Linear mappings preserving similarity on $B(H)$

by

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Abstract. Let $H$ be an infinite-dimensional complex Hilbert space. We give a characterization of surjective linear mappings on $B(H)$ that preserve similarity in both directions.

The problem we consider is one of the so-called linear preserver problems that have attracted a lot of attention in recent decades. In these problems one is interested in characterizing linear mappings on some algebra of operators that leave certain functions, subsets, relations, etc., invariant. A lot of results of this kind can be found in the survey papers [9], [1], [2] and the references therein.

Let $M_n(F)$ denote the algebra of all $n \times n$ matrices with entries in $F$. Linear similarity-preserving mappings on $M_n(F)$ (i.e. such that the similarity of $A, B \in M_n(F)$ implies the similarity of $\phi(A)$ and $\phi(B)$) were characterized by Hiai [3] and Lim [10].

Theorem 1. Let $\phi$ be a linear mapping on $M_n(F)$, where $F$ is an infinite field such that $\text{char} \ F = 0$ or $\text{char} \ F$ does not divide $n$. Then $\phi$ is similarity-preserving if and only if one of the following holds:

(a) there exists $B \in M_n(F)$ such that
$$\phi(X) = \text{tr}(X)B, \quad X \in M_n(F);$$

(b) there exist a nonsingular matrix $A \in M_n(F)$ and $c, d \in F$ such that either
$$\phi(X) = cAXA^{-1} + d \text{tr}(X)I, \quad X \in M_n(F),$$
or
$$\phi(X) = cAX^tA^{-1} + d \text{tr}(X)I, \quad X \in M_n(F).$$

Here $\text{tr}(X)$ denotes the trace and $X^t$ the transpose. Similar problems of preserving other equivalence relations (unitary equivalence, consimilarity etc.) were considered in [4].

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It turns out that similarity preservers are connected with nilpotent preservers. If a linear map $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ preserves similarity, then it preserves nilpotents [10]. Indeed, if $N$ is nilpotent, then $N \sim 2N$, which implies $\phi(N) \sim 2\phi(N)$. Then the only eigenvalue of $\phi(N)$ is zero, and therefore $\phi(N)$ is nilpotent.

We will study similarity-preserving mappings on $B(H)$, the algebra of all bounded linear operators on the infinite-dimensional complex Hilbert space $H$. The operators $A$ and $B$ are similar ($A \sim B$) if there exists an invertible operator $S \in B(H)$ such that $A = SBS^{-1}$. We say that a linear mapping $\phi : B(H) \to B(H)$ preserves similarity if the similarity of any $A$ and $B$ implies the similarity of $\phi(A)$ and $\phi(B)$, and it preserves similarity in both directions whenever $A \sim B$ if and only if $\phi(A) \sim \phi(B)$. In the case that $\phi$ is bijective, $\phi$ preserves similarity in both directions if $\phi$ and $\phi^{-1}$ preserve similarity. The set of operators that are similar to a given $A$ will be called the similarity orbit and denoted by $S(A)$.

In this note, operators of rank one, especially nilpotents of rank one will be of importance. By $x \otimes y$ we denote the bounded linear operator acting as $(x \otimes y)z = \langle z, y \rangle x$, $x \in H$. Clearly, $x \otimes y$ is a nilpotent of rank one if and only if $x$ and $y$ are non-zero and $\langle x, y \rangle = 0$. It is also easy to see that all rank one nilpotents are similar to each other.

Let us now state our main theorem.

**Theorem 2.** Let $H$ be an infinite-dimensional complex Hilbert space. If a surjective linear map $\phi : B(H) \to B(H)$ preserves similarity in both directions then there exist a non-zero complex number $c$ and a bounded bijective linear operator $A : H \to H$ such that either

(i) $\phi(T) = cATA^{-1}$ for every $T \in B(H)$, or

(ii) $\phi(T) = cAT^tA^{-1}$ for every $T \in B(H)$,

where $T^t$ denotes the transpose of $T$ relative to a fixed but arbitrary orthonormal basis.

Many of the linear preserver problems can be translated to the problem of preserving rank one operators. For example, spectrum-preserving linear or additive maps on $B(X)$, the algebra of all bounded linear operators on a complex Banach space $X$, have been treated in [5] and [12] where the proof is based on spectrum characterizations of rank one operators. Similarly, in [14] a characterization of rank one nilpotents in terms of nilpotents was obtained.

We prove our theorem by characterizing rank one nilpotents in $B(H)$ in terms of similarity. From now on we assume that $\phi$ is as in Theorem 2. Let us first show that a surjective mapping $\phi$ that preserves similarity in both directions is bijective.
Lemma 3. If a linear map $\phi : B(H) \rightarrow B(H)$ preserves similarity in both directions, then it is one-to-one.

Proof. Suppose $\phi(X) = 0$. As $A \sim 0$ if and only if $A = 0$, we get $\phi(X) \sim 0 = \phi(0)$, which implies that $X \sim 0$, and hence $X = 0$. ■

In the finite-dimensional case, nilpotent matrices $N$ are exactly those that are similar to $2N$. For $N \in B(H)$ we will show that similarity of $N$ and $2N$ implies nilpotency of $N$. While the converse is also true if rank $N$ is finite we do not know whether it holds for nilpotents of infinite rank.

Lemma 4. Every $N \in B(H)$ which is similar to $2N$ is nilpotent.

Proof. Suppose $2N = S^{-1}NS$ for some invertible $S \in B(H)$. Then $2^kN^k = S^{-1}N^kS$ for every positive integer $k$. Taking the norms we arrive at

$$\left(2^k - \|S^{-1}\|\|S\|\right)\|N^k\| \leq 0$$

for every $k$, which implies that $N$ must be nilpotent. ■

In the next lemma we characterize rank one nilpotents in terms of similarity.

Lemma 5. For any non-zero operator $N \in B(H)$ the following assertions are equivalent.

(i) $N$ is a rank one nilpotent.

(ii) $N \sim 2N$, and for every $A \in S(N)$ which is not a multiple of $N$, if $A + N \sim N$ then $A + \lambda N \sim N$ for every $\lambda \in \mathbb{C}$.

Proof. Let $N = x \otimes y$ be a rank one nilpotent. Clearly, $N \sim 2N$. Let now $A \not\in \text{span}\{N\}$ be similar to $N$. This implies that $A$ is also a rank one nilpotent and can thus be written as $u \otimes v$ for some non-zero $u, v \in H$ satisfying $\langle u, v \rangle = 0$. Suppose $A + N \sim N$. From the fact that $A + N = u \otimes v + x \otimes y$ is a rank one operator it follows that either $x$ and $u$ are linearly dependent, or $v$ and $y$ are linearly dependent. Let $u = kx$ for some $k \neq 0$. Then $A + N = x \otimes (kv + y)$ and for every $\lambda \in \mathbb{C}$ we also have $A + \lambda N = x \otimes (kv + \lambda y)$, which is a non-zero rank one operator since $A$ is not proportional to $N$. Since $\langle x, kv + \lambda y \rangle = 0$ it follows that $A + \lambda N$ is a rank one nilpotent and thus similar to $N$. The case that $v$ and $y$ are linearly dependent is analogous. We have shown that a rank one nilpotent $N$ satisfies (ii).

Now, we prove that (ii) implies (i). If $N$ is not a rank one nilpotent and $N \sim 2N$ then it is a nilpotent of rank greater than 1. We will exhibit an $A \in S(N), A \not\in \text{span}\{N\}$, such that $A + N \sim N$, but $A - N \not\in S(N)$. 

First of all assume \( N^2 = 0 \) and decompose \( H = \text{Ker} \, N \oplus (\text{Ker} \, N)^\perp \).

Choose a non-zero \( y \in (\text{Ker} \, N)^\perp = Y \), let \( Y_0 = \text{span}\{y\} \) and split \( H = \text{Ker} \, N \oplus (Y \oplus Y_0) \oplus Y_0 \). The operator matrix form of \( N \) then is

\[
N = \begin{bmatrix}
0 & M_1 & m_2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}, \quad M_1 \neq 0, \ m_2 \neq 0.
\]

Note that \( m_2 \) is a rank one operator. Let

\[
A = \begin{bmatrix}
0 & M_1 & 2m_2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}.
\]

Clearly, \( A \) is bounded and similar to \( N \), for

\[
A = S^{-1}NS, \quad S = \begin{bmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & 2 \\
\end{bmatrix}.
\]

Also, the sum

\[
A + N = \begin{bmatrix}
0 & 2M_1 & 3m_2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} = (S')^{-1}NS', \quad S' = \begin{bmatrix}
I & 0 & 0 \\
0 & 2I & 0 \\
0 & 0 & 3 \\
\end{bmatrix},
\]

is similar to \( N \). However,

\[
A - N = \begin{bmatrix}
0 & 0 & m_2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

is not similar to \( N \) as \( \text{rank}(A - N) \) is one.

Assume now that \( N \) is a nilpotent of nilindex \( k > 2 \). Then there exists a \( u \neq 0 \) such that \( u, Nu, \ldots, N^{k-1}u \) are linearly independent. Let \( Y = \text{span}\{u, Nu, \ldots, N^{k-1}u\} \). According to the decomposition \( H = Y \oplus Y^\perp \) we can represent \( N \) as

\[
(1) \quad N = \begin{bmatrix}
J & N_2 \\
0 & N_3 \\
\end{bmatrix},
\]

where \( J \) is the transpose of a \( k \times k \) Jordan block.

Define \( S_X = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \) for any bounded linear operator \( X : Y^\perp \to Y \).

Evidently, \( S_X \) is invertible and \( S_X^{-1} = S_{-X} \). Let us compute
Let us show that there is a rank one operator $X$ such that $JX - XN_3 \neq 0$. If $N_3 = 0$ then take any unit vector $w \in Y^\perp$ and define $Xy = \langle y, w \rangle u$ for $y \in Y^\perp$. Then $(JX - XN_3)w = Ju = Nu \neq 0$. If $N_3 \neq 0$ there exists a unit vector $w \in \text{Im} N_3$ such that $w = N_3x$ for some $x \in Y^\perp$. Define $Xy = \langle y, w \rangle N^{k-1}u$ for $y \in Y^\perp$. Since $N^{k}u = 0$ we obtain
\[
(JX - XN_3)x = J\langle x, w \rangle N^{k-1}u - \langle N_3x, w \rangle N^{k-1}u = \langle x, w \rangle N^{k}u - N^{k-1}u = -N^{k-1}u \neq 0.
\]
Finally, by choosing
\[
A = N + \begin{bmatrix} 0 & 2JX - 2XN_3 \\ 0 & 0 \end{bmatrix} = S_{2X}^{-1}NS_{2X}
\]
we achieve that
\[
A + N = S_{X}^{-1}2NS_{X} \sim 2N \sim N
\]
but $A - N$ cannot be similar to $N$ as $(A - N)^2 = 0$ while $N^2 \neq 0$. ■

Let us see how we can make use of this characterization. We will show that $\phi$ preserves rank one nilpotents in both directions. Let $N$ be a nilpotent of rank one. Then $N \sim 2N$, which implies that $\phi(N)$ is similar to $2\phi(N)$, and consequently, $\phi(N)$ is nilpotent. Let $A \in B(H)$, $A \notin \text{span}\{\phi(N)\}$, $A \sim \phi(N)$ and $A + \phi(N) \sim \phi(N)$. Since $\phi$ is bijective there exists a $B \in B(H)$ such that $\phi(B) = A$ and $B$ is not a multiple of $N$. As $\phi$ preserves similarity in both directions, $B \sim 2B$ and therefore $B$ is nilpotent. The relations $A \sim \phi(N)$, $A + \phi(N) \sim \phi(N)$, linearity of $\phi$ and the fact that $\phi$ preserves similarity in both directions imply that $B \sim N$ and $B + N \sim N$. By Lemma 5(ii), $B + \lambda N \sim N$ for every $\lambda \in \mathbb{C}$. Thus $\phi(B + \lambda N) = A + \lambda \phi(N) \sim \phi(N)$ for every $\lambda \in \mathbb{C}$, which shows that $\phi(N)$ is a rank one nilpotent. As $\phi$ is bijective and preserves similarity in both directions, it preserves rank one nilpotents in both directions.

Before we proceed we need some more technical lemmas.

**Lemma 6.** Let $K$ be a complex Hilbert space (finite- or infinite-dimensional) and $B \in B(K)$. The following are equivalent:
(1) There is no finite-rank nilpotent \( N \in B(K) \) with nilindex 2 such that \( B + N \) is similar to \( B \).

(2) \( B = \beta I \) for some \( \beta \in \mathbb{C} \).

Proof. Clearly (2) implies (1). Suppose now that \( B \) is not scalar. Then obviously, \( \dim K > 1 \). If \( K \) is of finite dimension then up to similarity we can represent \( B \) in its Jordan form as

\[
B = \begin{bmatrix}
J_1 & 0 \\
0 & J_2
\end{bmatrix},
\]

where \( J_1 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \), \( a \neq b \), or \( J_1 \) is a \( k \times k \) Jordan block, \( 2 \leq k \leq \dim K \), and the matrix \( J_2 \) may be missing. In both cases, taking as \( N \) the matrix having 1 in (1,2)-position and zeros elsewhere gives the similarity of \( B + N \) and \( B \). Note that \( N \) is a rank one square-zero matrix.

Let now \( K \) be infinite-dimensional. As \( B \) is not scalar there exists \( x \in K \) such that \( x \) and \( Bx \) are linearly independent. Denote by \( U \) the orthogonal complement of \( \text{span}\{x,Bx\} \) in \( K \). According to this decomposition \( B \) has now the form

\[
B = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0 & * \\ 1 & * \end{bmatrix}, \quad B_{21}x = 0.
\]

Let us choose a unit vector \( y \in (\text{Im} B_{21})^\perp \) and define the operators

\[
X : U \to \text{span}\{x,Bx\}, \quad Xu = \langle u, y \rangle x, \quad S = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}.
\]

Then

\[
S^{-1}BS = B + \begin{bmatrix}
-XB_{21} & B_{11}X - XB_{21}X - XB_{22} \\
0 & B_{21}X
\end{bmatrix}.
\]

Simple calculation shows that \( XB_{21} = 0 \) and \( B_{21}X = 0 \), so

\[
S^{-1}BS = B + N, \quad N = \begin{bmatrix} 0 & B_{11}X - XB_{22} \\ 0 & 0 \end{bmatrix}.
\]

Applying linear independence of \( x \) and \( Bx \) we observe that \( N \) is non-zero,

\[
(B_{11}X - XB_{22})y = \langle y, y \rangle B_{11}x - \langle B_{22}y, y \rangle x = Bx - \langle B_{22}y, y \rangle x,
\]

and since \( N \) is a sum of two operators of rank at most one, its rank cannot exceed 2. Clearly, \( N^2 = 0 \). So, we have found a non-zero square-zero nilpotent \( N \) of rank at most 2 such that \( B + N \) is similar to \( B \). □

Lemma 7. Let \( P \in B(H) \) be a non-trivial \((\neq 0, I)\) idempotent and \( B \) a bounded linear operator on \( H \). If for every finite rank nilpotent \( N \),

\[
(2) \quad P + N \sim P \quad \text{if and only if} \quad B + N \sim B,
\]
then there exist constants $\alpha$, $\beta$, $\alpha \neq 0$, such that

$$B = \alpha P + \beta I.$$  

**Proof.** It is easy to see that every idempotent is similar to a projection. Therefore, there is no loss of generality in assuming that $P$ is a non-trivial projection. There exists a decomposition $H = U \oplus V$ into closed orthogonal non-trivial subspaces $U$ and $V$ such that

$$P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$  

We first show that if (2) holds then $B_{12}$ and $B_{21}$ must be zero. This is equivalent to the fact that $U$ and $V$ are invariant subspaces for $B$. Choose non-zero $e \in U$ and $f \in V$. Then $e \otimes f$ is a rank one nilpotent. Let $\beta \in \mathbb{C}$ and define

$$X_\beta : V \to U, \quad X_\beta v = \beta (e \otimes f)v = \beta \langle v, f \rangle e, \quad v \in V,$$

and

$$S_\beta = \begin{bmatrix} I & X_\beta \\ 0 & I \end{bmatrix}.$$  

It is easy to see that $P + \beta e \otimes f = S_\beta^{-1} P S_\beta$ and so, by (2), $B + \beta e \otimes f$ is similar to $B$ for every $\beta$. We will now use Lemma 4 from [5] which states that if $\alpha \not\in \sigma(B)$ then

$$\alpha$$ is an eigenvalue of $B + e \otimes f \iff \langle (\alpha I - B)^{-1} e, f \rangle = 1.$$  

Choose $\alpha > \lVert B \rVert$ and suppose that $\langle (\alpha I - B)^{-1} e, f \rangle \neq 0$. There exists a non-zero constant $\beta$ such that $\langle (\alpha I - B)^{-1} \beta e, f \rangle = 1$. This shows that $\alpha$ is an eigenvalue of $B + \beta e \otimes f$ and leads to a contradiction with $B + \beta e \otimes f \sim B$. So, for every $e \in U$ and $f \in V$ we have $\langle (\alpha I - B)^{-1} e, f \rangle = 0$. It follows that $(\alpha I - B)^{-1} U \subseteq V^\perp = U$, so $U$ is invariant for $(\alpha I - B)^{-1}$. Taking $e \in V$ and $f \in U$ and applying similar arguments we find that $V$ is also invariant for $(\alpha I - B)^{-1}$ for all $\alpha > \lVert B \rVert$. Therefore, $U$ and $V$ are both invariant for $\alpha I - B$, and finally for $B$. The operator $B$ has now the form

$$B = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix}.$$  

It remains to show that $B_{11}$ and $B_{22}$ are multiples of identity. We will prove this only for $B_{11} \in B(U)$; the argument for $B_{22}$ is similar. Assuming that $B_{11}$ is not scalar and applying Lemma 6 we can find a rank one or two nilpotent $N \in B(U)$, $N^2 = 0$, such that $B_{11} + N \sim B_{11}$, and consequently $B + (N \oplus 0) \sim B$; but $P + (N \oplus 0)$ is not idempotent, and therefore it cannot be similar to $P$. The fact that $\alpha \neq 0$ follows from non-triviality of $P$. \blacksquare
We already know that \( \phi \) preserves rank one nilpotents in both directions. By standard methods (see [14, pp. 531–533]) one can find a bounded invertible linear operator \( A \) and a non-zero constant \( c \) such that either

\[
\phi(x \otimes y) = cA(x \otimes y)A^{-1} \quad \text{for all } x, y \in H, \quad \langle x, y \rangle = 0,
\]

or

\[
\phi(x \otimes y) = cA(y \otimes x)A^{-1} \quad \text{for all } x, y \in H, \quad \langle x, y \rangle = 0.
\]

By linearity of \( \phi \) and by composing \( \phi \) with transposition if necessary, we may now assume with no loss of generality that

\[
\phi(N) = N
\]

for all finite rank nilpotents \( N \in B(H) \).

Let us show that for every non-trivial idempotent \( P \) there exist constants \( \alpha \) and \( \beta \) such that \( \phi(P) = \alpha P + \beta I \). Let \( N \) be any finite rank nilpotent. Then

\[
P + N \sim P \iff \phi(P) + \phi(N) \sim \phi(P) \iff \phi(P) + N \sim \phi(P),
\]

and Lemma 7 gives the desired conclusion.

Next, we show that \( \alpha = 1 \) for every finite rank idempotent \( P \). As \( P \) is not scalar there exists a finite rank nilpotent \( N \) such that \( P + N \sim P \). Therefore

\[
\phi(P) = \alpha P + \beta I, \quad \phi(P + N) = \alpha'(P + N) + \beta'I,
\]

and, on the other hand,

\[
\phi(N) = N = \phi(P + N) - \phi(P) = \alpha'N + (\alpha' - \alpha)P + (\beta' - \beta)I
\]

and \((1 - \alpha')N = (\alpha' - \alpha)P + (\beta' - \beta)I\), which implies \( \alpha = \alpha' = 1, \beta' = \beta \). We have thus arrived at

\[
\phi(P) = P + k\beta I
\]

for every rank \( k \) idempotent \( P \).

We also have \( \phi(P) = P + \mu P I \) for any idempotent \( P \neq I \) of infinite rank. To verify this, choose any non-zero finite-rank idempotent \( Q \) satisfying \( PQ = QP = Q \). Then \( P - Q \) is a non-trivial idempotent. So, there exist constants \( \alpha_1, \alpha_2, \mu_1, \mu_2, \mu_3 \) such that

\[
\phi(Q) = Q + \mu_1 I, \\
\phi(P) = \alpha_1 P + \mu_2 I, \\
\phi(P - Q) = \alpha_2(P - Q) + \mu_3 I.
\]

Applying linearity of \( \phi \) gives \( \alpha_1 = \alpha_2 = 1 \), and so

\[
\phi(P) = P + \mu P I
\]

for every non-trivial idempotent \( P \).
Finally, we use the result of Pearcy and Topping [13] which states that every operator on $B(H)$ is a sum of five idempotents, and infer that $\phi(X) = X + f(X)I$ for every $X \in B(H)$ and some linear functional $f$ on $B(H)$. Every operator on $B(H)$ is also a finite sum of square-zero operators [13]. Verifying that $f(N) = 0$ if $N^2 = 0$ then yields $\phi(X) = X$ for every $X \in B(H)$. Let us now show that every square-zero operator $N$ is similar to $2N$. If we decompose $H = \text{Ker} N \oplus (\text{Ker} N)^\perp$, then the nilpotent $N$ takes the form

$$N = \begin{bmatrix} 0 & N_1 \\ 0 & 0 \end{bmatrix}$$

as $\text{Im} N \subseteq \text{Ker} N$. Taking $S = I \oplus 2I$ we obtain $S^{-1}NS = 2N$, so $N \sim 2N$. Thus $\phi(N) \sim 2\phi(N)$, $\phi(N)$ is nilpotent, and therefore $f(N) = 0$. This completes the proof of Theorem 2.

Let us finally note that Theorem 2 does not hold with the Hilbert space $H$ replaced by a Banach space $X$. Recall that there exists an infinite-dimensional Banach space $X$ such that the algebra $B(X)$ has a non-zero multiplicative linear functional $f$ [11, 15]. Obviously, the mapping $\phi : B(X) \to B(X)$ defined by $\phi(X) = X + f(X)I$ preserves similarity in both directions.

Recently, Ji and Du [8] obtained a similar characterization of mappings preserving similarity in both directions, but their assumptions were stronger: they assumed that $H$ is also separable and that $\phi$ is bounded. Related material can also be found in [6, 7].

References


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