On the Kunen–Shelah properties in Banach spaces

by

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Abstract. We introduce and study the Kunen–Shelah properties KS_i, i = 0, 1, ..., 7.
Let us highlight some of our results for a Banach space X; (1) X* has a w*-nonseparable
equivalent dual ball iff X has an ω_1-polyhedron (i.e., a bounded family \{x_i\}_{i < ω_1} such
that x_j ∉ cl(\{x_i : i ∈ ω_1 \setminus \{j\}\}) for every j ∈ ω_1) iff X has an uncountable bounded
almost biorthogonal system (UBABS) of type η for some η ∈ [0, 1) (i.e., a bounded family
\{(x_α, f_α)\}_{1 ≤ α < ω_1} ⊂ X × X* such that f_α(x_α) = 1 and |f_α(x_β)| ≤ η if α ≠ β); (2) if X
has an uncountable ω-independent system then X has an UBABS of type η for every η ∈ (0, 1);
(3) if X does not have the property (C) of Corson, then X has an ω_1-polyhedron; (4) X
has no ω_1-polyhedron iff X has no convex right-separated ω_1-family (i.e., a bounded family
\{x_i\}_{i < ω_1} such that x_j ∉ cl(\{x_i : j < i < ω_1\}) for every j ∈ ω_1) iff every w*-closed convex
subset of X* is w*-separable iff every convex subset of X* is w*-separable iff µ(X) = 1,
µ(X) being the Finet–Godefroy index of X (see [1]).

1. Introduction. If X is a Banach space and θ an ordinal, a family
\{x_α : α < θ\} ⊂ X is said to be a θ-basic sequence if there exists 1 < K < ∞
such that for every n < m in N, any λ_i ∈ R, i = 1, ..., m, and α_1 < ... <
α_m < θ we have \| ∑_{i=1}^{n} λ_i x_{α_i} \| ≤ K \| ∑_{i=1}^{m} λ_i x_{α_i} \|. A family \{x_i\}_{i ∈ J} ⊂ X
is a basic sequence if it is a θ-basic sequence for some ordinal θ. If K = 1
the basic sequence is said to be monotone. A biorthogonal system in X is a
family \{(x_i, x_i^*) : i ∈ J\} ⊂ X × X* such that x_i^*(x_i) = 1 and x_i^*(x_j) = 0 for
i, j ∈ I, i ≠ j. A Markushevich system (for short, an M-system) in X is a
biorthogonal system \{(x_i, x_i^*) : i ∈ J\} in X such that \{x_i^* : i ∈ I\} is total on
\{x_i : i ∈ I\} (see [14]).

It is well known (see [14, p. 599]) that if the density of a Banach space
X satisfies Dens(X) ≥ ℵ_1, then X has a monotone ω_1-basic sequence. Also
if Dens(X) > ℵ_1, then X has a monotone ω_1-basic sequence, because in

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this case an easy calculation shows that $w^*\text{-Dens}(X^*) \geq \aleph_1$. However, if $\aleph_1 \leq \text{Dens}(X) \leq \mathfrak{c}$ and $w^*\text{-Dens}(X^*) \leq \aleph_0$, then $X$ can fail to have an uncountable basic sequence, even an uncountable biorthogonal system. Indeed, under the axiom $\diamondsuit_{\aleph_1}$ (which implies the continuum hypothesis (CH)), Shelah [13] constructed a nonseparable Banach space $S$ that fails to have an uncountable biorthogonal system. Later Kunen [8, p. 1123] constructed under (CH) a Hausdorff compact space $K$ such that $C(K)$ is nonseparable and has no uncountable biorthogonal system, among other interesting pathological properties.

A Banach space $X$ is said to have the Kunen–Shelah property $KS_0$ (resp. $KS_1$) if $X$ has no uncountable basic sequence (resp. uncountable Markushevich system). A Banach space $X$ is said to have the Kunen–Shelah property $KS_2$ if $X$ has no uncountable biorthogonal system. Clearly, $KS_2 \Rightarrow KS_1 \Rightarrow KS_0$.

The first example of a Banach space $X$ such that $X \in KS_0$ but $X \notin KS_2$ was given in [9]; it is the space of Johnson–Lindenstrauss $JL_2$ (see [5]). The properties $KS_2$ and $KS_1$ were separated in [2] (see also [1]), where it was proved that if a Banach space $X$ has the property (C) of Corson and $w^*\text{-Dens}(X^*) \leq \aleph_0$, then $X \in KS_1$.

**Question 1.** Does there exist a Banach space $X$ such that $X \in KS_0$ but $X \notin KS_1$?

In this paper we study some structures similar to uncountable biorthogonal systems, namely: uncountable $\omega$-independent families, $\omega_1$-polyhedrons, uncountable bounded almost biorthogonal systems (UBABS), etc. The lack of these structures defines the Kunen–Shelah properties $KS_3, KS_4$, etc.

In Section 2 we prove that a Banach space $X$ has an $\omega_1$-polyhedron iff $X$ has an UBABS iff $X^*$ has a $w^*$-nonseparable dual equivalent ball. Section 3 deals with uncountable $\omega$-independent families. In Section 4 it is proved that $X$ has no $\omega_1$-polyhedron iff every $w^*$-closed convex subset of $X^*$ is $w^*$-separable. In Section 5 we answer some questions posed by Finet and Godefroy [1] concerning the index $\mu(X)$. In Section 6 we prove that a space $X$ has no convex right-separated $\omega_1$-family iff every $w^*$-closed convex subset of $X^*$ is $w^*$-separable. Finally, in Section 7 we show that $X$ has an $\omega_1$-polyhedron iff $X$ has a convex right-separated $\omega_1$-family, whence every $w^*$-closed convex subset of $X^*$ is $w^*$-separable iff every convex subset of $X^*$ is so.

Let us introduce some notation. $\omega_1$ is the first uncountable ordinal, $|A|$ the cardinality of the set $A$, and $\mathfrak{c} = |\mathbb{R}|$. If $X$ is a Banach space, $X^*$ denotes its dual, $B(X)$ and $S(X)$ the closed unit ball and sphere of $X$, resp., and $B(x,r)$ the closed ball with radius $r$ and center $x$. If $A \subset X$ we denote by $[A]$ the linear subspace spanned by $A$. Recall that a Banach space $X$ is
said to have the property (C) of Corson (for short, \( X \in (C) \)) if \( \bigcap_{i \in I} C_i \neq \emptyset \) whenever \( \{C_i : i \in I \} \) is a family of closed bounded convex subsets of \( X \) with the countable intersection property, i.e., \( \emptyset \neq \bigcap_{i \in I} C_i \) for every countable subset \( J \subset I \).

2. UBABS and \( \omega_1 \)-polyhedrons. If \( X \) is a Banach space, a bounded family \( \{(x_\alpha, f_\alpha)\}_{1 \leq \alpha \leq \omega_1} \subset X \times X^* \) is said to be an uncountable bounded almost biorthogonal system (for short, an UBABS) if there exists a real number \( 0 \leq \eta < 1 \) such that \( f_\alpha(x_\alpha) = 1 \) and \( f_\alpha(x_\beta) \leq \eta \) if \( \alpha \neq \beta \). If in addition \( |f_\alpha(x_\beta)| \leq \eta \) for \( \alpha \neq \beta \), then the UBABS is said to be of type \( \eta \). Define the index \( \tau(X) \) as follows:

\[
\tau(X) = \inf\{0 \leq \eta < 1 : X \text{ has an UBABS of type } \eta\},
\]

where \( \inf\{\emptyset\} = 1 \). Clearly, \( \tau(X) \) is invariant under isomorphisms and: (1) if \( X \) has an uncountable biorthogonal system, then \( \tau(X) = 0 \); (2) \( \tau(X) < 1 \) iff \( X \) has an UBABS.

If \( \tau \) is a cardinal, a bounded family \( \{x_i\}_{i \in \tau} \) in a Banach space \( X \) is said to be a \( \tau \)-polyhedron iff \( x_j \not\in \overline{\sigma}\{\{x_i\}_{i \in \tau \setminus \{j\}}\} \) for every \( j \in \tau \). In a dual Banach space \( X^* \) one can define a \( w^* \)-\( \tau \)-polyhedron in an analogous way, using the \( w^* \)-topology instead of the \( w \)-topology.

**Proposition 2.1.** A Banach space \( X \) has an \( \omega_1 \)-polyhedron iff \( X^* \) has a \( w^* \)-\( \omega_1 \)-polyhedron.

**Proof.** Let \( \{x_\alpha\}_{\alpha < \omega_1} \subset B(X) \) be an \( \omega_1 \)-polyhedron. By the Hahn–Banach Theorem there exists \( f_\alpha \in S(X^*) \) such that

\[
f_\alpha(x_\alpha) > \sup\{f_\alpha(x_i) : i \in \omega_1 \setminus \{\alpha\}\} =: e_\alpha.
\]

By passing to a subsequence, we can suppose that there exist \( 0 < \varepsilon < \infty \) and \( r \in \mathbb{R} \) such that \( f_\alpha(x_\alpha) - e_\alpha \geq \varepsilon > 0 \) and \( |r - f_\alpha(x_\alpha)| \leq \varepsilon/4 \) for all \( \alpha < \omega_1 \). Hence, if \( \alpha, \beta < \omega_1 \) with \( \alpha \neq \beta \), we have

\[
f_\alpha(x_\alpha) \geq r - \varepsilon/4 > r - 3\varepsilon/4 \geq f_\beta(x_\beta) - \varepsilon \geq e_\beta \geq f_\beta(x_\alpha),
\]

which implies that \( \{f_\alpha\}_{\alpha < \omega_1} \) is a \( w^* \)-\( \omega_1 \)-polyhedron in \( X^* \).

The converse implication is analogous. \( \blacksquare \)

In the following proposition we give the relation between \( \omega_1 \)-polyhedrons and UBABS.

**Proposition 2.2.** For a Banach space \( X \) the following are equivalent:

1. \( X \) has an \( \omega_1 \)-polyhedron.
2. \( X \) has an UBABS of type \( \eta \) for some \( 0 \leq \eta < 1 \).
3. \( X \) has an UBABS.

**Proof.** (1) \( \Rightarrow \) (2). If \( w^* \)-Dens\((X^*) \geq \aleph_0 \), then \( X \) has an uncountable biorthogonal system and so \( X \) has an UBABS of type 0. Now assume that
\( w^*\text{-Dens}(X^*) \leq \aleph_0 \). Let \( \{x_\alpha\}_{1 \leq \alpha < \omega_1} \subset X \) be an \( \omega_1 \)-polyhedron. Assume that \( x_1 = 0 \) and \( \|x_\alpha\| \leq 1 \). For each \( 1 \leq \alpha < \omega_1 \) consider \( f_\alpha \in S(X^*) \) such that
\[
1 \geq f_\alpha(x_\alpha) > \sup\{f_\alpha(x_i) : 1 \leq i < \omega_1, i \neq \alpha\} =: \varrho_\alpha.
\]
Observe that \( \varrho_\alpha \geq 0 \) if \( \alpha \neq 1 \). By passing to an uncountable subsequence, it can be assumed that there are real numbers \( 0 < \varepsilon, r \leq 1 \) such that \( f_\alpha(x_\alpha) - \varrho_\alpha \geq \varepsilon \) and \( |r - f_\alpha(x_\alpha)| < \varepsilon/8 \) for every \( 2 \leq \alpha < \omega_1 \). Since \( w^*\text{-Dens}(X^*) \leq \aleph_0 \), by passing again to a subsequence, we assume that there exists \( z \in X^* \) such that \( z(x_\alpha) > 0 \) and \( |z(x_\alpha) - 1| < \varepsilon/8 \) for every \( 2 \leq \alpha < \omega_1 \). Then, if \( g_\alpha = f_\alpha + z(x_\alpha) \), \( 2 \leq \alpha < \omega_1 \), we have
\[
g_\alpha(x_\alpha) = f_\alpha(x_\alpha) + 1 \geq r - \varepsilon/8 + 1 > r - 6\varepsilon/8 + 1 \geq f_\alpha(x_\alpha) - 7\varepsilon/8 + 1 \geq \sup\{g_\alpha(x_\beta) : 2 \leq \beta < \omega_1, \beta \neq \alpha\}
\]
\[
\geq \inf\{g_\alpha(x_\beta) : 2 \leq \beta < \omega_1, \beta \neq \alpha\} \geq -\varepsilon/8.
\]
Define \( h_\alpha = g_\alpha/g_\alpha(x_\alpha) \). Then, for \( 2 \leq \alpha, \beta < \omega_1, \alpha \neq \beta \), we have \( h_\alpha(x_\alpha) = 1 \) and
\[
-\frac{\varepsilon/8}{r - \varepsilon/8 + 1} \leq -\frac{\varepsilon/8}{g_\alpha(x_\alpha)} \leq \frac{g_\alpha(x_\beta)}{g_\alpha(x_\alpha)} \leq \frac{r + 1 - 6\varepsilon/8}{r + 1 - \varepsilon/8}.
\]
So, \( \{(x_\alpha, h_\alpha) : 2 \leq \alpha < \omega_1\} \subset X \times X^* \) is an UBABS of type \( \eta \) such that
\[
0 \leq \eta = \max\left\{\frac{\varepsilon/8}{r - \varepsilon/8 + 1}, \frac{r + 1 - 6\varepsilon/8}{r + 1 - \varepsilon/8}\right\} < 1.
\]
(2)\(\Rightarrow\)(3) is obvious and (3)\(\Rightarrow\)(1) is clear, because if \( \{(x_\alpha, f_\alpha)\}_{1 \leq \alpha < \omega_1} \subset X \times X^* \) is an UBABS, then \( \{x_\alpha\}_{\alpha < \omega_1} \) is an \( \omega_1 \)-polyhedron. \( \blacksquare \)

Let us consider some results on representation of elements in polyhedrons, which we need later. If \( \{x_i\}_{i \in I} \) is a \( w^*-\tau \)-polyhedron in a dual Banach space \( X^* \) with \( \tau = \text{card}(I) \) and \( K = \overline{\text{co}}^{w^*}(\{x_i\}_{i \in I}) \), then the core of \( K \) is the set
\[
K_0 = \text{core}(K) = \bigcap\{\overline{\text{co}}^{w^*}(\{x_i\}_{i \in I\setminus A}) : A \subset I, A \text{ finite}\}.
\]
Define the function \( \lambda : K \to [0, 1] \) as follows: for \( k \in K \),
\[
\lambda(k) = \sup\{\lambda \in [0, 1] : \exists u \in K, \exists i \in I \text{ such that } k = \lambda x_i + (1 - \lambda)u\}.
\]
Let \( H = \{x \in K : \lambda(x) = 0\} \). Since for every finite subset \( A \subset I \), each \( x \in K \) has the expression \( x = \sum_{i \in A} \lambda_i x_i + (1 - \mu)u \) with \( u \in \overline{\text{co}}^{w^*}(\{x_i\}_{i \in I\setminus A}) \), \( \lambda_i \in [0, 1], i \in A, \mu = \sum_{i \in A} \lambda_i \leq 1 \), it can be easily seen that \( H \subset K_0 \).

**Lemma 2.3.** Let \( \{x_i\}_{i \in I} \) be a \( w^*-\tau \)-polyhedron in the dual Banach space \( X^* \), \( \tau = \text{card}(I) \), \( K = \overline{\text{co}}^{w^*}(\{x_i\}_{i \in I}) \), \( K_0 = \text{core}(K) \) and \( H = \{x \in K : \lambda(x) = 0\} \). If \( x \in K \), then there exist a sequence \( \{\mu_n\}_{n \geq 1} \) of positive numbers with \( 0 \leq \sum_{n \geq 1} \mu_n = \mu \leq 1 \), a sequence \( \{i_n\}_{n \geq 1} \subset I \) of indices (not necessarily distinct) and \( u \in H \) such that \( x = \sum_{n \geq 1} \mu_n x_{i_n} + (1 - \mu)u \).
Proof. Clearly the statement is true if \( x \in H \). Assume that \( x \notin H \), i.e., \( \lambda(x) > 0 \). Choose \( 0 < \frac{1}{2} \lambda(x) \leq \lambda_1 \leq 1 \), \( i_1 \in I \), and \( u_1 \in \overline{c}o^{w^*}\{x_i | i \in I \backslash \{i_1\}\} \) such that \( x = \lambda_1 x_i + (1 - \lambda_1) u_1 \). If \( u_1 \in H \), we are done. Otherwise, \( \lambda(u_1) > 0 \) and we choose \( 0 < \frac{1}{2} \lambda(u_1) \leq \lambda_2 \leq 1 \), \( i_2 \in I \), and \( u_2 \in \overline{c}o^{w^*}\{x_i | i \in I \backslash \{i_2\}\} \) such that \( u_1 = \lambda_2 x_i + (1 - \lambda_2) u_2 \). By reiteration, there are two possibilities:

(A) \( u_m \in H \) for some \( m \in \mathbb{N} \). Then we obtain the representation

\[
(1) \quad x = \sum_{k=1}^{m} \lambda_k P_{k-1} x_{i_k} + P_m u_m, \quad P_n = \prod_{k=1}^{n} (1 - \lambda_k), \quad P_0 = 1.
\]

(B) Always \( u_m \notin H \). As \( P_m \) decreases in (1), the limit \( \lim_{m \to 1} P_m = P \in [0,1] \) exists. We have two cases:

- \( P > 0 \). Observe that this happens iff \( \sum_{k \geq 1} \lambda_k < \infty \). In consequence, the series \( \sum_{k \geq 1} \lambda_k P_{k-1} x_{i_k} \) converges and \( u_m \to u \in K \) as \( m \to \infty \). So \( x = \sum_{k \geq 1} \lambda_k P_{k-1} x_{i_k} + Pu \). We claim that \( \lambda(u) = 0 \). Indeed, suppose that \( \mu := \lambda(u) > 0 \) and pick \( q \in \mathbb{N} \) such that \( P/P_q > 1/2 \), \( \lambda_{q+1} < \mu/8 \). Then

\[
u_q = \frac{1}{P_q} \left( \sum_{j \geq 1} \lambda_{q+j} P_{q+j-1} x_{q+j} + Pu \right),
\]

which implies that \( \lambda(u_q) \geq (P/P_q) \lambda(u) = (P/P_q) \mu > \mu/2 \). Since \( 0 < \frac{1}{2} \lambda(u_q) \leq \lambda_{q+1} \leq 1 \), we obtain \( \mu/8 > \lambda_{q+1} \geq \mu/4 \), a contradiction.

- \( P = 0 \). In this case \( P_m u_m \to 0 \) as \( m \to \infty \) and we obtain the representation \( x = \sum_{k \geq 1} \lambda_k P_{k-1} x_{i_k} \) with \( \sum_{k \geq 1} \lambda_k P_{k-1} = 1 \). \)

In order to connect the existence of an UBABS in a Banach space \( X \) with the \( w^* \)-nonseparability of dual equivalent unit balls of \( X^* \), we introduce the index \( \sigma(X) \). If \( K \subset X^* \) is a disc (i.e., a convex symmetric subset of \( X^* \)), define

\[
\sigma(K) = \max\{0 \leq t \leq 1 : \exists A \subset K, |A| \leq \aleph_0, tK \subset \overline{c}o^{w*}(A \cup (-A))\}.
\]

Observe that \( 0 \leq \sigma(K) < 1 \) iff \( K \) is \( w^* \)-nonseparable and that there exists a countable subset \( A \subset K \) such that \( \sigma(K) \cdot K \subset \overline{c}o^{w^*}(A \cup (-A)) \).

**Lemma 2.4.** Let \( X \) be a Banach space, \( K \subset X^* \) a \( w^* \)-nonseparable disc and \( \sigma(K) < \rho \leq 1 \). Then there exists \( \varepsilon = \varepsilon(\rho) > 0 \) (depending on \( \rho \)) such that for every countable subset \( A \subset K \) there exists \( k \in K \) satisfying \( \text{dist}(gk, \overline{c}o^{w^*}(A \cup (-A))) \geq \varepsilon \).

**Proof.** In the contrary case, there exist a sequence of real numbers \( \varepsilon_n \downarrow 0 \) and a sequence of countable subsets \( A_n \subset K, \ n \geq 1 \), such that every \( k \in K \) satisfies \( \text{dist}(gk, \overline{c}o^{w^*}(A_n \cup (-A_n))) < \varepsilon_n \). So, if \( A = \bigcup_{n \geq 1} A_n \) we have \( \rho K \subset \overline{c}o^{w^*}(A \cup (-A)) \), a contradiction. \)

Define the index \( \sigma(X) \), \( X \) a Banach space, as follows:

\[
\sigma(X) = \inf\{\sigma(K) : K \subset X^* \text{ a dual equivalent ball of } X^*\}.
\]
It is clear that $\sigma(X)$ is invariant under isomorphisms.

**Proposition 2.5.** For a Banach space $X$ we have

$$\sigma(X) = \inf \{ \sigma(K) : K \subset X^* \text{ a } w^*\text{-compact disc} \}.$$  

*Proof.* Obviously $\sigma(X) \geq \inf \{ \sigma(K) : K \subset X^* \text{ a } w^*\text{-compact disc} \}$. In order to prove the opposite inequality, it is enough to see that $\sigma(X) \leq \sigma(K)$ for any $w^*$-compact disc $K \subset X^*$. Assume that such a $K$ is $w^*$-nonseparable, pick $\sigma(K) < \varrho < 1$ and let $\varepsilon = \varepsilon(\varrho) > 0$ be given by Lemma 2.4. For $0 < \delta < \varepsilon$ such that $\varrho + \delta < 1$ consider $H_\delta = K + \delta B(X^*)$, which is an equivalent dual ball of $X^*$. We claim that $\sigma(H_\delta) \leq \varrho + \delta$. Indeed, let $\varrho + \delta < t \leq 1$ and $A \subset H_\delta$ be a countable subset. Then $A \subset A_1 + A_2$, where $A_1 \subset K$ and $A_2 \subset \delta B(X^*)$ are countable. Assume that $tH_\delta \subset \overline{co}^{w^*}(A \cup (-A))$. As $\overline{co}^{w^*}(A \cup (-A)) \subset \overline{co}^{w^*}(A_1 \cup (-A_1)) + \delta B(X^*)$, we get

$$tK \subset tH_\delta \subset \overline{co}^{w^*}(A_1 \cup (-A_1)) + \delta B(X^*),$$

which implies that $\text{dist}(tk, \overline{co}^{w^*}(A_1 \cup (-A_1))) \leq \delta$ for all $k \in K$. But by Lemma 2.4 there exists $k \in K$ such that $\text{dist}(tk, \overline{co}^{w^*}(A_1 \cup (-A_1))) \geq \varepsilon$. Thus $\text{dist}(tk, \overline{co}^{w^*}(A_1 \cup (-A_1))) > \delta$, a contradiction. Therefore, we have $tH_\delta \nsubseteq \overline{co}^{w^*}(A \cup (-A))$ and $\sigma(H_\delta) \leq \varrho + \delta$ for $0 < \delta < \varepsilon$. Hence, $\sigma(X) \leq \varrho$ for every $\sigma(X) < \varrho < 1$, and we conclude that $\sigma(X) \leq \sigma(K)$. $\blacksquare$

**Proposition 2.6.** If $X$ is a Banach space then $\sigma(X) \leq \tau(X)$.

*Proof.* Assume that $\tau(X) < \eta < 1$ and choose an UBABS $\{(x_\alpha, f_\alpha)\}_{\alpha < \omega_1} \subset X \times X^*$ of type $\eta$ such that $\|f_\alpha\| = 1$ and $\|x_\alpha\| \leq M$ for all $\alpha < \omega_1$, for some $0 < M < \infty$. Clearly, $\{(\pm f_\alpha)\}_{\alpha < \omega_1}$ is a $w^*$-$\omega_1$-polyhedron. Define $K = \overline{co}^{w^*}(\{(\pm f_\alpha)\}_{\alpha < \omega_1})$, $K_0 = \text{core}(K)$ and $H = \{z \in K : \lambda(z) = 0\}$. It is easy to see that $|z(x_\alpha)| \leq \eta$ for every $z \in K_0$ and $\alpha < \omega_1$. We claim that $\sigma(K) \leq \eta$. Indeed, let $A \subset K$ be countable. By Lemma 2.3 there exists $\gamma < \omega_1$ such that

$$A \subset \overline{co}(\{(\pm f_\alpha)\}_{\alpha \leq \gamma} \cup H) \subset \overline{co}^{w^*}(\{(\pm f_\alpha)\}_{\alpha \leq \gamma} \cup H).$$

Clearly, $\overline{co}^{w^*}(A \cup (-A)) \subset \overline{co}^{w^*}(\{(\pm f_\alpha)\}_{\alpha \leq \gamma} \cup H)$ and for every $\gamma < \varrho < \omega_1$ and every $z \in \overline{co}^{w^*}(\{(\pm f_\alpha)\}_{\alpha \leq \gamma} \cup H)$ we have $|z(x_\varrho)| \leq \eta$.

Hence, for every $\varrho < \omega_1$ and $\eta < t \leq 1$ we have $tf_\varrho \notin \overline{co}^{w^*}(A \cup (-A))$. So $\sigma(K) \leq \eta$ and we conclude that $\sigma(X) \leq \tau(X)$. $\blacksquare$

Now we prove for a Banach space $X$ that $\sigma(X) = 1$ iff $\tau(X) = 1$.

**Proposition 2.7.** A Banach space $X$ has an UBABS of type $\eta$ for some $\eta \in [0, 1)$ iff $X^*$ has a $w^*$-nonseparable equivalent dual unit ball. So, $\sigma(X) = 1$ iff $\tau(X) = 1$.

*Proof.* Firstly, if $X$ has an UBABS of type $\eta$ for some $\eta \in [0, 1)$ (i.e., $\tau(X) < 1$), then by Proposition 2.6 we have $\sigma(X) < 1$ (i.e., $X^*$ has a $w^*$-nonseparable equivalent dual unit ball).
Assume now that $X$ is a Banach space with $\sigma(X) < 1$ equipped with an equivalent norm such that $\sigma(B(X^*)) < 1$. Fix $0 < \rho < 1$. If $A \subset S(X)$ and $\varepsilon \geq 0$ we put

$$(A, \varepsilon)^\perp = \{ z \in X^* : |z(x)| \leq \varepsilon, \forall x \in A \}, \quad S((A, \varepsilon)^\perp) = S(X^*) \cap (A, \varepsilon)^\perp.$$  

Clearly $\varepsilon B(X^*) + A^\perp \subset (A, \varepsilon)^\perp$.

**Claim 0.** If $A \subset S(X)$ and $A^\perp \neq \{0\}$, then $\varepsilon S(X^*) \subset \text{co}(S((A, \varepsilon)^\perp))$ for $0 \leq \varepsilon < 1$.

Indeed, let $u \in \varepsilon S(X^*)$ and $v \in A^\perp \setminus \{0\}$. We can find $\lambda, \mu > 0$ such that $u + \lambda v, u - \mu v \in S(X^*)$. Thus, $u + \lambda v, u - \mu v \in S((A, \varepsilon)^\perp)$. Let $t \in (0, 1)$ be such that $t\lambda + (1-t)(-\mu) = 0$. Then $u = t(u + \lambda v) + (1-t)(u - \mu v) \in \text{co}(S((A, \varepsilon)^\perp))$.

**Claim 1.** For any countable subsets $A \subset S(X)$ and $F \subset S(X^*)$ there exists $f \in S((A, \sqrt{\rho})^\perp)$ such that $\sqrt{\rho} f \notin \overline{co}^{w^*}(F \cup (-F))$.

The opposite means that $\sqrt{\rho} S((A, \sqrt{\rho})^\perp) \subset \overline{co}^{w^*}(F \cup (-F))$. By Claim 0 we have $\sqrt{\rho} S(X^*) \subset \text{co}(S((A, \sqrt{\rho})^\perp))$. So

$$\sqrt{\rho} B(X^*) \subset \overline{co}^{w^*}(\sqrt{\rho} S(X^*)) \subset \overline{co}^{w^*}(\sqrt{\rho} S((A, \sqrt{\rho})^\perp)) \subset \overline{co}^{w^*}(F \cup (-F)),$$

a contradiction because $\sigma(B(X^*)) < \rho$. So, Claim 1 holds.

**Claim 2.** There exist $0 \leq \delta < \varepsilon \leq 1 - \sqrt{\rho}$ such that for any countable subsets $A \subset S(X)$ and $F \subset S(X^*)$ there exist $f_0 \in S((A, \sqrt{\rho})^\perp)$ and $x_0 \in S(X)$ such that $f_0(x_0) \geq 1 - \delta$ and $f(x_0) \leq 1 - \varepsilon$ for all $f \in F$.

Define $\mathcal{R} = \{ r = (r_1, r_2) \in \mathbb{Q} \times \mathbb{Q} : 0 < r_1 < r_2 \leq 1 - \sqrt{\rho} \}$. As $\mathcal{R}$ is countable, we can put $\mathcal{R} = \{ r_n \}_{n \geq 1}$. If Claim 2 is false, for every pair $r_n = (r_{n1}, r_{n2}) \in \mathcal{R}$ we can choose countable subsets $A_n \subset S(X)$, $F_n \subset S(X^*)$, $n \geq 1$, such that for every $g \in S((A_n, \sqrt{\rho})^\perp)$ and every $x \in S(X)$, either $g(x) < 1 - r_{n1}$ or there exists $f \in F_n$ with $f(x) > 1 - r_{n2}$. Let $A = \bigcup_{n \geq 1} A_n$, $F = \bigcup_{n \geq 1} F_n$. By Claim 1 there exists $f_0 \in S((A, \sqrt{\rho})^\perp)$ such that $\sqrt{\rho} f_0 \notin \overline{co}^{w^*}(F \cup (-F))$. By the Hahn–Banach Theorem there exists $y \in S(X)$ such that

$$\sqrt{\rho} f_0(y) > \sup \{ |f(y)| : f \in F \} = \gamma_0 \geq 0.$$  

Choose a sequence $\{ z_n \}_{n \geq 1} \subset S(X)$ such that

$$(2) \quad \lim_{n \to \infty} f_0(z_n) = \| f_0 \| = 1, \quad 1 - f_0(z_n) < \frac{1}{n} (f_0(y) - \gamma_0), \quad n \geq 1.$$  

Then

$$f_0 \left( \frac{z_n + \frac{1}{n} y}{\| z_n + \frac{1}{n} y \|} \right) = 1 - \delta_n$$
with
\[
0 \leq \delta_n = \frac{\|z_n + \frac{1}{n}y\| - f_0(z_n) - \frac{1}{n}f_0(y)}{\|z_n + \frac{1}{n}y\|} \leq \frac{1 - f_0(z_n) + \frac{1}{n}(1 - f_0(y))}{\|z_n + \frac{1}{n}y\|}.
\]
Hence, \( \lim_{n \to \infty} \delta_n = 0 \). On the other hand, for every \( f \in F \),
\[
f \left( \frac{z_n + \frac{1}{n}y}{\|z_n + \frac{1}{n}y\|} \right) \leq \frac{1 + \frac{1}{n}\gamma_0}{\|z_n + \frac{1}{n}y\|} = 1 - \varepsilon_n,
\]
where
\[
\varepsilon_n = \frac{\|z_n + \frac{1}{n}y\| - 1 - \frac{1}{n}\gamma_0}{\|z_n + \frac{1}{n}y\|} \leq \frac{1 + \frac{1}{n} - 1 - \frac{1}{n}\gamma_0}{\|z_n + \frac{1}{n}y\|} = \frac{1}{n}(1 - \gamma_0)
\]
and
\[
\varepsilon_n > \frac{\|z_n + \frac{1}{n}y\| - f_0(z_n) - \frac{1}{n}f_0(y)}{\|z_n + \frac{1}{n}y\|} = \delta_n \geq 0
\]
by (2). Pick any \( n \in \mathbb{N} \) such that \( \frac{1}{n}(1 - \gamma_0)/\|z_n + \frac{1}{n}y\| \leq 1 - \sqrt{d} \). Then
\( 0 \leq \delta_n < \varepsilon_n \leq 1 - \sqrt{d} \) and there is some \( m \in \mathbb{N} \) such that \( \delta_n \leq r_{m1} < r_{m2} \leq \varepsilon_n \). Let \( x_0 = (z_n + \frac{1}{n}y)/\|z_n + \frac{1}{n}y\| \in S(X) \) and observe that \( f_0 \in S((A_m, \sqrt{d})^\perp) \), \( f_0(x_0) \geq 1 - \delta_n \) and \( f(x_0) \leq 1 - \varepsilon_n \) for all \( f \in F \). Then \( f_0 \in S((A_m, \sqrt{d})^\perp) \), \( f_0(x_0) \geq 1 - r_{m1} \) and \( f(x_0) \leq 1 - \varepsilon \) for all \( f \in F_m \), a contradiction. So, Claim 2 holds.

Let \( 0 \leq \delta < \varepsilon \leq 1 - \sqrt{d} \) be from Claim 2. We will construct a transfinite sequence \( \{(x_\alpha, f_\alpha)\}_{\alpha < \omega_1} \subset S(X) \times S(X^*) \) so that for every \( \alpha < \omega_1 \),
\[
(3) \quad f_\alpha(x_\alpha) \geq 1 - \delta,
\]
\[
(4) \quad f_\alpha(x_\beta) \leq 1 - \varepsilon \quad \text{if } \alpha \neq \beta.
\]
In the first step, we take \( x_1 \in S(X) \) and \( f_1 \in S(X^*) \) such that \( f_1(x_1) = 1 \). Let \( 1 < \alpha_0 < \omega_1 \) and suppose we have constructed a family \( \{(x_\alpha, f_\alpha) : \alpha < \alpha_0\} \) satisfying (3) and (4). Apply Claim 2, with \( F = \{f_\alpha : \alpha < \alpha_0\} \) and \( A = \{x_\alpha : \alpha < \alpha_0\} \). Denote the resulting elements \( x_0 \) and \( f_0 \) by \( x_{\alpha_0} \) and \( f_{\alpha_0} \). The inequality (3) for \( \alpha = \alpha_0 \) is satisfied by construction. The inequality (4) for \( \alpha = \alpha_0 \) and \( \beta < \alpha_0 \) holds because \( f_0 \in S((A, \sqrt{d})^\perp) \) and \( \varepsilon \leq 1 - \sqrt{d} \). For \( \beta = \alpha_0 \) and \( \alpha < \alpha_0 \), it follows because \( \sup\{f(x_0) : f \in F\} \leq 1 - \varepsilon \). Now the set \( \{(x_\alpha, f_\alpha)\}_{\alpha < \omega_1} \), where \( x_\alpha = x_\alpha, \ f_\alpha = f_\alpha/f_\alpha(x_\alpha), \ 1 \leq \alpha < \omega_1 \), is an uncountable bounded (by \( (1 - \delta)^{-1} \)) almost biorthogonal system.

**Proposition 2.8.** Let \( X \) be a Banach space such that \( \sigma(X) < 1/3 \). Then \( \tau(X) \leq 2\sigma(X)/(1 - \sigma(X)) \). So, for every Banach space \( X \):

1. \( \sigma(X) = 0 \) iff \( \tau(X) = 0 \).
2. \( \sigma(X) = 0 \) whenever \( X \) has an uncountable biorthogonal system.

**Proof.** (A) Let \( \| \cdot \| \) be an equivalent norm on \( X \) such that the corresponding dual unit ball \( B(X^*) \) satisfies \( \sigma(B(X^*)) < 1/3 \). It is enough to
prove that for every \( \sigma(B(X^*)) < a < 1/3 \) there exists in \( X \) an UBABS of type \( \eta \leq 2a/(1 - a) \). So, fix such an \( a \). By induction we choose a family \( \{ (x_\alpha, f_\alpha) \}_{\alpha < \omega_1} \subset S(X) \times S(X^*) \) such that

\[
(5) \quad f_\alpha(x_\alpha) > \frac{1 - a}{2}, \quad |f_\alpha(x_\beta)| < a \quad \text{if} \ \alpha \neq \beta.
\]

Pick \((x_1, f_1) \in S(X) \times S(X^*)\) satisfying \( f_1(x_1) = 1 \). Let \( \alpha < \omega_1 \) and assume that we have chosen \( \{ (x_\beta, f_\beta) \}_{\beta < \alpha} \subset S(X) \times S(X^*) \) satisfying (5). Set

\[
A_\alpha = [\{ x_\beta : \beta < \alpha \}], \quad F_\alpha = \overline{co}^w (\{ \pm f_\beta : \beta < \alpha \} \cup G_0),
\]

where \( G_0 \subset B(X^*) \) is a countable symmetric subset 1-norming on \( A_\alpha \). By [15, Lemma 4.3] there exists \( x_\alpha \in S(X) \) such that \( \sup\{ |f(x_\alpha)| : f \in F_\alpha \} < a \).

We claim that \( \text{dist}(x_\alpha, A_\alpha) > (1 - a)/2 \). Indeed, pick \( z \in A_\alpha \) and observe that if \( ||z|| < (1 + a)/2 \), then clearly \( ||z - x_\alpha|| > (1 - a)/2 \), and if \( ||z|| \geq (1 + a)/2 \), then

\[
||z - x_\alpha|| \geq \sup\{ f(z - x_\alpha) : f \in F_\alpha \} \\
\geq ||z|| - \sup\{ f(x_\alpha) : f \in F_\alpha \} > \frac{1 + a}{2} - a = \frac{1 - a}{2}.
\]

This means that if \( Q : X \to X/A_\alpha \) is the canonical quotient mapping, then \( ||Q(x_\alpha)|| > (1 - a)/2 \). So, as \( (X/A_\alpha)^* = A_\alpha^\perp \) there exists \( f_\alpha \in S(X^*) \cap A_\alpha^\perp \) such that \( f_\alpha(x_\alpha) > (1 - a)/2 \). Thus we have chosen the pair \( (x_\alpha, f_\alpha) \), and this completes the induction.

Now put \( \tilde{f}_\alpha = f_\alpha / f_\alpha(x_\alpha) \), consider the family \( \tilde{\mathcal{F}} = \{ (x_\alpha, \tilde{f}_\alpha) \}_{\alpha < \omega_1} \) and observe that:

(a) \( \tilde{\mathcal{F}} \) is bounded because \( ||x_\alpha|| = 1 \) and

\[
||\tilde{f}_\alpha|| = \frac{||f_\alpha||}{|f_\alpha(x_\alpha)|} < \frac{1}{(1 - a)/2} = \frac{2}{1 - a} < \frac{2}{1 - 1/3} = 3.
\]

(b) \( \tilde{f}_\alpha(x_\alpha) = 1 \) and

\[
|\tilde{f}_\alpha(x_\beta)| = \frac{|f_\alpha(x_\beta)|}{f_\alpha(x_\alpha)} < \frac{a}{(1 - a)/2} = \frac{2a}{1 - a} < 1 \quad \text{if} \ \alpha \neq \beta.
\]

So, \( \tilde{\mathcal{F}} \) is an UBABS of type \( \eta \leq 2a/(1 - a) \).

(B) (1) follows from (A) and Proposition 2.6; (2) follows from the definition of \( \tau(X) \) and (1). \( \blacksquare \)

3. On \( \omega \)-independence. The Kunen–Shelah property KS\( _3 \). A family \( \{ x_i \}_{i \in I} \) in a Banach space \( X \) is said to be \( \omega \)-independent if for every sequence \( (i_n)_{n \geq 1} \subset I \) of distinct indices, and every sequence \( (\lambda_n)_{n \geq 1} \subset \mathbb{R} \), the series \( \sum_{n=1}^{\infty} \lambda_n x_{i_n} \) converges (in norm) to 0 if \( \lambda_n = 0 \) for every \( n \geq 1 \) (see [6], [12]). A Banach space \( X \) is said to have the Kunen–Shelah property KS\( _3 \)
if $X$ has no uncountable $\omega$-independent family. Of course, every biorthogonal family is $\omega$-independent (i.e., $\text{KS}_3 \Rightarrow \text{KS}_2$), but there are $\omega$-independent families which are not merely biorthogonal systems. Here is an example: $X = C([0,1]^{\omega_1})$ and $\{f_\alpha^n\}_{\alpha<\omega_1, n \geq 1}$ defined as

$$f_\alpha^n((t_\gamma)_{\gamma<\omega_1}) = t_\alpha^n$$

for every $x = (t_\gamma)_{\gamma<\omega_1} \in [0,1]^{\omega_1}$. This family is $\omega$-independent but not a biorthogonal system by the Theorem of Müntz–Szász (see [11, Th. 15.26]).

**Question 2.** Does a Banach space have an uncountable biorthogonal system whenever it has an uncountable $\omega$-independent family?

Unfortunately, the indices $\sigma(X)$, $\tau(X)$ do not separate the properties $\text{KS}_2$ and $\text{KS}_3$, because as we prove in the following, if $X \in \text{KS}_3$, then $\sigma(X) = 0$.

**Lemma 3.1.** Let $X$ be a Banach space, $\{x_i\}_{1 \leq i < \omega_1} \subset X$ an uncountable bounded $\omega$-independent family, $H \subset X$ a closed separable subspace and $N \in \mathbb{N}$. Then there exist ordinal numbers $\rho < \gamma < \omega_1$ such that $x_\rho \not\in \overline{co}(H \cup \{\pm N x_i\}_{\gamma \leq i < \omega_1})$.

**Proof.** Without loss of generality suppose that $\|x_i\| \leq 1$ for all $i < \omega_1$. Assume that for every pair of ordinal numbers $\rho, \gamma$ such that $\rho < \gamma < \omega_1$, we have $x_\rho \in \overline{co}(H \cup \{\pm N x_i\}_{\gamma \leq i < \omega_1})$. For $n \in \mathbb{N}$ and $\rho < \gamma < \omega_1$, define $D_\gamma = \text{co}(\{\pm N x_i\}_{\gamma \leq i < \omega_1})$ and

$$H(\rho, \gamma, n) = \left\{(u, \lambda) \in H \times (0,1]: \exists v \in D_\gamma \text{ with } \|\lambda u + (1-\lambda)v - x_\rho\| < \frac{1}{2n}\right\}.$$ 

If $\rho < \gamma < \gamma' < \omega_1$ and $n \geq 1$, then by hypothesis and definition, we have $H(\rho, \gamma, n) \neq \emptyset$ and $H(\rho, \gamma, n+1) \subset H(\rho, \gamma, n) \supset H(\rho, \gamma', n)$.

For $\beta < \omega_1$ and $n \geq 1$ define

$$H(\beta, n) = \text{cl}\left(\bigcup\{H(\rho, \gamma, n): \beta \leq \rho < \gamma < \omega_1\}\right)$$

where “cl” means closure in $H \times (0,1]$. Clearly, for $\beta < \beta'$ and $n \geq 1$ we have

$$\emptyset \neq H(\beta', n) \subset H(\beta, n) \supset H(\beta, n+1).$$

Since $H \times (0,1]$ is hereditarily Lindelöf, for each $n \geq 1$ there exists $\beta_n < \omega_1$ such that for every $\beta_n \leq \beta < \omega_1$ we have $H(\beta, n) = H(\beta_n, n)$. So, for every $(u, \lambda) \in H(\beta_n, n)$ and every $\beta_n \leq \beta < \omega_1$ we have $(u, \lambda) \in H(\beta, n)$, which implies that there exist $\beta \leq \rho < \gamma < \omega_1$ and $v \in D_\gamma$ such that

$$\|x_\rho - (\lambda u + (1-\lambda)v)\| < 1/n.$$ 

Let $\beta_0 = \sup_{n \geq 1} \beta_n$ and fix $\beta_0 \leq \rho < \gamma < \omega_1$ and $n \geq 1$. Pick $(u, \mu) \in H(\rho, \gamma, n)$ and $w \in D_\gamma$ such that $\|x_\rho - (\mu u + (1-\mu)w)\| < 1/(2n)$. Since
$(u, \mu) \in H(\beta_0, n) = H(\gamma, n)$, there exist $\gamma \leq \sigma < \theta < \omega_1$ and $v \in D_\theta$ such that $\|x_\sigma - (\mu u + (1 - \mu)v)\| < 1/n$.

Set $T = x_\sigma - (\mu u + (1 - \mu)v)$. Then $\mu u = x_\sigma - T - (1 - \mu)v$ and 

$$\|x_\sigma - (x_\sigma - T - (1 - \mu)v + (1 - \mu)w)\| < \frac{1}{2n}.$$ 

Since $\|T\| < 1/n$, we obtain 

$$\|x_\sigma - (x_\sigma - (1 - \mu)v + (1 - \mu)w)\| = \|x_\sigma - (x_\sigma - T - (1 - \mu)v + (1 - \mu)w) - T\| \leq \|x_\sigma - (x_\sigma - T - (1 - \mu)v + (1 - \mu)w)\| + \|T\| < \frac{1}{2n} + \frac{1}{n} = \frac{3}{2n}.$$ 

Taking into account that $\mu v + (1 - \mu)w, -v \in D_\gamma$, $x_\sigma \in (1/N)D_\gamma$ and that $\|S\| < 3/(2n)$, we finally get $x_\sigma \in \text{cl}((1 + 1/N)D_\gamma + D_\gamma) = \text{cl}((2 + 1/N)D_\gamma)$. So, $x_\sigma$ is an accumulation point of $F_\gamma := (2 + 1/N)D_\gamma$ (because $x_\sigma \in F_\gamma \setminus F_\gamma$).

In consequence, we can conclude that every $x_i$, $\beta_0 \leq i < \omega_1$, is an accumulation point of every $F_\gamma$ for $\gamma < \omega_1$.

Let $(a_n)_{n \geq 1}$ be a sequence of positive numbers such that $\lim_{n \to \infty} a_n = 0$, $\sum_{n \geq 1} a_n = \infty$, and let $b_n = \sup_{m \geq n} a_m$. Fix $\beta_0 < \tau < \omega_1$. Using the proof of [6, Th. 3], as in [12], we can construct inductively a sequence $\{\varepsilon_n\}_{n \geq 1}$ of signs, a sequence $\{\lambda_n^\tau\}_{n \geq 1, 1 \leq r \leq k(n)}$ of real numbers and a sequence $\{\gamma_n^\tau\}_{n \geq 1, 1 \leq r \leq k(n)}$ of ordinals such that:

1. $\sum_{r=1}^{k(n)} |\lambda_r^\tau| \leq 2N + 1$ for every $n \geq 1$.
2. $\tau < \gamma_1^\tau < \ldots < \gamma_{k(n)}^\tau < \omega_1$ for every $n \geq 1$.
3. $x_\tau + \sum_{n \geq 1} a_n \varepsilon_n y_n = 0$, where $y_n = \sum_{r=1}^{k(n)} \lambda_r^\tau x_{\gamma_r^\tau}$.

Let us see the first two steps of this argument. Set $K = \{x_i\}_{\tau < i < \omega_1}$.

**Step 1.** By the proof of [6, Th. 3] we can find $p_1 \in \mathbb{N}$, a finite sequence $\{h_n\}_{1 \leq n \leq p_1}$ of (not necessarily distinct) elements of $K$ and a finite sequence $\{\varepsilon_n\}_{1 \leq n \leq p_1}$ of signs such that 

$$\|x_\tau + \sum_{n=1}^{p_1} a_n \varepsilon_n h_n\| < 2^{-1},$$  

$$\|x_\tau + \sum_{n=1}^{j} a_n \varepsilon_n h_n\| < b_1 + 2^{-1} \quad \text{for} \quad 1 \leq j \leq p_1.$$
Since $h_n \in \text{cl}(F_\beta)$ for $\beta_0 \leq \beta < \omega_1$, we can find, for $1 \leq n \leq p_1$, real numbers \( \{\lambda^n_r\}_{1 \leq r \leq k(n)} \) with $\sum_{r=1}^{k(n)} |\lambda^n_r| \leq 2N + 1$, and ordinals $\{\gamma^n_r\}_{r=1}^{k(n)}$ such that:

(a) $\tau < \gamma^n_1 < \ldots < \gamma^n_{k(n)} < \gamma^n_{1} < \ldots < \omega_1$.

(b) $\|x_{\tau} + \sum_{n=1}^{p_1} a_n \varepsilon_n (\sum_{r=1}^{k(n)} \lambda^n_r x_{\gamma^n_r})\| < 2^{-1}$.

(c) $\|x_{\tau} + \sum_{n=1}^{j} a_n \varepsilon_n (\sum_{r=1}^{k(n)} \lambda^n_r x_{\gamma^n_r})\| < b_1 + 1 + 2^{-1}$ for $1 \leq j \leq p_1$.

**STEP 2.** Let $u_1 = x_{\tau} + \sum_{n=1}^{p_1} a_n \varepsilon_n (\sum_{r=1}^{k(n)} \lambda^n_r x_{\gamma^n_r})$. By the proof of [6, Th. 3] we can find $p_1 < p_2 \in \mathbb{N}$, a finite sequence $\{h_n\}_{p_1+1 \leq n \leq p_2}$ of (not necessarily distinct) elements of $K$ and a finite sequence $\{\varepsilon_n\}_{p_1+1 \leq n \leq p_2}$ of signs such that 

$$\|u_1 + \sum_{n=p_1+1}^{p_2} a_n \varepsilon_n h_n\| < 2^{-2},$$

$$\|u_1 + \sum_{n=p_1+1}^{j} a_n \varepsilon_n h_n\| < b_1 + 2^{-1} + 2^{-2} \text{ for } p_1 + 1 \leq j \leq p_2.$$

Since $h_n \in \text{cl}(F_\beta)$ for $\beta_0 \leq \beta < \omega_1$, we can find, for $1 < n < p_2$, real numbers $\{\lambda^n_r\}_{1 \leq r \leq k(n)}$ with $\sum_{r=1}^{k(n)} |\lambda^n_r| \leq 2N + 1$, and ordinals $\{\gamma^n_r\}_{r=1}^{k(n)}$ such that:

(a) $\gamma^n_{k(p_1)} < \gamma^n_1 < \ldots < \gamma^n_{k(n)} < \gamma^n_{1} < \ldots < \omega_1$.

(b) $\|u_1 + \sum_{n=p_1+1}^{p_2} a_n \varepsilon_n (\sum_{r=1}^{k(n)} \lambda^n_r x_{\gamma^n_r})\| < 2^{-2}$.

(c) $\|u_1 + \sum_{n=p_1+1}^{j} a_n \varepsilon_n (\sum_{r=1}^{k(n)} \lambda^n_r x_{\gamma^n_r})\| < b_1 + 2^{-1} + 2^{-2}$ for $p_1 < j \leq p_2$.

Now by reiteration we obtain the complete construction. It is easy to see that the series $x_{\tau} + \sum_{n \geq 1} a_n \varepsilon_n (\sum_{r=1}^{k(n)} \lambda^n_r x_{\gamma^n_r})$ converges to zero. This proves that $\{x_i\}_{i \leq \omega_1}$ is not $\omega$-independent, a contradiction. So, we can choose $\varrho < \gamma < \omega_1$ such that $x_\varrho \notin \mathcal{C}_0(H \cup \{\pm N x_i\}_{\gamma \leq i < \omega_1})$.  

**PROPOSITION 3.2.** Let a Banach space $X$ have an uncountable $\omega$-independent family $\{x_\alpha\}_{1 \leq \alpha < \omega_1}$. Then for every $0 < \eta < 1$, there exist an uncountable subsequence $\{\alpha_i\}_{i < \omega_1} \subset \omega_1$ and an UBABS $\{(z_i, f_i)\}_{i < \omega_1} \subset X \times X^*$ of type $\eta$ such that $z_i = x_{\alpha_i}$ and $f_i(z_j) = 0$ for $j < i < \omega_1$. So, $\tau(X) = 0$ and $X$ has an $\omega_1$-polyhedron.

**Proof.** Let $\{x_i\}_{1 \leq i < \omega_1} \subset X$ be an uncountable $\omega$-independent family and suppose, without loss of generality, that $\|x_i\| \leq 1$ for every $i < \omega_1$. Let $N \in \mathbb{N}$ be such that $1/N \leq \eta$. In the following we choose by induction two subsequences $\{i_\alpha, j_\alpha\}_{\alpha < \omega_1}$ of ordinal numbers, with $i_\alpha < j_\alpha \leq i_\beta < j_\beta < \omega_1$ for $\alpha < \beta < \omega_1$, such that

$$x_{i_\alpha} \notin \mathcal{C}_0(\{x_{j_\beta} : \beta < \alpha\} \cup \{\pm N x_j\}_{j_\alpha \leq j < \omega_1}).$$
Indeed, let $\alpha < \omega_1$ and assume that we have chosen $\{i_\beta, j_\beta\}_{\beta < \alpha}$ satisfying (6). Put $H = \{[x_{i_\beta}]_{\beta < \alpha}\}$ and $\nu = \sup_{\beta < \alpha} \{j_\beta\}$ (if $\alpha = 1$, put $H = \emptyset$ and $\nu = 1$). By Lemma 3.1 there exist $\nu \leq \varrho < \gamma < \omega_1$ such that $x_\varrho \notin \overline{co}(H \cup \{\pm N x_i\}_{\gamma \leq \omega_1})$. So, we put $i_\alpha = \varrho$, $j_\alpha = \gamma$, and this completes the induction. Let $z_\alpha = x_{i_\alpha}$ for $\alpha < \omega_1$. By (6) we have $z_\alpha \notin \overline{co}([z_\beta : \beta < \alpha]) \cup \{\pm N z_j\}_{\alpha < j < \omega_1}$. So, by the Hahn–Banach Theorem there exists $f_\alpha \in X^*$ such that

$$1 = f_\alpha(z_\alpha) > \sup \{f_\alpha(x) : x \in \overline{co}([z_\beta : \beta < \alpha]) \cup \{\pm N z_j\}_{\alpha < j < \omega_1}\}.$$

Clearly, $f_\alpha(z_\beta) = 0$ if $\beta < \alpha$, and $|f_\alpha(N z_\beta)| < 1$, i.e., $|f_\alpha(z_\beta)| < 1/N$, if $\alpha < \beta < \omega_1$. Finally, if we choose an uncountable subsequence $A \subset \omega_1$ with $\{|f_\alpha| : \alpha \in A\}$ bounded, then $\{(z_\alpha, f_\alpha) : \alpha \in A\}$ is the UBABS of type $\eta$ we are looking for. 

4. The Kunen–Shelah property $KS_4$. A Banach space $X$ is said to have the Kunen–Shelah property $KS_4$ if $X$ has no $\omega_1$-polyhedron. The implication $KS_4 \Rightarrow KS_3$ was proved in [3]. It also follows from Proposition 3.2 and from Proposition 7.3 and a result of Sersouri [12].

PROPOSITION 4.1. Let $Z$ be a Banach space and $X \subset Z$ a closed subspace such that $Z/X$ is separable. Then the following are equivalent:

(a) $Z \in KS_4$.
(b) $X \in KS_4$.

Proof. (a)$\Rightarrow$(b). This is obvious.

(b)$\Rightarrow$(a). Assume that $Z \notin KS_4$; we will prove that $X \notin KS_4$. By Proposition 2.2 there exists in $Z$ an UBABS $\{(z_\alpha, f_\alpha) : \alpha < \omega_1\}$ of type $\eta \in [0,1)$ with $\|f_\alpha\| \leq M$ for all $\alpha < \omega_1$, for some $0 < M < \omega_1$. Set $\varepsilon := 1 - \eta$. Since $Z/X$ is separable, there exists an uncountable subset $I \subset \omega_1$ such that if $Q : Z \rightarrow Z/X$ is the canonical quotient mapping, then $\|Q z_\alpha - Q z_\beta\| < \varepsilon/(4M)$ for every $\alpha, \beta \in I$. Fix $\tau \in I$ and define $y_\alpha = z_\alpha - z_\tau$ for $\alpha \in I$. Since $\|Q y_\alpha\| < \varepsilon/(4M)$, there exists $x_\alpha \in X$ such that $\|x_\alpha - y_\alpha\| < \varepsilon/(4M)$ for all $\alpha \in I$. Then for any $\alpha, \beta \in I$, $\alpha \neq \beta$, we have

$$f_\alpha(x_\alpha) = f_\alpha(y_\alpha) + f_\alpha(x_\alpha - y_\alpha) \geq f_\alpha(y_\alpha) - M \frac{\varepsilon}{4M} = f_\alpha(z_\alpha) - f_\alpha(z_\tau) - \frac{\varepsilon}{4}$$

$$= 1 - f_\alpha(z_\tau) - \frac{\varepsilon}{4} > \eta - f_\alpha(z_\tau) + \frac{\varepsilon}{4} \geq f_\alpha(z_\beta) - f_\alpha(z_\tau) + \frac{\varepsilon}{4}$$

$$= f_\alpha(y_\beta) + \frac{\varepsilon}{4} = f_\alpha(y_\beta) + M \frac{\varepsilon}{4M} \geq f_\alpha(x_\beta),$$

which implies that $\{x_\alpha : \alpha \in I\}$ is an uncountable polyhedron in $X$, i.e., $X \notin KS_4$. 

In the following we obtain some characterizations of the property $KS_4$. We first prove some lemmas.

**Lemma 4.2.** Let $X$ be a locally convex topological space, $\tau = \sigma(X, X^*)$, $f \in X^* \setminus \{0\}$, $C \subset f^{-1}(1)$ a bounded convex subset and $B = \text{co}(C \cup (-C))$. Then $C$ is $\tau$-separable iff $B$ is $\tau$-separable.

*Proof.* Clearly, $B$ is $\tau$-separable whenever $C$ is. For the converse, suppose that $B$ is $\tau$-separable and choose a countable subset $A \subset C$ such that $D := \{tx - (1-t)y : x, y \in A, t \in [0, 1]\}$ is $\tau$-dense in $B$. Now it is an easy exercise to prove that $C \subset \tau\text{-cl}(A)$, i.e., $C$ is $\tau$-separable. $\blacksquare$

**Lemma 4.3.** Let $X$ be a locally convex topological space, $\tau = \sigma(X, X^*)$, and $C \subset X$ a convex subset such that for some $f \in X^*$ there exists a countable subset $R \subset \mathbb{R}$ satisfying:

1. $\emptyset \neq (\inf \{f(x) : x \in C\}, \sup \{f(x) : x \in C\}) \subset \overline{R}$.
2. $C_r := \{x \in C : f(x) = r\}$ is $\tau$-separable for each $r \in R$.

Then $C$ is $\tau$-separable.

*Proof.* By hypothesis $\inf \{f(x) : x \in C\} < \sup \{f(x) : x \in C\}$. For each $r \in R$, choose a countable subset $A_r \subset C_r$ such that $C_r \subset \tau\text{-cl}(A_r)$. Let $A = \bigcup_{r \in R} A_r$, a countable subset of $C$. We claim that $A$ is $\tau$-dense in $C$. Indeed, pick $z_0 \in C$ arbitrarily and let $U$ be a $\tau$-neighborhood of $z_0$ in $C$. By hypothesis, there exists some $r \in R$ such that $C_r \cap U \neq \emptyset$. So, $A_r \cap U \neq \emptyset$, whence $A \cap U \neq \emptyset$. $\blacksquare$

**Proposition 4.4.** Let $X$ be a Banach space. The following are equivalent:

1. $X \in KS_4$.
2. $K \subset X^*$ is $w^*$-separable whenever $K$ is a $w^*$-compact convex symmetric subset such that $\| \cdot \|\text{-int}(K) \neq \emptyset$.
3. $K \subset X^*$ is $w^*$-separable whenever $K$ is a $w^*$-compact convex symmetric subset, i.e., $\sigma(X) = 1 = \tau(X)$.
4. $K \subset X^*$ is $w^*$-separable whenever $K$ is a $w^*$-closed convex symmetric subset.
5. $K \subset X^*$ is $w^*$-separable whenever $K$ is a $w^*$-closed convex subset.

*Proof.* (1)$\Rightarrow$(2). This follows from Propositions 2.7 and 2.2, because if $K \subset X^*$ is a $w^*$-compact convex symmetric subset such that $\| \cdot \|\text{-int}(K) \neq \emptyset$, then $K$ is the dual unit ball of $X^*$ when $X$ is equipped with the equivalent norm $\| \cdot \|$ such that $|x| = \sup \{|x^*(x) : x^* \in K\}$ for every $x \in X$.

(2)$\Rightarrow$(3). Let $K \subset X^*$ be a $w^*$-compact convex symmetric subset and set $K_n = K + \frac{1}{n}B(X^*)$, which is a $w^*$-compact convex symmetric subset of $X^*$ with nonempty interior. By (2) there is a countable family $\{x_{n,m}\}_{m \geq 1} \subset K_n$
such that $K_n = \{x_{n,m} : m \geq 1\}^{w*}$ for every $n \geq 1$. Pick $k_{n,m} \in K$ such that
$\|k_{n,m} - x_{n,m}\| \leq 1/n$. Then it is easy to see that $K = \{k_{n,m} : n, m \geq 1\}^{w*}$.

$(3) \Rightarrow (4)$. Let $K \subset X^*$ be a $w^*$-closed convex symmetric subset and define $K_n = K \cap nB(X^*)$. By $(3)$, $K_n$ is $w^*$-separable and hence so is $K$, because $K = \bigcup_{n \geq 1} K_n$.

$(4) \Rightarrow (5)$. It is enough to prove that if $K \subset X^*$ is a $w^*$-compact convex subset, then $K$ is $w^*$-separable. Without loss of generality, assume that $0 \notin K$. Let $f \in X$ be such that $0 < \min\{f(k) : k \in K\} \leq \max\{f(k) : k \in K\} < \infty$. If $t \in [\min\{f(k) : k \in K\}, \max\{f(k) : k \in K\}]$, define $K_t = \{k \in K : f(k) = t\}$ and $C_t = \overline{w^*}(K_t \cup (-K_t))$. By $(4)$ and Lemma 4.2 each $C_t$ is $w^*$-separable. So, from Lemma 4.3 we conclude that $K$ is $w^*$-separable.

$(5) \Rightarrow (1)$. Suppose that there exists in $X$ a bounded $\omega_1$-polyhedron $\{x_i\}_{i < \omega_1}$. By Proposition 2.2, there exists in $X$ an UBABS $\{x_\alpha, f_\alpha\}_{\alpha < \omega_1} \subset X \times X^*$ such that $\|f_\alpha\| = 1$, $\|x_\alpha\| \leq M$, $f_\alpha(x_\alpha) = 1$ and $f_\alpha(x_\beta) \leq 1 - \varepsilon$ for every $\alpha, \beta < \omega_1$, $\alpha \neq \beta$, and some $1 \geq \varepsilon > 0$, $1 \leq M < \infty$. Let $K = \overline{w^*}([f_\alpha : \alpha < \omega_1])$. Consider the $w^*$-open slices $U_\alpha = \{k \in K : k(x_\alpha) > 1 - \varepsilon/3\}$ for all $\alpha < \omega_1$. Then $U_\alpha$ is a $w^*$-open neighborhood of $f_\alpha$ in $K$ and we can easily see that $U_\alpha \cap U_\beta = \emptyset$ whenever $\alpha \neq \beta$. Thus $K$ is $w^*$-nonseparable, a contradiction to $(5)$. So, $X \in KS_4$.

**Question 3.** Let $X$ be a Banach space. If $\tau(X) < 1$, is $\tau(X) = 0$? If $\tau(X) = 0$, does $X$ have an uncountable $\omega$-independent family?

5. The Finet–Godefroy indices. If $X$ is a Banach space, the *Finet–Godefroy indices* $d_{\infty}(X)$ and $\mu(X)$ were introduced in [1] and defined as follows:

$$d_{\infty}(X) = \inf\{d(X, Y) : Y \text{ a subspace of } \ell_{\infty}(\mathbb{N})\},$$

where $d(X, Y)$ is the Banach–Mazur distance. Clearly, $d_{\infty}(X)$ depends upon the norm $\|\cdot\|$ of $X$ and we see easily that: (i) $d_{\infty}(X) \in [1, \infty]$; (ii) $d_{\infty}(X) < \infty$ iff $X$ is isomorphic to a subspace of $\ell_{\infty}(\mathbb{N})$; (iii) $d_{\infty}(X, \|\cdot\|) = 1$ iff $(X, \|\cdot\|)$ is isometric to a subspace of $\ell_{\infty}(\mathbb{N})$ iff the dual unit ball $B(X^*)$ is $w^*$-separable. The corresponding isomorphic invariant index is

$$\mu(X) = \sup\{d_{\infty}(X, \|\cdot\|)\},$$

where the supremum is computed over the set of equivalent norms on $X$.

**Proposition 5.1.** Let $X$ be a Banach space. Then:

1. $\mu(X) = \sigma(X)^{-1}$ ($0^{-1} = \infty$).
2. If $X$ has an uncountable $\omega$-independent system, then $\mu(X) = \infty$.

**Proof.** (1) This follows from [1, Lemma III.1] and a simple calculation.

(2) By Proposition 3.2 and 2.8 we find that $\sigma(X) = 0$. Now apply (1).
The following questions are proposed in [1]:

(1) It is clear that $\mu(X) = 1$ if $X$ is separable. Is the converse true?
(2) Does there exist a nonseparable Banach space $X$ such that every quotient of $X$ is isometric to a subspace of $\ell_\infty(\mathbb{N})$?

In the following we answer these questions.

**Proposition 5.2.** Let $X$ be a Banach space. The following are equivalent:

(1) $X \in \text{KS}_4$.
(2) Every quotient of $(X, | \cdot |)$ is isometric to a subspace of $\ell_\infty(\mathbb{N})$, for every equivalent norm $| \cdot |$ on $X$.
(3) $\mu(X) = 1$.
(4) Every quotient of $X$ has the property $\text{KS}_4$.

**Proof.** (1)⇒(2). Let $| \cdot |$ be an equivalent norm on $X$, $Y \subset X$ a closed subspace and $Z = (X/Y, | \cdot |)$ the corresponding quotient space. Clearly, $(B(Z^*), w^*) = (B(Y^\perp), w^*)$. But $(B(Y^\perp), w^*)$ is $w^*$-separable by Proposition 4.4. So, $Z$ is isometric to a subspace of $\ell_\infty(\mathbb{N})$.

(2)⇒(3). By (2), $d_\infty(X, | \cdot |) = 1$ for every equivalent norm $| \cdot |$ on $X$. So, $\mu(X) = 1$.

(3)⇒(4). Since $\mu(X/Y) \leq \mu(X)$ for every quotient $X/Y$ (see [1, Th. III-2]), (3) implies that $\mu(X/Y) = 1$, i.e., $\sigma(X/Y) = 1$. So, by Proposition 4.4 we infer that $X/Y \in \text{KS}_4$.

(4)⇒(1). This is obvious. □

**Corollary 5.3.** If $X$ is either the space $C(K)$, under CH and $K$ being the Kunen compact space, or the space $S$ of Shelah, under $\diamondsuit_{\aleph_1}$, then $X$ is nonseparable, $\mu(X) = 1$ and every quotient of $(X, | \cdot |)$ is isometric to a subspace of $\ell_\infty(\mathbb{N})$, for every equivalent norm $| \cdot |$ of $X$.

**Proof.** This follows from Proposition 5.2 since in both cases $X \in \text{KS}_4$ (see Section 6). □

**Remarks.** (1) The fact that every quotient of $(X, | \cdot |)$ is isometric to a subspace of $\ell_\infty(\mathbb{N})$ for every equivalent norm $| \cdot |$ of $X$, when $X = C(K)$, $K$ being the Kunen compact, was shown in [4, Cor. 4.5].

(2) In [1] it is asked if $\mu(X) = \infty$ whenever the Banach space $X$ satisfies $\mu(X) > 1$. In fact, no Banach space $X$ with $1 < \mu(X) < \infty$ is known. Observe that $1 < \mu(X) < \infty$ implies that $X \in \text{KS}_3$ but $X \not\in \text{KS}_4$, because: (i) $1 < \mu(X) < \infty$ iff $1 > \sigma(X) > 0$ by Proposition 5.1; (ii) $1 > \sigma(X)$ iff $X \not\in \text{KS}_4$ by Proposition 4.4; and (iii) $\sigma(X) > 0$ implies $X \in \text{KS}_3$ by Propositions 3.2 and 2.8.
6. The Kunen–Shelah property $KS_5$. Let $\theta$ be an ordinal. A convex right-separated $\theta$-family in a Banach space $X$ is a bounded family $\{x_i\}_{i<\theta} \subseteq X$ such that $x_j \notin \overline{co}(\{x_i : j < i < \theta\})$ for every $j \in \theta$. A family $\{C_\alpha\}_{\alpha<\theta}$ of convex closed bounded subsets in $X$ is said to be a contractive (resp. expansive) $\theta$-onion iff $C_\alpha \subsetneq C_\beta$ (resp. $C_\beta \subsetneq C_\alpha$) whenever $\beta < \alpha < \theta$. It is easy to prove that $X$ has a contractive $\theta$-onion iff $X$ has a convex right-separated $\theta$-family. In the dual Banach space $X^*$ one can define a contractive (resp. expansive) $w^*$-$\theta$-onion in an analogous way, using the $w^*$-topology instead of the $w$-topology.

A Banach space $X$ is said to have the Kunen–Shelah property $KS_5$ if $X$ has no contractive uncountable onion. If $X$ has a $\tau$-polyhedron $\{x_\alpha : \alpha < \tau\}$, it is clear that $\{C_\alpha : \alpha < \tau\}$, where $C_\alpha = \overline{co}(\{x_\beta : \alpha < \beta < \tau\})$, is a contractive $\tau$-onion. So, the property $KS_5$ implies $KS_4$, whence by Proposition 3.2 we get $KS_5 \Rightarrow KS_3$, a result proved by Sersouri in [12].

**Proposition 6.1.** Let $X$ be a Banach space. Then:

1. $X$ has a contractive $\omega_1$-onion iff $X^*$ has an expansive $w^*$-$\omega_1$-onion.
2. $X$ has an expansive $\omega_1$-onion iff $X^*$ has a contractive $w^*$-$\omega_1$-onion.
3. $X$ is nonseparable iff $X^*$ has a contractive $w^*$-$\omega_1$-onion.

**Proof.** (1) Assume that $X$ has a contractive $\omega_1$-onion, i.e., there exists a sequence $\{x_\alpha\}_{\alpha<\omega_1} \subset B(X)$ such that $x_\alpha \notin \overline{co}(\{x_\beta\}_{\alpha<\beta<\omega_1})$. By the Hahn–Banach Theorem there exists $f_\alpha \in X^*$ such that

$$f_\alpha(x_\alpha) > \sup\{f_\alpha(x_\beta) : \alpha < \beta < \omega_1\} =: e_\alpha.$$

By passing to a subsequence, we can suppose that there exist $0 < \varepsilon, M < \infty$ and $r \in \mathbb{R}$ such that $\|f_\alpha\| \leq M$, $f_\alpha(x_\alpha) - e_\alpha \geq \varepsilon > 0$ and $|r - f_\alpha(x_\alpha)| \leq \varepsilon/4$ for all $\alpha < \omega_1$. Hence, if $\beta < \alpha < \omega_1$, we have

$$f_\alpha(x_\alpha) \geq r - \varepsilon/4 > r - 3\varepsilon/4 \geq f_\beta(x_\beta) - \varepsilon \geq e_\beta \geq f_\beta(x_\alpha),$$

which implies that $f_\alpha \notin \overline{co}^w(\{f_\beta : \beta < \alpha\}) =: K_\alpha$, i.e., $\{K_\alpha : \alpha < \omega_1\}$ is an expansive $w^*$-$\omega_1$-onion in $X^*$.

The converse implication is analogous.

(2) Use the same argument as in (1).

(3) Apply (2) and the fact that $X$ has an expansive $\omega_1$-onion iff $X$ is nonseparable. 

A Banach space has the property HL(1) (for short, $X \in HL(1)$) whenever for every family $\{U_i\}_{i \in I}$ of open semi-spaces of $X$ there exists a countable subset $\{i_n\}_{n \geq 1} \subset I$ such that $\bigcup_{n \geq 1} U_{i_n} = \bigcup_{i \in I} U_i$, i.e., every closed convex subset of $X$ is the intersection of a countable family of closed semi-spaces of $X$. 

Proposition 6.2. Let $X$ be a Banach space. Then the following are equivalent:

1. $X \in \text{KS}_5$.
2. Every convex subset of $X^*$ is $w^*$-separable.
3. $X \in \text{HL}(1)$.

Proof. (1)$\Leftrightarrow$(2). By Proposition 6.1, $X$ has no contractive uncountable onion iff $X^*$ has no expansive uncountable $w^*$-onion, and it is trivial to prove that this occurs iff every convex subset of $X^*$ is $w^*$-separable.

(2)$\Rightarrow$(3). Suppose that $X \not\in \text{HL}(1)$ and let $\mathfrak{F} = \{U_i\}_{i<\omega_1}$ be an uncountable family of open semi-spaces of $X$ such that $\mathfrak{F}$ has no countable subcover. Assume that $U_i = \{x \in X : x_i^*(x) < a_i\}$ with $a_i \neq 0$ for all $i < \omega_1$ (if $a_i = 0$ for some $i < \omega_1$, we replace $U_i$ by the family $U_{in} = \{x \in X : x_i^*(x) < -1/n\}$, $n \geq 1$). Dividing by $|a_i|$, we can suppose that each $U_i$ has the expression $U_i = \{x \in X : y_i^*(x) < \varepsilon_i\}$ with $\varepsilon_i = \pm 1$ and $y_i^* = x_i^*/|a_i|$. Set $\mathfrak{F}_1 = \{U_i \in \mathfrak{F} : \varepsilon_i = +1\}$ and $\mathfrak{F}_2 = \{U_i \in \mathfrak{F} : \varepsilon_i = -1\}$. It is clear that either $\mathfrak{F}_1$ or $\mathfrak{F}_2$ has no countable subcover.

Assume that $\mathfrak{F}_1$ does not admit a countable subcover (the argument for $\mathfrak{F}_2$ is similar). So, there exists an uncountable family $\{V_\alpha : \alpha < \omega_1\} \subset \mathfrak{F}_1$, $V_\alpha = \{x \in X : z_\alpha^*(x) < 1\}$, such that there exist $x_\alpha \in V_\alpha \setminus \bigcup_{\beta < \alpha} V_\beta$ for each $\alpha < \omega_1$. Put $A = \text{co}\{z_\alpha^*\}_{i<\omega_1}$, which is $w^*$-separable by hypothesis. Thus, we can find $\varrho < \omega_1$ such that $A \subset \overline{\text{co}}^{w^*}\{z_\alpha^*\}_{i \leq \varrho}$. Pick $\varrho < \alpha < \omega_1$. As $x_\alpha \in V_\alpha \setminus \bigcup_{\beta < \alpha} V_\beta$, we see that $z_\alpha^*(x_\alpha) < 1$ and $z_\beta^*(x_\alpha) \geq 1$ for every $\beta < \alpha$. Let $C = \{x^* \in X^* : x^*(x_\alpha) \geq 1\}$, which is a convex $w^*$-closed subset of $X^*$. Since $z_\alpha^* \in C$ for all $i \leq \varrho$, it follows that $A \subset C$. So, $z_\alpha^* \not\in C$ and $z_\alpha^* \in A$, a contradiction which proves (3).

(3)$\Rightarrow$(1). Suppose that $X$ has a contractive $\omega_1$-onion $\{C_\alpha\}_{\alpha < \omega_1}$. We choose vectors $x_\alpha \in C_\alpha \setminus C_{\alpha+1}$ and a sequence $\{U_\alpha\}_{\alpha < \omega_1}$ of open semi-spaces such that $x_\alpha \in U_\alpha$ and $U_\alpha \cap C_{\alpha+1} = \emptyset$. Clearly, no countable subfamily of $\{U_\alpha\}_{\alpha < \omega_1}$ covers $\{x_\alpha\}_{\alpha < \omega_1}$, which contradicts (3).

Remark. If $X$ is a Banach space, we write $X \in \text{L}(1)$ if from every cover of $X$ by open semi-spaces we can choose a countable subcover. Clearly, $X$ has the property (C) of Corson iff $X \in \text{L}(1)$. Since $X \in \text{HL}(1) \Rightarrow X \in \text{L}(1)$, we find that $X \in \text{KS}_5$ implies $X \in \text{(C)}$.

Proposition 6.3. If $X$ is either the space $C(K)$, under CH and $K$ being the Kunen compact space, or the space $S$ of Shelah, under $\diamondsuit_{\aleph_1}$, then $X \in \text{KS}_5$.

Proof. The space $C(K)$, $K$ being the Kunen compact space, satisfies $C(K) \in \text{KS}_5$ because for every uncountable family $\{x_i : i \in I\} \subset C(K)$, there exists $j \in I$ such that $x_j \in \text{wcl}(\{x_i : i \in I \setminus \{j\}\})$ (wcl = weak closure). It is clear that a space with this property cannot have an $\omega_1$-onion.
The space $S$ of Shelah has the property (see [13, Lemma 5.2]) that if \( \{y_i\}_{i<\omega_1} \subseteq S \) is an uncountable sequence, then for every $\varepsilon > 0$ and $n \geq 1$, there exist $i_0 < i_1 < \ldots < i_n < \omega_1$ such that

\[
\left\| y_{i_0} - \frac{1}{n} (y_{i_1} + \ldots + y_{i_n}) \right\| \leq \frac{1}{n} \left\| y_{i_0} \right\| + \varepsilon.
\]

Assume that $S$ has an $\omega_1$-onion $\{C_\alpha : 1 \leq \alpha < \omega_1\}$, with $C_1 \subseteq B(S)$. Choose $x_\alpha \in C_\alpha \setminus C_{\alpha+1}$ and let $\eta_\alpha := \text{dist}(x_\alpha, C_{\alpha+1})$, which satisfies $\eta_\alpha > 0$. By passing to a subsequence, it can be assumed that $\eta_\alpha \geq \eta > 0$ for all $\alpha < \omega_1$. Let $m \in \mathbb{N}$ satisfy $1/m < \eta/2$. By (7) there exist $i_0 < i_1 < \ldots < i_m < \omega_1$ such that

\[
\left\| x_{i_0} - \frac{1}{m} (x_{i_1} + \ldots + x_{i_m}) \right\| \leq \frac{1}{m} \left\| x_{i_0} \right\| + \frac{\eta}{2} < \eta.
\]

Since $\frac{1}{m}(x_{i_1} + \ldots + x_{i_m}) \in C_{i_0+1}$ and $\text{dist}(x_{i_0}, C_{i_0+1}) \geq \eta$, we get a contradiction which proves that $S \in \text{KS}_5$. \[\square\]

**7. KS\(_4\) and KS\(_5\) are equivalent.** If $X$ is Asplund or has the property (C) of Corson, it is easy to prove that $X \in \text{KS}_4 \iff X \in \text{KS}_5$. In the following we prove the equivalence $\text{KS}_5 \iff \text{KS}_4$ in general. A sequence $\{C_\alpha : \alpha < \omega_1\}$ of convex closed bounded subsets of a Banach space $X$ is said to be a **generalized $\omega_1$-onion** if $\emptyset \neq C_\alpha \subset C_\beta$ for $\beta < \alpha$, and there exists a subsequence $\{\alpha_\beta\}_{\beta < \omega_1} \subset \omega_1$, with $\alpha_\beta_1 < \alpha_\beta_2$ if $\beta_1 < \beta_2$, such that $C_{\alpha_\beta_1} \neq C_{\alpha_\beta_2}$, i.e., $\{C_\beta : \beta < \omega_1\}$ is an $\omega_1$-onion. For $C \subset X$, denote by $\text{cone}(C)$ the closed convex cone generated by $C$. Observe that if $C$ is convex, then $\text{cone}(C) = \text{cl}(\bigcup_{\lambda \geq 0} \lambda C)$.

**Lemma 7.1.** Let $X$ be a Banach space, $C \subset X$ a convex closed separable subset and $\{C_\alpha : 1 \leq \alpha < \omega_1\}$ a generalized $\omega_1$-onion in $X$.

1. If $\text{dist}(C, C_\alpha) = 0$ for every $\alpha < \omega_1$, then for every $\varepsilon > 0$ there exists $c_\varepsilon \in C$ such that $\text{dist}(c_\varepsilon, C_\alpha) \leq \varepsilon$ for every $\alpha < \omega_1$.

2. There are two mutually exclusive alternatives: either

   A. there exist two ordinals $\beta < \alpha < \omega_1$ and $z \in C_\beta$ such that $z \not\in \overline{\text{co}}([C] \cup \text{cone}(C_\alpha))$ or

   B. for every pair of ordinals $\beta < \alpha < \omega_1$ we have $C_\beta \subset \overline{\text{co}}([C] \cup \text{cone}(C_\alpha))$. In this case,

\[
\overline{\text{co}}([C] \cup \text{cone}(C_\alpha)) = \overline{\text{co}}([C] \cup \text{cone}(C_\beta)), \quad \forall \alpha, \beta < \omega_1,
\]

and for every $\varepsilon > 0$ there exists $c_\varepsilon \in X$ such that $\text{dist}(c_\varepsilon, C_\alpha) \leq \varepsilon$ for every $\alpha < \omega_1$. \[\square\]
Proof. (1) For every $\alpha < \omega_1$ and $n \geq 1$ consider $C(\alpha, n) = \{x \in C : \dist(x, C_\alpha) \leq 1/n\}$. Then $\{C(\alpha, n) : \alpha < \omega_1\}$ is a family of nonempty closed convex subsets such that $C(\alpha, n) \supseteq C(\beta, n)$ if $\alpha < \beta$, with the countable intersection property. Since $C$ is separable, we conclude that $\bigcap_{\alpha < \omega_1} C(\alpha, n) \neq \emptyset$ for every $n \geq 1$. So, if for every $n \geq 1$ we pick $c_n \in \bigcap_{\alpha < \omega_1} C(\alpha, n)$, then $\dist(c_n, C_\alpha) \leq 1/n$ for every $\alpha < \omega_1$.

(2) Clearly, the alternatives (A) and (B) are mutually exclusive. Suppose that (B) holds. Since $[C]$ is separable there exist two ordinals $\beta_0 < \alpha_0 < \omega_1$ and $z_0 \in C_{\beta_0} \setminus C_{\alpha_0}$ such that $z_0 \not\in [C]$ but $z_0 \in \overline{\cup}(\sigma([C] \cup \text{cone}(C_\alpha)))$ for every $\alpha < \omega_1$.

Claim. If $H = [C \cup \{z_0\}]$, then $\dist(H, C_\alpha) = 0$ for every $\alpha < \omega_1$.

Indeed, let $\varepsilon_0 = \dist(z_0, [C])$ and $n_0 \geq 1$ be such that $2/n_0 < \varepsilon_0$. Observe that for every $\alpha < \omega_1$ and $\varepsilon > 0$ we can choose $\lambda \in [0, 1)$, $\mu > 0$, $w \in [C]$ and $v \in C_\alpha$ such that

$$
(8) \quad \|\lambda w + (1 - \lambda)\mu v - z_0\| \leq \varepsilon.
$$

Let $M > 0$ be such that $C_1 \subset B(0, M)$. We claim that if we pick $\alpha < \omega_1$, $n \geq n_0$, $\lambda \in [0, 1)$, $\mu > 0$, $w \in [C]$ and $v \in C_\alpha$ satisfying (8) with $\varepsilon = 1/n$, then $(1 - \lambda)\mu \geq 1/(n_0 M)$. Indeed, otherwise

$$
\varepsilon_0 \leq \|\lambda w - z_0\| = \|\lambda w + (1 - \lambda)\mu v - z_0 - (1 - \lambda)\mu v\|
$$

$$
\leq \|\lambda w + (1 - \lambda)\mu v - z_0\| + \|(1 - \lambda)\mu v\|
$$

$$
\leq \frac{1}{n_0} + \frac{1}{n_0} < \varepsilon_0,
$$

which is a contradiction. So, for every $\alpha$, $n$, $\lambda$, $\mu$, $w$ and $v$ as above we have

$$
\left\| \frac{z_0}{(1 - \lambda)\mu} - \frac{\lambda}{(1 - \lambda)\mu} w - v \right\| \leq \frac{1}{(1 - \lambda)\mu n} \leq \frac{n_0 M}{n},
$$

and this proves that $\dist(H, C_\alpha) = 0$ for every $\alpha < \omega_1$.

As $H$ is separable, given $\varepsilon > 0$, applying (1) we can choose $c_\varepsilon \in X$ such that $\dist(c_\varepsilon, C_\alpha) \leq \varepsilon$ for every $\alpha < \omega_1$, and this completes the proof. 

Proposition 7.2. Let $X$ be a Banach space without the property (C) of Corson. Then there exists a sequence $\{(y_\alpha, y_\alpha^*) : \alpha < \omega_1\} \subset X \times X^*$ such that $y_\alpha^*(y_\alpha) = 1$ for all $\alpha < \omega_1$ but $y_\beta^*(y_\beta) = 0$ if $\beta < \alpha$, and $y_\alpha^*(y_\beta) \leq 0$ if $\beta > \alpha$. So, $X$ has an $\omega_1$-polyhedron and $X \notin KS_4$.

Proof. Since $X$ fails (C), it is easy to see that there exists in $X$ an $\omega_1$-onion $\{C_\alpha : \alpha < \omega_1\}$ such that $\bigcap_{\alpha < \omega_1} C_\alpha = \emptyset$. Using transfinite induction with $\omega_1$ steps we construct:
(1) A sequence \( \{n_\alpha : \alpha < \omega_1\} \subset \{0, 1\} \) such that if \( p(\alpha) = \left| \{\beta \leq \alpha : n_\beta = 1\} \right| \) then \( p(\alpha) < \aleph_0 \).

(2) Two sequences \( \{\varrho_\gamma, \tau_\gamma : \gamma < \omega_1\} \) of ordinals such that \( 1 \leq \varrho_\gamma < \tau_\gamma \leq \varrho_\beta < \omega_1 \) if \( \gamma < \beta < \omega_1 \).

(3) For each \( \alpha < \omega_1 \), a generalized \( \omega_1 \)-onion \( \{C^{(\alpha)}_\beta : \varrho_\alpha \leq \beta < \omega_1\} \) such that \( C_\gamma \supset C^{(\alpha)}_\gamma \supset C^{(\beta)}_\gamma \neq \emptyset \) if \( \alpha \leq \beta < \omega_1 \) and \( \varrho_\beta \leq \gamma < \omega_1 \).

(4) For each \( \alpha \) with \( n_\alpha = 0 \), an element \( y_\alpha \in C^{(\alpha)}_{\varrho_\alpha} \) such that if \( H_\alpha = [\{y_\beta : \beta < \alpha, n_\beta = 0\}] \) then \( y_\alpha \notin \overline{\text{co}}(H_\alpha \cup \text{cone}(C^{(\alpha)}_{\tau_\alpha})) \). Also, in this case we demand that \( C^{(\alpha)}_{\gamma} = \bigcap_{\beta < \alpha} C^{(\beta)}_{\gamma} \) for every \( \varrho_\alpha \leq \gamma < \omega_1 \).

(5) For each \( \alpha \) with \( n_\alpha = 1 \), a vector \( a_{p(\alpha)} \in X \) such that \( C^{(\alpha)}_\beta \subset B(a_{p(\alpha)}, 2^{-p(\alpha)}) \) for every \( \tau_\alpha \leq \beta < \omega_1 \), which will imply that \( \text{diam}(C^{(\alpha)}_\beta) \leq 2^{-p(\alpha) + 1} \), \( \text{dist}(a_{p(\alpha)}, C^{(\alpha)}_\beta) \leq 2^{-p(\alpha)}, \forall \tau_\alpha \leq \beta < \omega_1 \).

**STEP 1.** We choose \( n_1 = 0, \varrho_1 = 1, \tau_1 = 2, C^{(1)}_\beta = C_\beta \) for every \( 1 \leq \beta < \omega_1 \), \( y_1 \in C_1 \setminus C_2 \) arbitrary and \( H_1 = \{0\} \).

**STEP \( \alpha + 1 < \omega_1 \).** Suppose all the steps \( \beta \leq \alpha \) satisfying the above requirements are constructed. By hypothesis \( \{C^{(\alpha)}_\beta : \tau_\alpha \leq \beta < \omega_1\} \) is a generalized \( \omega_1 \)-onion. By Lemma 7.1 there are two mutually exclusive alternatives:

(A) There exist two ordinals \( \tau_\alpha \leq \beta_0 < \alpha_0 < \omega_1 \) and a vector \( z_0 \in C^{(\alpha)}_{\beta_0} \) such that \( z_0 \notin \overline{\text{co}}(H_\alpha \cup \text{cone}(C^{(\alpha)}_{\alpha_0})) \). Then we set \( \varrho_{\alpha + 1} = \beta_0, \tau_{\alpha + 1} = \alpha_0, n_{\alpha + 1} = 0, y_{\alpha + 1} = z_0 \) and \( C^{(\alpha + 1)}_\beta = C^{(\alpha)}_\beta \) for every \( \varrho_{\alpha + 1} \leq \beta < \omega_1 \).

(B) If (A) does not hold, there exists \( c \in X \) such that \( \text{dist}(c, C^{(\alpha)}_\beta) \leq 2^{-p(\alpha) + 2} \) for every \( \tau_\alpha \leq \beta < \omega_1 \). In this case we set \( n_{\alpha + 1} = 1, p(\alpha + 1) = p(\alpha) + 1, \varrho_{\alpha + 1} = \tau_\alpha, \tau_{\alpha + 1} = \tau_\alpha + 1, a_{p(\alpha + 1)} = c \) and \( C^{(\alpha + 1)}_\beta = B(a_{p(\alpha + 1)}, 2^{-p(\alpha + 1)}) \cap C^{(\alpha)}_\beta \) for every \( \varrho_{\alpha + 1} \leq \beta < \omega_1 \). Since \( n_{\alpha + 1} = 1 \) we do not choose \( y_{\alpha + 1} \).

**STEP \( \alpha < \omega_1 \), \( \alpha \) a limit ordinal.** Let \( \alpha < \omega_1 \) be a limit ordinal, and suppose all the steps \( \beta < \alpha \) satisfying the above requirements are constructed.

**CLAIM.** \( \left| \{\beta < \alpha : n_\beta = 1\} \right| < \aleph_0 \).

Indeed, otherwise we would have a sequence of ordinals \( \{\beta_m\}_{m \geq 1} \uparrow \alpha \), with \( \beta_m < \beta_{m + 1} < \alpha \), such that \( n_{\beta_m} = 1 \) for every \( m \geq 1 \). Obviously \( p(\beta_m) \uparrow +\infty \) as \( m \to \infty \). The sequence \( \{a_{p(\beta_m)}\}_{m \geq 1} \) is a Cauchy sequence. Indeed, if \( r < s \) are two integers, then for every \( \tau_{\beta_s} \leq \beta < \omega_1 \), since \( C^{(\beta_s)}_\beta \subset C^{(\beta_r)}_\beta \), we have
\[ \text{dist}(a_p(\beta_r), a_p(\beta_s)) \leq \text{dist}(a_p(\beta_r), C_\beta^{(\beta_r)}) + \text{diam}(C_\beta^{(\beta_r)}) + \text{dist}(a_p(\beta_s), C_\beta^{(\beta_r)}) \]

\[ \leq 2^{-p(\beta_r)} + 2^{-p(\beta_r)+1} + 2^{-p(\beta_s)} \quad \tau, s \to \infty. \]

Let \( a_0 := \lim_{m \to \infty} a_p(\beta_m) \) and \( \gamma_0 = \sup\{\tau_\beta : \beta < \alpha\} \). Then \( a_0 \in C_\gamma \) for every \( \gamma_0 \leq \gamma < \omega_1 \) because

\[ \text{dist}(a_0, C_\gamma) \leq \text{dist}(a_0, a_p(\beta_m)) + \text{dist}(a_p(\beta_m), C_\gamma^{(\beta_m)}) \xrightarrow{m \to \infty} 0. \]

Hence \( \bigcap_{\alpha < \omega_1} C_\alpha \neq \emptyset \), a contradiction which proves the Claim.

Define as above \( \gamma_0 = \sup\{\tau_\beta : \beta < \alpha\} \) and let \( D_\gamma := \bigcap_{\beta < \alpha} C_\gamma^{(\beta)} \) for every \( \gamma_0 \leq \gamma < \omega_1 \). By the Claim and the construction of the previous steps we have:

(a) There exists an ordinal \( \delta_0 < \alpha \) such that \( n_\delta = 0 \) for every \( \delta_0 \leq \delta < \alpha \). So, \( p(\delta) = p(\delta_0) \) for every \( \delta \in [\delta_0, \alpha) \).

(b) For every \( \gamma_0 \leq \gamma < \omega_1 \) we have \( D_\gamma = C_\gamma^{(\delta_0)} \), which by the induction hypothesis implies that \( \{D_\gamma : \gamma_0 \leq \gamma < \omega_1\} \) is a generalized \( \omega_1 \)-onion.

If \( H_\alpha := \{\{y_\beta : \beta < \alpha, n_\beta = 0\}\} \), by Lemma 7.1 we have the following mutually exclusive alternatives:

(A) There are two ordinals \( \gamma_0 \leq \beta < \alpha_0 < \omega_1 \) and a vector \( z_0 \in D_\beta_0 \) such that \( z_0 \notin \overline{\text{co}}(H_\alpha \cup \text{cone}(D_{\alpha_0})) \). In this case we set \( \varrho_\alpha = \beta_0, \tau_\alpha = \alpha_0, n_\alpha = 0, y_\alpha = z_0 \) and \( C_\alpha^{(\alpha)} = D_\beta \) for every \( \varrho_\alpha \leq \beta < \omega_1 \).

(B) If (A) does not hold, there exists \( c \in X \) such that \( \text{dist}(c, D_\gamma) \leq 2^{-p(\delta_0)+2} \) for every \( \gamma_0 \leq \gamma < \omega_1 \). In this case we set \( n_\alpha = 1, p(\alpha) = p(\delta_0) + 1, \varrho_\alpha = \gamma_0, \tau_\alpha = \varrho_\alpha + 1, a_p(\alpha) = c \) and \( C_\alpha^{(\alpha)} = B(a_p(\alpha), 2^{-p(\alpha)}) \cap D_\gamma \) for \( \gamma_0 \leq \gamma < \omega_1 \). Since \( n_\alpha = 1 \) we do not choose \( y_\alpha \).

This completes the induction.

Obviously, there exists \( \varrho < \omega_1 \) such that \( n_\varrho = 0 \) for every \( \varrho \leq \alpha < \omega_1 \), which gives us the sequence \( \{y_\alpha : \varrho \leq \alpha < \omega_1\} \) such that

\[ y_\alpha \notin \overline{\text{co}}(\{y_\beta : \varrho \leq \beta < \alpha\}) \cup \text{cone}(\{y_\beta : \alpha < \beta < \omega_1\}) =: K_\alpha \]

for every \( \varrho \leq \alpha < \omega_1 \). Therefore, by the Hahn–Banach Theorem there exists \( y_\alpha^* \in X^* \) such that \( y_\alpha^*(y_\alpha) = 1 \) but \( \sup\{y_\alpha^*(y) : y \in K_\alpha\} < 1 \). In particular, \( y_\alpha^*(y_\beta) = 0 \) if \( \varrho \leq \beta < \alpha \), and \( y_\alpha^*(y_\beta) \leq 0 \) if \( \alpha < \beta < \omega_1 \).  

**Proposition 7.3.** Let \( X \) be a Banach space. We have:

1. If \( X \in KS_4 \), then \( X \in (C) \).
2. \( X \in KS_4 \) iff \( X \in KS_5 \).

**Proof.** (1) This follows from Proposition 7.2 where it is proved that if \( X \notin (C) \) then \( X \) has an \( \omega_1 \)-polyhedron.
(2) Clearly, $X \in \text{KS}_5$ implies $X \in \text{KS}_4$. Assume that $X \in \text{KS}_4$. By (1) we see that $X \in (C)$. In order to prove that $X \in \text{KS}_5$, by Proposition 6.2 it is enough to prove that every convex subset $C \subset X^*$ is $w^*$-separable. Since $X \in \text{KS}_4$, $\overline{C}^{w^*}$ is $w^*$-separable by Proposition 4.4. So, there exists a countable family $\{z_n : n \geq 1\} \subset \overline{C}^{w^*} w^*$-dense in $\overline{C}^{w^*}$. Since $X \in (C)$, by \cite[10, p. 147]{10} there exists a countable family $\{z_{nm} : n, m \geq 1\} \subset C$ such that $z_n \in \overline{C}^{w^*}(\{z_{nm} : m \geq 1\})$ for every $n \geq 1$. So, $C$ is $w^*$-separable. 

Remarks. A nonseparable Banach space $X$ has the Kunen–Shelah property $\text{KS}_6$ if for every uncountable family $\{x_i\}_{i \in I} \subset X$ there exists $j \in I$ such that $x_j \in \text{wel}(\{x_i\}_{i \in I \setminus \{j\}})$. Clearly, $\text{KS}_6 \Rightarrow \text{KS}_5$. It seems that the only known example of a Banach space $X$ such that $X \in \text{KS}_6$ is the space $X = C(K)$, $K$ being the Kunen compact space \cite[8, p. 1123]{8} constructed by Kunen under CH. This space $C(K)$ of Kunen has more interesting pathological properties. For example, $((C(K))^n, w^n)$ is hereditarily Lindelöf for every $n \in \mathbb{N}$.

In view of this situation, we can introduce the property $\text{KS}_7$. A Banach space $X$ is said to have the Kunen–Shelah property $\text{KS}_7$ if $(X^n, w^n)$ is hereditarily Lindelöf for every $n \in \mathbb{N}$. It can be easily proved that $\text{KS}_7 \Rightarrow \text{KS}_6$.

We know neither if the Shelah space $S$ has the property $\text{KS}_7$ nor if the properties $\text{KS}_5$, $\text{KS}_6$ and $\text{KS}_7$ are inequivalent.

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