2.3–2.6 we obtain the following theorem.

THEOREM 2.7. *Let α be a complex number with $\Re(\alpha) > 0$, T a (C,α) bounded operator on a complex Banach space X, and $E \in B(X)$. Then the following assertions are equivalent*:

(1) $\operatorname*{so-lim}_{n\to\infty} M_n^\alpha(T) = E$.

$$(2) \begin{cases} \text{(a)} \ \operatorname*{so-lim}_{\lambda\to 1^+}(\lambda - 1)R(\lambda, T) = E, \\ \text{(b)} \ \operatorname*{so-lim}_{n} \dfrac{T^n}{n^\alpha} = 0. \end{cases}$$

$$(3) \begin{cases} \text{(a)} \ X = \overline{R(I-T)} \oplus N(I-T), \\ \text{(b)} \ \operatorname*{so-lim}_{n\to\infty}(T-I)^l M_n^\alpha(T) = 0 \textit{ for some integer } l \ge 1. \end{cases}$$

Next we mention the corresponding results for the weak operator topology. For a given $x \in X$ if $\operatorname{w-lim}_{n\to\infty}(T^n/n^\alpha)x = 0$ for some $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ then an easy observation gives

$$\operatorname*{w-lim}_{n\to\infty} \max_{0\le k\le n} \frac{T^k}{n^\alpha} x = 0.$$

In addition, it can be checked that Lemma 1.1 and Propositions 2.3–2.6 hold with the strong operator topology replaced by the weak operator topology. This yields the corresponding theorem in the weak operator topology. Here we only state the result without proof.

THEOREM 2.8. *Let α be a complex number with $\Re(\alpha) > 0$, T a (C,α) bounded operator on a complex Banach space X into itself*, *and $E \in B(X)$. Then*

(1) $\operatorname*{w-lim}_{n\to\infty} M_n^\alpha x = Ex \quad$ *for all $x \in X$*

if and only if

(2) $\operatorname*{w-lim}_{\lambda\to 1^+}(\lambda - 1)R(\lambda, T)x = Ex \quad$ *for all $x \in X$*

and

(3) $\operatorname*{w-lim}_{n\to\infty} \dfrac{T^n}{n^\alpha} x = 0 \quad$ *for all $x \in X$.*

3. Finally, we give some corollaries and an example.

COROLLARY 3.1. *If $\alpha > 0$ and T is (C,α) strongly ergodic in a Banach space X, then T is (C,β) strongly ergodic for any $\beta \ge \alpha$.*

PROPOSITION 3.2. *Let $\alpha > 0$ and T a (C,α) bounded operator in a reflexive Banach space X satisfying $(T^n/n^\alpha)x \to 0$ as $n \to \infty$ for all $x \in X$. Then T is (C,α) strongly ergodic.*

Proof. Let $x \in X$. Since X is reflexive and T is (C,β) bounded for any $\beta > \alpha$ (see [D, Lemma 1]), there exists a subsequence $(M^\beta_{n_k})_k$ such that $M^\beta_{n_k} x$ converges weakly to some element in X; call it Ex. Also [D, Lemma 1] asserts that $(T-I)M^\beta_n \to 0$ as $n \to \infty$. It follows that $(I-T)E = E(I-T) = 0$, $N(I-T) = R(E)$, $E^2 = E$ and $\overline{R(I-T)} \subset N(E)$. Now, for $x \in N(E)$ clearly x is a weak cluster point of the sequence

$$\{(I - M^\beta_n)x\}_n = \left\{(I-T)\frac{1}{A^\beta_n}\sum_{k=0}^n A^{\beta-1}_{n-k}(I + \ldots + T^{k-1})x\right\}_n.$$

Since every weakly closed convex subset of X is norm closed, it follows that $x \in \overline{R(I-T)}$. Then we have the decomposition $X = \overline{R(I-T)} \oplus N(I-T)$ and by assumption $(T^n/n^\alpha)x \to 0$, so T satisfies condition $S(l,\alpha)$ for some integer $l \geq 1$, and we apply Theorem 2.7 to obtain the desired result.

Remark. There exists a (C,α) bounded operator T in a reflexive Banach space which is (C,β) strongly ergodic for every $\beta > \alpha$ but not (C,α) strongly ergodic.

To see this, we take $X = \mathbb{C}^2$ and we consider the operator T defined on X by $T = \left[\begin{smallmatrix} -1 & -1 \\ 0 & -1 \end{smallmatrix}\right]$. We check easily that

$$T^n = \begin{bmatrix} (-1)^n & n(-1)^n \\ 0 & (-1)^n \end{bmatrix} \quad \text{for } n = 1,2,\ldots$$

So $n \leq \|T^n\| \leq n+2$. Obviously T^n/n does not converge to 0 as $n \to \infty$ and consequently T cannot be $(C,1)$ strongly ergodic. However, T is not power bounded but it is $(C,1)$ bounded, that is, $M^1_n = (n+1)^{-1}\sum_{k=0}^n T^k$ is bounded and thus it is (C,β) bounded for every $\beta \geq 1$. Hence it follows from Proposition 3.2 that T is (C,β) strongly ergodic for every $\beta > 1$.

Proposition 3.3. *Let $\alpha > 0$ and T be a (C,α) bounded operator in a reflexive Banach space X satisfying* w-$\lim_n (T^n/n^\alpha)x = 0$ *for all $x \in X$. Then T is (C,α) weakly ergodic.*

We conclude this paper by the following example of an operator T which is (C,α) weakly ergodic but not (C,α) strongly ergodic.

Example. Let X be the Hilbert space l_2 with the canonical basis $\{e_k\}$. We consider the operator T defined on X by $Te_k = \omega_k e_{k+1}$ for every $k \geq 1$ where $\omega_k = 1 + 1/k^2$, $k = 1,2,\ldots$ We check easily that $T^n e_k = (\omega_k \ldots \omega_{k+n-1})e_{k+n}$ for $k = 1,2,\ldots$ Let

$$P_n = \omega_1 \ldots \omega_n = \left(1 + \frac{1}{1^2}\right)\left(1 + \frac{1}{2^2}\right)\ldots\left(1 + \frac{1}{n^2}\right).$$

Obviously P_n converges; we denote by P its limit. Then $1 \le \omega_k \dots \omega_{k+n-1} \le P$ for all $k, n \ge 1$. It follows that $\sup_n \|T^n\| \le P$ and $\|(T^n/n)x\| \le (P/n)\|x\| \to 0$ for all $x \in l_2$. It is also clear that $\sup_n \|M_n^1(T)\| < \infty$.

We now consider the operator S defined on the Hilbert space $l_2 \times l_2$ by

$$S = \begin{bmatrix} T & T - I \\ 0 & T \end{bmatrix}.$$

It is not hard to check that

$$S^n = \begin{bmatrix} T^n & n(T^n - T^{n-1}) \\ 0 & T^n \end{bmatrix} \qquad \text{(see [T-Z])}$$

and

$$M_n^1(S) = \begin{bmatrix} M_n^1(T) & \frac{n}{n+1}(T^n - M_{n-1}^1(T)) \\ 0 & M_n^1(T) \end{bmatrix}.$$

Clearly

$$\frac{S^n}{n} = \begin{bmatrix} \frac{T^n}{n} & T^n - T^{n-1} \\ 0 & \frac{T^n}{n} \end{bmatrix},$$

and

$$\left\| \frac{S^n}{n}(e_k \oplus e_k) \right\|^2 \ge \|(T^n - T^{n-1})e_k\|^2 \ge 1.$$

Thus S^n/n does not converge strongly to 0 in the operator topology. So S is not strongly ergodic. However, $(S^n/n)(x \oplus y)$ converges weakly to 0 for every $x \oplus y \in l_2 \times l_2$ (since $T^n y$ is weakly convergent to 0). Moreover, S is also $(C, 1)$ bounded. We can now apply Proposition 3.3 to conclude that S is $(C, 1)$ weakly ergodic.

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