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Linear mappings preserving similarity on $B(H)$

by

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Abstract. Let H be an infinite-dimensional complex Hilbert space. We give a characterization of surjective linear mappings on $B(H)$ that preserve similarity in both directions.

The problem we consider is one of the so-called linear preserver problems that have attracted a lot of attention in recent decades. In these problems one is interested in characterizing linear mappings on some algebra of operators that leave certain functions, subsets, relations, etc., invariant. A lot of results of this kind can be found in the survey papers [9], [1], [2] and the references therein.

Let $M_n(\mathbb{F})$ denote the algebra of all $n \times n$ matrices with entries in $\mathbb{F}$. Linear similarity-preserving mappings on $M_n(\mathbb{F})$ (i.e. such that the similarity of $A, B \in M_n(\mathbb{F})$ implies the similarity of $\phi(A)$ and $\phi(B)$) were characterized by Hiai [3] and Lim [10].

THEOREM 1. *Let ϕ be a linear mapping on $M_n(\mathbb{F})$, where $\mathbb{F}$ is an infinite field such that* $\operatorname{char} \mathbb{F} = 0$ *or* $\operatorname{char} \mathbb{F}$ *does not divide n. Then ϕ is similarity-preserving if and only if one of the following holds*:

(a) *there exists $B \in M_n(\mathbb{F})$ such that*
$$\phi(X) = \operatorname{tr}(X)B, \qquad X \in M_n(\mathbb{F});$$

(b) *there exist a nonsingular matrix $A \in M_n(\mathbb{F})$ and $c, d \in \mathbb{F}$ such that either*
$$\phi(X) = cAXA^{-1} + d\operatorname{tr}(X)I, \qquad X \in M_n(\mathbb{F}),$$
or
$$\phi(X) = cAX^tA^{-1} + d\operatorname{tr}(X)I, \qquad X \in M_n(\mathbb{F}).$$

Here $\operatorname{tr}(X)$ denotes the trace and X^t the transpose. Similar problems of preserving other equivalence relations (unitary equivalence, consimilarity etc.) were considered in [4].

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It turns out that similarity preservers are connected with nilpotent preservers. If a linear map $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ preserves similarity, then it preserves nilpotents [10]. Indeed, if N is nilpotent, then $N \sim 2N$, which implies $\phi(N) \sim 2\phi(N)$. Then the only eigenvalue of $\phi(N)$ is zero, and therefore $\phi(N)$ is nilpotent.

We will study similarity-preserving mappings on $B(H)$, the algebra of all bounded linear operators on the infinite-dimensional complex Hilbert space H. The operators A and B are *similar* ($A \sim B$) if there exists an invertible operator $S \in B(H)$ such that $A = SBS^{-1}$. We say that a linear mapping $\phi : B(H) \to B(H)$ *preserves similarity* if the similarity of any A and B implies the similarity of $\phi(A)$ and $\phi(B)$, and it *preserves similarity in both directions* whenever $A \sim B$ if and only if $\phi(A) \sim \phi(B)$. In the case that ϕ is bijective, ϕ preserves similarity in both directions if ϕ and ϕ^{-1} preserve similarity. The set of operators that are similar to a given A will be called the *similarity orbit* and denoted by $\mathcal{S}(A)$.

In this note, operators of rank one, especially nilpotents of rank one will be of importance. By $x \otimes y$ we denote the bounded linear operator acting as $(x \otimes y)z = \langle z, y\rangle x$, $x \in H$. Clearly, $x \otimes y$ is a nilpotent of rank one if and only if x and y are non-zero and $\langle x, y\rangle = 0$. It is also easy to see that all rank one nilpotents are similar to each other.

Let us now state our main theorem.

THEOREM 2. *Let H be an infinite-dimensional complex Hilbert space. If a surjective linear map $\phi : B(H) \to B(H)$ preserves similarity in both directions then there exist a non-zero complex number c and a bounded bijective linear operator $A : H \to H$ such that either*

(i) $\phi(T) = cATA^{-1}$ *for every* $T \in B(H)$, *or*
(ii) $\phi(T) = cAT^tA^{-1}$ *for every* $T \in B(H)$,

where T^t denotes the transpose of T relative to a fixed but arbitrary orthonormal basis.

Many of the linear preserver problems can be translated to the problem of preserving rank one operators. For example, spectrum-preserving linear or additive maps on $B(X)$, the algebra of all bounded linear operators on a complex Banach space X, have been treated in [5] and [12] where the proof is based on spectrum characterizations of rank one operators. Similarly, in [14] a characterization of rank one nilpotents in terms of nilpotents was obtained.

We prove our theorem by characterizing rank one nilpotents in $B(H)$ in terms of similarity. From now on we assume that ϕ is as in Theorem 2. Let us first show that a surjective mapping ϕ that preserves similarity in both directions is bijective.

LEMMA 3. *If a linear map $\phi : B(H) \to B(H)$ preserves similarity in both directions, then it is one-to-one.*

Proof. Suppose $\phi(X) = 0$. As $A \sim 0$ if and only if $A = 0$, we get $\phi(X) \sim 0 = \phi(0)$, which implies that $X \sim 0$, and hence $X = 0$. ∎

In the finite-dimensional case, nilpotent matrices N are exactly those that are similar to $2N$. For $N \in B(H)$ we will show that similarity of N and $2N$ implies nilpotency of N. While the converse is also true if rank N is finite we do not know whether it holds for nilpotents of infinite rank.

LEMMA 4. *Every $N \in B(H)$ which is similar to $2N$ is nilpotent.*

Proof. Suppose $2N = S^{-1}NS$ for some invertible $S \in B(H)$. Then $2^k N^k = S^{-1} N^k S$ for every positive integer k. Taking the norms we arrive at

$$(2^k - \|S^{-1}\| \, \|S\|)\|N^k\| \le 0 \quad \text{for every } k,$$

which implies that N must be nilpotent. ∎

In the next lemma we characterize rank one nilpotents in terms of similarity.

LEMMA 5. *For any non-zero operator $N \in B(H)$ the following assertions are equivalent.*

(i) *N is a rank one nilpotent.*

(ii) *$N \sim 2N$, and for every $A \in \mathcal{S}(N)$ which is not a multiple of N, if $A + N \sim N$ then $A + \lambda N \sim N$ for every $\lambda \in \mathbb{C}$.*

Proof. Let $N = x \otimes y$ be a rank one nilpotent. Clearly, $N \sim 2N$. Let now $A \notin \operatorname{span}\{N\}$ be similar to N. This implies that A is also a rank one nilpotent and can thus be written as $u \otimes v$ for some non-zero $u, v \in H$ satisfying $\langle u, v \rangle = 0$. Suppose $A + N \sim N$. From the fact that $A + N = u \otimes v + x \otimes y$ is a rank one operator it follows that either x and u are linearly dependent, or v and y are linearly dependent. Let $u = kx$ for some $k \neq 0$. Then $A + N = x \otimes (kv + y)$ and for every $\lambda \in \mathbb{C}$ we also have $A + \lambda N = x \otimes (kv + \lambda y)$, which is a non-zero rank one operator since A is not proportional to N. Since $\langle x, kv + \lambda y \rangle = 0$ it follows that $A + \lambda N$ is a rank one nilpotent and thus similar to N. The case that v and y are linearly dependent is analogous. We have shown that a rank one nilpotent N satisfies (ii).

Now, we prove that (ii) implies (i). If N is not a rank one nilpotent and $N \sim 2N$ then it is a nilpotent of rank greater than 1. We will exhibit an $A \in \mathcal{S}(N)$, $A \notin \operatorname{span}\{N\}$, such that $A + N \sim N$, but $A - N \notin \mathcal{S}(N)$.

First of all assume $N^2 = 0$ and decompose $H = \operatorname{Ker} N \oplus (\operatorname{Ker} N)^\perp$. Choose a non-zero $y \in (\operatorname{Ker} N)^\perp = Y$, let $Y_0 = \operatorname{span}\{y\}$ and split $H = \operatorname{Ker} N \oplus (Y \ominus Y_0) \oplus Y_0$. The operator matrix form of N then is

$$N = \begin{bmatrix} 0 & M_1 & m_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad M_1 \neq 0,\ m_2 \neq 0.$$

Note that m_2 is a rank one operator. Let

$$A = \begin{bmatrix} 0 & M_1 & 2m_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly, A is bounded and similar to N, for

$$A = S^{-1}NS, \qquad S = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Also, the sum

$$A + N = \begin{bmatrix} 0 & 2M_1 & 3m_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = (S')^{-1}NS', \qquad S' = \begin{bmatrix} I & 0 & 0 \\ 0 & 2I & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

is similar to N. However,

$$A - N = \begin{bmatrix} 0 & 0 & m_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is not similar to N as $\operatorname{rank}(A - N)$ is one.

Assume now that N is a nilpotent of nilindex $k > 2$. Then there exists a $u \neq 0$ such that $u, Nu, \ldots, N^{k-1}u$ are linearly independent. Let $Y = \operatorname{span}\{u, Nu, \ldots, N^{k-1}u\}$. According to the decomposition $H = Y \oplus Y^\perp$ we can represent N as

$$N = \begin{bmatrix} J & N_2 \\ 0 & N_3 \end{bmatrix}, \tag{1}$$

where J is the transpose of a $k \times k$ Jordan block.

Define $S_X = \left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right]$ for any bounded linear operator $X : Y^\perp \to Y$. Evidently, S_X is invertible and $S_X^{-1} = S_{-X}$. Let us compute

$$S_X^{-1} N S_X = N + \begin{bmatrix} 0 & JX - XN_3 \\ 0 & 0 \end{bmatrix},$$
$$S_{2X}^{-1} N S_{2X} = N + \begin{bmatrix} 0 & 2JX - 2XN_3 \\ 0 & 0 \end{bmatrix},$$
$$S_X^{-1} 2N S_X = 2N + \begin{bmatrix} 0 & 2JX - 2XN_3 \\ 0 & 0 \end{bmatrix}.$$

Let us show that there is a rank one operator X such that $JX - XN_3 \neq 0$. If $N_3 = 0$ then take any unit vector $w \in Y^\perp$ and define $Xy = \langle y, w\rangle u$ for $y \in Y^\perp$. Then $(JX - XN_3)w = Ju = Nu \neq 0$. If $N_3 \neq 0$ there exists a unit vector $w \in \operatorname{Im} N_3$ such that $w = N_3 x$ for some $x \in Y^\perp$. Define $Xy = \langle y, w\rangle N^{k-1}u$ for $y \in Y^\perp$. Since $N^k u = 0$ we obtain

$$\begin{aligned}(JX - XN_3)x &= J\langle x, w\rangle N^{k-1}u - \langle N_3 x, w\rangle N^{k-1}u \\ &= \langle x, w\rangle N^k u - N^{k-1}u = -N^{k-1}u \neq 0.\end{aligned}$$

Finally, by choosing

$$A = N + \begin{bmatrix} 0 & 2JX - 2XN_3 \\ 0 & 0 \end{bmatrix} = S_{2X}^{-1} N S_{2X}$$

we achieve that

$$A + N = S_X^{-1} 2N S_X \sim 2N \sim N$$

but $A - N$ cannot be similar to N as $(A - N)^2 = 0$ while $N^2 \neq 0$. ■

Let us see how we can make use of this characterization. We will show that ϕ preserves rank one nilpotents in both directions. Let N be a nilpotent of rank one. Then $N \sim 2N$, which implies that $\phi(N)$ is similar to $2\phi(N)$, and consequently, $\phi(N)$ is nilpotent. Let $A \in B(H)$, $A \notin \operatorname{span}\{\phi(N)\}$, $A \sim \phi(N)$ and $A + \phi(N) \sim \phi(N)$. Since ϕ is bijective there exists a $B \in B(H)$ such that $\phi(B) = A$ and B is not a multiple of N. As ϕ preserves similarity in both directions, $B \sim 2B$ and therefore B is nilpotent. The relations $A \sim \phi(N)$, $A + \phi(N) \sim \phi(N)$, linearity of ϕ and the fact that ϕ preserves similarity in both directions imply that $B \sim N$ and $B + N \sim N$. By Lemma 5(ii), $B + \lambda N \sim N$ for every $\lambda \in \mathbb{C}$. Thus $\phi(B + \lambda N) = A + \lambda\phi(N) \sim \phi(N)$ for every $\lambda \in \mathbb{C}$, which shows that $\phi(N)$ is a rank one nilpotent. As ϕ is bijective and preserves similarity in both directions, it preserves rank one nilpotents in both directions.

Before we proceed we need some more technical lemmas.

Lemma 6. *Let K be a complex Hilbert space* (*finite- or infinite-dimensional*) *and $B \in B(K)$. The following are equivalent*:

(1) *There is no finite-rank nilpotent* $N \in B(K)$ *with nilindex* 2 *such that* $B+N$ *is similar to* B.

(2) $B = \beta I$ *for some* $\beta \in \mathbb{C}$.

Proof. Clearly (2) implies (1). Suppose now that B is not scalar. Then obviously, $\dim K > 1$. If K is of finite dimension then up to similarity we can represent B in its Jordan form as

$$B = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix},$$

where $J_1 = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$, $a \neq b$, or J_1 is a $k \times k$ Jordan block, $2 \leq k \leq \dim K$, and the matrix J_2 may be missing. In both cases, taking as N the matrix having 1 in $(1,2)$-position and zeros elsewhere gives the similarity of $B+N$ and B. Note that N is a rank one square-zero matrix.

Let now K be infinite-dimensional. As B is not scalar there exists $x \in K$ such that x and Bx are linearly independent. Denote by U the orthogonal complement of $\operatorname{span}\{x, Bx\}$ in K. According to this decomposition B has now the form

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \qquad B_{11} = \begin{bmatrix} 0 & * \\ 1 & * \end{bmatrix}, \qquad B_{21}x = 0.$$

Let us choose a unit vector $y \in (\operatorname{Im} B_{21})^{\perp}$ and define the operators

$$X : U \to \operatorname{span}\{x, Bx\}, \qquad Xu = \langle u, y \rangle x, \qquad S = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}.$$

Then

$$S^{-1}BS = B + \begin{bmatrix} -XB_{21} & B_{11}X - XB_{21}X - XB_{22} \\ 0 & B_{21}X \end{bmatrix}.$$

Simple calculation shows that $XB_{21} = 0$ and $B_{21}X = 0$, so

$$S^{-1}BS = B + N, \qquad N = \begin{bmatrix} 0 & B_{11}X - XB_{22} \\ 0 & 0 \end{bmatrix}.$$

Applying linear independence of x and Bx we observe that N is non-zero,

$$(B_{11}X - XB_{22})y = \langle y, y \rangle B_{11}x - \langle B_{22}y, y \rangle x = Bx - \langle B_{22}y, y \rangle x,$$

and since N is a sum of two operators of rank at most one, its rank cannot exceed 2. Clearly, $N^2 = 0$. So, we have found a non-zero square-zero nilpotent N of rank at most 2 such that $B+N$ is similar to B. ∎

LEMMA 7. *Let* $P \in B(H)$ *be a non-trivial* $(\neq 0, I)$ *idempotent and* B *a bounded linear operator on* H. *If for every finite rank nilpotent* N,

$$P + N \sim P \quad \text{if and only if} \quad B + N \sim B, \tag{2}$$

then there exist constants α, β, $\alpha \neq 0$, *such that*

$$B = \alpha P + \beta I.$$

Proof. It is easy to see that every idempotent is similar to a projection. Therefore, there is no loss of generality in assuming that P is a non-trivial projection. There exists a decomposition $H = U \oplus V$ into closed orthogonal non-trivial subspaces U and V such that

$$P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

We first show that if (2) holds then B_{12} and B_{21} must be zero. This is equivalent to the fact that U and V are invariant subspaces for B. Choose non-zero $e \in U$ and $f \in V$. Then $e \otimes f$ is a rank one nilpotent. Let $\beta \in \mathbb{C}$ and define

$$X_\beta : V \to U, \qquad X_\beta v = \beta(e \otimes f)v = \beta\langle v, f\rangle e, \qquad v \in V,$$

and

$$S_\beta = \begin{bmatrix} I & X_\beta \\ 0 & I \end{bmatrix}.$$

It is easy to see that $P + \beta e \otimes f = S_\beta^{-1} P S_\beta$ and so, by (2), $B + \beta e \otimes f$ is similar to B for every β. We will now use Lemma 4 from [5] which states that if $\alpha \notin \sigma(B)$ then

$$\alpha \text{ is an eigenvalue of } B + e \otimes f \;\Leftrightarrow\; \langle(\alpha I - B)^{-1}e, f\rangle = 1.$$

Choose $\alpha > \|B\|$ and suppose that $\langle(\alpha I - B)^{-1}e, f\rangle \neq 0$. There exists a non-zero constant β such that $\langle(\alpha I - B)^{-1}\beta e, f\rangle = 1$. This shows that α is an eigenvalue of $B + \beta e \otimes f$ and leads to a contradiction with $B + \beta e \otimes f \sim B$. So, for every $e \in U$ and $f \in V$ we have $\langle(\alpha I - B)^{-1}e, f\rangle = 0$. It follows that $(\alpha I - B)^{-1}U \subseteq V^\perp = U$, so U is invariant for $(\alpha I - B)^{-1}$. Taking $e \in V$ and $f \in U$ and applying similar arguments we find that V is also invariant for $(\alpha I - B)^{-1}$ for all $\alpha > \|B\|$. Therefore, U and V are both invariant for $\alpha I - B$, and finally for B. The operator B has now the form

$$B = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix}.$$

It remains to show that B_{11} and B_{22} are multiples of identity. We will prove this only for $B_{11} \in B(U)$; the argument for B_{22} is similar. Assuming that B_{11} is not scalar and applying Lemma 6 we can find a rank one or two nilpotent $N \in B(U)$, $N^2 = 0$, such that $B_{11} + N \sim B_{11}$, and consequently $B + (N \oplus 0) \sim B$; but $P + (N \oplus 0)$ is not idempotent, and therefore it cannot be similar to P. The fact that $\alpha \neq 0$ follows from non-triviality of P. ∎

We already know that ϕ preserves rank one nilpotents in both directions. By standard methods (see [14, pp. 531–533]) one can find a bounded invertible linear operator A and a non-zero constant c such that either

$$\phi(x \otimes y) = cA(x \otimes y)A^{-1} \quad \text{for all } x, y \in H,\ \langle x, y\rangle = 0,$$

or

$$\phi(x \otimes y) = cA(y \otimes x)A^{-1} \quad \text{for all } x, y \in H,\ \langle x, y\rangle = 0.$$

By linearity of ϕ and by composing ϕ with transposition if necessary, we may now assume with no loss of generality that

$$\phi(N) = N$$

for all finite rank nilpotents $N \in B(H)$.

Let us show that for every non-trivial idempotent P there exist constants α and β such that $\phi(P) = \alpha P + \beta I$. Let N be any finite rank nilpotent. Then

$$\begin{aligned} P + N \sim P \;&\Leftrightarrow\; \phi(P) + \phi(N) \sim \phi(P) \\ &\Leftrightarrow\; \phi(P) + N \sim \phi(P), \end{aligned}$$

and Lemma 7 gives the desired conclusion.

Next, we show that $\alpha = 1$ for every finite rank idempotent P. As P is not scalar there exists a finite rank nilpotent N such that $P + N \sim P$. Therefore

$$\phi(P) = \alpha P + \beta I, \qquad \phi(P + N) = \alpha'(P + N) + \beta' I,$$

and, on the other hand,

$$\begin{aligned} \phi(N) = N &= \phi(P + N) - \phi(P) \\ &= \alpha' N + (\alpha' - \alpha)P + (\beta' - \beta)I \end{aligned}$$

and $(1 - \alpha')N = (\alpha' - \alpha)P + (\beta' - \beta)I$, which implies $\alpha = \alpha' = 1$, $\beta' = \beta$. We have thus arrived at

$$\phi(P) = P + k\beta I$$

for every rank k idempotent P.

We also have $\phi(P) = P + \mu_P I$ for any idempotent $P \neq I$ of infinite rank. To verify this, choose any non-zero finite-rank idempotent Q satisfying $PQ = QP = Q$. Then $P - Q$ is a non-trivial idempotent. So, there exist constants α_1, α_2, μ_1, μ_2, μ_3 such that

$$\begin{aligned} \phi(Q) &= Q + \mu_1 I, \\ \phi(P) &= \alpha_1 P + \mu_2 I, \\ \phi(P - Q) &= \alpha_2(P - Q) + \mu_3 I. \end{aligned}$$

Applying linearity of ϕ gives $\alpha_1 = \alpha_2 = 1$, and so

$$\phi(P) = P + \mu_P I$$

for every non-trivial idempotent P.

Finally, we use the result of Pearcy and Topping [13] which states that every operator on $B(H)$ is a sum of five idempotents, and infer that $\phi(X) = X + f(X)I$ for every $X \in B(H)$ and some linear functional f on $B(H)$. Every operator on $B(H)$ is also a finite sum of square-zero operators [13]. Verifying that $f(N) = 0$ if $N^2 = 0$ then yields $\phi(X) = X$ for every $X \in B(H)$. Let us now show that every square-zero operator N is similar to $2N$. If we decompose $H = \operatorname{Ker} N \oplus (\operatorname{Ker} N)^{\perp}$, then the nilpotent N takes the form

$$N = \begin{bmatrix} 0 & N_1 \\ 0 & 0 \end{bmatrix}$$

as $\operatorname{Im} N \subseteq \operatorname{Ker} N$. Taking $S = I \oplus 2I$ we obtain $S^{-1}NS = 2N$, so $N \sim 2N$. Thus $\phi(N) \sim 2\phi(N)$, $\phi(N)$ is nilpotent, and therefore $f(N) = 0$. This completes the proof of Theorem 2.

Let us finally note that Theorem 2 does not hold with the Hilbert space H replaced by a Banach space X. Recall that there exists an infinite-dimensional Banach space X such that the algebra $B(X)$ has a non-zero multiplicative linear functional f [11, 15]. Obviously, the mapping $\phi : B(X) \to B(X)$ defined by $\phi(X) = X + f(X)I$ preserves similarity in both directions.

Recently, Ji and Du [8] obtained a similar characterization of mappings preserving similarity in both directions, but their assumptions were stronger: they assumed that H is also separable and that ϕ is bounded. Related material can also be found in [6, 7].

References

[1] M. Brešar and P. Šemrl, *Linear preservers on B(X)*, in: Banach Center Publ. 38, Inst. Math., Polish Acad. Sci., 1997, 49–58.

[2] A. Guterman, C.-K. Li and P. Šemrl, *Some general techniques on linear preserver problems*, Linear Algebra Appl. 315 (2000), 61–81.

[3] F. Hiai, *Similarity preserving linear maps on matrices*, ibid. 97 (1987), 127–139.

[4] R. Horn, C.-K. Li and N.-K. Tsing, *Linear operators preserving certain equivalence relations on matrices*, SIAM J. Matrix Anal. Appl. 12 (1991), 195–204.

[5] A. A. Jafarian and A. R. Sourour, *Spectrum-preserving linear maps*, J. Funct. Anal. 66 (1986), 225–261.

[6] G. X. Ji, *Similarity-preserving linear maps on $\mathcal{B}(\mathcal{H})$*, Linear Algebra Appl. 360 (2003), 249–257.

[7] —, *Asymptotic similarity-preserving linear maps on $\mathcal{B}(\mathcal{H})$*, ibid. 368 (2003), 371–378.

[8] G. X. Ji and H. K. Du, *Similarity-invariant subspaces and similarity-preserving linear maps*, Acta Math. Sinica 18 (2002), 489–498.

[9] C.-K. Li and N.-K. Tsing, *Linear preserver problems: A brief introduction and some special techniques*, Linear Algebra Appl. 162–164 (1992), 217–235.

[10] M. H. Lim, *A note on similarity preserving linear maps on matrices*, ibid. 190 (1993) 229–233.

[11] B. S. Mityagin and I. S. Edel'shteĭn, *Homotopy type of linear groups for two classes of Banach algebras*, Funktsional. Anal. i Prilozhen. 4 (1970), no. 3, 61–72 (in Russian).
[12] M. Omladič and P. Šemrl, *Spectrum-preserving additive maps*, Linear Algebra Appl. 153 (1991), 67–72.
[13] C. Pearcy and D. Topping, *Sums of small numbers of idempotents*, Michigan Math. J. 14 (1967), 453–465.
[14] P. Šemrl, *Linear maps that preserve the nilpotent operators*, Acta Sci. Math. (Szeged) 61 (1995), 523–534.
[15] A. Wilansky, *Subalgebras of* $B(X)$, Proc. Amer. Math. Soc. 29 (1971), 355–360.

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