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H^∞ functional calculus for sectorial and bisectorial operators

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Abstract. We give a concise exposition of the basic theory of H^∞ functional calculus for N-tuples of sectorial or bisectorial operators, with respect to operator-valued functions; moreover we restate and prove in our setting a result of N. Kalton and L. Weis about the boundedness of the operator $f(T_1, \ldots, T_N)$ when f is an R-bounded operator-valued holomorphic function.

1. Introduction. The H^∞ functional calculus for sectorial operators was introduced by A. McIntosh [10] (in the Hilbert space setting) in the '80s. The aim was to give a meaning to $f(T)$ when T is a sectorial operator and f is a complex-valued holomorphic function on a sector containing the spectrum of T, with suitable assumptions on the growth of f at 0 and at ∞. Besides the natural generalization to the Banach space case (see [4]), it was extended in two directions, either by replacing T with an N-tuple of operators $(T_1, \ldots, T_N)$, or by assuming that f is operator-valued (see [1], [9]). These extensions are not trivial, in the sense that it is much harder to find sufficient conditions for the boundedness of an operator of the type $f(T_1, \ldots, T_N)$ (with f operator-valued). A result in this direction can be found in the paper [7], where, however, the proof is carried out in the case of a single sectorial operator; moreover we notice that the definition of functional calculus there is slightly different from ours.

This result of Kalton and Weis is used in our paper [6], where, however, the operators are not sectorial, but bisectorial (see the definition in the next section); moreover we need not only the boundedness of the operator $f(T_1, \ldots, T_N)$, but also an estimate of $\|f(T_1, \ldots, T_N)\|$, which is not explicitly stated in [7].

The aim of this paper is therefore to restate and prove the result of Kalton and Weis in the more general situation mentioned above. To this end we make a concise exposition of the basic theory in the most general

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case; we think that also this part of the paper can be useful, since in the existing literature it is difficult to find explicit proofs of some of these results.

The paper is organized as follows: in §2 we give some basic definitions; in §§3 and 4 we introduce and study the functional calculus; in §5 we prove some technical results; §6 is devoted to the proof of the Kalton–Weis theorem.

2. Sectorial and bisectorial operators. In the whole paper X is a complex (nontrivial) Banach space. We say that a linear operator T *acts* in X if its domain $\mathcal{D}(T)$ and its range $\mathcal{R}(T)$ are subspaces of X; for such an operator $\sigma(T)$ denotes the spectrum and $\varrho(T)$ the resolvent set. As usual, $\mathcal{L}(X)$ denotes the Banach algebra of bounded linear operators from X to X.

The symbol $\mathbb{R}^+$ denotes the set of positive (i.e. > 0) real numbers, and $\mathbb{N}$ the set of positive integers; "arg" denotes the principal argument and takes values in $]-\pi, \pi[$.

We shall work with sectors and "double sectors" of the complex plane. Their definition is the following.

Definition 2.1. (a) For $\theta \in]0,\pi[$ we set $S_\theta = \{re^{i\alpha};\ r \in \mathbb{R}^+,\ \alpha \in]-\theta, \theta[\}$.
(b) For $\delta \in]0, \pi/2[$ we set $\varSigma_\delta = \{\varrho\, e^{i\alpha};\ \varrho \in \mathbb{R}\setminus\{0\},\ \alpha \in]\pi/2-\delta, \pi/2+\delta[\}$.

Thus S_θ is the open sector around $\mathbb{R}^+$ with half-opening θ, and $\varSigma_\delta$ is the open "double sector" around the imaginary axis, with half-opening δ.

Definition 2.2. Let T be a linear operator acting in X. We say that T is *sectorial* if:

(i) $\mathcal{D}(T)$ and $\mathcal{R}(T)$ are dense in X

and there is a $\theta \in]0, \pi[$ such that

(ii) $\sigma(T) \subseteq \overline{S}_\theta$,
(iii) $\sup_{\lambda \in \mathbb{C}\setminus \overline{S}_{\theta'}} \|\lambda(\lambda - T)^{-1}\| < \infty$ for all $\theta' \in]\theta, \pi[$.

The greatest lower bound of the θ's satisfying (ii) and (iii) will be called the *spectral angle* of T.

Definition 2.3. Let T be a linear operator acting in X. We say that T is *bisectorial* if

(i) $\mathcal{D}(T)$ and $\mathcal{R}(T)$ are dense in X

and there is a $\delta \in]0, \pi/2[$ such that

(ii) $\sigma(T) \subseteq \overline{\varSigma}_\delta$,
(iii) $\sup_{\lambda \in \mathbb{C}\setminus \overline{\varSigma}_{\delta'}} \|\lambda(\lambda - T)^{-1}\| < \infty$ for all $\delta' \in]\delta, \pi/2[$.

The greatest lower bound of the δ's satisfying (ii) and (iii) will be called the *spectral angle* of T.

It is obvious that a bisectorial operator T with spectral angle $\delta \in]0, \pi/2[$ is also a sectorial operator with spectral angle $\delta + \pi/2$ (and that in both cases the operator is closed, since its resolvent set is not empty). The reason why we want to consider bisectorial operators is that we are going to study a functional calculus for operators that can be either sectorial or bisectorial; however, in the bisectorial case we want to use holomorphic functions that are defined only on a double sector.

Bisectorial operators also appear in the papers [3] and [11].

We will need the following result (for a proof we refer to [8, Ths. 2.1, 3.1] or to [4, Th. 3.8]).

LEMMA 2.4. *Let T be a linear operator acting in X. Assume that there is a subset Ω of $\varrho(T)$ such that* 0 *and* ∞ *are accumulation points of Ω and* $\sup_{\lambda\in\Omega} \|\lambda(\lambda - T)^{-1}\| < \infty$. *Then*:

(a) $\overline{\mathcal{D}(T)} = \{x \in X;\ \lim_{\lambda\to\infty,\, \lambda\in\Omega} \lambda(\lambda - T)^{-1}x = x\}$;
(b) $\overline{\mathcal{R}(T)} = \{x \in X;\ \lim_{\lambda\to 0,\, \lambda\in\Omega} \big(x + T(\lambda - T)^{-1}x\big) = 0\}$;
(c) $\ker T \cap \overline{\mathcal{R}(T)} = \{0\}$.

By Lemma 2.4(c) any (bi)sectorial operator is injective; moreover, if $0 \neq z \in \varrho(T)$, then $z^{-1} \in \varrho(T^{-1})$ and $(z^{-1} - T^{-1})^{-1} = -zT(z - T)^{-1}$. Therefore it is easy to prove that T^{-1} is (bi)sectorial simultaneously with T, with the same spectral angle.

The following converse of the triangle inequality will be useful.

LEMMA 2.5. *Let $\alpha, \beta \in \mathbb{R}$ and $w, z \in \mathbb{C}$; assume that $w = |w|e^{i\alpha}$, $z = |z|e^{i\beta}$. Then*

$$|w + z| \geq (|w| + |z|)|\cos((\alpha - \beta)/2)|,$$
$$|w - z| \geq (|w| + |z|)|\sin((\alpha - \beta)/2)|.$$

Proof.

$$\begin{aligned}|w \pm z|^2 &= |w|^2 + |z|^2 \pm 2|w|\,|z| \cos(\alpha - \beta)\\ &= (|w| \pm |z|)^2 \cos^2((\alpha - \beta)/2) + (|w| \mp |z|)^2 \sin^2((\alpha - \beta)/2). \ \blacksquare\end{aligned}$$

Let us introduce now a useful family of functions.

DEFINITION 2.6. (a) For each positive integer n we let ψ_n be the function on $\mathbb{C} \setminus \{-n, -n^{-1}\}$ to $\mathbb{C}$ defined by

$$\psi_n(z) = \frac{n^2 z}{(1 + nz)(n + z)}.$$

We denote ψ_1 by ψ.

(b) If n and N are positive integers, we set

$$\Psi_{n,N} : (\mathbb{C} \setminus \{-n, \, -n^{-1}\})^N \to \mathbb{C}, \qquad \Psi_{n,N}(z_1, \ldots, z_N) = \prod_{k=1}^{N} \psi_n(z_k).$$

If N is understood, we shall write Ψ instead of $\Psi_{1,N}$.

Note that

$$\Psi(z) = \prod_{k=1}^{N} \frac{z_k}{(1+z_k)^2}.$$

LEMMA 2.7. *Let $n \in \mathbb{N}$ and $\theta \in \,]0, \pi[$. Then for all $z \in S_\theta$ we have*

$$|\psi_n(z)| \le (\cos(\theta/2))^{-2} \min\{1, n|z|, n|z|^{-1}\}.$$

Proof. If $z \in S_\theta$, then $z = |z|e^{i\alpha}$ with $\alpha \in \,]-\theta, \theta[$. Therefore, taking into account Lemma 2.5, we obtain

$$\begin{aligned} |\psi_n(z)| &= \frac{n^2|z|}{|1+nz|\,|n+z|} \le \frac{n^2|z|}{\cos^2(\alpha/2)(1+n|z|)(n+|z|)} \\ &\le \frac{1}{\cos^2(\theta/2)} \, \frac{n|z|}{1+n|z|} \, \frac{n}{n+|z|} \end{aligned}$$

and the desired inequality follows easily. ∎

In the next sections, we are going to define and study operators that we shall call $f(T)$, in the following two situations:

(i) T is a sectorial operator with spectral angle $\theta \in [0, \pi[$ and f is a holomorphic function defined on a sector $S_{\theta'}$ with $\theta < \theta' < \pi$;

(ii) T is a bisectorial operator with spectral angle $\theta \in [0, \pi/2[$ and f is a holomorphic function defined on a bisector $\Sigma_{\theta'}$ with $\theta < \theta' < \pi/2$.

We shall also deal with operators $f(T_1, \ldots, T_N)$, where each T_k is sectorial or bisectorial, and f is a (possibly operator-valued) holomorphic function of N variables defined on the cartesian product of N sectors (or double sectors). In order to handle this situation we need another definition.

DEFINITION 2.8. Let $\varphi \in \,]0, \pi[$. We define Γ_φ to be the curve parametrized by

$$\mathbb{R} \setminus \{0\} \ni \varrho \mapsto |\varrho| e^{-i\varphi \operatorname{sgn} \varrho}$$

and oriented according to the increasing values of ϱ (i.e. according to the decreasing imaginary parts). If $0 < \theta < \varphi$, we will also say that Γ_φ is an *admissible curve* for S_θ.

REMARK 2.9. Let $\varphi \in \,]0, \pi[$. It is obvious that, as a set, $\Gamma_{\pi-\varphi} = -\Gamma_\varphi$. However, when we write "$-\Gamma_\varphi$" it is understood that the orientation of the curve is opposite to the one provided by Definition 2.8, i.e. $-\Gamma_\varphi$ is oriented according to the increasing imaginary parts.

DEFINITION 2.10. Let $\theta \in \,]0, \pi/2[$. An *admissible curve* for Σ_θ is any curve of the type

$$\Gamma_\varphi \cup (-\Gamma_\varphi), \quad \text{where} \quad \theta + \pi/2 < \varphi < \pi.$$

3. H_0^∞ functional calculus. In what follows, it is understood that:

(i) each of the operators $T_1, \ldots, T_N$ is either sectorial or bisectorial and T_k has spectral angle $\le \theta_k$, where θ_k belongs to $]0, \pi[$ or $]0, \pi/2[$ according to the case;

(ii) for all $j, k \in \{1, \ldots, N\}$ the resolvent operators of T_j and T_k commute;

(iii) $G_k = S_{\theta_k}$ if T_k is treated as a sectorial operator; $G_k = \Sigma_{\theta_k}$ if T_k is treated as a bisectorial operator; $G := \prod_{k=1}^N G_k$;

(iv) Ω_k is a set of the same type as G_k, with an angle greater than θ_k, and we set $\Omega = \prod_{k=1}^N \Omega_k$;

(v) Γ^k is an admissible curve for G_k contained in Ω_k, and we set $\Gamma = \prod_{k=1}^N \Gamma^k$.

We write $\mathbf{T}$ instead of $(T_1, \ldots, T_N)$, whenever convenient, and we let $\mathcal{B} = \mathcal{B}(\mathbf{T})$ be the commutator of the set $\bigcup_{k=1}^N \{(\lambda - T_k)^{-1};\ \lambda \in \varrho(T_k)\}$, that is, the closed subalgebra of the Banach algebra $\mathcal{L}(X)$ consisting of the operators that commute with the resolvent operators of $T_1, \ldots, T_N$. Note that $\mathcal{B} \supseteq \{\lambda I_X;\ \lambda \in \mathbb{C}\}$; functions with values in $\{\lambda I_X;\ \lambda \in \mathbb{C}\}$ will be naturally identified with complex-valued functions.

DEFINITION 3.1. Let Y be a complex Banach space. We denote by

- $H(\Omega, Y)$ the vector space of Y-valued holomorphic functions on Ω;
- $H^\infty(\Omega, Y)$ the Banach space of Y-valued bounded holomorphic functions on Ω, with the norm $\|f\|_\infty := \sup_{z \in \Omega} \|f(z)\|_Y$;
- $H_0^\infty(\Omega, Y)$ the space of holomorphic functions $f : \Omega \to Y$ satisfying the following condition: there are $s, C > 0$ such that for all $z = (z_1, \ldots, z_N) \in \Omega$,

$$\|f(z)\|_Y \le C \prod_{j=1}^N (\min\{|z_j|, |z_j|^{-1}\})^s;$$

- $H_{\mathrm{P}}(\Omega, Y)$ the space of holomorphic functions $f : \Omega \to Y$ with polynomial growth at 0 and at ∞, that is, satisfying the following condition: there are $s \in \mathbb{R}$ and $C > 0$ such that for all $z = (z_1, \ldots, z_N) \in \Omega$,

$$\|f(z)\|_Y \le C \prod_{j=1}^N (\max\{|z_j|, |z_j|^{-1}\})^s.$$

The mention of Y will be omitted when $Y = \mathbb{C}$.

REMARKS 3.2. (a) It follows from Definition 3.1 that

$$H_0^\infty(\Omega, Y) \subseteq H^\infty(\Omega, Y) \subseteq H_{\mathrm{P}}(\Omega, Y) \subseteq H(\Omega, Y).$$

(b) If Y is a Banach algebra, then so is $H^\infty(\Omega, Y)$, and $H_0^\infty(\Omega, Y)$ is a two-sided ideal of $H^\infty(\Omega, Y)$. Moreover $H_{\mathrm{P}}(\Omega, Y)$ is an algebra.

(c) If $f \in H_{\mathrm{P}}(\Omega, Y)$, then it follows from Lemma 2.7 that there is $m \in \mathbb{N}$ such that $\Psi_{n,N}^m f \in H_0^\infty(\Omega, Y)$ for all $n \in \mathbb{N}$.

THEOREM 3.3. *Let $f \in H_0^\infty(\Omega, \mathcal{B})$. Then the integral*

$$\int_\Gamma f(z) \prod_{k=1}^N (z_k - T_k)^{-1}\, dz$$

converges in the norm of $\mathcal{L}(X)$ and does not depend on Γ.

Proof. For suitable $C, s \in \mathbb{R}^+$ we have

$$\begin{aligned}\Big\| f(z) \prod_{k=1}^N (z_k - T_k)^{-1} \Big\| &\le C \prod_{k=1}^N |z_k|^{-1} \prod_{k=1}^N (\min\{|z_k|, |z_k|^{-1}\})^s \\ &= C \prod_{k=1}^N \min\{|z_k|^{s-1}, |z_k|^{-s-1}\}.\end{aligned}$$

This proves the integrability. The same estimate yields the independence of the integral from the system of curves, by means of standard arguments of holomorphic function theory. ∎

DEFINITION 3.4. Let $f \in H_0^\infty(\Omega, \mathcal{B})$. We set

$$f(\mathbf{T}) = (2\pi i)^{-N} \int_\Gamma f(z) \prod_{k=1}^N (z_k - T_k)^{-1}\, dz.$$

THEOREM 3.5. *$f \mapsto f(\mathbf{T})$ is an algebra homomorphism from $H_0^\infty(\Omega, \mathcal{B})$ to $\mathcal{B}$.*

Proof. The only nontrivial thing to check is that $(fg)(\mathbf{T}) = f(\mathbf{T})g(\mathbf{T})$ whenever $f, g \in H_0^\infty(\Omega, \mathcal{B})$. We choose two systems of admissible curves $\Gamma = \prod_{k=1}^n \Gamma^k$ and $\widetilde{\Gamma} = \prod_{k=1}^n \widetilde{\Gamma}^k$ with the property that $\widetilde{\Gamma}^k$ "lies outside Γ^k", in the sense that every point of Γ^k belongs to the boundary of a (double) sector with respect to which $\widetilde{\Gamma}^k$ is admissible. Then

$$\begin{aligned} f(\mathbf{T})g(\mathbf{T}) &= (2\pi i)^{-2N} \int_\Gamma \int_{\widetilde{\Gamma}} f(z)g(w) \prod_{k=1}^N (z_k - T_k)^{-1}(w_k - T_k)^{-1}\, dw\, dz \\ &= (2\pi i)^{-2N} \int_\Gamma \int_{\widetilde{\Gamma}} f(z)g(w) \prod_{k=1}^N (w_k - z_k)^{-1}((z_k - T_k)^{-1} - (w_k - T_k)^{-1})\, dw\, dz.\end{aligned}$$

By developing the product one gets 2^N summands, so that we have 2^N integrals, each in $2N$ variables, and in each integral we can exchange the order of integration as we like. But it follows from Cauchy's theorem that for all $k \in \{1, \ldots, N\}$ and $w \in \widetilde{\Gamma}$,

$$\int_{\Gamma^k} f(z)(w_k - z_k)^{-1}\, dz_k = 0,$$

since the function of z_k is holomorphic "inside Γ^k". Therefore the $2^N - 1$ integrals in which not all the resolvents are of kind $(z_k - T_k)^{-1}$ are 0; hence

$$\begin{aligned} f(\mathbf{T})g(\mathbf{T}) &= (2\pi i)^{-2N} \int_{\Gamma}\int_{\widetilde{\Gamma}} f(z)g(w) \prod_{k=1}^{N} (w_k - z_k)^{-1}(z_k - T_k)^{-1}\, dw\, dz \\ &= (2\pi i)^{-N} \int_{\Gamma} f(z)g(z) \prod_{k=1}^{N} (z_k - T_k)^{-1}\, dz \\ &\qquad \text{(by an iterated application of the residue theorem)} \\ &= (fg)(\mathbf{T}). \ \blacksquare \end{aligned}$$

Corollary 3.6. *If $f, g \in H_0^\infty(\Omega, \mathcal{B})$ and $f(z)$ commutes with $g(z)$ for all $z \in \Omega$, then $f(\mathbf{T})$ commutes with $g(\mathbf{T})$.*

Proof. Indeed, $f(\mathbf{T})g(\mathbf{T}) = (fg)(\mathbf{T}) = (gf)(\mathbf{T}) = g(\mathbf{T})f(\mathbf{T})$. ■

In the following we shall only meet products of the type $f(\mathbf{T})g(\mathbf{T})$ when f or g is scalar-valued, and so Corollary 3.6 applies.

Theorem 3.7. *Assume that $N = 1$, $f \in H_0^\infty(\Omega, \mathcal{B})$, and $\alpha \in \mathbb{C} \setminus \overline{\Omega}$. Set $r_\alpha(z) = (\alpha - z)^{-1}$. Then $r_\alpha f \in H_0^\infty(\Omega, \mathcal{B})$ and $(r_\alpha f)(T) = f(T)(\alpha - T)^{-1}$.*

Proof. By the resolvent equation

$$\int_{\Gamma} f(z)(z - T)^{-1}\, dz\, (\alpha - T)^{-1} = \int_{\Gamma} \frac{f(z)}{\alpha - z}\, (z - T)^{-1}\, dz - \int_{\Gamma} \frac{f(z)}{\alpha - z}\, dz\, (\alpha - T)^{-1}$$

and the last integral vanishes by Cauchy's theorem. ■

Theorem 3.8. *For positive integers n and N,*

$$\Psi_{n,N}(\mathbf{T}) = \prod_{k=1}^{N} (nT_k(n^{-1} + T_k)^{-1}(n + T_k)^{-1}).$$

Proof. By Fubini's theorem we have $\Psi_{n,N}(\mathbf{T}) = \prod_{k=1}^{N} \psi_n(T_k)$; therefore it is not restrictive to take $N = 1$, and hence we only prove that

$$\psi_n(T) = nT(n^{-1} + T)^{-1}(n + T)^{-1}.$$

When $n = 1$, by the residue theorem we get

$$\begin{aligned}\psi(T) &= \frac{1}{2\pi i}\int_\Gamma \frac{z}{(1+z)^2}\,(z-T)^{-1}\,dz = -\frac{d}{dz}(z(z-T)^{-1})\Big|_{z=-1}\\ &= (1+T)^{-1} - (1+T)^{-2} = T(1+T)^{-2}.\end{aligned}$$

When $n \geq 2$,

$$\begin{aligned}\psi_n(T) &= \frac{1}{2\pi i}\int_\Gamma \frac{n\,z}{(n^{-1}+z)(n+z)}\,(z-T)^{-1}\,dz\\ &= \frac{n^2}{n^{-1}-n}\,(-n-T)^{-1} + \frac{1}{n-n^{-1}}\,(-n^{-1}-T)^{-1}\\ &= \frac{n}{1-n^2}\,(n^2(-n^{-1}-T) - (-n-T))(-n-T)^{-1}(-n^{-1}-T)^{-1}\\ &= nT(n^{-1}+T)^{-1}(n+T)^{-1}. \ \blacksquare\end{aligned}$$

REMARK 3.9. As the operators $T_1, \ldots, T_N$ are (bi)sectorial we obtain

$$\sup_{n\geq 1} \|\Psi_{n,N}(\mathbf{T})\| < \infty$$

and $\Psi_{n,N}(\mathbf{T})$ is injective. Moreover Lemma 2.4 implies that $\Psi_{n,N}(\mathbf{T})^m x \to x$ as $n \to \infty$ for all $x \in X$ and $m \in \mathbb{N}$.

THEOREM 3.10. *For all $m \in \mathbb{N}$ the range of $\Psi_{n,N}(\mathbf{T})^m$ is independent of n, and is dense in X.*

Proof. Let p, n be positive integers. Then

$$\begin{aligned}\psi_p(T_k) &= pT_k(p^{-1}+T_k)^{-1}(p+T_k)^{-1}\\ &= nT_k(n^{-1}+T_k)^{-1}(n+T_k)^{-1}pn^{-1}(n^{-1}+T_k)\\ &\quad\times (p^{-1}+T_k)^{-1}(n+T_k)(p+T_k)^{-1}\\ &= \psi_n(T_k)pn^{-1}((n^{-1}-p^{-1})(p^{-1}+T_k)^{-1}+1)((n-p)(p+T_k)^{-1}+1)\end{aligned}$$

and hence, by the commutativity of the resolvent operators of $T_1, \ldots, T_N$, we get $\mathcal{R}(\Psi_{p,N}(\mathbf{T})^m) \subseteq \mathcal{R}(\Psi_{n,N}(\mathbf{T})^m)$, and by symmetry $\mathcal{R}(\Psi_{p,N}(\mathbf{T})^m) = \mathcal{R}(\Psi_{n,N}(\mathbf{T})^m)$. Now Remark 3.9 implies immediately that this common range is dense in X. $\blacksquare$

In particular, in the case $N = 1$, we have

THEOREM 3.11. *$\mathcal{R}(\psi(T)^m) = \mathcal{D}(T^m) \cap \mathcal{R}(T^m)$.*

Proof. For all $x \in X$ we have

$$\psi(T)^m x = T^m(1+T)^{-2m}x = (1+T)^{-m}T^m(1+T)^{-m}x,$$

hence $\psi(T)^m x \in \mathcal{D}(T^m) \cap \mathcal{R}(T^m)$. Conversely, assume that $y \in \mathcal{D}(T^m) \cap \mathcal{R}(T^m)$. Then $y = T^m z$ with $z \in \mathcal{D}(T^{2m})$; therefore there is an $x \in X$ such that $z = (1+T)^{-2m}x$, and $y = T^m(1+T)^{-2m}x = \psi(T)^m x$. $\blacksquare$

REMARK 3.12. Let $f \in H_0^\infty(\Omega, \mathcal{B})$. By Corollary 3.6, the operators $f(\mathbf{T})$ and $\Psi_{n,N}(\mathbf{T})$ commute; therefore it is well known that for all $m \in \mathbb{N}$,

$$f(\mathbf{T})\Psi_{n,N}(\mathbf{T})^{-m} \subseteq \Psi_{n,N}(\mathbf{T})^{-m} f(\mathbf{T}).$$

Hence by Theorem 3.10, the domain of $\Psi_{n,N}(\mathbf{T})^{-m} f(\mathbf{T})$ is dense in X. Moreover $\Psi_{n,N}(\mathbf{T})^{-m} f(\mathbf{T})$ is closed, as $f(\mathbf{T})$ is bounded and $\Psi_{n,N}(\mathbf{T})^{-m}$ is closed.

4. H^∞ functional calculus. We are interested in building a functional calculus defined on $H^\infty(\Omega, \mathcal{B})$ for N-tuples of (bi)sectorial operators; however, technical reasons suggest defining the functional calculus on the larger space $H_{\mathrm{P}}(\Omega, \mathcal{B})$.

DEFINITION 4.1. Let $f \in H_{\mathrm{P}}(\Omega, \mathcal{B})$, and let $m \in \mathbb{N}$ be such that $\Psi^m f \in H_0^\infty(\Omega, \mathcal{B})$. We set

$$f(\mathbf{T}) = \Psi(\mathbf{T})^{-m}(\Psi^m f)(\mathbf{T}).$$

REMARKS 4.2. (a) Definition 4.1 is meaningful, as $\Psi(\mathbf{T})^{-m}(\Psi^m f)(\mathbf{T})$ does not depend on m. Indeed, by Theorem 3.5 we have

$$(\Psi^{m+1} f)(\mathbf{T}) = \Psi(\mathbf{T})(\Psi^m f)(\mathbf{T}),$$

and by applying $\Psi(\mathbf{T})^{-m-1}$ to both sides we obtain

$$\Psi(\mathbf{T})^{-m-1}(\Psi^{m+1} f)(\mathbf{T}) = \Psi(\mathbf{T})^{-m}(\Psi^m f)(\mathbf{T}).$$

Moreover, this argument, with $m = 0$, proves that Definition 4.1 extends Definition 3.4.

(b) By Remark 3.12, for every $f \in H_{\mathrm{P}}(\Omega, \mathcal{B})$, $f(\mathbf{T})$ is a closed operator with dense domain.

THEOREM 4.3. *If $f \in H_{\mathrm{P}}(\Omega, \mathcal{B})$ and $\lambda \in \mathbb{C}\backslash\{0\}$, then $\lambda f(\mathbf{T}) = (\lambda f)(\mathbf{T})$.*

Proof. Trivial. ∎

THEOREM 4.4. *If $f : \Omega \to \mathcal{B}$ is constant, and $S \in \mathcal{B}$ is its unique value, then $f(\mathbf{T}) = S$.*

Proof. Indeed, taking into account that S commutes with the resolvent operators of the T_j's, one can check easily that $(\Psi f)(\mathbf{T}) = \Psi(\mathbf{T})S$. ∎

THEOREM 4.5. *Let $f, g \in H_{\mathrm{P}}(\Omega, \mathcal{B})$. Then $f(\mathbf{T}) + g(\mathbf{T}) \subseteq (f+g)(\mathbf{T})$ and $f(\mathbf{T})g(\mathbf{T}) \subseteq (fg)(\mathbf{T})$.*

Proof. For a suitably large $m \in \mathbb{N}$ we have $f(\mathbf{T}) = \Psi(\mathbf{T})^{-m}(\Psi^m f)(\mathbf{T})$ and $g(\mathbf{T}) = \Psi(\mathbf{T})^{-m}(\Psi^m g)(\mathbf{T})$.

Let $x \in \mathcal{D}(f(\mathbf{T}) + g(\mathbf{T})) = \mathcal{D}(f(\mathbf{T})) \cap \mathcal{D}(g(\mathbf{T}))$. Then $(\Psi^m f)(\mathbf{T})x$ and $(\Psi^m g)(\mathbf{T})x$ belong to $\mathcal{D}(\Psi(\mathbf{T})^{-m})$, so that

$$(\Psi^m f)(\mathbf{T})x + (\Psi^m g)(\mathbf{T})x \in \mathcal{D}(\Psi(\mathbf{T})^{-m}).$$

But, by Theorem 3.5, we have $(\Psi^m f)(\mathbf{T}) + (\Psi^m g)(\mathbf{T}) = (\Psi^m(f+g))(\mathbf{T})$. Therefore $(\Psi^m(f+g))(\mathbf{T})x \in \mathcal{D}(\Psi(\mathbf{T})^{-m})$, i.e. $x \in \mathcal{D}((f+g)(\mathbf{T}))$, and

$$(f+g)(\mathbf{T})x = \Psi(\mathbf{T})^{-m}((\Psi^m f)(\mathbf{T})x + (\Psi^m g)(\mathbf{T})x) = f(\mathbf{T})x + g(\mathbf{T})x.$$

This proves that $f(\mathbf{T}) + g(\mathbf{T}) \subseteq (f+g)(\mathbf{T})$.

Let $x \in \mathcal{D}(f(\mathbf{T})g(\mathbf{T}))$. Then $(\Psi^m g)(\mathbf{T})x \in \mathcal{D}(\Psi(\mathbf{T})^{-m})$ and

$$(\Psi^m f)(\mathbf{T})\Psi(\mathbf{T})^{-m}(\Psi^m g)(\mathbf{T})x \in \mathcal{D}(\Psi(\mathbf{T})^{-m}).$$

From Remark 3.12 we get $\Psi(\mathbf{T})^{-m}(\Psi^m f)(\mathbf{T})(\Psi^m g)(\mathbf{T})x \in \mathcal{D}(\Psi(\mathbf{T})^{-m})$, that is (by Theorem 3.5),

$$(\Psi^{2m} fg)(\mathbf{T})x = (\Psi^m f)(\mathbf{T})(\Psi^m g)(\mathbf{T})x \in \mathcal{D}(\Psi(\mathbf{T})^{-2m}).$$

Therefore $x \in \mathcal{D}((fg)(\mathbf{T}))$ and

$$\begin{aligned}(fg)(\mathbf{T})x &= \Psi(\mathbf{T})^{-2m}(\Psi^{2m} fg)(\mathbf{T})x \\ &= \Psi(\mathbf{T})^{-m}(\Psi^m f)(\mathbf{T})\Psi(\mathbf{T})^{-m}(\Psi^m g)(\mathbf{T})x = f(\mathbf{T})g(\mathbf{T})x. \quad \blacksquare\end{aligned}$$

Corollary 4.6. *Let $f, g \in H_{\mathrm{P}}(\Omega, \mathcal{B})$.*

(a) *If $g(\mathbf{T}) \in \mathcal{L}(X)$, then $(f+g)(\mathbf{T}) = f(\mathbf{T}) + g(\mathbf{T})$.*
(b) *If $g(\mathbf{T}) \in \mathcal{L}(X)$, then $(fg)(\mathbf{T}) = f(\mathbf{T})g(\mathbf{T})$.*

Proof. (a) By Theorem 4.5, $f(\mathbf{T}) = (f+g-g)(\mathbf{T}) \supseteq (f+g)(\mathbf{T}) - g(\mathbf{T})$. Since $g(\mathbf{T}) \in \mathcal{L}(X)$, we have $\mathcal{D}((f+g)(\mathbf{T}) - g(\mathbf{T})) = \mathcal{D}((f+g)(\mathbf{T}))$. Hence $f(\mathbf{T}) + g(\mathbf{T}) \supseteq (f+g)(\mathbf{T})$.

(b) For a suitably large $m \in \mathbb{N}$ the operators $(\Psi^m fg)(\mathbf{T})$ and $(\Psi^m f)(\mathbf{T})$ are bounded together with $g(\mathbf{T})$; therefore as a consequence of Theorem 4.5 we obtain $(\Psi^m f)(\mathbf{T})g(\mathbf{T}) = (\Psi^m fg)(\mathbf{T})$. Hence

$$f(\mathbf{T})g(\mathbf{T}) = \Psi(T)^{-m}(\Psi^m f)(\mathbf{T})g(\mathbf{T}) = \Psi(T)^{-m}(\Psi^m fg)(\mathbf{T}) = (fg)(\mathbf{T}). \quad \blacksquare$$

Theorem 4.7. *Let $f \in H^\infty(\Omega, \mathcal{B})$. A necessary and sufficient condition for $f(\mathbf{T})$ to be a bounded operator is that $\sup_{n\in\mathbb{N}} \|(\Psi_{n,N} f)(\mathbf{T})\| < \infty$. In this case $f(\mathbf{T})x = \lim_{n\to\infty}(\Psi_{n,N} f)(\mathbf{T})x$ for all $x \in X$.*

Proof. If $f(\mathbf{T}) \in \mathcal{L}(X)$, then

$$\sup_{n\in\mathbb{N}} \|(\Psi_{n,N} f)(\mathbf{T})\| \le (\sup_{n\in\mathbb{N}} \|\Psi_{n,N}(\mathbf{T})\|)\|f(\mathbf{T})\| < \infty$$

by Remark 3.9. Moreover, by Corollary 4.6 (and Remark 3.9), we have

$$(\Psi_{n,N} f)(\mathbf{T})x = \Psi_{n,N}(\mathbf{T})f(\mathbf{T})x \underset{n\to\infty}{\longrightarrow} f(\mathbf{T})x.$$

Conversely, assume that $\sup_{n\in\mathbb{N}} \|(\Psi_{n,N} f)(\mathbf{T})\| = M < \infty$. If $x \in \mathcal{D}(f(\mathbf{T}))$, then, by Theorem 4.5 and Remark 3.9,

$$(\Psi_{n,N} f)(\mathbf{T})x = \Psi_{n,N}(\mathbf{T})f(\mathbf{T})x \underset{n\to\infty}{\longrightarrow} f(\mathbf{T})x.$$

Therefore $\|f(\mathbf{T})x\| \le M\|x\|$ for all $x \in \mathcal{D}(f(\mathbf{T}))$. Since $f(\mathbf{T})$ is closed and densely defined, this proves that $f(\mathbf{T}) \in \mathcal{L}(X)$. $\blacksquare$

THEOREM 4.8. *Let $\mathcal{A}$ be a closed subalgebra of $\mathcal{B}$, and let $H^\infty_{\mathbf{T}}(\Omega,\mathcal{A})$ denote the set of bounded holomorphic functions $f:\Omega\to\mathcal{A}$ such that $f(\mathbf{T})\in\mathcal{L}(X)$. Then*:

(a) *$H^\infty_{\mathbf{T}}(\Omega,\mathcal{A})$ is a subalgebra of $H^\infty(\Omega,\mathcal{A})$ containing $H^\infty_0(\Omega,\mathcal{A})$;*
(b) *the operator $H^\infty_{\mathbf{T}}(\Omega,\mathcal{A})\ni f\mapsto f(\mathbf{T})\in\mathcal{L}(X)$ is closed in the norm of $H^\infty(\Omega,\mathcal{A})\times\mathcal{L}(X)$.*

Proof. (a) Trivial (see Corollary 4.6).

(b) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $H^\infty_{\mathbf{T}}(\Omega,\mathcal{A})$ convergent to f in $H^\infty(\Omega,\mathcal{A})$ such that $f_n(\mathbf{T})\to S$ in $\mathcal{L}(X)$ as $n\to\infty$; we have to show that $S=f(\mathbf{T})$. We have

$$\begin{aligned}&\|(\Psi f_n)(\mathbf{T})-(\Psi f)(\mathbf{T})\|\\ &\qquad\le(2\pi)^{-N}\int_\Gamma|\Psi(z)|\,\|f_n(z)-f(z)\|\prod_{k=1}^N\|(z_k-T_k)^{-1}\|\,d|z|\\ &\qquad\le(2\pi)^{-N}\|f_n-f\|_\infty\int_\Gamma|\Psi(z)|\prod_{k=1}^N\|(z_k-T_k)^{-1}\|\,d|z|\underset{n\to\infty}{\longrightarrow}0.\end{aligned}$$

Then for all $x\in X$ we have

$$\Psi(\mathbf{T})f_n(\mathbf{T})x=(\Psi f_n)(\mathbf{T})x\underset{n\to\infty}{\longrightarrow}(\Psi f)(\mathbf{T})x,$$
$$\Psi(\mathbf{T})^{-1}\Psi(\mathbf{T})f_n(\mathbf{T})x=f_n(\mathbf{T})x\underset{n\to\infty}{\longrightarrow}Sx.$$

Since $\Psi(\mathbf{T})^{-1}$ is closed, this proves that $Sx=\Psi(\mathbf{T})^{-1}(\Psi f)(\mathbf{T})x=f(\mathbf{T})x$. ∎

THEOREM 4.9. *Let $\mathcal{A}$ be a closed subalgebra of $\mathcal{B}$. The following statements are equivalent*:

(a) *$f(\mathbf{T})\in\mathcal{L}(X)$ for all $f\in H^\infty(\Omega,\mathcal{A})$;*
(b) *there exists $C\in\mathbb{R}^+$ such that $\|f(\mathbf{T})\|\le C\|f\|_\infty$ for all $f\in H^\infty_0(\Omega,\mathcal{A})$.*

Proof. (a)⇒(b). This follows immediately from Theorem 4.8 and the closed graph theorem.

(b)⇒(a). Let $f\in H^\infty(\Omega,\mathcal{A})$. Then $\Psi_{n,N}f\in H^\infty_0(\Omega,\mathcal{A})$, and so

$$\|(\Psi_{n,N}f)(\mathbf{T})\|\le C\|\Psi_{n,N}f\|_\infty\le C'\|f\|_\infty$$

(see Lemma 2.7). Hence, by Theorem 4.7, we have $f(\mathbf{T})\in\mathcal{L}(X)$. ∎

REMARK 4.10. If T is an operator in X and $\alpha\in\mathbb{C}\setminus\{0\}$, then it is obvious that $\mathcal{D}(\alpha T)=\mathcal{D}(T)$, $\mathcal{R}(\alpha T)=\mathcal{R}(T)$, and $\varrho(\alpha T)=\alpha\varrho(T)$ with

$$(\alpha\lambda-\alpha T)^{-1}=\alpha^{-1}(\lambda-T)^{-1}.$$

Hence, if $\alpha\in\mathbb{R}^+$ and T is (bi)sectorial with spectral angle θ, then the same is true for αT. More precisely, if $\|\lambda(\lambda-T)^{-1}\|\le M$ in a certain domain which is invariant under multiplication by positive real numbers, then in the same domain we also have $\|\lambda(\lambda-\alpha T)^{-1}\|=\|\alpha^{-1}\lambda(\alpha^{-1}\lambda-T)^{-1}\|$

$\leq M$. In particular, if $\alpha_1, \dots, \alpha_N \in \mathbb{R}^+$, then $\alpha_1 T_1, \dots, \alpha_N T_N$ have the same properties as $T_1, \dots, T_N$, and their resolvent operators commute.

THEOREM 4.11. *Let $\alpha_1, \dots, \alpha_N \in \mathbb{R}^+$. If $f \in H_{\mathrm{P}}(\Omega, \mathcal{B})$ and g is the function defined by $g(\zeta_1, \dots, \zeta_N) = f(\alpha_1\zeta_1, \dots, \alpha_N\zeta_N)$, then*

$$f(\alpha_1 T_1, \dots, \alpha_N T_N) = g(T_1, \dots, T_N).$$

Proof. If $f \in H_0^\infty$ the equality is obtained by a change of variables in the integral that defines $f(\alpha_1 T_1, \dots, \alpha_N T_N)$. In the general case we set $\mu(z_1, \dots, z_N) = (\alpha_1 z_1, \dots, \alpha_N z_N)$, so that $g = f \circ \mu$. We also write $\alpha\mathbf{T}$ for $(\alpha_1 T_1, \dots, \alpha_N T_N)$. Then

$$f(\alpha\mathbf{T}) = \Psi(\alpha\mathbf{T})^{-m}((\Psi \circ \mu)^m g)(\mathbf{T}).$$

However,

$$\Psi(\mathbf{T})^m((\Psi \circ \mu)^m g)(\mathbf{T}) = (\Psi^m g(\Psi \circ \mu)^m)(\mathbf{T}) = (\Psi \circ \mu)(\mathbf{T})^m(\Psi^m g)(\mathbf{T}).$$

Hence

$$f(\alpha\mathbf{T}) = \Psi(\mathbf{T})^{-m}(\Psi^m g)(\mathbf{T}) = g(\mathbf{T}). \ \blacksquare$$

THEOREM 4.12. *Let $f \in H_{\mathrm{P}}(\Omega_j, \mathcal{B})$, and let $g : \Omega \to \mathcal{B}$ be defined by $g(z) = f(z_j)$. Then $g \in H_{\mathrm{P}}(\Omega, \mathcal{B})$ and $g(\mathbf{T}) = f(T_j)$.*

Proof. Since the first statement is trivial, we prove that $g(\mathbf{T}) = f(T_j)$. Let m be a suitably large integer. From Fubini's theorem it follows that

$$(\Psi^m g)(\mathbf{T}) = \Big(\prod_{k \neq j} \psi(T_k)^m\Big)(\psi^m f)(T_j) = \Psi(\mathbf{T})^m \psi(T_j)^{-m}(\psi^m f)(T_j).$$

Hence

$$g(\mathbf{T}) = \Psi(\mathbf{T})^{-m}(\Psi^m g)(\mathbf{T}) = \psi(T_j)^{-m}(\psi^m f)(T_j) = f(T_j). \ \blacksquare$$

5. Some results in the case of a single operator. The results of this section concern the case $N = 1$, so that we will write T instead of $\mathbf{T}$ or T_j. Note that, by means of Theorem 4.12, these results can be easily extended to the case of several operators when the function f actually depends on one variable only.

DEFINITION 5.1. For $w \in \mathbb{C}$ we denote by p_w the function defined on $\mathbb{C} \setminus\,]-\infty, 0]$ by $p_w(z) = z^w = e^{w(\log|z| + i \arg z)}$.

LEMMA 5.2. *If f and $p_1 f$ belong to $H_0^\infty(\Omega)$, then $(p_1 f)(T) = Tf(T)$.*

Proof. From the assumptions it follows that f is integrable along Γ, with $\int_\Gamma f = 0$. As $T(z - T)^{-1}$ is bounded along Γ and T is closed we get

$$Tf(T) = \frac{1}{2\pi i} \int_\Gamma f(z) T(z - T)^{-1}\, dz = \frac{1}{2\pi i} \int_\Gamma z f(z)(z - T)^{-1}\, dz. \ \blacksquare$$

THEOREM 5.3. *If* $k \in \mathbb{Z}$, *then* $p_k(T) = T^k$.

Proof. The case $k = 0$ follows immediately from Theorem 4.4. Assume that $k > 0$. Take $m \in \mathbb{N}$ with $m > k$ so that $p_k \psi^m \in H_0^\infty$. By an iterated application of Lemma 5.2 we have $(\psi^m p_k)(T) = T^k \psi(T)^m$, hence

$$p_k(T) = \psi(T)^{-m} T^k \psi(T)^m = (1+T)^{2m} T^{-m} T^k T^m (1+T)^{-2m} = T^k.$$

If $k < 0$ and $m > |k|$, then from the equality $\psi(T)^m = (p_{-k} p_k \psi^m)(T) = T^{-k}(p_k \psi^m)(T)$ we obtain $(p_k \psi^m)(T) = T^k \psi(T)^m$. Hence

$$p_k(T) = \psi(T)^{-m}(p_k \psi^m)(T) = \psi(T)^{-m} T^k \psi(T)^m.$$

Now we remark that T^{-1} has the same properties as T, and that $\psi(T^{-1}) = \psi(T)$; therefore $p_k(T) = \psi(T^{-1})^{-m}(T^{-1})^{-k}\psi(T^{-1})^m = T^k$ by the same argument as above. ∎

As $p_w \in H_{\mathrm{P}}(\Omega)$ we can give the following

DEFINITION 5.4. For $w \in \mathbb{C}$ we set $T^w := p_w(T)$.

Observe that by Theorem 5.3 this definition extends the case of the integer exponents.

As in Theorem 3.7 we denote by r_λ the function $z \mapsto (\lambda - z)^{-1}$.

THEOREM 5.5. *Let* $\lambda \in \mathbb{C} \setminus \overline{G}$. *Then* $T^w(\lambda - T)^{-1} = (p_w r_\lambda)(T)$ *for all* $w \in \mathbb{C}$. *In particular*, *if* $0 < \operatorname{Re} w < 1$ *and* $\lambda \notin \overline{\Omega}$, *then* $T^w(\lambda - T)^{-1} \in \mathcal{L}(X)$ *and*

$$\|T^w(\lambda - T)^{-1}\| \le C_{w,\Omega} |\lambda|^{\operatorname{Re} w - 1}.$$

Proof. We fix $w \in \mathbb{C}$ and $m \in \mathbb{N}$ with $m > |\operatorname{Re} w|$. Then

$$\begin{aligned}(p_w r_\lambda)(T) &= \psi(T)^{-m}(p_w r_\lambda \psi^m)(T)\\ &= \psi(T)^{-m}(p_w \psi^m)(T)(\lambda - T)^{-1} \qquad \text{(by Theorem 3.7)}\\ &= T^w(\lambda - T)^{-1}.\end{aligned}$$

This proves the first statement. Now we assume that $0 < \operatorname{Re} w < 1$. Then

$$\begin{aligned}\|T^w(\lambda - T)^{-1}\| = \|(p_w r_\lambda)(T)\| &= \left\| \frac{1}{2\pi i} \int_\Gamma \frac{z^w}{(\lambda - z)} (z - T)^{-1}\, dz \right\|\\ &\le C \int_\Gamma \frac{|z|^{\operatorname{Re} w - 1}}{|\lambda - z|}\, d|z| \le C C_0^{-1} \int_\Gamma \frac{|z|^{\operatorname{Re} w - 1}}{|\lambda| + |z|}\, d|z|\\ &= C C_0^{-1} |\lambda|^{\operatorname{Re} w - 1} \int_\Gamma \frac{|\zeta|^{\operatorname{Re} w - 1}}{1 + |\zeta|}\, d|\zeta|\end{aligned}$$

(where Lemma 2.5 yields the constant $C_0 = C_0(\Omega)$ when $\lambda \notin \overline{\Omega}$). ∎

REMARK 5.6. The constant $C_{w,\Omega}$ that appears in Theorem 5.5 obviously depends also on the operator T. However, looking at the proof of that theorem, one can see that this dependence concerns only the supremum of $\|z(z-T)^{-1}\|$ on Γ. In particular (see Remark 4.10) the constant does not change if one replaces T with αT ($\alpha \in \mathbb{R}^+$).

6. The Kalton–Weis theorem. In this section we go back to the general situation described at the beginning of Section 3.

LEMMA 6.1. *If $t, s \in \mathbb{R}^+$, then*

$$\sum_{k\in\mathbb{Z}} (\min\{(2^k t)^s, (2^k t)^{-s}\}) \le \frac{2^{s+1}}{2^s - 1}.$$

Proof. We set $k_t = \min\{k \in \mathbb{Z};\ 2^k t \ge 1\}$. Then

$$\sum_{k\in\mathbb{Z}} (\min\{(2^k t)^s, (2^k\, t)^{-s}\}) = \sum_{k<k_t} (2^k t)^s + \sum_{k\ge k_t} (2^k t)^{-s}$$
$$= (2^{k_t-1} t)^s \sum_{j=0}^{\infty} 2^{-js} + (2^{k_t}\, t)^{-s} \sum_{j=0}^{\infty} 2^{-js} \le 2 \sum_{j=0}^{\infty} 2^{-js} = \frac{2}{1-2^{-s}} = \frac{2^{s+1}}{2^s - 1}. \ \blacksquare$$

LEMMA 6.2. *Let $s \in]0,1[$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$, $\lambda = |\lambda| e^{i\beta}$, with $0 < |\beta| < \pi$.*

(a) *If $0 < \varepsilon < |\beta|$, then there exists $g \in H_0^\infty(S_{|\beta|-\varepsilon})$ such that*

$$g(z)^2 = z^s (z-\lambda)^{-1} \quad \text{for all } z \in S_{|\beta|-\varepsilon}.$$

(b) *If $|\beta| \ne \pi/2$ and $0 < \varepsilon < ||\beta| - \pi/2|$, then there exists $g \in H_0^\infty(\Sigma_{||\beta|-\pi/2|-\varepsilon})$ such that*

$$g(z)^2 = z^s (z-\lambda)^{-1} \quad \text{for all } z \in \Sigma_{||\beta|-\pi/2|-\varepsilon}.$$

Proof. From $g(z)^2 = z^s(z-\lambda)^{-1}$ one gets $|g(z)| = |z^s(z-\lambda)^{-1}|^{1/2}$ in both cases; since z does not approach λ, if g is holomorphic then $g \in H_0^\infty$.

(a) If $r \in \mathbb{R}^+$, then

$$|\arg(\lambda - r)| = \arg(|\lambda| e^{i|\beta|} - r) = \operatorname{arccot} \frac{|\lambda|\cos|\beta| - r}{|\lambda| \sin|\beta|} > |\beta|$$

so that $\lambda - r \notin S_{|\beta|}$. Therefore for all $z \in S_{|\beta|-\varepsilon}$ one has $z, z-\lambda \notin]-\infty, 0]$, and we can set $g(z) = z^{s/2}(z-\lambda)^{-1/2}$.

(b) If $|\beta| > \pi/2$, the assertion follows from (a), as $\Sigma_{|\beta|-\pi/2-\varepsilon} \subseteq S_{|\beta|-\varepsilon}$. If $|\beta| < \pi/2$, then in a way similar to (a) it can be proved that for all $z \in \Sigma_{\pi/2-|\beta|-\varepsilon}$ we have $z, \lambda - z \notin]-\infty, 0]$, and therefore we can set $g(z) = iz^{s/2}(\lambda - z)^{-1/2}$. $\blacksquare$

THEOREM 6.3. *Let $f \in H_0^\infty(\Omega, \mathcal{B})$ and $r \in]0,1[$. Then*

$$f(\mathbf{T}) = (2\pi i)^{-N} \int_\Gamma f(z) \prod_{j=1}^N z_j^{-r} T_j^r (z_j - T_j)^{-1}\, dz,$$

and the integral converges in the norm of $\mathcal{L}(X)$.

Proof. First, we remark that the operators $T_j^r(z_j - T_j)^{-1}$ commute (by Theorem 5.5 and Corollary 3.6). Next, for $m \in \{0, \ldots, N\}$ we set

$$S_m = (2\pi i)^{-N} \int_\Gamma f(z) \prod_{j=1}^m z_j^{-r} T_j^r (z_j - T_j)^{-1} \prod_{k=m+1}^N (z_k - T_k)^{-1}\, dz.$$

This integral is convergent in the norm of $\mathcal{L}(X)$ because, by Theorem 5.5, we have (for $z \in \Gamma$)

$$\begin{aligned}
&\Big\| f(z) \prod_{j=1}^m z_j^{-r} T_j^r (z_j - T_j)^{-1} \prod_{k=m+1}^N (z_k - T_k)^{-1} \Big\| \\
&\quad \le C \prod_{j=1}^N (\min\{|z_j|, |z_j|^{-1}\})^s \prod_{j=1}^N |z_j|^{-1} = C \prod_{j=1}^N \min\{|z_j|^{s-1}, |z_j|^{-s-1}\}
\end{aligned}$$

and this is an integrable function on Γ. Since $S_0 = f(\mathbf{T})$ and S_N is the right-hand side of the equality to be proved, it is sufficient to show that $S_m = S_{m-1}$ when $1 \le m \le N$.

Let $m \in \{1, \ldots, N\}$ be fixed, and let Δ be a curve of the same type as Γ^m, lying "inside Γ^m". Then (by Theorem 5.5, which also enables us to change the order of integration)

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{\Gamma^m} f(z) z_m^{-r} T_m^r (z_m - T_m)^{-1}\, dz_m \\
&\quad = \frac{1}{2\pi i} \int_{\Gamma^m} f(z) z_m^{-r} \frac{1}{2\pi i} \int_\Delta \frac{\zeta^r}{(z_m - \zeta)^{-1}} (\zeta - T_m)^{-1}\, d\zeta\, dz_m \\
&\quad = \frac{1}{2\pi i} \int_\Delta \zeta^r \frac{1}{2\pi i} \int_{\Gamma^m} f(z) \frac{z_m^{-r}}{(z_m - \zeta)^{-1}}\, dz_m\, (\zeta - T_m)^{-1}\, d\zeta \\
&\quad = \frac{1}{2\pi i} \int_\Delta f(z_1, \ldots, z_{m-1}, \zeta, z_{m+1}, \ldots, z_N) (\zeta - T_m)^{-1}\, d\zeta \\
&\quad = \frac{1}{2\pi i} \int_{\Gamma^m} f(z) (z_m - T_m)^{-1}\, dz_m.
\end{aligned}$$

This proves that $S_m = S_{m-1}$. ∎

LEMMA 6.4. *Assume that $f(\mathbf{T}) \in \mathcal{L}(X)$ for all $f \in H^\infty(\Omega)$. Then for every $f \in H_0^\infty(\Omega)$ there is a $C > 0$ such that if $(\alpha_k)_{k\in\mathbb{Z}^N}$ is a family of complex numbers with $\{k \in \mathbb{Z}^N;\ \alpha_k \neq 0\}$ finite, and $(t_1, \ldots, t_N) \in (\mathbb{R}^+)^N$, then*

$$\Big\| \sum_{k\in\mathbb{Z}^N} \alpha_k f(2^{k_1} t_1 T_1, \ldots, 2^{k_N} t_N T_N) \Big\| \le C \max_{k\in\mathbb{Z}^N} |\alpha_k|.$$

Proof. It is not restrictive to assume that $\max_{k\in\mathbb{Z}^N} |\alpha_k| = 1$. By Theorem 4.11, if $f \in H_0^\infty(\Omega)$ then

$$\sum_{k\in\mathbb{Z}^N} \alpha_k f(2^{k_1} t_1 T_1, \ldots, 2^{k_N} t_N T_N) = g(\mathbf{T}),$$

where $g(z) = \sum_{k\in\mathbb{Z}^N} \alpha_k f(2^{k_1} t_1 z_1, \ldots, 2^{k_N} t_N z_N)$. Therefore by Theorem 4.9 (with $\mathcal{A} = \mathbb{C}$)

$$\Big\| \sum_{k\in\mathbb{Z}^N} \alpha_k f(2^{k_1} t_1 T_1, \ldots, 2^{k_N} t_N T_N) \Big\| = \|g(\mathbf{T})\| \le C_0 \|g\|_\infty,$$

where C_0 does not depend on $(\alpha_k)_{k\in\mathbb{Z}^N}$, $(t_1, \ldots, t_N)$ and f. Now, by the geometric properties of Ω,

$$\begin{aligned}
\|g\|_\infty &= \sup_{z\in\Omega} |g(z)| = \sup_{z\in\Omega} \Big| \sum_{k\in\mathbb{Z}^N} \alpha_k f(2^{k_1} z_1, \ldots, 2^{k_N} z_N) \Big| \\
&\le \sup_{z\in\Omega} \sum_{k\in\mathbb{Z}^N} |f(2^{k_1} z_1, \ldots, 2^{k_N} z_N)| \\
&\le C_f \sup_{z\in\Omega} \sum_{k\in\mathbb{Z}^N} \prod_{j=1}^N \min\{(2^{k_j}|z_j|)^s, (2^{k_j}|z_j|)^{-s}\} \\
&= C_f \sup_{z\in\Omega} \prod_{j=1}^N \sum_{k\in\mathbb{Z}} \min\{(2^{k}|z_j|)^s, (2^{k}|z_j|)^{-s}\} \\
&\le C_f \left(\frac{2^{s+1}}{2^s - 1} \right)^N
\end{aligned}$$

(by Lemma 6.1). ■

Now we are going to state and prove the theorem of Kalton and Weis that we mentioned in the introduction. In their paper [7] the main tool is a property called U-boundedness. We use a slightly stronger one, which is more common in the literature.

A subset $\mathcal{U}$ of $\mathcal{L}(X)$ is said to be *R-bounded* if there is a $C > 0$ such that for any positive integer N and for arbitrary $T_1, \ldots, T_N \in \mathcal{U}$ and $x_1, \ldots, x_N \in X$,

$$(6.5)\qquad \Big(\sum_{\varepsilon\in\{-1,1\}^N}\Big\|\sum_{k=1}^N \varepsilon_k T_k\, x_k\Big\|^2\Big)^{1/2} \le C\Big(\sum_{\varepsilon\in\{-1,1\}^N}\Big\|\sum_{k=1}^N \varepsilon_k\, x_k\Big\|^2\Big)^{1/2}$$

(and it is known that the exponent 2 can be replaced by any other exponent $p > 0$ with a suitable modification of the constant C). The best constant that can be put in (6.5) is called the R_2-*bound* of $\mathcal{U}$, and is denoted by $\mathcal{R}_2(\mathcal{U})$. It is easy to see that any R-bounded set of operators is bounded, and that the elementary operations on sets preserve the R-boundedness. Moreover the "contraction principle" says that (6.5) holds when $T_k = \lambda_k I_X$, with $C = 2\max_{1\le k\le N}|\lambda_k|$. Hence for any R-bounded subset of $\mathcal{L}(X)$ and $M \in \mathbb{R}^+$,

$$(6.6)\qquad \mathcal{R}_2(\{\lambda T;\ \lambda\in\mathbb{C},\ |\lambda|\le M,\ T\in\mathcal{U}\}) \le 2M\mathcal{R}_2(\mathcal{U}).$$

For details about R-boundedness we refer to the papers [2], [5], [12].

THEOREM 6.7 (Kalton–Weis). *Let $H_R^\infty(\Omega,\mathcal{B})$ be the vector space of holomorphic functions $g:\Omega\to\mathcal{B}$ with R-bounded range. Assume that for some open set Ω' of the same type as Ω, but smaller, $f(\mathbf{T})\in\mathcal{L}(X)$ for all $f\in H^\infty(\Omega')$. Then $g(\mathbf{T})\in\mathcal{L}(X)$ for all $g\in H_R^\infty(\Omega,\mathcal{B})$. Moreover there exists $C(\mathbf{T})>0$ such that $\|g(\mathbf{T})\|\le C(\mathbf{T})\mathcal{R}_2(g(\Omega))$ for all $g\in H_R^\infty(\Omega,\mathcal{B})$.*

Proof. We fix $g\in H_R^\infty(\Omega,\mathcal{B})$ and set $g_n = \Psi_{n,N} g$. We will show that $\sup_{n\in\mathbb{N}}\|g_n(\mathbf{T})\|\le C\,\mathcal{R}_2(g(\Omega))$; by Theorem 4.7 that will prove the theorem. Since $g\in H^\infty(\Omega,\mathcal{B})$, and hence $g_n\in H_0^\infty(\Omega,\mathcal{B})$, by Theorem 6.3 we can write (with $r\in\,]0,1[$)

$$g_n(\mathbf{T}) = (2\pi i)^{-N}\int_\Gamma g_n(z)\prod_{j=1}^N z_j^{-r}T_j^r(z_j-T_j)^{-1}\,dz,$$

where Γ is a curve lying in $\Omega\setminus\overline{\Omega'}$. Notice that the integral with respect to z_j is performed on Γ^j, which is the union of two or four half-lines with origin at 0; therefore we obtain the sum of a finite number of terms of the form

$$(2\pi i)^{-N}\int_{\widetilde{\Gamma}^1}\cdots\int_{\widetilde{\Gamma}^N} g_n(z_1,\dots,z_N)\prod_{j=1}^N z_j^{-r}T_j^r(z_j-T_j)^{-1}dz_N\cdots dz_1,$$

where $\widetilde{\Gamma}^j$ is one of the (two or four) branches of Γ^j. Thus it is enough to estimate each of these integrals.

Let us introduce the abbreviation $2^{\mathbf{k}}\mathbf{t}e^{i\beta}$ for $(2^{k_1}t_1e^{i\beta_1},\dots,2^{k_N}t_Ne^{i\beta_N})$. If $\widetilde{\Gamma}^j$ is the half-line through $e^{i\beta_j}$ with $|\beta_j|<\pi$, we get

$$\int\limits_{\widetilde{\Gamma}^1} \cdots \int\limits_{\widetilde{\Gamma}^N} g_n(z_1,\ldots,z_N) \prod_{j=1}^N z_j^{-r} T_j^r (z_j - T_j)^{-1}\, dz_N \cdots dz_1$$

$$= \int\limits_0^\infty \cdots \int\limits_0^\infty g_n(\mathbf{t}e^{i\beta}) \prod_{j=1}^N t_j^{-1} e^{i(1-r)\beta_j} (t_j^{-1}T_j)^r (e^{i\beta_j} - t_j^{-1}T_j)^{-1}\, dt_N \cdots dt_1$$

$$= \sum_{k_1\in\mathbb{Z}} \cdots \sum_{k_N\in\mathbb{Z}} \int\limits_{2^{k_1}}^{2^{1+k_1}} \cdots \int\limits_{2^{k_N}}^{2^{1+k_N}} g_n(\mathbf{t}e^{i\beta})$$

$$\cdot \prod_{j=1}^N t_j^{-1} e^{i(1-r)\beta_j} (t_j^{-1}T_j)^r (e^{i\beta_j} - t_j^{-1}T_j)^{-1}\, dt_N \cdots dt_1$$

$$= \int\limits_{[1,2]^N} \sum_{k_1\in\mathbb{Z}} \cdots \sum_{k_N\in\mathbb{Z}} g_n(2^{\mathbf{k}}\mathbf{t}e^{i\beta})$$

$$\cdot \prod_{j=1}^N t_j^{-1} e^{i(1-r)\beta_j}\, (2^{-k_j} t_j^{-1}T_j)^r (e^{i\beta_j} - 2^{-k_j} t_j^{-1}T_j)^{-1}\, dt_N \cdots dt_1.$$

Here we have carried the sums inside the integrals: this is allowed by the inequalities that follow. We have also applied Theorem 4.11 which implies that $t_j^{-r}T_j^r = (t_j^{-1}T_j)^r$. Now, taking into account Theorem 5.5 and Remark 5.6 we get

$$\sup_{\mathbf{t}\in[1,2]^N} \sup_{\mathbf{k}\in\mathbb{Z}^N} \Big\| \prod_{j=1}^N t_j^{-1} e^{i(1-r)\beta_j} (2^{-k_j} t_j^{-1}T_j)^r (e^{i\beta_j} - 2^{-k_j} t_j^{-1}T_j)^{-1} \Big\| = C(\mathbf{T}) < \infty$$

while (as $g_n \in H_0^\infty(\Omega,\mathcal{B})$), for a suitable $s \in \mathbb{R}^+$ and all $\mathbf{t} \in [1,2]^N$,

$$\|g_n(2^{\mathbf{k}}\mathbf{t}e^{i\beta})\| \le C(g_n) \prod_{j=1}^N \min\{(2^{k_j}t_j)^s, (2^{k_j}t_j)^{-s}\}$$

so that, by Lemma 6.1, we get

$$\sup_{\mathbf{t}\in[1,2]^N} \sum_{k\in\mathbb{Z}^N} \Big\| g_n(2^{\mathbf{k}}\,\mathbf{t}\,e^{i\beta}) \prod_{j=1}^N t_j^{-1} e^{i(1-r)\beta_j} (2^{-k_j} t_j^{-1}T_j)^r (e^{i\beta_j} - 2^{-k_j} t_j^{-1}T_j)^{-1} \Big\| \le C(g_n) C(\mathbf{T}) \left(\frac{2^{s+1}}{2^s - 1} \right)^N.$$

The proof will be concluded if we show that on the right-hand side of the last inequality we can put $C\,\mathcal{R}_2(g(\Omega))$. By Fatou's lemma it is enough to estimate in the same way the norm of

$$M_{n,E}(t) := \sum_{\mathbf{k}\in E} g_n(2^{\mathbf{k}}\mathbf{t}e^{i\beta}) \prod_{j=1}^{N} t_j^{-1} e^{i(1-r)\beta_j} (2^{-k_j} t_j^{-1} T_j)^r (e^{i\beta_j} - 2^{-k_j} t_j^{-1} T_j)^{-1}$$

for any finite subset E of $\mathbb{Z}^N$. Therefore we fix $n \in \mathbb{N}$, $\mathbf{t} \in [1,2]^N$, a finite subset E of $\mathbb{Z}^N$, and moreover $x \in X$ with $\|x\| \le 1$ and $x^* \in X^*$ with $\|x^*\| \le 1$. By Lemma 6.2 and Theorem 5.5 we can write

$$(2^{-k_j}\, t_j^{-1} T_j)^r (e^{i\beta_j} - 2^{-k_j}\, t_j^{-1} T_j)^{-1} = h_j (2^{-k_j}\, t_j^{-1} T_j)^2,$$

where $h_j \in H_0^\infty(\Omega_j')$. Therefore

$$\prod_{j=1}^{N} (2^{-k_j}\, t_j^{-1} T_j)^r (e^{i\beta_j} - 2^{-k_j} t_j^{-1} T_j)^{-1} = h(2^{-k_1} t_1^{-1} T_1, \ldots, 2^{-k_N} t_N^{-1} T_N)^2$$

with $h \in H_0^\infty(\Omega')$. The operator $h(2^{-k_1}\, t_1^{-1} T_1, \ldots, 2^{-k_N}\, t_N^{-1} T_N)$ belongs to the closed subalgebra of $\mathcal{L}(X)$ spanned by the resolvents of the operators T_j, hence commutes with the values of g_n since the range of g_n is contained in $\mathcal{B}$. Hence

$$\begin{aligned} M_{n,E}(t) = \sum_{k\in E} \Big(\prod_{j=1}^{N} t_j^{-1} e^{i(1-r)\beta_j}\Big) & h(2^{-k_1} t_1^{-1} T_1, \ldots, 2^{-k_N}\, t_N^{-1} T_N) \\ & \cdot g_n(2^{\mathbf{k}}\mathbf{t}e^{i\beta}) h(2^{-k_1} t_1^{-1} T_1, \ldots, 2^{-k_N} t_N^{-1} T_N). \end{aligned}$$

Thus, if q is the cardinality of E, as $\sum_{\varepsilon\in\{-1,1\}^E} \varepsilon_k \varepsilon_l = 2^q \delta_{k,l}$ we get

$$\begin{aligned} &|\langle M_{n,E}(t)x, x^*\rangle| \\ &\quad = \Big(\prod_{j=1}^{N} t_j^{-1}\Big) \Big| \sum_{k\in E} \langle g_n(2^{\mathbf{k}}\mathbf{t}e^{i\beta}) h(2^{-k_1} t_1^{-1} T_1, \ldots, 2^{-k_N} t_N^{-1} T_N)x, \\ &\qquad\qquad\qquad\qquad\qquad\qquad h(2^{-k_1} t_1^{-1} T_1, \ldots, 2^{-k_N} t_N^{-1} T_N)^* x^*\rangle \Big| \\ &\quad \le 2^{-q} \Big| \sum_{\varepsilon\in\{-1,1\}^E} \sum_{k,l\in E} \langle \varepsilon_k\, g_n(2^{\mathbf{k}}\mathbf{t}e^{i\beta}) h(2^{-k_1}\, t_1^{-1} T_1, \ldots, 2^{-k_N} t_N^{-1} T_N)x, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \varepsilon_l h(2^{-l_1} t_1^{-1} T_1, \ldots, 2^{-l_N} t_N^{-1} T_N)^* x^*\rangle \Big| \\ &\quad \le 2^{-q} \Big(\sum_{\varepsilon\in\{-1,1\}^E} \Big\| \sum_{k\in E} \varepsilon_k g_n(2^{\mathbf{k}}\mathbf{t}e^{i\beta}) h(2^{-k_1} t_1^{-1} T_1, \ldots, 2^{-k_N} t_N^{-1} T_N)x \Big\|^2 \Big)^{1/2} \\ &\qquad \cdot \Big(\sum_{\varepsilon\in\{-1,1\}^E} \Big\| \sum_{l\in E} \varepsilon_l h(2^{-l_1} t_1^{-1} T_1, \ldots, 2^{-l_N} t_N^{-1} T_N)^* x^* \Big\|^2 \Big)^{1/2}. \end{aligned}$$

From Lemma 2.7 and (6.6), it follows that $\mathcal{R}_2(g_n(\Omega)) \leq C\,\mathcal{R}_2(g(\Omega))$, where C only depends on Ω. Therefore

$$\begin{aligned}
&|\langle M_{n,E}(t)x, x^*\rangle| \\
&\quad \leq 2^{-q} C\mathcal{R}_2(g(\Omega))\Big(\sum_{\varepsilon\in\{-1,1\}^E}\Big\|\sum_{k\in E}\varepsilon_k h(2^{-k_1}t_1^{-1}T_1,\ldots,2^{-k_N}t_N^{-1}T_N)x\Big\|^2\Big)^{1/2} \\
&\qquad \cdot \Big(\sum_{\varepsilon\in\{-1,1\}^E}\Big\|\sum_{k\in E}\varepsilon_k\, h(2^{-k_1}\,t_1^{-1}T_1,\ldots,2^{-k_N}\,t_N^{-1}T_N)^* x^*\Big\|^2\Big)^{1/2} \\
&\quad \leq 2^{-q} C\mathcal{R}_2(g(\Omega))\sum_{\varepsilon\in\{-1,1\}^E}\Big\|\sum_{k\in E}\varepsilon_k h(2^{-k_1}t_1^{-1}T_1,\ldots,2^{-k_N}t_N^{-1}T_N)\Big\|^2.
\end{aligned}$$

Finally, we apply Lemma 6.4, which is possible by the assumption that $f(\mathbf{T}) \in \mathcal{L}(X)$ for all $f \in H^\infty(\Omega')$. By that lemma,

$$\Big\|\sum_{k\in E}\varepsilon_k h(2^{-k_1}t_1^{-1}T_1,\ldots,2^{-k_N}t_N^{-1}T_N)\Big\|$$

is bounded above by a constant that depends only on h, and therefore only on Ω'. Thus

$$|\langle M_{n,E}(t)x, x^*\rangle| \leq 2^{-q}\sum_{\varepsilon\in\{-1,1\}^E} C\mathcal{R}_2(g) = C\mathcal{R}_2(g).$$

This concludes the proof. ∎

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