STUDIA MATHEMATICA 166 (3) (2005)

Position dependent random maps in one and higher dimensions

by

WAEL BAHSOUN (Victoria) and PAWEŁ GÓRA (Montreal)

Abstract. A random map is a discrete-time dynamical system in which one of a number of transformations is randomly selected and applied on each iteration of the process. We study random maps with position dependent probabilities on the interval and on a bounded domain of $\mathbb{R}^n$. Sufficient conditions for the existence of an absolutely continuous invariant measure for a random map with position dependent probabilities on the interval and on a bounded domain of $\mathbb{R}^n$ are the main results.

1. Introduction. Let $\tau_1, \ldots, \tau_K$ be a collection of transformations from X to X. Usually, the random map T is defined by choosing τ_k with constant probability p_k, $p_k > 0$, $\sum_{k=1}^{K} p_k = 1$. The ergodic theory of such dynamical systems was studied in [9] and in [8] (see also [7]).

There is a rich literature on random maps with position dependent probabilities with $\tau_1, \ldots, \tau_K$ being continuous contracting transformations (see [10]).

In this paper, we deal with piecewise monotone transformations $\tau_1, \ldots, \tau_K$ and position dependent probabilities $p_k(x)$, $k = 1, \ldots, K$, where $p_k(x) > 0$, $\sum_{k=1}^{K} p_k(x) = 1$, i.e., the p_k's are functions of position. We point out that studying such dynamical systems was begun in [5], where sufficient conditions for the existence of an absolutely continuous invariant measure were given. The conditions in [5] are applicable only when $\tau_1, \ldots, \tau_K$ are C^2 expanding transformations (see [5] for details). In this paper, we prove the existence of an absolutely continuous invariant measure for a random map T on $[a, b]$ under milder conditions (see Section 4, Conditions (A) and (B)). Moreover, we prove the existence of an absolutely continuous invariant measure for a random map T on a bounded domain of $\mathbb{R}^n$ (see Section 6, Condition (C)).

2000 *Mathematics Subject Classification*: Primary 37A05, 37E05.

Key words and phrases: random map, absolutely continuous invariant measure, Perron–Frobenius operator.

The research of P.G. was supported by NSERC grant. W.B. is a recipient of PIMS postdoctoral fellowship.

The paper is organized in the following way: In Section 2, following the ideas of [5], we formulate the definition of a random map T with position dependent probabilities and introduce its Perron–Frobenius operator. In Section 3, we prove some properties of the Perron–Frobenius operator of T. In Section 4, we prove the existence of an absolutely continuous invariant measure for T on $[a,b]$. In Section 5, we give an example of a random map T which does not satisfy the conditions of [5]; yet, it preserves an absolutely continuous invariant measure under conditions (A) and (B). In Section 6, we prove the existence of an absolutely continuous invariant measure for T on a bounded domain of $\mathbb{R}^n$. In Section 7, we give an example of a random map in $\mathbb{R}^n$ that preserves an absolutely continuous invariant measure.

2. Preliminaries. Let $(X, \mathfrak{B}, \lambda)$ be a measure space, where λ is an underlying measure. Let $\tau_k : X \to X$, $k = 1, \ldots, K$, be piecewise one-to-one, non-singular transformations on a common partition $\mathcal{P}$ of X: $\mathcal{P} = \{I_1, \ldots, I_q\}$ and $\tau_{k,i} = \tau_k|_{I_i}$, $i = 1, \ldots, q$, $k = 1, \ldots, K$ ($\mathcal{P}$ can be found by considering finer partitions). We define the *transition function* for the random map $T = \{\tau_1, \ldots \tau_K; p_1(x), \ldots, p_K(x)\}$ as follows:

$$\mathbb{P}(x, A) = \sum_{k=1}^{K} p_k(x)\chi_A(\tau_k(x)),$$

where A is any measurable set and $\{p_k(x)\}_{k=1}^{K}$ is a set of position dependent measurable probabilities, i.e., $\sum_{k=1}^{K} p_k(x) = 1$, $p_k(x) \geq 0$ for any $x \in X$, and χ_A denotes the characteristic function of the set A. We define $T(x) = \tau_k(x)$ with probability $p_k(x)$ and $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)$ with probability $p_{k_N}(\tau_{k_{N-1}} \circ \cdots \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ \cdots \circ \tau_{k_1}(x)) \cdots p_{k_1}(x)$. The transition function $\mathbb{P}$ induces an operator $\mathbb{P}_*$ on measures on $(X, \mathfrak{B})$ defined by

$$\begin{aligned}\mathbb{P}_*\mu(A) &= \int \mathbb{P}(x, A)\, d\mu(x) = \sum_{k=1}^{K} \int p_k(x)\chi_A(\tau_k(x))\, d\mu(x) \\ &= \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x)\, d\mu(x) = \sum_{k=1}^{K}\sum_{i=1}^{q} \int_{\tau_{k,i}^{-1}(A)} p_k(x)\, d\mu(x).\end{aligned}$$

We say that the measure μ is T-*invariant* iff $\mathbb{P}_*\mu = \mu$, i.e.,

$$\mu(A) = \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x)\, d\mu(x), \qquad A \in \mathfrak{B}.$$

If μ has density f with respect to λ, then $\mathbb{P}_*\mu$ also has a density which we denote by $P_T f$. By change of variables, we obtain

$$\int_A P_T f(x)\,d\lambda(x) = \sum_{k=1}^{K}\sum_{i=1}^{q}\int_{\tau_{k,i}^{-1}(A)} p_k(x)f(x)\,d\lambda(x)$$
$$= \sum_{k=1}^{K}\sum_{i=1}^{q}\int_A p_k(\tau_{k,i}^{-1}x)f(\tau_{k,i}^{-1}x)\,\frac{1}{J_{k,i}(\tau_{k,i}^{-1})}\,d\lambda(x),$$

where $J_{k,i}$ is the Jacobian of $\tau_{k,i}$ with respect to λ. Since this holds for any measurable set A we obtain an a.e. equality

$$(P_T f)(x) = \sum_{k=1}^{K}\sum_{i=1}^{q} p_k(\tau_{k,i}^{-1}x)f(\tau_{k,i}^{-1}x)\,\frac{1}{J_{k,i}(\tau_{k,i}^{-1})}\,\chi_{\tau_k(I_i)}(x)$$

or

$$(P_T f)(x) = \sum_{k=1}^{K} P_{\tau_k}(p_k f)(x),$$

where P_{τ_k} is the Perron–Frobenius operator corresponding to the transformation τ_k (see [1] for details). We call P_T the *Perron–Frobenius operator* of the random map T. It is main tool in this paper, with very useful properties.

3. Properties of the Perron–Frobenius operator of T. The properties of P_T resemble the properties of the classical Perron–Frobenius operator of a single transformation.

LEMMA 3.1. *P_T has the following properties*:

(i) *P_T is linear*;
(ii) *P_T is non-negative*; *i.e.*, $f \geq 0 \Rightarrow P_T f \geq 0$;
(iii) $P_T f = f \Leftrightarrow \mu = f\cdot\lambda$ *is T-invariant*;
(iv) $\|P_T f\|_1 \leq \|f\|_1$, *where* $\|\cdot\|_1$ *denotes the L^1 norm*;
(v) $P_{T\circ R} = P_R \circ P_T$. *In particular*, $P_T^N f = P_{T^N} f$.

Proof. The proofs of (i)–(iv) are analogous to those for a single transformation. For the proof of (v), let T and R be two random maps corresponding to $\{\tau_1,\ldots,\tau_K; p_1,\ldots,p_K\}$ and $\{\zeta_1,\ldots,\zeta_L; r_1,\ldots,r_L\}$ respectively. We define $\{\tau_k\}_{k=1}^K$ and $\{\zeta_l\}_{l=1}^L$ on a common partition $\mathcal{P}$. We have

$$P_R(P_T f) = P_R\Big(\sum_{k=1}^{K} P_{\tau_k}(p_k f)\Big)$$
$$= \sum_{l=1}^{L}\sum_{k=1}^{K} P_{\zeta_l}(r_l P_{\tau_k}(p_k f))$$
$$= \sum_{l=1}^{L}\sum_{k=1}^{K}\sum_{i=1}^{q} r_l(\zeta_{l,i}^{-1})[P_{\tau_k}(p_k f)](\zeta_{l,i}^{-1})\,\frac{1}{J_{\zeta,l,i}(\zeta_{l,i}^{-1})}\,\chi_{\zeta_{l,i}(I_i)}$$

$$= \sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{j=1}^{q}\sum_{i=1}^{q} r_l(\zeta_{l,i}^{-1})p_k(\tau_{k,j}^{-1}\circ\zeta_{l,i}^{-1})f(\tau_{k,j}^{-1}\circ\zeta_{l,i}^{-1})$$

$$\times \frac{1}{J_{\tau,k,j}(\tau_{k,j}^{-1}\circ\zeta_{l,i}^{-1})}\frac{1}{J_{\zeta,l,i}(\zeta_{l,i}^{-1})}\chi_{\tau_k(I_j)}(\zeta_{l,i}^{-1})\chi_{\zeta_{l,i}(I_i)}$$

$$= \sum_{k=1}^{K}\sum_{l=1}^{L} P_{\tau_k\circ\zeta_l}(p_k(\zeta_l)r_l f) = P_{T\circ R}f. \ \blacksquare$$

4. The existence of an absolutely continuous invariant measure on $[a,b]$**.** Let $(I,\mathfrak{B},\lambda)$ be a measure space, where λ is normalized Lebesgue measure on $I=[a,b]$. Let $\tau_k : I\to I$, $k=1,\dots,K$, be piecewise one-to-one and differentiable, non-singular transformations on a partition $\mathcal{P}$ of I: $\mathcal{P}=\{I_1,\dots,I_q\}$ and $\tau_{k,i}=\tau_k|_{I_i}$, $i=1,\dots,q$, $k=1,\dots,K$. Denote by $V(\cdot)$ the standard one-dimensional variation of a function, and by $\mathrm{BV}(I)$ the space of functions of bounded variation on I equipped with the norm $\|\cdot\|_{\mathrm{BV}}=V(\cdot)+\|\cdot\|_1$.

Let $g_k(x)=p_k(x)/|\tau_k'(x)|$, $k=1,\dots,K$. We assume the following conditions:

Condition (A). $\sum_{k=1}^{K} g_k(x)<\alpha<1$, $x\in I$.

Condition (B). $g_k\in\mathrm{BV}(I)$, $k=1,\dots,K$.

Under the above conditions our goal is to prove

$$V_I P_T^n f \le AV_I f + B\|f\|_1 \tag{4.1}$$

for some $n\ge 1$, where $0<A<1$ and $B>0$. The inequality (4.1) guarantees the existence of a T-invariant measure absolutely continuous with respect to Lebesgue measure and the quasi-compactness of the operator P_T with all the consequences of this fact (see [1]). We will need a number of lemmas:

Lemma 4.1. *Let* $f\in\mathrm{BV}(I)$. *Suppose* $\tau: I\to J$ *is differentiable and* $\tau'(x)\neq 0$, $x\in I$. *Set* $\phi=\tau^{-1}$ *and let* $g(x)=p(x)/|\tau'(x)|\in\mathrm{BV}(I)$. *Then*

$$V_J(f(\phi)g(\phi)) \le (V_I f+\sup_I f)(V_I g+\sup_I g).$$

Proof. First, note that we have dropped all the k,i indices to simplify the notation. The proof follows in the same way as in Lemma 3 of [9]. $\blacksquare$

Lemma 4.2. *Let* T *satisfy conditions* (A) *and* (B). *Then for any* $f\in\mathrm{BV}(I)$,

$$V_I P_T f \le AV_I f + B\|f\|_1,$$

where

$$A = 3\alpha + \max_{1\le i\le q}\sum_{k=1}^{K} V_{I_i} g_k, \qquad B = 2\beta\alpha + \beta\max_{1\le i\le q}\sum_{k=1}^{K} V_{I_i} g_k,$$

with $\beta = \max_{1\le i\le q}(\lambda(I_i))^{-1}$.

Proof. First, we will refine the partition $\mathcal{P}$ to satisfy an additional condition. Let $\eta > 0$ be such that $\sum_{k=1}^{K}(g_k(x)+\varepsilon_k) < \alpha$ whenever $|\varepsilon_k| < \eta$, $k = 1,\dots,K$. Since g_k, $k = 1,\dots,K$, are of bounded variation we can find a finite partition $\mathcal{K}$ such that for any $k = 1,\dots,K$,

$$|g_k(x) - g_k(y)| < \eta$$

for x, y in the same element of $\mathcal{K}$. Without loss of generality, we can assume that our original partition is the join $\mathcal{P}\vee\mathcal{K}$. Then

$$\max_{1\le i\le q}\sum_{k=1}^{K}\sup_{x\in I_i} g_k(x) < \alpha.$$

We have $V_I(P_T f) = V_I(\sum_{k=1}^{K} P_{\tau_k}(p_k f))$. We will estimate this variation. Let $\phi_{k,i} = \tau_{k,i}^{-1}$, $k = 1,\dots,K$, $i = 1,\dots,q$. We have

$$\begin{aligned}
(4.2)\qquad V_I\Big(\sum_{k=1}^{K} P_{\tau_k}(p_k f)\Big) &= V_I\Big(\sum_{k=1}^{K}\sum_{i=1}^{q} f(\phi_{k,i}) g_k(\phi_{k,i})\chi_{\tau_k(I_i)}\Big)\\
&\le \sum_{k=1}^{K}\sum_{i=1}^{q}[|f(a_{i-1})|\,|g_k(a_{i-1})| + |f(a_i)|\,|g_k(a_i)|]\\
&\quad + \sum_{k=1}^{K}\sum_{i=1}^{q} V_{\tau_k(I_i)}[f(\phi_{k,i}) g_k(\phi_{k,i})].
\end{aligned}$$

First, we estimate the first sum on the right hand side of (4.2):

$$\begin{aligned}
(4.3)\qquad &\sum_{k=1}^{K}\sum_{i=1}^{q}[|f(a_{i-1})|\,|g_k(a_{i-1})| + |f(a_i)|\|g_k(a_i)|]\\
&= \sum_{i=1}^{q}\Big[|f(a_{i-1})|\Big(\sum_{k=1}^{K}|g_k(a_{i-1})|\Big) + |f(a_i)|\Big(\sum_{k=1}^{K}|g_k(a_i)|\Big)\Big]\\
&\le \alpha\Big(\sum_{i=1}^{q}(|f(a_{i-1})| + |f(a_i)|)\Big)\\
&\le \alpha\Big(\sum_{i=1}^{q}\Big(V_{I_i} f + (\lambda(I_i))^{-1}\int_{I_i} f\,d\lambda\Big)\Big) = \alpha(V_I f + \beta\|f\|_1).
\end{aligned}$$

We now estimate the second term on the right hand side of (4.2). Using Lemma 4.1 we obtain

$$\begin{aligned}(4.4)\quad &\sum_{k=1}^{K}\sum_{i=1}^{q} V_{\tau_k(I_i)}[f(\phi_{k,i})g_k(\phi_{k,i})]\\ &\le \sum_{k=1}^{K}\sum_{i=1}^{q}(V_{I_i}f+\sup_{I_i} f)(V_{I_i}g_k+\sup_{I_i} g_k)\\ &\le \sum_{i=1}^{q}\Big(2V_{I_i}f+\beta\int_{I_i} f\,d\lambda\Big)\Big(\max_{1\le i\le q}\sum_{k=1}^{K}(V_{I_i}g_k+\sup_{I_i} g_k)\Big)\\ &\le (2V_I f+\beta\|f\|_1)\Big(\max_{1\le i\le q}\sum_{k=1}^{K}V_{I_i}g_k+\alpha\Big).\end{aligned}$$

Thus, using (4.3) and (4.4), we obtain

$$V_I P_T f \le \Big(3\alpha+\max_{1\le i\le q}\sum_{k=1}^{K}V_{I_i}g_k\Big)V_I f+\Big(2\beta\alpha+\beta\max_{1\le i\le q}\sum_{k=1}^{K}V_{I_i}g_k\Big)\|f\|_1. \quad ■$$

In the following two lemmas we show that the constants α as well as $\max_{1\le i\le q}\sum_{k=1}^{K}V_{I_i}g_k$ decrease when we consider higher iterations T^n instead of T. The constant β obviously increases, but this is not important.

Lemma 4.3. *Let T be a random map which satisfies condition* (A). *Then, for $x\in I$,*

$$(4.5)\qquad \sum_{w\in\{1,\dots,K\}^N}\frac{p_w(x)}{|T'_w(x)|}<\alpha^N,$$

where $T_w(x)=\tau_{k_N}\circ\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x)$ and $p_w(x)=p_{k_N}(\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x))\cdot p_{k_{N-1}}(\tau_{k_{N-2}}\circ\cdots\circ\tau_{k_1}(x))\cdots p_{k_1}(x)$ define the random map T^N.

Proof. We have

$$T^N(x)=\tau_{k_N}\circ\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x)$$

with probability

$$p_{k_N}(\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x))\cdot p_{k_{N-1}}(\tau_{k_{N-2}}\circ\cdots\circ\tau_{k_1}(x))\cdots p_{k_1}(x).$$

The maps defining T^N may be indexed by $w\in\{1,\dots,K\}^N$. Set

$$T_w(x)=\tau_{k_N}\circ\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x),$$

where $w=(k_1,\dots,k_N)$, and

$$p_w(x)=p_{k_N}(\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x))\cdot p_{k_{N-1}}(\tau_{k_{N-2}}\circ\cdots\circ\tau_{k_1}(x))\cdots p_{k_1}(x).$$

Then

$$T'_w(x)=\tau'_{k_N}(\tau_{k_{N-1}}\circ\cdots\circ\tau_{k_1}(x))\tau'_{k_{N-1}}(\tau_{k_{N-2}}\circ\cdots\circ\tau_{k_1}(x))\cdots\tau'_{k_1}(x).$$

Suppose that T satisfies condition (A). We will prove (4.5) using induction on N. For $N=1$, we have

$$\sum_{w\in\{1,\dots,K\}} \frac{p_w(x)}{|T_w'(x)|} < \alpha$$

by condition (A). Assume (4.5) is true for $N-1$. Then

$$\sum_{w\in\{1,\dots,K\}^N} \frac{p_w(x)}{|T_w'(x)|} = \sum_{\overline{w}\in\{1,\dots,K\}^{N-1}} \sum_{k=1}^{K} \frac{p_k(x)p_{\overline{w}}(\tau_k(x))}{|\tau_k'(x)|\,|T_{\overline{w}}'(\tau_k(x))|}$$
$$\le \bigg(\sum_{k=1}^{K} \frac{p_k(x)}{|\tau_k'(x)|}\bigg)\bigg(\sum_{\overline{w}\in\{1,\dots,K\}^{N-1}} \frac{p_{\overline{w}}(\tau_k(x))}{|T_{\overline{w}}'(\tau_k(x))|}\bigg) < \alpha\cdot\alpha^{N-1} = \alpha^N. \ ■$$

LEMMA 4.4. *Let $g_w = p_w/|T_w'|$, where T_w and p_w are defined in Lemma 4.3 and $w\in\{1,\dots,K\}^n$. Define*

$$W_1 \equiv \max_{1\le i\le q} \sum_{k=1}^{K} V_{I_i} g_k, \qquad W_n \equiv \max_{J\in\mathcal{P}^{(n)}} \sum_{w\in\{1,\dots,K\}^n} V_J g_w,$$

where $\mathcal{P}^{(n)}$ is the common monotonicity partition for all T_w. Then, for all $n\ge 1$,

$$W_n \le n\alpha^{n-1} W_1,$$

where α is defined in condition (A).

Proof. We prove the lemma by induction on n. For $n=1$ the assertion is true by definition of W_n. Assume that it is true for n, i.e.,

$$W_n \le n\alpha^{n-1} W_1.$$

Let $J\in\mathcal{P}^{(n+1)}$ and $x_0 < x_1 < \cdots < x_l$ be a sequence of points in J. Then

$$\sum_w \sum_{j=0}^{l-1} |g_w(x_{j+1}) - g_w(x_j)| = \sum_{j=0}^{l-1} \sum_{w\in\{1,\dots,K\}^{n+1}} |g_w(x_{j+1}) - g_w(x_j)|$$
$$\le \sum_{j=0}^{l-1} \sum_{\overline{w}\in\{1,\dots,K\}^n} \sum_{k=1}^{K} |g_{\overline{w}}(\tau_k(x_{j+1}))g_k(x_{j+1}) - g_{\overline{w}}(\tau_k(x_j))g_k(x_j)|$$
$$\le \sum_{j=0}^{l-1} \sum_{\overline{w}\in\{1,\dots,K\}^n} \sum_{k=1}^{K} |g_{\overline{w}}(\tau_k(x_{j+1}))g_k(x_{j+1}) - g_{\overline{w}}(\tau_k(x_{j+1}))g_k(x_j)|$$
$$+ \sum_{j=0}^{l-1} \sum_{\overline{w}\in\{1,\dots,K\}^n} \sum_{k=1}^{K} |g_{\overline{w}}(\tau_k(x_{j+1}))g_k(x_j) - g_{\overline{w}}(\tau_k(x_j))g_k(x_j)|$$

$$
\begin{aligned}
&\le \sum_{j=0}^{l-1}\sum_{k=1}^{K} |g_k(x_{j+1}) - g_k(x_j)| \sum_{\overline{w}\in\{1,\dots,K\}^n} g_{\overline{w}}(\tau_k(x_{j+1}))\\
&\quad + \sum_{j=0}^{l-1}\sum_{k=1}^{K} g_k(x_j) \sum_{\overline{w}\in\{1,\dots,K\}^n} |g_{\overline{w}}(\tau_k(x_{j+1})) - g_{\overline{w}}(\tau_k(x_j))|\\
&\le \alpha^n \sum_{j=0}^{l-1}\sum_{k=1}^{K} |g_k(x_{j+1}) - g_k(x_j)|\\
&\quad + \alpha \sum_{j=0}^{l-1} \sum_{\overline{w}\in\{1,\dots,K\}^n} |g_{\overline{w}}(\tau_k(x_{j+1})) - g_{\overline{w}}(\tau_k(x_j))|\\
&\le \alpha^n W_1 + \alpha W_n \le \alpha^n W_1 + n\alpha^n W_1 = (n+1)\alpha^n W_1.
\end{aligned}
$$

We used condition (A) and Lemma 4.3. ∎

THEOREM 4.5. *Let T be a random map which satisfies conditions* (A) *and* (B). *Then T preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator P_T is quasi-compact on* $\mathrm{BV}(I)$ (see [1]).

Proof. Let N be such that $A_N = 3\alpha^N + W_N < 1$. Then, by Lemma 4.3,

$$\sum_{w\in\{1,\dots,K\}^N} g_w(x) < \alpha^N, \qquad x \in I.$$

We refine the partition $\mathcal{P}^{(N)}$ as in the proof of Lemma 4.2, to have

$$\max_{J\in\mathcal{P}^N} \sum_{w\in\{1,\dots,K\}^N} \sup_J g_w < \alpha^N.$$

Then, by Lemma 4.2, we get

$$\|P_T^N f\|_{\mathrm{BV}} \le A_N \|f\|_{\mathrm{BV}} + B_N \|f\|_1,$$

where $B_N = \beta_N(2\alpha^N + W_N)$, $\beta_N = \max_{J\in\mathcal{P}^N}(\lambda(J))^{-1}$. The theorem follows by the standard technique (see [1]). ∎

REMARK 4.6. It is enough to assume that condition (A) is satisfied for some iterate $T^m, m \ge 1$.

REMARK 4.7. The number of absolutely continuous invariant measures for random maps has been studied in [4]. The proof of [4], which uses graph theoretic methods, goes through analogously in our case, i.e., when T is a random map with position dependent probabilities.

5. Example. We present an example of a random map T which does not satisfy the conditions of [5], yet it preserves an absolutely continuous invariant measure under conditions (A) and (B).

EXAMPLE 5.1. Let T be a random map which is given by $\{\tau_1, \tau_2; p_1(x), p_2(x)\}$, where

$$\tau_1(x) = \begin{cases} 2x & \text{for } 0 \le x \le 1/2, \\ x & \text{for } 1/2 < x \le 1, \end{cases} \qquad p_1(x) = \begin{cases} 2/3 & \text{for } 0 \le x \le 1/2, \\ 1/3 & \text{for } 1/2 < x \le 1, \end{cases}$$

$$\tau_2(x) = \begin{cases} x + 1/2 & \text{for } 0 \le x \le 1/2, \\ 2x - 1 & \text{for } 1/2 < x \le 1, \end{cases} \qquad p_2(x) = \begin{cases} 1/3 & \text{for } 0 \le x \le 1/2, \\ 2/3 & \text{for } 1/2 < x \le 1. \end{cases}$$

Then $\sum_{k=1}^{2} g_k(x) = 2/3 < 1$. Therefore, T satisfies conditions (A) and (B). Consequently, by Theorem 4.5, T preserves an invariant measure absolutely continuous with respect to Lebesgue measure. Notice that τ_1, τ_2 are piecewise linear Markov maps defined on the same Markov partition $\mathcal{P}$: $\{[0, 1/2], [1/2, 1]\}$. For such maps the Perron–Frobenius operator reduces to a matrix (see [1]). The corresponding matrices are:

$$P_{\tau_1} = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}, \qquad P_{\tau_2} = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}.$$

Their invariant densities are $f_{\tau_1} = [0, 2]$ and $f_{\tau_2} = [2/3, 4/3]$. The Perron–Frobenius operator of the random map T is given by

$$P_T = \begin{pmatrix} 2/3 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1/3 & 0 \\ 0 & 2/3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix}.$$

If the invariant density of T is $f = [f_1, f_2]$, normalized by $f_1 + f_2 = 2$ and satisfying the equation $fP_T = f$, then $f_1 = 2/3$ and $f_2 = 4/3$.

6. The existence of an absolutely continuous invariant measure in $\mathbb{R}^n$. Let S be a bounded region in $\mathbb{R}^n$ and λ_n be Lebesgue measure on S. Let $\tau_k : S \to S$, $k = 1, \ldots, K$, be piecewise one-to-one and C^2, non-singular transformations on a partition $\mathcal{P}$ of S, $\mathcal{P} = \{S_1, \ldots, S_q\}$ and $\tau_{k,i} = \tau_k|_{S_i}$, $i = 1, \ldots, q$, $k = 1, \ldots, K$. Suppose each S_i is a bounded closed domain having a piecewise C^2 boundary of finite $(n-1)$-dimensional measure. We assume that the faces of ∂S_i meet at angles bounded uniformly away from 0. We will also assume that the probabilities $p_k(x)$ are piecewise C^1 functions on the partition $\mathcal{P}$. Let $D\tau_{k,i}^{-1}(x)$ be the derivative matrix of $\tau_{k,i}^{-1}$ at x. We assume:

Condition (C).

$$\max_{1\le i\le q}\sum_{k=1}^{K} p_k(x)\|D\tau_{k,i}^{-1}(\tau_{k,i}(x))\| < \sigma < 1.$$

Let $\sup_{x\in\tau_{k,i}(S_i)}\|D\tau_{k,i}^{-1}(x)\| =: \sigma_{k,i}$ and $\sup_{x\in S_i} p_k(x) =: \pi_{k,i}$. Using the smoothness of $D\tau_{k,i}^{-1}$'s and p_k's we can refine the partition $\mathcal{P}$ to satisfy

Condition (C′).

$$\sum_{k=1}^{K}\max_{1\le i\le q}\sigma_{k,i}\pi_{k,i} < \sigma < 1.$$

Under this condition, our goal is to prove the existence of an a.c.i.m. for the random map $T = \{\tau_1,\dots,\tau_K; p_1,\dots,p_K\}$. The main tool of this section is the multidimensional notion of variation defined using derivatives in the distributional sense (see [3]):

$$V(f) = \int_{\mathbb{R}^n}\|Df\| = \sup\Big\{\int_{\mathbb{R}^n} f\,\mathrm{div}(g)\,d\lambda_n : g = (g_1,\dots,g_n)\in C_0^1(\mathbb{R}^n,\mathbb{R}^n)\Big\},$$

where $f\in L_1(\mathbb{R}^n)$ has bounded support, Df denotes the gradient of f in the distributional sense, and $C_0^1(\mathbb{R}^n,\mathbb{R}^n)$ is the space of continuously differentiable functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ having a compact support. We will use the following property of variation which is derived from [3, Remark 2.14]: If $f = 0$ outside a closed domain A whose boundary is Lipschitz continuous, $f|_A$ is continuous, $f|_{\mathrm{int}(A)}$ is C^1, then

$$V(f) = \int_{\mathrm{int}(A)}\|Df\|\,d\lambda_n + \int_{\partial A}|f|\,d\lambda_{n-1},$$

where λ_{n-1} is the $n-1$-dimensional measure on the boundary of A. In this section we shall consider the Banach space (see [3, Remark 1.12])

$$\mathrm{BV}(S) = \{f\in L_1(S) : V(f) < \infty\},$$

with the norm $\|f\|_{\mathrm{BV}} = V(f) + \|f\|_1$. We adapt the following two lemmas from [6]. Their proofs are exactly the same as in [6].

Lemma 6.1. *Consider $S_i\in\mathcal{P}$. Let x be a point in ∂S_i and $y = \tau_k(x)$ a point in $\partial(\tau_k(S_i))$. Let $J_{k,i}$ be the Jacobian of $\tau_k|_{S_i}$ at x and $J_{k,i}^0$ be the Jacobian of $\tau_k|_{\partial S_i}$ at x. Then*

$$\frac{J_{k,i}^0}{J_{k,i}} \le \sigma_{k,i}. \ \blacksquare$$

Fix $1\le i\le q$. Let Z denote the set of singular points of ∂S_i. Let us construct for any $x\in Z$ the largest cone with vertex at x and which lies completely in S_i. Let $\theta(x)$ denote the vertex angle of this cone. Then define

$$\beta(S_i) = \min_{x\in Z} \theta(x).$$

Since the faces of ∂S_i meet at angles bounded away from 0, we have $\beta(S_i) > 0$. Let $\alpha(S_i) = \pi/2 + \beta(S_i)$ and

$$a(S_i) = |\cos(\alpha(S_i))|.$$

Now we will construct a C^1 field of segments L_y, $y \in \partial S_i$, every L_y being a central ray of a regular cone contained in S_i, with vertex angle at y greater than or equal to $\beta(S_i)$.

We start at points $y \in Z$ where the minimal angle $\beta(S_i)$ is attained, defining L_y to be central rays of the largest regular cones contained in S_i. Then we extend this field of segments to the C^1 field we want, making L_y short enough to avoid overlapping. Let $\delta(y)$ be the length of L_y, $y \in \partial S_i$. By the compactness of ∂S_i we have

$$\delta(S_i) = \inf_{y\in\partial S_i} \delta(y) > 0.$$

Now, we shorten the L_y of our field, making them all of length $\delta(S_i)$.

LEMMA 6.2. *For any S_i, $i = 1, \ldots, q$, if f is a C^1 function on S_i, then*

$$\int_{\partial S_i} f(y)\, d\lambda_{n-1}(y) \le \frac{1}{a(S_i)}\left(\frac{1}{\delta(S_i)} \int_{S_i} f\, d\lambda_n + V_{\mathrm{int}(S_i)}(f)\right). \ \blacksquare$$

Our main technical result is the following:

THEOREM 6.3. *If T is a random map which satisfies condition* (C), *then*

$$V(P_T f) \le \sigma(1 + 1/a)V(f) + \left(M + \frac{\sigma}{a\delta}\right)\|f\|_1,$$

where $a = \min\{a(S_i) : i = 1, \ldots, q\} > 0$, $\delta = \min\{\delta(S_i) : i = 1, \ldots, q\} > 0$, *and*

$$M_{k,i} = \sup_{x\in S_i}\left(Dp_k(x) - \frac{DJ_{k,i}}{J_{k,i}}\, p_k(x)\right), \qquad M = \sum_{k=1}^{K} \max_{1\le i\le q} M_{k,i}.$$

Proof. We have $V(P_T f) \le \sum_{k=1}^{K} V(P_{\tau_k}(p_k f))$. To estimate $V(P_{\tau_k}(p_k f))$, let

$$F_{k,i} = \frac{f(\tau_{k,i}^{-1})p_k(\tau_{k,i}^{-1})}{J_{k,i}(\tau_{k,i}^{-1})}, \qquad R_{k,i} = \tau_{k,i}(S_i), \qquad i = 1, \ldots, q,\ k = 1, \ldots, K.$$

Then

$$\begin{aligned}\int_{\mathbb{R}^n} \|DP_{\tau_k}(p_k f)\|\, d\lambda_n &\le \sum_{i=1}^{q} \int_{\mathbb{R}^n} \|D(F_{k,i}\chi_{R_i})\|\, d\lambda_n \\ &\le \sum_{i=1}^{q}\Big(\int_{\mathbb{R}^n} \|D(F_{k,i})\chi_{R_i}\|\, d\lambda_n + \int_{\mathbb{R}^n} \|F_{k,i}(D\chi_{R_i})\|\, d\lambda_n\Big).\end{aligned}$$

Now, for the first integral we have

$$\begin{aligned}\int\limits_{\mathbb{R}^n} \|D(F_{k,i})\chi_{R_i}\|\,d\lambda_n &= \int\limits_{R_i} \|D(F_{k,i}p_k)\|\,d\lambda_n \\ &\le \int\limits_{R_i} \left\| D(f(\tau_{k,i}^{-1}))\,\frac{p_k(\tau_{k,i}^{-1})}{J_{k,i}(\tau_{k,i}^{-1})}\right\| d\lambda_n + \int\limits_{R_i} \left\| f(\tau_{k,i}^{-1}) D\bigg(\frac{p_k(\tau_{k,i}^{-1})}{J_{k,i}(\tau_{k,i}^{-1})}\bigg)\right\| d\lambda_n \\ &\le \int\limits_{R_i} \|Df(\tau_{k,i}^{-1})\|\,\|D\tau_{k,i}^{-1}\|\,\frac{p_k(\tau_{k,i}^{-1})}{J_{k,i}(\tau_{k,i}^{-1})}\,d\lambda_n + \int\limits_{R_i} \|f(\tau_{k,i}^{-1})\|\,\frac{M_k}{J_{k,i}(\tau_{k,i}^{-1})}\,d\lambda_n \\ &\le \sigma_{k,i}\pi_{k,i}\int\limits_{S_i} \|Df\|\,d\lambda_n + M_k \int\limits_{S_i} \|f\|\,d\lambda_n.\end{aligned}$$

For the second integral we have

$$\int\limits_{\mathbb{R}^n} \|F_{k,i}(D\chi_{R_i})\|\,d\lambda_n = \int\limits_{\partial R_i} |f(\tau_{k,i}^{-1}))|\,\frac{p_k(\tau_{k,i}^{-1})}{J_{k,i}(\tau_{k,i}^{-1})}\,d\lambda_{n-1} = \int\limits_{\partial S_i} |f| p_k\,\frac{J_{k,i}^0}{J_{k,i}}\,d\lambda_{n-1}.$$

By Lemma 4.3, $J_{k,i}^0/J_{k,i} \le \sigma_{k,i}$. Using Lemma 4.2, we get

$$\begin{aligned}\int\limits_{\mathbb{R}^n} \|F_{k,i}(D\chi_{R_i})\|\,d\lambda_n &\le \sigma_{k,i}\pi_{k,i}\int\limits_{\partial S_i} |f|\,d\lambda_{n-1} \\ &\le \frac{\sigma_{k,i}\pi_{k,i}}{a}\,V_{S_i}(f) + \frac{\sigma_{k,i}\pi_{k,i}}{a\delta}\int\limits_{S_i} |f|\,d\lambda_n.\end{aligned}$$

Using Condition (C′), summing first over i, we obtain

$$\begin{aligned}V(P_{\tau_k}(p_k f)) &\le (\max_{1\le i\le q} \sigma_{k,i}\pi_{k,i})(1+1/a)V(f) \\ &\quad + \Big(\max_{1\le i\le q} M_{k,i} + \frac{\max_{1\le i\le q}\sigma_{k,i}\pi_{k,i}}{a\delta}\Big)\|f\|_1.\end{aligned}$$

Then summing over k yields the assertion. ∎

THEOREM 6.4. *Let T be a random map which satisfies condition* (C). *If $\sigma(1+1/a)<1$, then T preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator P_T is quasi-compact on* BV(S) (see [1]).

Proof. This follows by the standard technique (see [1]). ∎

7. Example in $\mathbb{R}^2$. In this section, we present an example of a random map which satisfies condition (C) of Theorem 6.3 and thus it preserves an absolutely continuous invariant measure.

EXAMPLE 7.1. Let T be a random map given by $\{\tau_1, \tau_2; p_1(x), p_2(x)\}$, where $\tau_1, \tau_2 : I^2 \to I^2$ are defined by

$$\tau_1(x_1, x_2) = \begin{cases} (3x_1, 2x_2) & \text{for } (x_1, x_2) \in S_1 = \{0 \le x_1, x_2 \le 1/3\}, \\ (3x_1 - 1, 2x_2) & \text{for } (x_1, x_2) \in S_2 = \{1/3 < x_1 \le 2/3; 0 \le x_2 \le 1/3\}, \\ (3x_1 - 2, 2x_2) & \text{for } (x_1, x_2) \in S_3 = \{2/3 < x_1 \le 1; 0 \le x_2 \le 1/3\}, \\ (3x_1, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_4 = \{0 < x_1 \le 1/3; 1/3 < x_2 \le 2/3\}, \\ (3x_1 - 1, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_5 = \{1/3 < x_1, x_2 \le 2/3\}, \\ (3x_1 - 2, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_6 = \{2/3 < x_1 \le 1; 1/3 < x_2 \le 2/3\}, \\ (3x_1, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_7 = \{0 \le x_1 \le 1/3; 2/3 < x_2 \le 1\}, \\ (3x_1 - 1, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_8 = \{1/3 < x_1 \le 2/3; 2/3 < x_2 \le 1\}, \\ (3x_1 - 2, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_9 = \{2/3 < x_1 \le 1; 2/3 < x_2 \le 1\}, \end{cases}$$

$$\tau_2(x_1, x_2) = \begin{cases} (3x_1, 3x_2) & \text{for } (x_1, x_2) \in S_1, \\ (2 - 3x_1, 3x_2) & \text{for } (x_1, x_2) \in S_2, \\ (3x_1 - 2, 3x_2) & \text{for } (x_1, x_2) \in S_3, \\ (3x_1, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_4, \\ (2 - 3x_1, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_5, \\ (3x_1 - 2, 3x_2 - 1) & \text{for } (x_1, x_2) \in S_6, \\ (3x_1, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_7, \\ (2 - 3x_1, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_8, \\ (3x_1 - 2, 3x_2 - 2) & \text{for } (x_1, x_2) \in S_9,, \end{cases}$$

and

$$p_1(x) = \begin{cases} 0.215 & \text{for } (x_1, x_2) \in S_1, \\ 0.216 & \text{for } (x_1, x_2) \in S_2, \\ 0.216 & \text{for } (x_1, x_2) \in S_3, \\ 0.216 & \text{for } (x_1, x_2) \in S_4, \\ 0.215 & \text{for } (x_1, x_2) \in S_5, \\ 0.216 & \text{for } (x_1, x_2) \in S_6, \\ 0.216 & \text{for } (x_1, x_2) \in S_7, \\ 0.216 & \text{for } (x_1, x_2) \in S_8, \\ 0.215 & \text{for } (x_1, x_2) \in S_9, \end{cases} \quad p_2(x) = \begin{cases} 0.785 & \text{for } (x_1, x_2) \in S_1, \\ 0.784 & \text{for } (x_1, x_2) \in S_2, \\ 0.784 & \text{for } (x_1, x_2) \in S_3, \\ 0.784 & \text{for } (x_1, x_2) \in S_4, \\ 0.785 & \text{for } (x_1, x_2) \in S_5, \\ 0.784 & \text{for } (x_1, x_2) \in S_6, \\ 0.784 & \text{for } (x_1, x_2) \in S_7, \\ 0.784 & \text{for } (x_1, x_2) \in S_8, \\ 0.785 & \text{for } (x_1, x_2) \in S_9. \end{cases}$$

The derivative matrix of $\tau_{1,i}^{-1}$ is

$$\begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix},$$

and the derivative matrix of $\tau_{2,i}^{-1}$ is

$$\begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1/3 & 0 \\ 0 & 1/3 \end{pmatrix}.$$

Therefore, the Euclidean matrix norm $\|D\tau_{1,i}^{-1}\|$ is $\sqrt{2}/3$ or $\sqrt{13}/6$, and the Euclidean matrix norm $\|D\tau_{2,i}^{-1}\|$ is $\sqrt{2}/3$. Then

$$\max_{1\le i\le q}\sum_{k=1}^{K} p_k(x)\|D\tau_{k,i}^{-1}(\tau_{k,i}(x))\| \le 0.216\,\frac{\sqrt{13}}{6} + 0.785\frac{\sqrt{2}}{3}.$$

For this partition $\mathcal{P}$, we have $a = 1$, which implies

$$\sigma(1+1/a) = 2\left(0.216\,\frac{\sqrt{13}}{6} + 0.785\,\frac{\sqrt{2}}{3}\right) \approx 0.9998 < 1.$$

Therefore, by Theorem 6.4, the random map T admits an absolutely continuous invariant measure. Notice that τ_1, τ_2 are piecewise linear Markov maps defined on the same Markov partition $\mathcal{P} = \{S_1, \ldots, S_9\}$. For such maps the Perron–Frobenius operator reduces to a matrix and the invariant density is constant on the elements of the partition (see [1]). The Perron–Frobenius operator of T is represented by the matrix

$$M = \Pi_1 M_1 + \Pi_2 M_2,$$

where M_1, M_2 are the matrices of P_{τ_1} and P_{τ_2} respectively, and Π_1, Π_2 are the diagonal matrices of $p_1(x)$ and $p_2(x)$ respectively. Then M is given by

$$M = p_1\,\mathbf{Id}_9 \cdot \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 \\ 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \end{pmatrix}$$

$$
+ p_2 \,\mathbf{Id}_9 \cdot \begin{pmatrix}
1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\
1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9
\end{pmatrix}
$$

$$
= \begin{pmatrix}
a & a & a & a & a & a & b & b & b \\
c & c & c & c & c & c & d & d & d \\
c & c & c & c & c & c & d & d & d \\
e & e & e & e & e & e & e & e & e \\
e & e & e & e & e & e & e & e & e \\
e & e & e & e & e & e & e & e & e \\
e & e & e & e & e & e & e & e & e \\
e & e & e & e & e & e & e & e & e \\
e & e & e & e & e & e & e & e & e
\end{pmatrix},
$$

where

$$
\begin{aligned}
p_1 &= (0.215, 0.216, 0.216, 0.216, 0.215, 0.216, 0.216, 0.216, 0.215), \\
p_2 &= (0.785, 0.784, 0.784, 0.784, 0.785, 0.784, 0.784, 0.784, 0.785),
\end{aligned}
$$

$\mathbf{Id}_9$ is the 9×9 identity matrix and

$$
a = 0.12306, \qquad b = 0.087222, \qquad c = 0.12311,
$$
$$
d = 0.087111, \qquad e = 0.11111.
$$

The invariant density of T is

$$
f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9), \qquad f_i = f_{|S_i}, \qquad i = 1, \dots, 9,
$$

normalized by

$$
f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + f_9 = 9,
$$

and satisfying the equation $fM = f$. Then

$$
f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = \frac{9}{6.29739}, \qquad f_7 = f_8 = f_9 = \frac{0.29739}{3} f_1.
$$

References

[1] A. Boyarsky and P. Góra, *Laws of Chaos*, Birkhäuser, Boston, 1997.

[2] N. Dunford and J. T. Schwartz, *Linear Operators*, *Part I: General Theory*, Interscience, New York, 1964.

[3] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhaüser, Boston, 1984.

[4] P. Góra, *Graph-theoretic bound on number of a.c.i.m. for random transformations*, Proc. Amer. Math. Soc. 116 (1992), 401–410.

[5] P. Góra and A. Boyarsky, *Absolutely continuous invariant measures for random maps with position dependent probabilities*, J. Math. Anal. Appl. 278 (2003), 225–242.

[6] —, —, *Absolutely continuous invariant measures for piecewise expanding C^2 transformations in $\mathbb{R}^N$*, Israel J. Math. 67 (1989), 272–286.

[7] Y. Kifer, *Ergodic Theory of Random Transformations*, Birkhäuser, Boston, 1986.

[8] T. Morita, *Random iteration of one-dimensional transformations*, Osaka J. Math. 22 (1985), 489–518.

[9] S. Pelikan, *Invariant densities for random maps of the interval*, Trans. Amer. Math. Soc. 281 (1984), 813–825.

[10] Ö. Stenflo, *Uniqueness of invariant measures for place-dependent random iterations of functions*, in: IMA Vol. Math. Appl. 132, Springer, 2002, 13–32.

Department of Mathematics and Statistics
University of Victoria
PO BOX 3045 STN CSC
Victoria, B.C., V8W 3P4, Canada
E-mail: wab@math.uvic.ca

Department of Mathematics and Statistics
Concordia University
7141 Sherbrooke Street West
Montreal, Quebec H4B 1R6, Canada
E-mail: pgora@vax2.concordia.ca

Received December 19, 2003
Revised version August 23, 2004 (5338)