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Compact operators between K- and J-spaces

by

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Dedicated to Professor Hans Triebel on the occasion of his 65th birthday

Abstract. The paper establishes necessary and sufficient conditions for compactness of operators acting between general K-spaces, general J-spaces and operators acting from a J-space into a K-space. Applications to interpolation of compact operators are also given.

1. Introduction. Interpolation of compact operators is one of the most active research areas in interpolation theory. Many authors have worked on this subject since the beginning of abstract interpolation theory in the early 1960s. During the last twenty years, new tools have been developed which are intimately related to the type of the interpolation method under consideration. Nowadays, still a lot of work is being done along different lines.

Talking only about the real method, it was shown in the joint papers of one of the present authors with Edmunds and Potter [7], with Fernandez [8] and with Peetre [11] that properties of the vector-valued sequence spaces that come up when defining the real interpolation space $(A_0, A_1)_{\theta,q}$ are very useful to study the behaviour of compact operators under interpolation. These efforts culminated with Cwikel's [13] proof that if $T : \bar{A} \to \bar{B}$ with $T : A_0 \to B_0$ compact then $T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ is also compact. Later, the approach developed in [7, 8, 11] was used by Cobos, Kühn and Schonbek [9] to give a broad generalization of Cwikel's result, including a function parameter version and even compactness theorems for other interpolation methods. Techniques related to vector-valued sequence spaces have also turned out to be useful to study compactness in the multidimensional case and in the case of infinite families, as can be seen in the papers by Cobos and Peetre [12] and Carro and Peetre [5], respectively.

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A different aspect of these ideas has been studied very recently by another of the present authors in [17], where she has characterized compact operators between real interpolation spaces in terms of weaker compactness conditions and convergence of certain sequences of operators involving projections on vector-valued sequence spaces. For the complex interpolation method, a somewhat similar problem has also been studied recently by Schonbek [27].

In the present paper, we continue the research of [17] working now with general K-functors and general J-functors. The interest of these interpolation methods has been pointed out by many authors. See, for example, the books by Peetre [26], Ovchinnikov [25] and Brudnyĭ and Krugljak [4], as well as the papers by Cwikel and Peetre [14], Janson [21], Nilsson [24] and Evans and Opic [16].

We establish here necessary and sufficient conditions for compactness of operators acting between K-spaces, between J-spaces and from a J-space into a K-space. The characterizations consist of a weaker compactness condition and convergence of certain sequences of operators involving the K- and J-functionals. They are based on the methods developed in [7–9, 11]. We also show by means of examples that conditions required on the sequence space that define the K- and J-functors are essential for the results.

When we specialize the results we recover the theorems of [17], but we also obtain new information. In particular, we get a characterization of compact operators between real interpolation spaces that blends conditions found in [17]. This last result shows the optimality of the arguments used by Cobos, Kühn and Schonbek in [9, Thm. 1.3].

As another application of our characterizations we obtain extended versions of compactness results of [7–9, 11, 13]. Our new approach provides a clear explanation of the different assumptions needed for them, and gives a better understanding of the original results. We also compare the behaviour under interpolation of compact operators with the behaviour of weakly compact operators. As is well known (see the book by Beauzamy [1] or the paper by Mastyło [22]), if $T : A_0 \cap A_1 \to B_0 + B_1$ is weakly compact, then the interpolated operator is weakly compact as well. This is not the case in general for compact operators. Nevertheless, our characterizations enable us to determine when compactness of $T : A_0 \cap A_1 \to B_0 + B_1$ transmits to the interpolated operator.

The organization of the paper is as follows. In Section 2 we recall definitions of general K- and J-spaces and, for later use, we establish a number of auxiliary results. Theorems for K-spaces are contained in Section 3 and those for J-spaces are in Section 4. In Section 5 we deal with operators acting from a J-space into a K-space. Finally, in Section 6, we compare the behaviour under interpolation of weakly compact operators with the behaviour of compact operators.

2. K- and J-spaces. We start by recalling several notions from interpolation theory (cf. [2–4, 28]). Let $\bar{A} = (A_0, A_1)$ be a Banach couple, that is to say, A_0 and A_1 are Banach spaces continuously embedded in some Hausdorff topological vector space. We endow $A_0 + A_1$ [respectively $A_0 \cap A_1$] with the norm $K(1, \cdot)$ [respectively $J(1, \cdot)$] where for $t > 0$ we put

$$K(t, a) = K(t, a; A_0, A_1) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1,\ a_i \in A_i\}$$

and $J(t, a) = J(t, a; A_0, A_1) = \max\{\|a\|_{A_0},\ t\|a\|_{A_1}\}$.

Let ℓ_q ($1 \le q \le \infty$) and c_0 be the usual spaces of q-summable [respectively, null] scalar sequences with $\mathbb{Z}$ as index set. Given any sequence $\{\omega_m\}$ of positive numbers, we put $\ell_q(\omega_m) = \{\xi = \{\xi_m\} : \{\omega_m \xi_m\} \in \ell_q\}$ and define $c_0(\omega_m)$ similarly. We denote by $\bar{\ell}_q$ the Banach couple $(\ell_q, \ell_q(2^{-m}))$.

Let Γ be a Banach space of real-valued sequences with $\mathbb{Z}$ as index set. Assume that Γ contains all sequences with only finitely many non-zero coordinates, and that whenever $|\xi_m| \le |\mu_m|$ for each $m \in \mathbb{Z}$ and $\{\mu_m\} \in \Gamma$, then $\{\xi_m\} \in \Gamma$ and $\|\{\xi_m\}\|_\Gamma \le \|\{\mu_m\}\|_\Gamma$.

Following the terminology of Nilsson [24], Γ is said to be *K-non-trivial* if

$$\{\min(1, 2^m)\} \in \Gamma. \tag{2.1}$$

We say that the space Γ is *J-non-trivial* if

$$\sup\Big\{ \sum_{m=-\infty}^{\infty} \min(1, 2^{-m})|\xi_m| : \|\xi\|_\Gamma \le 1\Big\} < \infty. \tag{2.2}$$

Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let Γ be a K-non-trivial sequence space. The *K-space* $\bar{A}_{\Gamma;K} = (A_0, A_1)_{\Gamma;K}$ consists of all $a \in A_0 + A_1$ such that $\{K(2^m, a)\} \in \Gamma$. We put $\|a\|_{\bar{A}_{\Gamma;K}} = \|\{K(2^m, a)\}\|_\Gamma$.

If Γ is J-non-trivial, the *J-space* $\bar{A}_{\Gamma;J} = (A_0, A_1)_{\Gamma;J}$ is defined as the collection of all sums $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$), where $\{u_m\} \subseteq A_0 \cap A_1$ and $\{J(2^m, u_m)\} \in \Gamma$. We put

$$\|a\|_{\bar{A}_{\Gamma;J}} = \inf\Big\{\|\{J(2^m, u_m)\}\|_\Gamma : a = \sum_{m=-\infty}^{\infty} u_m\Big\}.$$

The spaces $\bar{A}_{\Gamma;K}$ and $\bar{A}_{\Gamma;J}$ are Banach spaces. Conditions (2.1) and (2.2) are essential to get meaningful definitions (see [24] and [4]).

Example 2.1. For $\Gamma = \ell_q(2^{-\theta m})$ with $1 \le q \le \infty$ and $0 < \theta < 1$, K- and J-spaces agree and they are equal to the classical real interpolation space

$$(A_0, A_1)_{\ell_q(2^{-\theta m});K} = (A_0, A_1)_{\ell_q(2^{-\theta m});J} = (A_0, A_1)_{\theta,q} \quad \text{(see [2–4, 28])}.$$

EXAMPLE 2.2. Let $f : (0,\infty) \to (0,\infty)$ be a *function parameter*, that is, $f(t)$ increases from 0 to ∞, $f(t)/t$ decreases from ∞ to 0 and, for every $t > 0$, $s_f(t) = \sup\{f(tu)/f(u) : u > 0\}$ is finite and $s_f(t) = o(\max\{1,t\})$ as $t \to 0$ and $t \to \infty$ (see [19, 18]). For $\Gamma = \ell_q(1/f(2^m))$ with $1 \le q \le \infty$, K- and J-spaces coincide again. The resulting space is now the real interpolation space with a function parameter

$$(A_0, A_1)_{\ell_q(1/f(2^m));K} = (A_0, A_1)_{\ell_q(1/f(2^m));J} = (A_0, A_1)_{f,q} \text{ (see [26, 21, 18])}.$$

When $f(t) = t^\theta$ we recover the spaces $(A_0, A_1)_{\theta,q}$ of Example 2.1.

In other interesting examples K- and J-spaces do not coincide in general.

EXAMPLE 2.3. It is not difficult to verify that

$$(A_0, A_1)_{\ell_\infty(\min\{1,2^{-m}\});K} = A_0 + A_1, \quad (A_0, A_1)_{\ell_1(\max\{1,2^{-m}\});J} = A_0 \cap A_1.$$

EXAMPLE 2.4. Let $A_i^\sim$ be the *Gagliardo completion* of A_i, that is, the space of all those $a \in A_0 + A_1$ for which there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ in some bounded subset of A_i which converges to a in $A_0 + A_1$. The norm $\|\cdot\|_{A_i^\sim}$ in $A_i^\sim$ is given by $\|a\|_{A_i^\sim} = \inf_{\{a_n\}}\{\sup_{n\in\mathbb{N}}\{\|a_n\|_{A_i}\}\}$. It is easy to show (see, for example, [2, Thm. 5.1.4]) that $\|a\|_{A_0^\sim} = \lim_{t\to\infty} K(t,a)$ and $\|a\|_{A_1^\sim} = \lim_{t\to 0} K(t,a)/t$. Hence

$$A_0^\sim = (A_0, A_1)_{\ell_\infty;K}, \quad A_1^\sim = (A_0, A_1)_{\ell_\infty(2^{-m});K}.$$

EXAMPLE 2.5. Another distinguished example is A_i°, the closed subspace of A_i generated by $A_0 \cap A_1$. It turns out that $A_0^\circ = (A_0, A_1)_{\ell_1;J}$ and $A_1^\circ = (A_0, A_1)_{\ell_1(2^{-m});J}$.

If $\overline{B} = (B_0, B_1)$ is another Banach couple, we write $T : \bar{A} \to \overline{B}$ to mean that T is a linear operator from $A_0 + A_1$ into $B_0 + B_1$ whose restriction to each A_i defines a bounded operator from A_i into B_i. We put $\|T\|_{\bar{A},\overline{B}} = \max\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\}$. Clearly, if $T : \bar{A} \to \overline{B}$, then the restrictions

$$T : (A_0, A_1)_{\Gamma;K} \to (B_0, B_1)_{\Gamma;K} \quad \text{and} \quad T : (A_0, A_1)_{\Gamma;J} \to (B_0, B_1)_{\Gamma;J}$$

are bounded with norms less than or equal to $\|T\|_{\bar{A},\overline{B}}$. This information is very rough for working with interpolated operators. In fact, in the case of the classical real method (Example 2.1) the well known convexity inequality $\|T\|_{\bar{A}_{\theta,q},\overline{B}_{\theta,q}} \le C\|T\|_{A_0,B_0}^{1-\theta}\|T\|_{A_1,B_1}^{\theta}$ is an indispensable tool for developing the theory. For the real method with a function parameter (Example 2.2) the corresponding inequality reads

$$\|T\|_{\bar{A}_{f,q},\overline{B}_{f,q}} \le C\|T\|_{A_0,B_0}\, s_f(\|T\|_{A_1,B_1}/\|T\|_{A_0,B_0}).$$

In the general setting where we are working, we can get extra information about the norms of interpolated operators if we know the behaviour of shift operators on the sequence space Γ. For $k \in \mathbb{Z}$, the *shift operator* τ_k is defined by

$$\tau_k \xi = \{\xi_{m+k}\}_{m\in\mathbb{Z}} \quad \text{for } \xi = \{\xi_m\}_{m\in\mathbb{Z}}.$$

The following result can be easily established.

LEMMA 2.6. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples, let $T : \bar{A} \to \bar{B}$, let Γ be a K-non-trivial sequence space and let Λ be a J-non-trivial sequence space. Then, for each $n \in \mathbb{N}$:*

(i) *if $\|T\|_{A_0,B_0} \le 2^{-n}$ and $\|T\|_{A_1,B_1} \le 1$, then*
$$\|T\|_{\bar{A}_{\Gamma;K},\bar{B}_{\Gamma;K}} \le 2^{-n}\|\tau_n\|_{\Gamma,\Gamma} \quad \text{and} \quad \|T\|_{\bar{A}_{\Lambda;J},\bar{B}_{\Lambda;J}} \le 2^{-n}\|\tau_n\|_{\Lambda,\Lambda};$$
(ii) *if $\|T\|_{A_0,B_0} \le 1$ and $\|T\|_{A_1,B_1} \le 2^{-n}$, then*
$$\|T\|_{\bar{A}_{\Gamma;K},\bar{B}_{\Gamma;K}} \le \|\tau_{-n}\|_{\Gamma,\Gamma} \quad \text{and} \quad \|T\|_{\bar{A}_{\Lambda;J},\bar{B}_{\Lambda;J}} \le \|\tau_{-n}\|_{\Lambda,\Lambda}.$$

In view of Lemma 2.6, it will be useful to assume later that shift operators satisfy $\lim_{n\to\infty} 2^{-n}\|\tau_n\|_{\Gamma,\Gamma} = 0$ and/or $\lim_{n\to\infty} \|\tau_{-n}\|_{\Gamma,\Gamma} = 0$. Both conditions are satisfied by sequence spaces of Examples 2.2 and 2.1 because for $\Gamma = \ell_q(1/f(2^m))$, we have $2^{-n}\|\tau_n\|_{\Gamma,\Gamma} \le 2^{-n}s_f(2^n)$ and $\|\tau_{-n}\|_{\Gamma,\Gamma} \le s_f(2^{-n})$. The sequence spaces of Example 2.3 do not satisfy any of these two conditions. The spaces of Examples 2.4 and 2.5 satisfy only one of them.

Let A be any of the spaces $(A_0, A_1)_{\Gamma;K}$, $(A_0, A_1)_{\Gamma;J}$. Following [6], we define the functions ψ and ϱ by

$$\psi(t) = \psi(t, A, \bar{A}) = \sup\{K(t, a) : \|a\|_A = 1\},$$
$$\varrho(t) = \varrho(t, A, \bar{A}) = \inf\{J(t, a) : a \in A_0 \cap A_1,\ \|a\|_A = 1\}.$$

For compactness theorems of the following sections we shall need results of [6]. To apply them, we need to know the behaviour of ψ and ϱ at 0 and at ∞. Next we show that this behaviour can be controlled by shift operators.

LEMMA 2.7. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let Γ be a K-non-trivial sequence space and let Λ be a J-non-trivial sequence space. Then there exist constants $C_1, C_2 > 0$ such that for each $n \in \mathbb{N}$:*

(i) $\psi(2^n, \bar{A}_{\Gamma;K}, \bar{A}) \le C_1\|\tau_n\|_{\Gamma,\Gamma}$, $\psi(2^n, \bar{A}_{\Lambda;J}, \bar{A}) \le C_1\|\tau_n\|_{\Lambda,\Lambda}$;
(ii) $\varrho(2^{-n}, \bar{A}_{\Gamma;K}, \bar{A}) \ge 1/C_2\|\tau_n\|_{\Gamma,\Gamma}$, $\varrho(2^{-n}, \bar{A}_{\Lambda;J}, \bar{A}) \ge 1/C_2\|\tau_n\|_{\Lambda,\Lambda}$.

Proof. Given any $a \in (A_0, A_1)_{\Gamma;K}$ and any $n \in \mathbb{N}$, using the Hahn–Banach theorem we can find $g : \bar{A} \to \bar{\mathbb{K}} = (\mathbb{K}, \mathbb{K})$ so that $g(a) = 2^{-n}K(2^n, a)$, $\|g\|_{A_0'} \le 2^{-n}$ and $\|g\|_{A_1'} \le 1$. Let C_1 be the norm of the identity operator from $(\mathbb{K}, \|\cdot\|_{\bar{\mathbb{K}}_{\Gamma;K}})$ into $\mathbb{K}$. According to Lemma 2.6, we obtain

$$\|g\|_{\bar{A}_{\Gamma;K},\mathbb{K}} \le C_1\|g\|_{\bar{A}_{\Gamma;K},\bar{\mathbb{K}}_{\Gamma;K}} \le C_1 2^{-n}\|\tau_n\|_{\Gamma,\Gamma}.$$

This implies that $2^{-n}K(2^n,a)/\|a\|_{\bar{A}_{\Gamma;K}} \le C_1 2^{-n}\|\tau_n\|_{\Gamma,\Gamma}$ and so

$$\psi(2^n,\bar{A}_{\Gamma;K},\bar{A}) = \sup\{K(2^n,a) : \|a\|_{\bar{A}_{\Gamma;K}} = 1\} \le C_1\|\tau_n\|_{\Gamma,\Gamma}.$$

The case of $\bar{A}_{\Lambda;J}$ is similar but we interpolate now by the J-method.

To establish (ii) take any $a \in A_0 \cap A_1$ and any $n \in \mathbb{N}$, and let $T : \overline{\mathbb{K}} \to \bar{A}$ be the operator defined by $T(\lambda) = \lambda a/2^n J(2^{-n},a)$. Then $\|T\|_{\mathbb{K},A_0} \le 2^{-n}$ and $\|T\|_{\mathbb{K},A_1} \le 1$. So, if we write C_2 for the norm of the identity operator from $\mathbb{K}$ into $(\mathbb{K},\|\cdot\|_{\overline{\mathbb{K}}_{\Gamma;K}})$, using again Lemma 2.6 we derive that $\|T\|_{\mathbb{K},\bar{A}_{\Gamma;K}} \le C_2 2^{-n}\|\tau_n\|_{\Gamma,\Gamma}$. It follows that $\|a\|_{\bar{A}_{\Gamma;K}}/2^n J(2^{-n},a) \le C_2 2^{-n}\|\tau_n\|_{\Gamma,\Gamma}$ and thus $1/C_2\|\tau_n\|_{\Gamma,\Gamma} \le J(2^{-n},a)/\|a\|_{\bar{A}_{\Gamma;K}}$. Taking the infimum, we conclude that $1/C_2\|\tau_n\|_{\Gamma,\Gamma} \le \varrho(2^{-n},\bar{A}_{\Gamma;K},\bar{A})$. The case of $\bar{A}_{\Lambda;J}$ can be treated analogously. ∎

In the next sections we shall need to work with vector-valued sequence spaces. Given any sequence space Γ, any sequence $\{W_m\}$ of Banach spaces and any sequence $\{\lambda_m\}$ of positive numbers, we put

$$\Gamma(\lambda_m W_m) = \{u = \{u_m\} : u_m \in W_m \text{ and } \|u\|_{\Gamma(\lambda_m W_m)} = \|\{\lambda_m\|u_m\|_{W_m}\}\|_\Gamma < \infty\}.$$

When $\lambda_m = 1$ for all $m \in \mathbb{Z}$, we write simply $\Gamma(W_m)$.

3. Compactness and K-spaces. Given any Banach couple (B_0,B_1), we put $F_m = (B_0+B_1, K(2^m,\cdot))$, $m \in \mathbb{Z}$. Note that the K-space $(B_0,B_1)_{\Gamma;K}$ is isometric to the diagonal of the vector-valued sequence space $\Gamma(F_m)$. More precisely, the map $j : (B_0,B_1)_{\Gamma;K} \to \Gamma(F_m)$ which to each element $b \in B_0 + B_1$ associates the constant sequence $jb = \{\dots,b,b,b,\dots\}$ is a metric injection.

For characterizing compact operators between K-spaces we shall need operators $\{P_n\}_{n\in\mathbb{N}}$ defined by

$$P_n\{\xi_m\} = \{\dots,0,0,\xi_{-n},\xi_{-n+1},\dots,\xi_{n-1},\xi_n,0,0,\dots\}.$$

The symbol I stands for the identity operator.

Theorem 3.1. *Let Γ be a K-non-trivial sequence space such that*

$$\lim_{n\to\infty}\|\xi - P_n\xi\|_\Gamma = 0 \quad \textit{for all } \xi \in \Gamma. \tag{3.1}$$

Let $\bar{A} = (A_0,A_1)$, $\overline{B} = (B_0,B_1)$ be Banach couples and let $T : \bar{A} \to \overline{B}$. Then the interpolated operator $T : (A_0,A_1)_{\Gamma;K} \to (B_0,B_1)_{\Gamma;K}$ is compact if and only if the following conditions hold.

(a) $T : (A_0,A_1)_{\Gamma;K} \to B_0 + B_1$ *is compact.*
(b) $\sup\{\|(I-P_n)\{K(2^m,Ta)\}\|_\Gamma : \|a\|_{\bar{A}_{\Gamma;K}} \le 1\} \to 0$ *as* $n \to \infty$.

Proof. First note that the operators P_n can also be defined on $\Gamma(F_m)$, $\ell_\infty(F_m)$ or $\ell_\infty(2^{-m}F_m)$ and they act boundedly with norms equal to 1. Moreover, for each $n \in \mathbb{N}$,

$$P_n : \ell_\infty(F_m) + \ell_\infty(2^{-m}F_m) \to \ell_\infty(F_m) \cap \ell_\infty(2^{-m}F_m)$$

is bounded with norm less than or equal to 2^n.

In order to show that the conditions are sufficient we observe that, for each $n \in \mathbb{N}$, the operator $P_n jT : (A_0, A_1)_{\Gamma;K} \to \Gamma(F_m)$ is compact. This follows from (a) and the factorization

$$\begin{aligned} \bar{A}_{\Gamma;K} \xrightarrow{T} B_0 + B_1 \xrightarrow{j} \ell_\infty(F_m) + \ell_\infty(2^{-m}F_m) \\ \xrightarrow{P_n} \ell_\infty(F_m) \cap \ell_\infty(2^{-m}F_m) \hookrightarrow \Gamma(F_m). \end{aligned}$$

Since (b) means that $\{P_n jT\}$ converges to jT in $\mathfrak{L}(\bar{A}_{\Gamma;K}, \Gamma(F_m))$, it follows that jT is compact. This yields compactness of $T : (A_0, A_1)_{\Gamma;K} \to (B_0, B_1)_{\Gamma;K}$ because j is a metric injection.

Conversely, if $T : (A_0, A_1)_{\Gamma;K} \to (B_0, B_1)_{\Gamma;K}$ is compact, then (a) follows from the factorization

$$(A_0, A_1)_{\Gamma;K} \xrightarrow{T} (B_0, B_1)_{\Gamma;K} \hookrightarrow B_0 + B_1.$$

On the other hand, given any $\varepsilon > 0$, by compactness of $T : \bar{A}_{\Gamma;K} \to \bar{B}_{\Gamma;K}$, we can find a finite set $\{b_1, \ldots, b_s\} \subset (B_0, B_1)_{\Gamma;K}$ so that for any $a \in (A_0, A_1)_{\Gamma;K}$ with $\|a\|_{\bar{A}_{\Gamma;K}} \le 1$, we have

$$\min\{\|Ta - b_r\|_{\bar{B}_{\Gamma;K}} : 1 \le r \le s\} \le \varepsilon/3.$$

By (3.1), there exists $N \in \mathbb{N}$ such that if $n \ge N$ then

$$\|\{K(2^m, b_r)\} - P_n\{K(2^m, b_r)\}\|_\Gamma \le \varepsilon/3 \quad \text{for } r = 1, \ldots, s.$$

Hence, if $n \ge N$, given any $a \in (A_0, A_1)_{\Gamma;K}$ with $\|a\|_{\bar{A}_{\Gamma;K}} \le 1$, if we choose r so that $\|Ta - b_r\|_{\bar{B}_{\Gamma;K}} \le \varepsilon/3$ we get

$$\begin{aligned} &\|(I - P_n)\{K(2^m, Ta)\}\|_\Gamma = \|jTa - P_n jTa\|_{\Gamma(F_m)} \\ &\quad \le \|jTa - jb_r\|_{\Gamma(F_m)} + \|jb_r - P_n jb_r\|_{\Gamma(F_m)} + \|P_n jb_r - P_n jTa\|_{\Gamma(F_m)} \\ &\quad\quad \le 2\|Ta - b_r\|_{\bar{B}_{\Gamma;K}} + \varepsilon/3 \le \varepsilon. \end{aligned}$$

This gives (b) and completes the proof. ∎

Theorem 3.1 can also be derived from a result of Dmitriev [15]. When we write Theorem 3.1 for $\Gamma = \ell_q(2^{-\theta m})$ with $1 \le q < \infty$ and $0 < \theta < 1$ (Example 2.1), we obtain [17, Thm. 4.1].

REMARK 3.2. Assumption (3.1) has only been used to show that compactness of $T : \bar{A}_{\Gamma;K} \to \bar{B}_{\Gamma;K}$ implies condition (b). Without (3.1) this is not true in general as the following example shows.

EXAMPLE 3.3. Take $\Gamma = \ell_\infty$, which clearly fails (3.1), let $\bar{A} = (\ell_\infty, \ell_1)$, $\overline{B} = (\ell_\infty(\min\{1, 2^{-m}\}), \ell_1)$ and let T be the operator defined by

$$T\xi = \{\ldots, 0, 0, \xi_1, \xi_2, \xi_3, \ldots\} \quad \text{for } \xi = \{\xi_m\}.$$

As indicated in Example 2.4, $(\ell_\infty, \ell_1)_{\ell_\infty;K} = \ell_\infty^\sim = \ell_\infty$ and

$$(\ell_\infty(\min\{1, 2^{-m}\}), \ell_1)_{\ell_\infty;K} = \ell_\infty(\min\{1, 2^{-m}\})^\sim = \ell_\infty(\min\{1, 2^{-m}\}).$$

The operator $T : \ell_\infty \to \ell_\infty(\min\{1, 2^{-m}\})$ is compact because it is the limit of a sequence of finite rank operators. Thus, the interpolated operator $T : (\ell_\infty, \ell_1)_{\ell_\infty;K} \to (\ell_\infty(\min\{1, 2^{-m}\}), \ell_1)_{\ell_\infty;K}$ is compact. Nevertheless, for each $m \geq 0$ and $\xi \in \ell_\infty$, since $\ell_1 \hookrightarrow \ell_\infty(\min\{1, 2^{-m}\})$ with norm 1, we have $K(2^m, T\xi) = \|T\xi\|_{\ell_\infty(\min\{1,2^{-m}\})}$. Hence,

$$\begin{aligned}\sup\{\|(I - P_n)\{K(2^m, T\xi)\}\|_{\ell_\infty} : \|\xi\|_{\bar{A}_{\ell_\infty;K}} \leq 1\} \\ \geq \sup\{\|T\xi\|_{\ell_\infty(\min\{1,2^{-m}\})} : \|\xi\|_{\ell_\infty} \leq 1\} = 1/2,\end{aligned}$$

that is to say, condition (b) does not hold.

Next we shall use Theorem 3.1 to derive extended versions of compactness results established by Cobos, Edmunds and Potter in [7]. We start with the following auxiliary result which follows from an idea of Nilsson [24, p. 295]. We write $\overline{\ell_\infty(F)}$ for the couple $(\ell_\infty(F_m), \ell_\infty(2^{-m}F_m))$.

LEMMA 3.4. *Let Γ be a K-non-trivial sequence space and let $\{F_m\}$ be a sequence of Banach spaces. Then*

$$\overline{\ell_\infty(F)}_{\Gamma;K} = (\ell_\infty(F_m), \ell_\infty(2^{-m}F_m))_{\Gamma;K} \hookrightarrow \Gamma(F_m).$$

Given a Banach couple $\bar{A} = (A_0, A_1)$, we put $\bar{A}^\circ = (A_0^\circ, A_1^\circ)$.

THEOREM 3.5. *Let Γ be a K-non-trivial sequence space such that*

$$2^{-n}\|\tau_n\|_{\Gamma,\Gamma} \to 0 \quad \textit{and} \quad \|\tau_{-n}\|_{\Gamma,\Gamma} \to 0 \quad \textit{as } n \to \infty. \tag{3.2}$$

Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be Banach couples and let $T : \overline{A} \to \overline{B}$ be such that $T : A_0 \to B_0$ and $T : A_1 \to B_1$ are compact. Then $T : (A_0^\circ, A_1^\circ)_{\Gamma;K} \to (B_0^\circ, B_1^\circ)_{\Gamma;K}$ is also compact.

Proof. According to Theorem 3.1 and Remark 3.2, it suffices to show two things:

(a) $T : (A_0^\circ, A_1^\circ)_{\Gamma;K} \to B_0^\circ + B_1^\circ$ is compact.
(b) $\|jT - P_n jT\|_{\bar{A}^\circ_{\Gamma;K}, \Gamma(F_m)} \to 0$ as $n \to \infty$, where

$$F_m = (B_0^\circ + B_1^\circ, K(2^m, \cdot)).$$

Since $T : A_0^\circ \to B_0^\circ$ and $T : A_1^\circ \to B_1^\circ$ are compact, an easy direct argument shows that $T : A_0^\circ + A_1^\circ \to B_0^\circ + B_1^\circ$ is also compact. As $(A_0^\circ, A_1^\circ)_{\Gamma;K} \hookrightarrow$

$A_0^\circ + A_1^\circ$, (a) follows. To check (b), we shall use the operators

$$Q_n^+\{u_m\} = \{\dots, 0, 0, u_{n+1}, u_{n+2}, \dots\},$$
$$Q_n^-\{u_m\} = \{\dots, u_{-n-2}, u_{-n-1}, 0, 0, \dots\}.$$

Note that, for any $n \in \mathbb{N}$, the identity operator I can be written as $I = P_n + Q_n^+ + Q_n^-$. The operators Q_n^+ and Q_n^- have norm 1 on $\ell_\infty(F_m)$, $\ell_\infty(2^{-m}F_m)$ and $\Gamma(F_m)$. Moreover, for each $n \in \mathbb{N}$,

$$\|Q_n^+\|_{\ell_\infty(F_m),\ell_\infty(2^{-m}F_m)} \le 2^{-n}, \quad \|Q_n^-\|_{\ell_\infty(2^{-m}F_m),\ell_\infty(F_m)} \le 2^{-n}. \tag{3.3}$$

Using Lemma 3.4, we have

$$\begin{aligned}\|jT - P_n jT\|_{\bar{A}^\circ_{\Gamma;K},\Gamma(F_m)} &\le \|jT - P_n jT\|_{\bar{A}^\circ_{\Gamma;K},\overline{\ell_\infty(F)}_{\Gamma;K}} \\ &\le \|Q_n^+ jT\|_{\bar{A}^\circ_{\Gamma;K},\overline{\ell_\infty(F)}_{\Gamma;K}} + \|Q_n^- jT\|_{\bar{A}^\circ_{\Gamma;K},\overline{\ell_\infty(F)}_{\Gamma;K}}.\end{aligned}$$

In order to establish (b) we show that the last two terms go to 0 as n goes to ∞. By (3.2) and Lemma 2.6, it suffices to check that

$$\|Q_n^- jT\|_{A_0^\circ,\ell_\infty(F_m)} \to 0 \quad \text{as } n \to \infty, \tag{3.4}$$

$$\|Q_n^+ jT\|_{A_1^\circ,\ell_\infty(2^{-m}F_m)} \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Given $\varepsilon > 0$, by compactness of $T : A_0^\circ \to B_0^\circ$, there exists a finite subset $\{a_1, \dots, a_s\} \subseteq A_0 \cap A_1$ with $\|a_r\|_{A_0} \le 1$ for $r = 1, \dots, s$ such that for any $a \in A_0^\circ$ with $\|a\|_{A_0} \le 1$, we have $\min\{\|Ta - Ta_r\|_{B_0} : r = 1, \dots, s\} \le \varepsilon/2$. The set $\{jTa_1, \dots, jTa_s\}$ is contained in $\ell_\infty(F_m) \cap \ell_\infty(2^{-m}F_m)$. Using (3.3), we can find $N \in \mathbb{N}$ so that for any $n \ge N$,

$$\|Q_n^-\|_{\ell_\infty(2^{-m}F_m),\ell_\infty(F_m)} \max\{\|jTa_r\|_{\ell_\infty(2^{-m}F_m)} : r = 1, \dots, s\} \le \varepsilon/2.$$

Then $\|Q_n^- jT\|_{A_0^\circ,\ell_\infty(F_m)} \le \varepsilon$ for $n \ge N$. Indeed, given any $a \in A_0^\circ$ with $\|a\|_{A_0} \le 1$, choosing a_r such that $\|Ta - Ta_r\|_{B_0} \le \varepsilon/2$ we get

$$\begin{aligned}&\|Q_n^- jTa\|_{\ell_\infty(F_m)} \\ &\quad \le \|Q_n^- j(Ta - Ta_r)\|_{\ell_\infty(F_m)} + \|Q_n^- jTa_r\|_{\ell_\infty(F_m)} \\ &\quad \le \|Ta - Ta_r\|_{B_0} + \|Q_n^-\|_{\ell_\infty(2^{-m}F_m),\ell_\infty(F_m)} \|jTa_r\|_{\ell_\infty(2^{-m}F_m)} \le \varepsilon.\end{aligned}$$

The proof of (3.5) is similar but using now compactness of $T : A_1^\circ \to B_1^\circ$. ∎

If $\bar{A}$ is an ordered couple, then compactness of $T : A_1 \to B_1$ is not required to derive that the interpolated operator is compact:

THEOREM 3.6. *Under the same assumptions on Γ, $\bar{A}$ and $\bar{B}$ as in the previous theorem, assume also that A_1 is continuously embedded in A_0 and that $T : \bar{A} \to \bar{B}$ with $T : A_0 \to B_0$ compact. Then $T : (A_0^\circ, A_1^\circ)_{\Gamma;K} \to (B_0^\circ, B_1^\circ)_{\Gamma;K}$ is compact.*

Proof. We proceed as in Theorem 3.5. We should only modify the arguments given to establish (a) and (3.5) because we used compactness of T :

$A_1 \to B_1$ there. Now, $T : A_0^\circ \to B_0^\circ + B_1^\circ$ is compact and $T : A_1^\circ \to B_0^\circ + B_1^\circ$ is bounded. Moreover, by Lemma 2.7,

$$\lim_{t\to\infty} \frac{\psi(t, \bar{A}^\circ_{\Gamma;K}, \bar{A}^\circ)}{t} = \lim_{n\to\infty} \frac{\psi(2^n, \bar{A}^\circ_{\Gamma;K}, \bar{A}^\circ)}{2^n} \le \lim_{n\to\infty} \frac{C_1\|\tau_n\|_{\Gamma,\Gamma}}{2^n} = 0.$$

Hence, compactness of $T : (A_0^\circ, A_1^\circ)_{\Gamma;K} \to B_0^\circ + B_1^\circ$ follows from [6, Thm. 3.1]. To establish (3.5) we consider the diagram

$$A_1^\circ \hookrightarrow A_0^\circ \xrightarrow{T} B_0^\circ \xrightarrow{j} \ell_\infty(F_m) \xrightarrow{Q_n^+} \ell_\infty(2^{-m}F_m)$$

to get

$$\begin{aligned}\|Q_n^+ jT\|_{A_1^\circ,\ell_\infty(2^{-m}F_m)} &\le \|T\|_{A_1^\circ,B_0^\circ}\|Q_n^+\|_{\ell_\infty(F_m),\ell_\infty(2^{-m}F_m)} \\ &\le 2^{-n}\|T\|_{A_1^\circ,B_0^\circ} \to 0 \quad \text{as } n\to\infty. \ \blacksquare\end{aligned}$$

If the couple $\bar{B}$ is ordered, then we can even dispense with the assumption on the behaviour of $\|\tau_{-n}\|_{\Gamma,\Gamma}$.

THEOREM 3.7. *Let Γ be a K-non-trivial sequence space such that*

$$2^{-n}\|\tau_n\|_{\Gamma,\Gamma} \to 0 \quad \text{as } n\to\infty. \tag{3.6}$$

Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples with B_1 continuously embedded in B_0, and let $T : \bar{A} \to \bar{B}$ be such that $T : A_0 \to B_0$ is compact. Then $T : (A_0^\circ, A_1^\circ)_{\Gamma;K} \to (B_0^\circ, B_1^\circ)_{\Gamma;K}$ is also compact.

Proof. We proceed as in Theorem 3.5 and we derive (a) as in Theorem 3.6. Since we do not suppose now that $\|\tau_{-n}\|_{\Gamma,\Gamma} \to 0$ as $n\to\infty$, to complete the proof we should check that $\|Q_n^+ jT\|_{\bar{A}^\circ_{\Gamma;K},\overline{\ell_\infty(F)}_{\Gamma;K}} \to 0$ as $n\to\infty$. According to the diagram

$$A_1^\circ \xrightarrow{T} B_1^\circ \hookrightarrow B_0^\circ \xrightarrow{j} \ell_\infty(F_m) \xrightarrow{Q_n^+} \ell_\infty(2^{-m}F_m)$$

and (3.3), we get $\|Q_n^+ jT\|_{A_1^\circ,\ell_\infty(2^{-m}F_m)} \le 2^{-n}\|T\|_{A_1^\circ,B_0^\circ}$. On the other hand, the embedding $B_1 \hookrightarrow B_0$ implies that $\ell_\infty(2^{-m}F_m) \hookrightarrow \ell_\infty(F_m)$. The factorization

$$A_0^\circ \xrightarrow{T} B_0^\circ \xrightarrow{j} \ell_\infty(F_m) \xrightarrow{Q_n^+} \ell_\infty(2^{-m}F_m) \hookrightarrow \ell_\infty(F_m)$$

and (3.3) yield $\|Q_n^+ jT\|_{A_0^\circ,\ell_\infty(F_m)} \le 2^{-n}\|T\|_{A_0^\circ,B_0^\circ}$. Consequently,

$$\begin{aligned}\|Q_n^+ jT\|_{\bar{A}^\circ_{\Gamma;K},\overline{\ell_\infty(F)}_{\Gamma;K}} &\le \max\{\|Q_n^+ jT\|_{A_0^\circ,\ell_\infty(F_m)}, \|Q_n^+ jT\|_{A_1^\circ,\ell_\infty(2^{-m}F_m)}\} \\ &\le 2^{-n}\max\{\|T\|_{A_0^\circ,B_0^\circ}, \|T\|_{A_1^\circ,B_0^\circ}\} \to 0 \quad \text{as } n\to\infty. \ \blacksquare\end{aligned}$$

Theorem 3.7 can also be derived from a result of Mastyło [23, Cor. 3.2]. For $\Gamma = \ell_q(2^{-\theta m})$, Theorems 3.5 and 3.7 give the Banach case of [7, Thms. 3.1 and 3.2], while Theorem 3.6 gives a result of [11]. Writing down Theorems 3.5 and 3.7 for the case $\Gamma = \ell_q(1/f(2^m))$ we recover [7, Thm. 3.3].

We close this section with an example which shows that the assumption (3.6) is essential in Theorem 3.7.

EXAMPLE 3.8. Let $A_0 = A_1 = B_1 = c_0$, $B_0 = \ell_\infty(\min\{1, 2^{-m}\})$ and let T be the operator defined by $T\xi = \{\ldots, 0, 0, \xi_1, \xi_2, \ldots\}$ for $\xi = \{\xi_m\}$. Then $B_1 \hookrightarrow B_0$, $T : A_0 \to B_0$ is compact and $T : A_1 \to B_1$ is bounded. Moreover, $A_0^\circ = A_1^\circ = c_0$, $B_1^\circ = c_0$ and $B_0^\circ = c_0(\min\{1, 2^{-m}\})$. We now take $\Gamma = \ell_\infty(2^{-m})$ (Example 2.4). Then $\|\tau_n\|_{\Gamma,\Gamma} = 2^n$, so (3.6) does not hold. Interpolating with Γ we obtain $(A_0^\circ, A_1^\circ)_{\Gamma;K} = (A_1^\circ)^\sim = c_0$ and

$$(B_0^\circ, B_1^\circ)_{\Gamma;K} = (B_1^\circ)^\sim = \{\xi = \{\xi_m\} \in \ell_\infty : \lim_{m\to-\infty} \xi_m = 0\}$$

and it is clear that $T : c_0 \to (B_1^\circ)^\sim$ is not compact.

4. Compactness and J-spaces. Let G_m be the Banach space $A_0 \cap A_1$ endowed with the norm $J(2^m, \cdot)$. The J-space $(A_0, A_1)_{\Gamma;J}$ is related to the vector-valued sequence space $\Gamma(G_m)$ through the map $\pi : \Gamma(G_m) \to (A_0, A_1)_{\Gamma;J}$ defined by $\pi\{u_m\} = \sum_{m=-\infty}^{\infty} u_m$. Namely, the space $(A_0, A_1)_{\Gamma;J}$ is the quotient space of $\Gamma(G_m)$ given by the surjective map π.

In order to characterize compact operators between J-spaces we shall need the operators $\{P_n\}_{n\in\mathbb{N}}$, $\{Q_n^+\}_{n\in\mathbb{N}}$ and $\{Q_n^-\}_{n\in\mathbb{N}}$ introduced in Section 3. We shall consider these operators on the spaces $\Gamma(G_m)$, $\ell_1(G_m)$ and $\ell_1(2^{-m}G_m)$. They have similar properties to those in the case of spaces modelled on $A_0 + A_1$. In particular, for each $n \in \mathbb{N}$,

$$P_n : \ell_1(G_m) + \ell_1(2^{-m}G_m) \to \ell_1(G_m) \cap \ell_1(2^{-m}G_m) \quad \text{is bounded}$$

and

$$\|Q_n^+\|_{\ell_1(G_m),\ell_1(2^{-m}G_m)} \le 2^{-n}, \qquad \|Q_n^-\|_{\ell_1(2^{-m}G_m),\ell_1(G_m)} \le 2^{-n}. \tag{4.1}$$

We first state an easy inequality between norms of shift operators and norms of the projections Q_n^+, Q_n^-.

LEMMA 4.1. *Let Γ be a J-non-trivial sequence space and let D be the value of the supremum in* (2.2). *Then, for each $n \in \mathbb{N}$,*

(i) $\|Q_n^+\|_{\Gamma(G_m),\ell_1(2^{-m}G_m)} \le D2^{-n}\|\tau_n\|_{\Gamma,\Gamma}$;
(ii) $\|Q_n^-\|_{\Gamma(G_m),\ell_1(G_m)} \le D\|\tau_{-n}\|_{\Gamma,\Gamma}$.

THEOREM 4.2. *Let Γ be a J-non-trivial sequence space with*

$$2^{-n}\|\tau_n\|_{\Gamma,\Gamma} \to 0 \quad \text{and} \quad \|\tau_{-n}\|_{\Gamma,\Gamma} \to 0 \quad \text{as } n \to \infty. \tag{4.2}$$

Let $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ be Banach couples and let $T : \overline{A} \to \overline{B}$. Then the interpolated operator $T : (A_0, A_1)_{\Gamma;J} \to (B_0, B_1)_{\Gamma;J}$ is compact if and only if the following conditions hold.

(a) $T : A_0 \cap A_1 \to (B_0, B_1)_{\Gamma;J}$ *is compact.*

(b) $\sup\Big\{\Big\|T\Big(\sum_{|m|>n} u_m\Big)\Big\|_{\overline{B}_{\Gamma;J}} : \|\{J(2^m, u_m)\}\|_\Gamma \le 1\Big\} \to 0$ *as $n \to \infty$.*

Proof. Assume first that (a) and (b) hold. Using the factorization

$$\begin{array}{c} \Gamma(G_m) \hookrightarrow \ell_1(G_m) + \ell_1(2^{-m}G_m) \xrightarrow{P_n} \ell_1(G_m) \cap \ell_1(2^{-m}G_m) \xrightarrow{\pi} A_0 \cap A_1 \\ \hfill \downarrow T \\ \hfill (B_0,B_1)_{\Gamma;J} \end{array}$$

and condition (a), we deduce that, for each $n \in \mathbb{N}$, the operator $T\pi P_n : \Gamma(G_m) \to (B_0, B_1)_{\Gamma;J}$ is compact. According to (b),

$$\|T\pi - T\pi P_n\|_{\Gamma(G_m),\overline{B}_{\Gamma;J}} \to 0 \quad \text{as } n \to \infty.$$

Hence $T\pi : \Gamma(G_m) \to (B_0, B_1)_{\Gamma;J}$ is compact. Since π is a metric surjection, the compactness of $T : (A_0, A_1)_{\Gamma;J} \to (B_0, B_1)_{\Gamma;J}$ follows.

Conversely, suppose that $T : (A_0, A_1)_{\Gamma;J} \to (B_0, B_1)_{\Gamma;J}$ is compact. Since $A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{\Gamma;J}$, we infer that $T : A_0 \cap A_1 \to (B_0, B_1)_{\Gamma;J}$ is compact. Let us check that (b) also holds. It is not hard to verify that the sequence $\{\|T\pi - T\pi P_n\|_{\Gamma(G_m),\overline{B}_{\Gamma;J}}\}$ is non-increasing. Let δ be its limit. Our aim is to show that $\delta = 0$. Choose a sequence of vectors $\{w_n\}_{n\in\mathbb{N}}$ in the unit ball of $\Gamma(G_m)$ with $\delta = \lim_{n\to\infty}\|T\pi(I - P_n)w_n\|_{\overline{B}_{\Gamma;J}}$. Since $T\pi : \Gamma(G_m) \to (B_0, B_1)_{\Gamma;J}$ is compact and the sequence $\{(I - P_n)w_n\}$ is bounded in $\Gamma(G_m)$, we can find a subsequence $\{T\pi(I - P_{n'})w_{n'}\}$ converging to some b in $(B_0, B_1)_{\Gamma;J}$. So $\delta = \|b\|_{\overline{B}_{\Gamma;J}}$.

By (4.2) and Lemma 4.1,

$$\lim_{n'\to\infty} \|Q^+_{n'}\|_{\Gamma(G_m),\ell_1(2^{-m}G_m)} = 0 = \lim_{n'\to\infty} \|Q^-_{n'}\|_{\Gamma(G_m),\ell_1(G_m)}.$$

Moreover, the diagrams

$$\ell_1(2^{-m}G_m) \xrightarrow{\pi} A_1 \xrightarrow{T} B_1 \hookrightarrow B_0 + B_1, \quad \ell_1(G_m) \xrightarrow{\pi} A_0 \xrightarrow{T} B_0 \hookrightarrow B_0 + B_1$$

show that $T\pi \in \mathfrak{L}(\ell_1(2^{-m}G_m), B_0 + B_1) \cap \mathfrak{L}(\ell_1(G_m), B_0 + B_1)$. Hence, $\{T\pi Q^+_{n'}w_{n'}\}$ and $\{T\pi Q^-_{n'}w_{n'}\}$ are null sequences in $B_0 + B_1$. Since we have $T\pi(I - P_{n'})w_{n'} = T\pi Q^+_{n'}w_{n'} + T\pi Q^-_{n'}w_{n'}$ it follows that $\{T\pi(I - P_{n'})w_{n'}\}$ converges to 0 in $B_0 + B_1$. But $\{T\pi(I - P_{n'})w_{n'}\}$ converges to b in $\overline{B}_{\Gamma;J}$. By compatibility, we conclude that $b = 0$ and therefore $\delta = \|b\|_{\overline{B}_{\Gamma;J}} = 0$. ■

When $\Gamma = \ell_q(2^{-\theta m})$, $1 \le q \le \infty$, $0 < \theta < 1$, we recover [17, Thm. 3.1].

REMARK 4.3. Condition (4.2) has only been used to show that compactness of $T : \overline{A}_{\Gamma;J} \to \overline{B}_{\Gamma;J}$ implies (b). If (4.2) is not satisfied, this implication does not hold in general as the following example shows.

EXAMPLE 4.4. Let $\overline{A} = (\ell_1, \ell_\infty(\min\{1, 2^{-m}\}))$, $\overline{B} = (\ell_1(\min\{1, 2^{-m}\}), \ell_\infty(\min\{1, 2^{-m}\}))$ and, for $\xi = \{\xi_m\}$, put $T\xi = \{\ldots, 0, 0, \xi_1, \xi_2, \ldots\}$. Choose $\Gamma = \ell_1$. So, $\lim_{n\to\infty} 2^{-n}\|\tau_n\|_{\Gamma,\Gamma} = 0$ but $\lim_{n\to\infty} \|\tau_{-n}\|_{\Gamma,\Gamma} \neq 0$. As pointed out in Example 2.5, $(\ell_1, \ell_\infty(\min\{1, 2^{-m}\}))_{\ell_1;J} = \ell_1^\circ = \ell_1$ and

$$(\ell_1(\min\{1, 2^{-m}\}), \ell_\infty(\min\{1, 2^{-m}\}))_{\ell_1;J} = \ell_1(\min\{1, 2^{-m}\})^\circ \\ = \ell_1(\min\{1, 2^{-m}\}).$$

It is clear that the interpolated operator $T : \ell_1 \to \ell_1(\min\{1, 2^{-m}\})$ is compact. However, condition (b) does not hold. Indeed, since $A_0 = \ell_1 \hookrightarrow A_1 = \ell_\infty(\min\{1, 2^{-m}\})$ with norm 1, for any $m \le 0$ we have $G_m = A_0$ with $J(2^m, \cdot) = \|\cdot\|_{A_0}$. For $n \in \mathbb{N}$ fixed, given any $a \in A_0$ with $\|a\|_{A_0} = 1$, put $u = \{u_m\}$ with $u_m = a$ if $m = -n-1$ and $u_m = 0$ otherwise. Then

$$\|u\|_{\Gamma(G_m)} = \|a\|_{A_0} = 1, \quad \Big\|T\Big(\sum_{|m|>n} u_m\Big)\Big\|_{\overline{B}_{\Gamma;J}} = \|Ta\|_{\ell_1(\min\{1,2^{-m}\})}.$$

Hence

$$\sup\Big\{\Big\|T\Big(\sum_{|m|>n} u_m\Big)\Big\|_{\overline{B}_{\Gamma;J}} : \|\{J(2^m, u_m)\}\|_\Gamma \le 1\Big\} \\ \ge \sup\{\|Ta\|_{\ell_1(\min\{1,2^{-m}\})} : \|a\|_{\ell_1} = 1\} = 1/2.$$

From Theorem 4.2 we are going to obtain extended versions of results by Cobos and Fernandez [8] and Cobos and Peetre [11]. Again our approach gives a clear explanation of the different assumptions required for them. First we state an auxiliary result that can be easily proved. We write $\overline{\ell_1(G)}$ for the couple $(\ell_1(G_m), \ell_1(2^{-m}G_m))$.

Lemma 4.5. *Let Γ be a J-non-trivial sequence space and let $\{G_m\}$ be a sequence of Banach spaces. Then*

$$\Gamma(G_m) \hookrightarrow (\ell_1(G_m), \ell_1(2^{-m}G_m))_{\Gamma;J} = \overline{\ell_1(G)}_{\Gamma;J}.$$

Theorem 4.6. *Let Γ be a J-non-trivial sequence space satisfying* (4.2). *Let $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ be Banach couples and let $T : \overline{A} \to \overline{B}$ with $T : A_i \to B_i$ compact for $i = 0, 1$. Then the interpolated operator $T : (A_0, A_1)_{\Gamma;J} \to (B_0, B_1)_{\Gamma;J}$ is compact.*

Proof. By Theorem 4.2, it is enough to check that

(a) $T : A_0 \cap A_1 \to (B_0, B_1)_{\Gamma;J}$ is compact,
(b) $\lim_{n\to\infty} \|T\pi(I-P_n)\|_{\Gamma(G_m),\overline{B}_{\Gamma;J}} = 0$, where $G_m = (A_0 \cap A_1, J(2^m, \cdot))$.

(a) follows easily from the fact that $T : A_0 \cap A_1 \to B_i$ is compact for $i = 0, 1$. In order to establish (b), observe that $T\pi(I-P_n) = T\pi Q_n^+ + T\pi Q_n^-$. Thus, by Lemma 4.5,

$$\|T\pi(I-P_n)\|_{\Gamma(G_m),\overline{B}_{\Gamma;J}} \le \|T\pi Q_n^+\|_{\Gamma(G_m),\overline{B}_{\Gamma;J}} + \|T\pi Q_n^-\|_{\Gamma(G_m),\overline{B}_{\Gamma;J}} \\ \le \|T\pi Q_n^+\|_{\overline{\ell_1(G)}_{\Gamma;J},\overline{B}_{\Gamma;J}} + \|T\pi Q_n^-\|_{\overline{\ell_1(G)}_{\Gamma;J},\overline{B}_{\Gamma;J}}.$$

We are going to show that the last two terms go to 0 as $n \to \infty$. According to (4.2) and Lemma 2.6, it suffices to check that

$$\|T\pi Q_n^+\|_{\ell_1(G_m),B_0} \to 0 \quad \text{as } n \to \infty, \tag{4.3}$$

$$\|T\pi Q_n^-\|_{\ell_1(2^{-m}G_m),B_1} \to 0 \quad \text{as } n \to \infty. \tag{4.4}$$

The sequence $\{\|T\pi Q_n^+\|_{\ell_1(G_m),B_0}\}$ is non-increasing. Let δ be its limit. Choose a sequence $\{w_n\}_{n\in\mathbb{N}}$ of vectors in the unit ball of $\ell_1(G_m)$ such that $\delta = \lim_{n\to\infty}\|T\pi Q_n^+ w_n\|_{B_0}$. Since $T : A_0 \to B_0$ is compact and $\{\pi Q_n^+ w_n\}$ is bounded in A_0, we can find a subsequence $\{T\pi Q_{n'}^+ w_{n'}\}$ converging to some b in B_0. Hence $\delta = \|b\|_{B_0}$. On the other hand, since $\|Q_{n'}^+\|_{\ell_1(G_m),\ell_1(2^{-m}G_m)} \le 2^{-n'}$, we see that $\{T\pi Q_{n'}^+ w_{n'}\}$ converges to 0 in B_1. By compatibility $b = 0$ and so $\delta = \|b\|_{B_0} = 0$.

The proof of (4.4) is analogous but using compactness of $T : A_1 \to B_1$. ∎

If $B_0 \hookrightarrow B_1$ then we can derive the same conclusion without requiring that $T : A_1 \to B_1$ is compact:

Theorem 4.7. *Under the same assumptions on Γ, $\bar{A}$ and $\bar{B}$ as in the previous theorem, suppose also that $B_0 \hookrightarrow B_1$ and that $T : \bar{A} \to \bar{B}$ with $T : A_0 \to B_0$ compact. Then $T : (A_0, A_1)_{\Gamma;J} \to (B_0, B_1)_{\Gamma;J}$ is compact.*

Proof. We follow the same scheme as in Theorem 4.6. Taking into account that $2^{-n}\|\tau_n\|_{\Gamma,\Gamma} \to 0$ and using Lemma 2.7, we get

$$\lim_{t\to 0} \frac{t}{\varrho(t, \bar{B}_{\Gamma;J}, \bar{B})} = \lim_{n\to\infty} \frac{2^{-n}}{\varrho(2^{-n}, \bar{B}_{\Gamma;J}, \bar{B})} \le \lim_{n\to\infty} C_2 2^{-n}\|\tau_n\|_{\Gamma,\Gamma} = 0.$$

So, compactness of $T : A_0 \cap A_1 \to B_0$ and [6, Thm. 3.2], imply that $T : A_0 \cap A_1 \to (B_0, B_1)_{\Gamma;J}$ is compact. That is, (a) (of the previous proof) holds.

To establish (b), we observe that (4.3) follows with the same argument as in Theorem 4.6. To check (4.4), we use the factorization

$$\ell_1(2^{-m}G_m) \xrightarrow{Q_n^-} \ell_1(G_m) \xrightarrow{\pi} A_0 \xrightarrow{T} B_0 \hookrightarrow B_1$$

and (4.1). We obtain

$$\|T\pi Q_n^-\|_{\ell_1(2^{-m}G_m),B_1} \le 2^{-n}\|T\|_{A_0,B_1} \to 0 \quad \text{as } n \to \infty.$$

This completes the proof. ∎

If $A_0 \hookrightarrow A_1$, then we can even omit the assumption $\lim_{n\to\infty}\|\tau_{-n}\|_{\Gamma,\Gamma} = 0$:

Theorem 4.8. *Let Γ be a J-non-trivial sequence space such that*

$$2^{-n}\|\tau_n\|_{\Gamma,\Gamma} \to 0 \quad \text{as } n \to \infty. \tag{4.5}$$

Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples with A_0 continuously embedded in A_1 and let $T : \bar{A} \to \bar{B}$ be such that $T : A_0 \to B_0$ is compact. Then $T : (A_0, A_1)_{\Gamma;J} \to (B_0, B_1)_{\Gamma;J}$ is also compact.

Proof. Let $\mathbb{Z}^+ = \{m \in \mathbb{Z} : m \geq 0\}$ and let Γ^+ be the Banach lattice on $\mathbb{Z}^+$ consisting of all real-valued sequences $\xi = \{\xi_m\}_{m\in\mathbb{Z}^+}$ which have a finite norm $\|\xi\|_{\Gamma^+} = \|\{\dots, 0, 0, \xi_0, \xi_1, \xi_2, \dots\}\|_{\Gamma}$. Form the vector-valued space $\Gamma^+(G_m)$ over $\mathbb{Z}^+$. It is not hard to check that $\pi : \Gamma^+(G_m) \to (A_0, A_1)_{\Gamma;J}$ is still surjective and that the quotient norm it induces is equivalent to $\|\cdot\|_{\bar{A}_{\Gamma;J}}$. Working with the spaces $\ell_1^+(G_m)$, $\ell_1^+(2^{-m}G_m)$ and the operators $P_n^+\{\xi_m\} = \{\xi_0, \xi_1, \dots, \xi_{n-1}, \xi_n, 0, 0, \dots\}$, the argument in Theorem 4.2 can be repeated with only minor modifications, to deduce that if

(a) $T : A_0 \cap A_1 \to (B_0, B_1)_{\Gamma;J}$ is compact, and
(b) $\|T\pi Q_n^+\|_{\Gamma^+(G_m),\overline{B}_{\Gamma;J}} \to 0$ as $n \to \infty$,

then $T : (A_0, A_1)_{\Gamma;J} \to (B_0, B_1)_{\Gamma;J}$ is compact.

Since we still have $2^{-n}\|\tau_n\|_{\Gamma,\Gamma} \to 0$, we can derive (a) as in Theorem 4.7. To establish (b), note that

$$\|T\pi Q_n^+\|_{\Gamma^+(G_m),\overline{B}_{\Gamma;J}} \leq \|T\pi Q_n^+\|_{(\ell_1^+(G_m),\ell_1^+(2^{-m}G_m))_{\Gamma;J},\overline{B}_{\Gamma;J}}.$$

Using compactness of $T : A_0 \to B_0$, we can repeat the argument given in Theorem 4.6 to derive that $\|T\pi Q_n^+\|_{\ell_1^+(G_m),B_0} \to 0$. Then Lemma 2.6 yields (b). ∎

Writing down Theorem 4.8 for $\Gamma = \ell_q(2^{-\theta m})$ we get a result of Cobos and Fernandez [8, Thm. 2.1]. For this example, Theorems 4.6 and 4.7 give results of Cobos and Peetre [11, Example 2.4].

We end this section with an example which shows that without the assumption (4.5), Theorem 4.8 is not true in general.

EXAMPLE 4.9. Let $A_0 = \ell_1$, $A_1 = \ell_\infty$, $B_0 = \ell_\infty(\min\{1, 2^{-m}\})$, $B_1 = \ell_\infty$ and put $T\xi = \{\dots, 0, 0, \xi_1, \xi_2, \dots\}$ for $\xi = \{\xi_m\}$. Then $A_0 \hookrightarrow A_1$, $T : A_0 \to B_0$ is compact and $T : A_1 \to B_1$ is bounded. Choose $\Gamma = \ell_1(2^{-m})$. We find that $\|\tau_n\|_{\Gamma,\Gamma} = 2^n$, so (4.5) is not satisfied. By Example 2.5, $(A_0, A_1)_{\ell_1(2^{-m});J} = A_1^\circ = c_0$ and $(B_0, B_1)_{\ell_1(2^{-m});J} = B_1^\circ = \ell_\infty$, and obviously $T : c_0 \to \ell_\infty$ fails to be compact.

5. Compact operators from a J-space into a K-space. The main result of this section is a characterization of compact operators acting from a J-space into a K-space. The result is new even for the case of the classical real method, and it shows that the way followed by Cobos, Kühn and Schonbek to establish the one-sided compactness theorem [9, Thm. 1.3] is optimal. We shall work with the sequence spaces Γ and Λ such that for any Banach couple $\overline{A} = (A_0, A_1)$,

$$(A_0, A_1)_{\Gamma;J} \hookrightarrow (A_0, A_1)_{\Lambda;K}. \tag{5.1}$$

Under this assumption it is clear that whenever $T : \overline{A} \to \overline{B}$ then the restriction $T : \overline{A}_{\Gamma;J} \to \overline{B}_{\Lambda;K}$ is bounded.

We denote by G_m the space $(A_0 \cap A_1, J(2^m, \cdot))$ and by F_m the space $(B_0 + B_1, K(2^m, \cdot))$. We shall work again with the operators P_n, Q_n^+, Q_n^- introduced in Section 3, but now we shall use them on the two couples. To avoid misunderstanding, we denote by R_n, S_n^+, S_n^- the operators P_n, Q_n^+, Q_n^- when they act on $\Gamma(G_m)$ or on any other sequence space related to $(A_0, A_1)_{\Gamma;J}$.

THEOREM 5.1. *Let Γ be a J-non-trivial sequence space and let Λ be a K-non-trivial sequence space satisfying* (5.1). *Assume also that*

$$2^{-n}\|\tau_n\|_{\Gamma,\Gamma} \to 0 \quad \textit{and} \quad \|\tau_{-n}\|_{\Gamma,\Gamma} \to 0 \qquad \textit{as } n \to \infty. \tag{5.2}$$

Let $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ be Banach couples and let $T : \overline{A} \to \overline{B}$. Then $T : (A_0, A_1)_{\Gamma;J} \to (B_0, B_1)_{\Lambda;K}$ is compact if and only if the following conditions hold.

(a) $T : A_0 \cap A_1 \to (B_0, B_1)_{\Lambda;K}$ *is compact.*
(b) $T : (A_0, A_1)_{\Gamma;J} \to B_0 + B_1$ *is compact.*
(c) $\sup\Big\{\Big\|(I - P_n)\Big\{K\Big(2^m, T\Big(\sum_{|k|>n} u_k\Big)\Big)\Big\}\Big\|_\Lambda : \|\{J(2^m, u_m)\}\|_\Gamma \le 1\Big\} \to 0$ *as $n \to \infty$.*

Proof. Suppose first that (a)–(c) are satisfied. Take any $n \in \mathbb{N}$ and consider the operators $jT\pi R_n$, $P_n jT\pi(S_n^+ + S_n^-)$ acting from $\Gamma(G_m)$ into $\Lambda(F_m)$. The factorizations

$$\Gamma(G_m) \xrightarrow{R_n} \ell_1(G_m) \cap \ell_1(2^{-m}G_m) \xrightarrow{\pi} A_0 \cap A_1 \xrightarrow{T} (B_0, B_1)_{\Lambda;K} \xrightarrow{j} \Lambda(F_m)$$

and

$$\begin{array}{c} \Gamma(G_m) \xrightarrow{S_n^+ + S_n^-} \Gamma(G_m) \xrightarrow{\pi} (A_0, A_1)_{\Gamma;J} \xrightarrow{T} B_0 + B_1 \xrightarrow{j} \ell_\infty(F_m) + \ell_\infty(2^{-m}F_m) \\ \hfill \downarrow P_n \quad \\ \hfill \Lambda(F_m) \quad \end{array}$$

and conditions (a), (b) imply that $jT\pi R_n$ and $P_n jT\pi(S_n^+ + S_n^-)$ are compact. Since (c) means that $\|jT\pi - jT\pi R_n - P_n jT\pi(S_n^+ + S_n^-)\|_{\Gamma(G_m),\Lambda(F_m)} \to 0$ as $n \to \infty$, we find that $jT\pi : \Gamma(G_m) \to \Lambda(F_m)$ is compact. Hence, by the properties of π and j, compactness of $T : (A_0, A_1)_{\Gamma;J} \to (B_0, B_1)_{\Lambda;K}$ follows.

Conversely, suppose that $T : (A_0, A_1)_{\Gamma;J} \to (B_0, B_1)_{\Lambda;K}$ is compact. Then (a) and (b) can be easily derived from the embeddings $A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{\Gamma;J}$ and $(B_0, B_1)_{\Lambda;K} \hookrightarrow B_0 + B_1$. To establish (c) put

$$\delta_n = \|jT\pi - jT\pi R_n - P_n jT\pi(S_n^+ + S_n^-)\|_{\Gamma(G_m),\Lambda(F_m)}.$$

It is easy to see that $\delta_1 \ge \delta_2 \ge \cdots \ge 0$. Hence the sequence $\{\delta_n\}_{n\in\mathbb{N}}$ is convergent. Let δ be its limit. It remains to show that $\delta = 0$. With this aim, find vectors w_n in the unit ball of $\Gamma(G_m)$ such that

$$\begin{aligned}\delta &= \lim_{n\to\infty} \|(jT\pi - jT\pi R_n - P_n jT\pi(S_n^+ + S_n^-))w_n\|_{\Lambda(F_m)} \\ &= \lim_{n\to\infty} \|(jT\pi(S_n^+ + S_n^-) - P_n jT\pi(S_n^+ + S_n^-))w_n\|_{\Lambda(F_m)} \\ &= \lim_{n\to\infty} \|(Q_n^+ + Q_n^-)jT\pi(S_n^+ + S_n^-)w_n\|_{\Lambda(F_m)}.\end{aligned}$$

Since $\{\pi(S_n^+ + S_n^-)w_n\}$ is bounded in $(A_0, A_1)_{\Gamma;J}$, compactness of the operator $T : (A_0, A_1)_{\Gamma;J} \to (B_0, B_1)_{\Lambda;K}$ implies, by passing to a subsequence, that $\{T\pi(S_{n'}^+ + S_{n'}^-)w_{n'}\}$ converges to some b in $(B_0, B_1)_{\Lambda;K}$. The subsequence also converges to b in $B_0 + B_1$. But, by (5.2) and Lemma 4.1, we have

$$\lim_{n\to\infty} \|S_n^+\|_{\Gamma(G_m),\ell_1(2^{-m}G_m)} = 0 = \lim_{n\to\infty} \|S_n^-\|_{\Gamma(G_m),\ell_1(G_m)}.$$

It follows that $\{(S_n^+ + S_n^-)w_n\}$ has limit 0 in $\ell_1(G_m) + \ell_1(2^{-m}G_m)$ and so $\{T\pi(S_{n'}^+ + S_{n'}^-)w_{n'}\}$ converges to 0 in $B_0 + B_1$. This implies $b = 0$. Finally, since $\|Q_n^+ + Q_n^-\|_{\Lambda(F_m),\Lambda(F_m)} = 1$, we obtain

$$\begin{aligned}\delta &= \lim_{n'\to\infty} \|(Q_{n'}^+ + Q_{n'}^-)jT\pi(S_{n'}^+ + S_{n'}^-)w_{n'}\|_{\Lambda(F_m)} \\ &\le \lim_{n'\to\infty} \|T\pi(S_{n'}^+ + S_{n'}^-)w_{n'}\|_{\overline{B}_{\Lambda;K}} = \|b\|_{\overline{B}_{\Lambda;K}} = 0. \ \blacksquare\end{aligned}$$

Remark 5.2. Condition (5.2) has only been used to show that compactness of $T : (A_0, A_1)_{\Gamma;J} \to (B_0, B_1)_{\Lambda;K}$ implies (c). If it is not satisfied, this implication does not hold in general as the following example shows.

Example 5.3. Take $\Gamma = \ell_1$ and $\Lambda = \ell_\infty$. Then (5.1) holds because for any Banach couple $\overline{A} = (A_0, A_1)$ we have (see Examples 2.4 and 2.5)

$$(A_0, A_1)_{\ell_1;J} = A_0^\circ \hookrightarrow A_0^\sim = (A_0, A_1)_{\ell_\infty;K}.$$

But condition (5.2) fails because $\|\tau_{-n}\|_{\ell_1,\ell_1} = 1$. Let

$$\overline{A} = (\ell_1, \ell_\infty(\min\{1, 2^{-m}\})), \qquad \overline{B} = (\ell_1(\min\{1, 2^{-m}\}), \ell_\infty(\min\{1, 2^{-m}\}))$$

and let $T\xi = \{\ldots, 0, 0, \xi_1, \xi_2, \ldots\}$ for $\xi = \{\xi_m\}$. Since $\overline{A}_{\ell_1;J} = \ell_1^\circ = \ell_1$ and $\overline{B}_{\ell_\infty;K} = \ell_1(\min\{1, 2^{-m}\})^\sim = \ell_1(\min\{1, 2^{-m}\})$, we see that the interpolated operator $T : \overline{A}_{\ell_1;J} \to \overline{B}_{\ell_\infty;K}$ is compact. However, given any $n \in \mathbb{N}$ and any $\xi \in \ell_1$ with $\|\xi\|_{\ell_1} = 1$, if we put $u = \{u_m\}$ with $u_m = \xi$ if $m = -n-1$ and $u_m = 0$ otherwise, then $\|\{J(2^m, u_m)\}\|_{\ell_1} = J(2^{-n-1}, \xi) = \|\xi\|_{\ell_1} = 1$ and

$$\begin{aligned}&\Big\|(I - P_n)\Big\{K\Big(2^m, T\Big(\sum_{|k|>n} u_k\Big)\Big)\Big\}\Big\|_{\ell_\infty} \\ &\qquad\qquad \ge K(2^{n+1}, T\xi) \ge K(1, T\xi) = \|T\xi\|_{\ell_\infty(\min\{1,2^{-m}\})}.\end{aligned}$$

Therefore

$$\sup\Big\{\Big\|(I-P_n)\Big\{K\Big(2^m,T\Big(\sum_{|k|>n}u_k\Big)\Big)\Big\}\Big\|_\Lambda : \|\{J(2^m,u_m)\}\|_\Gamma\le 1\Big\}$$
$$\ge \|T\|_{\ell_1,\ell_\infty(\min\{1,2^{-m}\})}=1/2.$$

As can be seen in [24, Lemma 2.5], if the Calderón transform $\Omega\{\xi_m\} = \{\sum_{k=-\infty}^{\infty}\min(1,2^{m-k})|\xi_k|\}_{m\in\mathbb{Z}}$ is bounded from Γ into Λ, then (5.1) holds. In particular, this is the case for $\Gamma=\Lambda=\ell_q(1/f(2^m))$ (Example 2.2).

Next we derive from Theorem 5.1 an extension of the function parameter version of Cwikel's compactness theorem established by Cobos, Kühn and Schonbek in [9, Thm. 2.3].

THEOREM 5.4. *Let Γ be a J-non-trivial sequence space and let Λ be a K-non-trivial sequence space satisfying* (5.1). *Suppose also that Γ satisfies* (5.2). *Let $\overline{A}=(A_0,A_1)$, $\overline{B}=(B_0,B_1)$ be Banach couples and let $T:\overline{A}\to\overline{B}$ with $T:A_0\to B_0$ compactly. Then $T:(A_0,A_1)_{\Gamma;J}\to(B_0,B_1)_{\Lambda;K}$ is compact.*

Proof. By Theorem 5.1 it suffices to show that

(a) $T:A_0\cap A_1\to(B_0,B_1)_{\Lambda;K}$ is compact,
(b) $T:(A_0,A_1)_{\Gamma;J}\to B_0+B_1$ is compact,
(c) $\|(Q_n^+ + Q_n^-)jT\pi(S_n^+ + S_n^-)\|_{\Gamma(G_m),\Lambda(F_m)}\to 0$ as $n\to\infty$, where $G_m=(A_0\cap A_1,J(2^m,\cdot))$ and $F_m=(B_0+B_1,K(2^m,\cdot))$.

Using (5.2) and Lemma 2.7, we have

$$\lim_{t\to\infty}\frac{\psi(t,\overline{A}_{\Gamma;J},\overline{A})}{t}=0=\lim_{t\to 0}\frac{t}{\varrho(t,\overline{B}_{\Gamma;J},\overline{B})}.$$

Since $T:A_0\cap A_1\to B_0$ is compact and $T:A_0\cap A_1\to B_1$ is bounded, applying [6, Thm. 3.2] and assumption (5.1) we derive (a). On the other hand, as $T:A_0\to B_0+B_1$ is compact and $T:A_1\to B_0+B_1$ is bounded, (b) follows from [6, Thm. 3.1]. It remains to establish (c). By Lemmas 3.4 and 4.5, we get

$$\begin{aligned}
&\|(Q_n^+ + Q_n^-)jT\pi(S_n^+ + S_n^-)\|_{\Gamma(G_m),\Lambda(F_m)}\\
&\quad\le \|(Q_n^+ + Q_n^-)jT\pi(S_n^+ + S_n^-)\|_{\overline{\ell_1(G)}_{\Gamma;J},\overline{\ell_\infty(F)}_{\Lambda;K}}\\
&\quad\le \|Q_n^- jT\pi(S_n^+ + S_n^-)\|_{\overline{\ell_1(G)}_{\Gamma;J},\overline{\ell_\infty(F)}_{\Lambda;K}}\\
&\qquad+\|Q_n^+ jT\pi S_n^+\|_{\overline{\ell_1(G)}_{\Gamma;J},\overline{\ell_\infty(F)}_{\Lambda;K}}+\|Q_n^+ jT\pi S_n^-\|_{\overline{\ell_1(G)}_{\Gamma;J},\overline{\ell_\infty(F)}_{\Lambda;K}}.
\end{aligned}$$

Since the couple $\overline{A}$ is interpolated by a J-space and $A_0\cap A_1=A_0^\circ\cap A_1^\circ$, we may and do assume that $A_0=A_0^\circ$ and $A_1=A_1^\circ$. Then the argument given to establish (3.4) can be repeated to show that

$$\|Q_n^- jT\pi(S_n^+ + S_n^-)\|_{\ell_1(G_m),\ell_\infty(F_m)} \le \|Q_n^- jT\|_{A_0^\circ,\ell_\infty(F_m)} \to 0 \quad \text{as } n\to\infty.$$

The same argument as in (4.3) implies that

$$\|Q_n^+ jT\pi S_n^+\|_{\ell_1(G_m),\ell_\infty(F_m)} \le \|T\pi S_n^+\|_{\ell_1(G_m),B_0} \to 0 \quad \text{as } n\to\infty.$$

Furthermore, the factorization

$$\ell_1(2^{-m}G_m) \xrightarrow{S_n^-} \ell_1(G_m) \xrightarrow{jT\pi} \ell_\infty(F_m) \xrightarrow{Q_n^+} \ell_\infty(2^{-m}F_m)$$

gives

$$\|Q_n^+ jT\pi S_n^-\|_{\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}F_m)} \le 2^{-n}\|T\|_{A_0,B_0}2^{-n} \to 0 \quad \text{as } n\to\infty.$$

Consequently, using Lemma 2.6 and assumptions (5.2) and (5.1), we deduce that $\{\|Q_n^- jT\pi(S_n^+ + S_n^-)\|_{\overline{\ell_1(G)}_{\Gamma;J},\overline{\ell_\infty(F)}_{\Lambda;K}}\}$, $\{\|Q_n^+ jT\pi S_n^+\|_{\overline{\ell_1(G)}_{\Gamma;J},\overline{\ell_\infty(F)}_{\Lambda;K}}\}$ and $\{\|Q_n^+ jT\pi S_n^-\|_{\overline{\ell_1(G)}_{\Gamma;J},\overline{\ell_\infty(F)}_{\Lambda;K}}\}$ converge to 0 as $n\to\infty$. This gives (c) and completes the proof. ■

6. Compactness versus weak compactness. A well known result of Beauzamy [1] says that if the inclusion $I : A_0\cap A_1 \to A_0+A_1$ is weakly compact and $1<q<\infty$ then the identity of the classical real interpolation space $I : (A_0,A_1)_{\theta,q} \to (A_0,A_1)_{\theta,q}$ is also weakly compact (that is, $(A_0,A_1)_{\theta,q}$ is reflexive). This result has attracted considerable attention. A number of authors have extended it in several directions (see, for example, [20, 4, 22, 10]). The following theorem is very close to [22, Cor. 11] (see also [4, Thm. 4.6.8]). The proof is a slight modification of the arguments given in [22] by using Lemma 2.7.

THEOREM 6.1. *Let Γ be a K-non-trivial sequence space such that Γ is reflexive, $2^{-n}\|\tau_n\|_{\Gamma,\Gamma}\to 0$ and $\|\tau_{-n}\|_{\Gamma,\Gamma}\to 0$ as $n\to\infty$. Let $\overline{A}=(A_0,A_1)$, $\overline{B}=(B_0,B_1)$ be Banach couples and let $T:\overline{A}\to\overline{B}$. Then $T:(A_0,A_1)_{\Gamma;K}\to(B_0,B_1)_{\Gamma;K}$ is weakly compact if and only if $T:A_0\cap A_1\to B_0+B_1$ is weakly compact.*

A similar result does not hold for compact operators. Take, for example, $\overline{A}=\overline{B}=(\ell_2,\ell_2(\max\{1,2^m\}))$, let $T\xi=\{\ldots,0,0,\xi_1,\xi_2,\ldots\}$ for $\xi=\{\xi_m\}$ and choose $\Gamma=\ell_2(2^{-\theta m})$ with $0<\theta<1$. The space Γ is K-non-trivial and reflexive with shift operators having the required behaviour. We have $A_0\cap A_1=\ell_2(\max\{1,2^m\})$ and $B_0+B_1=\ell_2$. So, $T:A_0\cap A_1\to B_0+B_1$ is compact. However, $(A_0,A_1)_{\theta,2}=(B_0,B_1)_{\theta,2}=\ell_2(\max\{1,2^{\theta m}\})$ and $T:(A_0,A_1)_{\theta,2}\to(B_0,B_1)_{\theta,2}$ fails to be compact.

Next we shall use the results of Sections 3 and 4 to determine when compactness of $T:A_0\cap A_1\to B_0+B_1$ passes to the interpolated operator.

We shall assume that the sequence space Γ has the property that for any Banach couple $\bar{A} = (A_0, A_1)$,

$$(A_0, A_1)_{\Gamma;J} = (A_0, A_1)_{\Gamma;K}. \tag{6.1}$$

We denote by $(A_0, A_1)_\Gamma$ the common space in (6.1). The spaces G_m, F_m and the operators P_n, Q_n^+, Q_n^-, R_n, S_n^+, S_n^- are defined as in the previous section.

THEOREM 6.2. *Let Γ be a K- and J-non-trivial sequence space satisfying* (6.1). *Assume also that for any $\xi \in \Gamma$,*

$$\|\xi - P_n\xi\|_\Gamma \to 0 \quad \text{as } n \to \infty \tag{6.2}$$

and that

$$2^{-n}\|\tau_n\|_{\Gamma,\Gamma} \to 0 \quad \text{and} \quad \|\tau_{-n}\|_{\Gamma,\Gamma} \to 0 \quad \text{as } n \to \infty. \tag{6.3}$$

Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples and let $T : \bar{A} \to \bar{B}$. Then $T : (A_0, A_1)_\Gamma \to (B_0, B_1)_\Gamma$ is compact if and only if the following conditions hold.

(a) $T : A_0 \cap A_1 \to B_0 + B_1$ *is compact.*
(b) $\sup\{\|(I - P_n)\{K(2^m, Ta)\}\|_\Gamma : \|a\|_{\bar{A}_\Gamma} \le 1\} \to 0$ *as* $n \to \infty$.
(c) $\sup\{\|T(\sum_{|m|>n} u_m)\|_{\bar{B}_\Gamma} : \|\{J(2^m, u_m)\}\|_\Gamma \le 1\} \to 0$ *as* $n \to \infty$.

Proof. Suppose first that the three conditions hold. The factorization

$$\begin{array}{rcl} \Gamma(G_m) \xrightarrow{R_n} \ell_1(G_m) \cap \ell_1(2^{-m}G_m) \xrightarrow{\pi} A_0 \cap A_1 \xrightarrow{T} & B_0 + B_1 & \\ & \downarrow j & \\ & \ell_\infty(F_m) + \ell_\infty(2^{-m}F_m) & \\ & \downarrow P_n & \\ & \Gamma(F_m) & \end{array}$$

and (a) imply that, for any $n \in \mathbb{N}$, $P_n jT\pi R_n : \Gamma(G_m) \to \Gamma(F_m)$ is compact. Since

$$\begin{aligned} &\|jT\pi - P_n jT\pi R_n\|_{\Gamma(G_m),\Gamma(F_m)} \\ &\quad \le \|jT\pi - P_n jT\pi\|_{\Gamma(G_m),\Gamma(F_m)} + \|P_n jT\pi - P_n jT\pi R_n\|_{\Gamma(G_m),\Gamma(F_m)} \\ &\quad \le \|(I - P_n)jT\|_{\bar{A}_\Gamma,\Gamma(F_m)} + \|T\pi(I - R_n)\|_{\Gamma(G_m),\bar{B}_\Gamma}, \end{aligned}$$

using (b) and (c), we infer that $jT\pi : \Gamma(G_m) \to \Gamma(F_m)$ is compact. Hence $T : (A_0, A_1)_\Gamma \to (B_0, B_1)_\Gamma$ is compact. Note that this implication still holds when (6.2) and (6.3) are not satisfied.

Conversely, if $T : (A_0, A_1)_\Gamma \to (B_0, B_1)_\Gamma$ is compact, then (a) follows from the inclusions $A_0 \cap A_1 \hookrightarrow (A_0, A_1)_\Gamma$ and $(B_0, B_1)_\Gamma \hookrightarrow B_0 + B_1$. We obtain (b) by Theorem 3.1 and (c) by Theorem 4.2. ∎

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