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Automatic continuity of operators commuting with translations

by

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Abstract. Let τ_X and τ_Y be representations of a topological group G on Banach spaces X and Y, respectively. We investigate the continuity of the linear operators $\Phi\colon X \to Y$ with the property that $\Phi \circ \tau_X(t) = \tau_Y(t) \circ \Phi$ for each $t \in G$ in terms of the invariant vectors in Y and the automatic continuity of the invariant linear functionals on X.

1. Introduction. Numerous interesting operators arising in mathematical analysis, as well as in mathematical physics, have the property of commuting with a group of meaningful transformations. Specifically a lot of attention has been paid to the operators which commute with translations on the classical translation-invariant topological linear spaces $\mathfrak{F}(G)$ of functions (or distributions) over some locally compact group G. A basic task is to derive a suitable representation for all such operators. As a matter of fact, it is well known that those linear operators $\Phi\colon L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, $1 \le p, q \le \infty$, commuting with translations which, in addition, are continuous, are necessarily representable in the form $\Phi(f) = k \star f$ for a unique tempered distribution k (see [29, Section I.3]). Thus verifying the automatic continuity of operators of this kind has become a subject of lively interest which has been developed even for linear operators $\Phi\colon X \to Y$ which commute with translations, where X and Y are Banach spaces on which G acts as a group of continuous transformations. In this context, it is worth emphasizing the results by B. E. Johnson [16] and G. A. Willis [34]. Willis showed that *if G is a locally compact group which contains $\mathbb{F}_2$ as a closed subgroup, then every linear operator $\Phi\colon X \to L^1(G)$ which commutes with translations is continuous for each representation of G on a Banach space X.* Johnson

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considered the question whether, given bounded representations of a locally compact abelian group G on Banach spaces X and Y, every linear operator $\Phi\colon X \to Y$ which commutes with translations is continuous. Moreover, it is shown that *every linear operator $\Phi\colon X \to Y$ which commutes with translations is continuous if and only if every γ-covariant linear functional on X is continuous whenever γ is a character of G such that there is a non-zero vector in Y on which G acts by scalar multiplication corresponding to γ.*

The automatic continuity of linear operators commuting with translations sometimes subsumes the uniqueness of invariant norms. This is a subject which started with the seminal paper by K. Jarosz [13] on $L^1(\mathbb{R})$ and $L^p(\mathbb{T})$ $(1 \le p \le \infty)$ and it has been successfully carried out for arbitrary locally compact abelian groups [7, 31] and for non-abelian compact groups [8]. In [32] we investigated the uniqueness-of-invariant-norm problem for an arbitrary Banach space $(X, \|\cdot\|)$ on which a compact group G acts as a group of continuous transformations. The problem was to decide whether each complete norm $|\cdot|$ on X which is well-behaved with respect to the transformations of G is necessarily equivalent to the norm $\|\cdot\|$.

On the other hand, it turns out that both the automatic continuity of linear operators which commute with translations and the uniqueness-of-invariant-norm problem are closely related to the problem of determining whether or not there are discontinuous translation invariant linear functionals. This latter question was raised by G. H. Meisters [20] for the spaces $\mathfrak{F}(G)$ and has been much studied (see [21] and [23] for the best general source of information in this area). It is important to note here that Meisters asked in [21, Section 6] whether it is possible to establish a general result of this type: *if the translation-invariant linear functionals on $\mathfrak{F}(G)$ are automatically continuous, then so are the linear operators $\Phi\colon \mathfrak{F}(G) \to \mathfrak{F}(G)$ which commute with translations.* This connects to Johnson's result because the class of covariant linear functionals is just a little different from the class of invariant linear functionals and, in fact, we shall show later that for most of the spaces $\mathfrak{F}(G)$ the automatic continuity of the covariant linear functionals is equivalent to the automatic continuity of the invariant linear functionals.

We are interested in investigating the problem of the automatic continuity of the linear operators $\Phi\colon X \to Y$ which commute with translations in the sense that $\Phi \circ \tau_X(t) = \tau_Y(t) \circ \Phi$ for each t in a given group G, where τ_X and τ_Y are representations of G on the Banach spaces X and Y, respectively.

In Section 2 we review some of the standard facts on representations of groups on Banach spaces, the uniqueness-of-invariant-norm problem, and automatic continuity.

Section 3 deals with locally compact abelian groups. We begin by discussing how Johnson's theorem [16] characterizes the automatic continuity of linear operators which commute with translations in terms of the so-called scalar modules and covariant linear functionals. Then we derive an interesting characterization of the automatic continuity of those operators which are well-behaved with respect to a flow on a Riemannian manifold in terms of the equilibrium points of the dynamical system.

Section 4 is intended as an attempt to obtain the basic principle which characterizes the automatic continuity of linear operators which commute with translations in non-abelian context. We first provide what we consider the appropriate non-abelian version of the scalar modules and covariant linear functionals. For a representation τ of a topological group G on a Banach space X we consider the representations $\pi \otimes \tau$ of G on $H_\pi \otimes X$, where π ranges over the finite-dimensional irreducible unitary representations of G and H_π is the representation space of π. A discontinuous linear operator $\Phi\colon X \to Y$ which commutes with translations is shown to exist in the case when there is, for some π, a non-zero $\pi \otimes \tau$-invariant vector in $H_\pi \otimes Y$ and a discontinuous $\pi \otimes \tau$-invariant linear functional on $H_\pi \otimes X$. We then prove that the converse is even true for some spaces Y and some groups.

In Section 5 we examine how the compactness of the group affects the problem. It turns out that there is a strong dichotomy in the answer to the question depending on whether the group is compact or not. For a compact group we obtain interesting characterizations of the automatic continuity of linear operators which commute with translations, and we illustrate the usefulness of our characterization when considering the Banach spaces $L^p(\Omega)$, where Ω is a compact Hausdorff space equipped with a positive Radon measure on which G acts as a group of measure-preserving transformations. We also discuss the usefulness of Kazhdan's property (T) for investigating the problem.

Let us finally remark that we generalize a number of the main results from [7, 8, 32].

2. Preliminaries

2.1. *Representations.* Let $L(X)$ denote the Banach algebra of all continuous linear operators on a given non-zero complex Banach space X, and let X^* be the topological dual space of X. Let X_* be any linear subspace in X^*. As usual, $\sigma(X, X_*)$ stands for the coarsest topology on X for which each of the functionals in X_* is continuous. For every $T \in L(X)$, $T^* \in L(X^*)$ stands for the adjoint operator of T.

A *representation* of a group G on a Banach space X is a group homomorphism $\tau\colon G \to L(X)$ from G into the group of all invertible elements of $L(X)$. For all $t \in G$ and $x \in X$ we call $\tau(t)x$ the *translate* of x by t. The representation is said to be *bounded* if there exists a constant C such that $\|\tau(t)\| \leq C$ for each $t \in G$ and, in this situation, X becomes a Banach G-module in the sense of [16]. It is worth pointing out that by defining $|x| = \sup_{t\in G} \|\tau(t)x\|$ for each $x \in X$ we obtain a norm on X which is equivalent to $\|\cdot\|$ and, with respect to this new norm, $\tau(G)$ consists of isometries. In the case when G is a topological group, the representation is said to be *strongly continuous* if the map $t \mapsto \tau(t)x$ is continuous for each $x \in X$. Let X_* be a linear subspace of X^*. We call the representation $\sigma(X, X_*)$-*continuous* if the function $t \mapsto \xi(\tau(t)x)$ is continuous on G for all $x \in X$ and $\xi \in X_*$. Of course, every strongly continuous representation is $\sigma(X, X_*)$-continuous.

Let τ_X and τ_Y be representations of a group G on Banach spaces X and Y. Then a linear operator $\Phi\colon X \to Y$ is said to *commute with translations* if

$$\Phi(\tau_X(t)x) = \tau_Y(t)(\Phi(x))$$

for all $x \in X$ and $t \in G$.

In the representation theory of topological groups, the unitary representations play a predominant rôle. As usual, a *unitary representation* of a topological group G is a strongly continuous representation π of G on some Hilbert space H_π such that $\pi(G)$ consists of unitary operators. The unitary representation π is said to be *irreducible* if the only closed subspaces of H_π that are invariant under $\pi(G)$ are the trivial ones, that is, $\{0\}$ and H_π. Two unitary representations π and π' of G are said to be *equivalent* if there is a unitary operator $U : H_\pi \to H_{\pi'}$ such that $U\pi(t) = \pi'(t)U$ for each $t \in G$. We shall denote by $[\pi]$ the class of an irreducible unitary representation π of G. The set of equivalence classes of irreducible unitary representations of G is called the dual space of G and is denoted by $\widehat{G}$. We denote by $\widehat{G}_{\mathrm{FIN}}$ the subset of $\widehat{G}$ consisting of finite-dimensional representations.

Let G be a locally compact group. We denote by $M(G)$ the Banach space of all bounded complex-valued regular Borel measures on G. Recall that $M(G)$ is a Banach $*$-algebra with the product given by convolution $\star$ and involution given by $\mu^*(E) = \overline{\mu(E^{-1})}$ for all $\mu \in M(G)$ and $E \subset G$ measurable. As usual, the Banach algebra $L^1(G)$ of all (Haar-)integrable complex-valued functions (or rather, equivalence classes thereof) on G is identified with the two-sided ideal $M_{\mathrm{ac}}(G)$ of $M(G)$ consisting of measures which are absolutely continuous with respect to Haar measure. Let $M_{\mathrm{f}}(G)$ stand for the subalgebra of $M(G)$ consisting of discrete measures with finite support. For every $t \in G$, δ_t stands for the point mass measure at t. Every unitary representation π of G determines a norm-decreasing unital algebra $*$-homomorphism $\widetilde{\pi}$ from the Banach algebra $M(G)$ into the C^*-algebra $L(H_\pi)$ which is defined by

$$\widetilde{\pi}(\mu) = \int_G \pi(t)\,d\mu(t)$$

for each $\mu \in M(G)$. Furthermore, if $[\pi] \in \widehat{G}_{\mathrm{FIN}}$, then the restriction of $\widetilde{\pi}$ to $M_{\mathrm{f}}(G)$ gives an algebraically irreducible representation of $M_{\mathrm{f}}(G)$ on the finite-dimensional space H_π and so the Jacobson density theorem yields $\widetilde{\pi}(M_{\mathrm{f}}(G)) = L(H_\pi)$.

2.2. *The uniqueness-of-norm problem.* The uniqueness-of-norm problem is a classical topic in automatic continuity theory, which has been mainly developed in the context of Banach algebras. The most important result in this area is the famous theorem by B. E. Johnson [15] that every semisimple Banach algebra (such as $L^1(G)$ for any locally compact group G) carries a unique Banach algebra norm. Recently quite a lot of attention has been paid to the question of whether classical Banach spaces related to a locally compact group G, such as $L^p(G)$ with $1 \le p \le \infty$, carry a unique translation-invariant norm.

Let τ be a representation of a group G on a Banach space $(X, \|\cdot\|)$. We call a complete norm $|\cdot|$ on X *topologically invariant/invariant* if $\tau(G)$ is a subset/bounded subset of $L(X, |\cdot|)$. Following [32], we say that X carries a *unique topologically invariant/unique invariant* norm if every topologically invariant/invariant norm on X is necessarily equivalent to $\|\cdot\|$. We already know [7, 8, 32] that the uniqueness-of-invariant-norm problem is closely related to the existence of non-zero invariant elements and discontinuous invariant linear functionals. Recall that if τ is a representation of a topological group G on a Banach space X, then an element $x \in X$ is said to be τ*-invariant* if $\tau(t)x = x$ for each $t \in G$, and that a linear functional ϕ on X is said to be τ*-invariant* if $\phi(\tau(t)x) = \phi(x)$ for all $x \in X$ and $t \in G$. We shall sometimes use the term *invariant* instead of τ-invariant in the case where no confusion can arise about the meaning of the representation. On the other hand, when considering a left regular representation of G on a Banach space $\mathfrak{F}(G)$ of functions (or Radon measures) on G, it is customary to use the term *translation-invariant* rather than invariant.

Remark 1. It would be tempting to define the invariant norms as those complete norms $|\cdot|$ on X for which $\tau(G)$ is a subgroup of the group $\mathrm{Iso}(X, |\cdot|)$ of isometries of the Banach space $(X, |\cdot|)$. According to this temporary definition, one may ask whether it is now true that $|\cdot| = c\|\cdot\|$ for some constant $c > 0$. This is far from being true. It is known [13] that any complete norm on $L^2(\mathbb{T})$ making rotations continuous is equivalent to the classical norm $\|\cdot\|_2$. Nevertheless, the norm $|\cdot| = \|\cdot\|_2 + \|\cdot\|_1$ on $L^2(\mathbb{T})$ makes rotations isometric and the property $|\cdot| = c\|\cdot\|_2$ fails to be true for each $c > 0$. Thus, even when we restrict our attention to this narrower notion of invariance, the uniqueness of norm has to be understood up to equivalence. On the other hand, it should be taken into account that there is a complete norm $|\cdot|$ on X which is equivalent to $\|\cdot\|$ and such that $(X, |\cdot|)$ has only trivial isometries [14]. Finally, it is easily shown that the subgroups of $\mathrm{Iso}(X, |\cdot|)$ with $|\cdot|$ equivalent to $\|\cdot\|$ are nothing but the bounded subgroups of the group $\mathrm{Inv}(L(X, \|\cdot\|))$ of invertible operators of $L(X, \|\cdot\|)$.

2.3. *Automatic continuity.* When studying the automatic continuity of linear operators which commute with translations we become involved with the so-called *stability lemma*. Recall that the *separating space* $\mathfrak{S}(\Phi)$ of a linear map Φ from a Banach space X into a Banach space Y is defined by

$$\mathfrak{S}(\Phi) = \{y \in Y : \text{there exists } (x_n) \to 0 \text{ in } X \text{ with } (\Phi(x_n)) \to y\}.$$

The separating space is a closed subspace of Y. Moreover, it is an immediate restatement of the closed graph theorem that Φ is continuous if and only if $\mathfrak{S}(\Phi) = \{0\}$. Another standard fact that we shall use is that $\Psi\Phi$ is continuous if and only if $\Psi(\mathfrak{S}(\Phi)) = \{0\}$ whenever Ψ is any continuous linear operator from Y into another Banach space Z. We refer the reader to [4], where the basic properties of the separating space are explored.

Lemma 1 (Stability lemma). *Let X and Y be Banach spaces, and let (S_n) and (T_n) be sequences of continuous linear operators on X and Y, respectively. If Φ is a linear map from X into Y such that $T_n\Phi = \Phi S_n$ for each $n \in \mathbb{N}$, then there exists $N \in \mathbb{N}$ such that*

$$\overline{(T_1 \cdots T_n)(\mathfrak{S}(\Phi))} = \overline{(T_1 \cdots T_N)(\mathfrak{S}(\Phi))}$$

for each $n \geq N$.

In order to put the stability lemma into action we shall need the following result.

Lemma 2. *Let G be a locally compact group. If Σ is a non-empty set of finite-dimensional pairwise non-equivalent irreducible unitary representations of G, then one of the following assertions holds*:

(i) *There exists $\mu \in M_{\mathrm{f}}(G)$ such that the set $\{\pi \in \Sigma : \widetilde{\pi}(\mu) \neq 0\}$ is non-empty and finite.*
(ii) *There exist sequences (π_n) in Σ and (μ_n) in $M_{\mathrm{f}}(G)$ such that*

$$\widetilde{\pi}_n(\mu_n \star \cdots \star \mu_1) \neq 0 \quad \text{and} \quad \widetilde{\pi}_n(\mu_{n+1} \star \cdots \star \mu_1) = 0$$

for each $n \in \mathbb{N}$.

Proof. This result is derived from [32, Lemma 4.2]. However, there is one apparently technical difficulty: all results from [32] are stated for the situation when G is compact. Indeed, just a quick glance through [32, Subsection 4.1] shows that the same reasoning also applies to the case when we are concerned with finite-dimensional representations. Accepting this statement we can prove our lemma as follows.

If Σ is finite, then assertion (i) holds true with $\mu = \delta_e$, where e stands for the identity of G.

We now assume that Σ is infinite. Retaining the notation of [32, Lemma 4.2] we consider the set

$$\Pi = \{\pi \in \Sigma \colon \exists \mu \text{ with } \widetilde{\pi}(\mu) \neq 0 \text{ and } \widetilde{\sigma}(\mu) = 0 \ \forall \sigma \in \Sigma \setminus \{\pi\}\}.$$

It is clear that $\Pi = \emptyset$ in the case where the first assertion in the lemma fails to hold. In such a case, according to the proof of [32, Lemma 4.2], it follows that we can construct sequences (π_n) and (μ_n) satisfying the second assertion. ∎

3. Abelian groups and dynamical systems. In this section we are concerned with a locally compact abelian group G. Then every irreducible unitary representation of G is 1-dimensional and so it corresponds naturally to a group homomorphism from G into $\mathbb{T}$. Thus $\widehat{G}$ is identified with the group of continuous group homomorphisms from G into $\mathbb{T}$ (which is the usual meaning of $\widehat{G}$ when G is abelian). We write G_{DIS} for the group G viewed as a discrete group.

3.1. *Johnson Theorem.* B. E. Johnson introduced in [16] the notion of G-module. Recall that a *Banach G-module*, for a given locally compact abelian group G, is a Banach space X equipped with a mapping $(t, x) \mapsto t \cdot x$ from $G \times X$ into X such that

(i) $x \mapsto t \cdot x$ is linear on X for each $t \in G$;
(ii) $s \cdot (t \cdot x) = (st) \cdot x$ for all $s, t \in G$ and $x \in X$;
(iii) $e \cdot x = x$ for each $x \in X$, where e stands for the identity of G;
(iv) there exists $K > 0$ with $\|t \cdot x\| \leq K\|x\|$ for all $t \in G$ and $x \in X$.

A *G-submodule* of X is a closed linear subspace M of X with $G \cdot M \subset M$. The G-submodule M is said to be *scalar* if for each $t \in G$ there is $\gamma(t) \in \mathbb{C}$ with $t \cdot x = \gamma(t)x$ for all $t \in G$ and $x \in M$. It should be noted that γ is then a group homomorphism from G into $\mathbb{T}$. We call a vector $x \in X$ *γ-scalar* if $t \cdot x = \gamma(t)x$ for each $t \in G$. A linear functional ϕ on X is said to be *γ-covariant* if $\phi(t \cdot x) = \gamma(t)\phi(x)$ for all $x \in X$ and $t \in G$.

Let X and Y be Banach G-modules. Then a linear operator $\Phi \colon X \to Y$ is a *G-module homomorphism* if $\Phi(t \cdot x) = t \cdot \Phi(x)$ for all $t \in G$ and $x \in X$.

THEOREM 3 ([16, Theorem 4.1 and Corollary 4.2]). *Let G be a locally compact abelian group, let X, Y be Banach G-modules, and let $\Phi \colon X \to Y$ be a G-module homomorphism. Then $\mathfrak{S}(\Phi)$ is the direct sum of a finite number of scalar G-submodules of Y. Moreover, the following assertions are equivalent:*

(i) *Every G-module homomorphism $\Phi \colon X \to Y$ is continuous.*
(ii) *Every γ-covariant linear functional on X is continuous whenever $\gamma \in \widehat{G}_{\mathrm{DIS}}$ is such that there exists a non-zero γ-scalar vector in Y.*

It is clear that the notion of Banach G-module is equivalent to that of bounded representation of G on a Banach space. Of course, G-module homomorphisms are just the linear maps which commute with translations, and thus the last part of Theorem 3 can be rephrased as follows.

COROLLARY 4. *Let τ_X and τ_Y be bounded representations of a locally compact abelian group G on Banach spaces X and Y, respectively. Then the following assertions are equivalent*:

(i) *Every linear operator $\Phi: X \to Y$ which commutes with translations is continuous.*
(ii) *Every γ-covariant linear functional on X is continuous whenever $\gamma \in \widehat{G}_{\mathrm{DIS}}$ is such that there exists a non-zero γ-scalar vector in Y.*

COROLLARY 5. *Let τ_X be a bounded representation of a locally compact abelian group G on a Banach space X. Then the following assertions are equivalent*:

(i) *Every linear operator $\Phi: X \to Y$ which commutes with translations is continuous for each bounded representation τ_Y of G on a Banach space Y.*
(ii) *Every γ-covariant linear functional on X is continuous for each $\gamma \in \widehat{G}_{\mathrm{DIS}}$.*

Proof. By Corollary 4, it is clear that (ii) implies (i). On the other hand, if we consider the regular representation of G on $\ell^\infty(G)$, then $\gamma \in \ell^\infty(G)$ is γ-scalar for each $\gamma \in \widehat{G}_{\mathrm{DIS}}$. Thus in the case where (ii) fails Corollary 4 shows that there is a discontinuous linear operator $\Phi: X \to \ell^\infty(G)$ which commutes with translations. ∎

COROLLARY 6. *Let τ_Y be a bounded representation of a locally compact abelian group G on a Banach space Y. Then the following assertions are equivalent*:

(i) *Every linear operator $\Phi: X \to Y$ which commutes with translations is continuous for each bounded representation τ_X of G on a Banach space X.*
(ii) *The only γ-scalar vector of Y is $\{0\}$ for each $\gamma \in \widehat{G}_{\mathrm{DIS}}$.*

Proof. By Corollary 4, (ii) implies (i).

Let τ be the regular representation of $G\times\mathbb{Z}$ on $\ell^\infty(G\times\mathbb{Z})$. Then [36, Theorem 6] gives a discontinuous τ-invariant linear functional ϕ on $\ell^\infty(G \times \mathbb{Z})$. We now consider $\ell^\infty(G \times \mathbb{Z})$ equipped with the representation τ' of G on $\ell^\infty(G \times \mathbb{Z})$ given by $\tau'(t)x = \tau(t, 0)x$ for all $t \in G$ and $x \in \ell^\infty(G \times \mathbb{Z})$. Let $\gamma \in \widehat{G}_{\mathrm{DIS}}$. For every $x \in \ell^\infty(G \times \mathbb{Z})$ we define a function $\gamma \cdot x \in \ell^\infty(G \times \mathbb{Z})$ by $(\gamma \cdot x)(t, k) = \gamma(t)x(t, k)$ for each $(t, k) \in G \times \mathbb{Z}$. Our goal is to show that the linear functional ψ on $\ell^\infty(G \times \mathbb{Z})$ defined by $\psi(x) = \phi(\overline{\gamma} \cdot x)$ for each

$x \in \ell^\infty(G\times\mathbb{Z})$ is γ-covariant and discontinuous. If $t \in G$ and $x \in \ell^\infty(G\times\mathbb{Z})$, then

$$\psi(\tau'(t)x) = \phi(\overline{\gamma}\cdot\tau(t,0)x) = \phi(\gamma(t)\tau(t,0)(\overline{\gamma}\cdot x)) = \gamma(t)\phi(\overline{\gamma}\cdot x) = \gamma(t)\psi(x).$$

On the other hand, if (x_n) is a sequence in $\ell^\infty(G\times\mathbb{Z})$ with $\lim x_n = 0$ and $\lim\phi(x_n) \neq 0$, then it is easily seen that $\lim\gamma\cdot x_n = 0$ and that $\psi(\gamma\cdot x_n) = \phi(x_n)$, which gives the discontinuity of ψ. Hence, if (ii) fails, then Corollary 4 shows that there is a discontinuous linear operator $\Phi\colon \ell^\infty(G\times\mathbb{Z}) \to Y$ which commutes with translations. ∎

Corollary 7. *Let τ be a bounded representation of a locally compact abelian group G on a Banach space X. Then the following assertions are equivalent*:

(i) *X carries a unique invariant norm.*
(ii) *Every γ-covariant linear functional on X is continuous whenever $\gamma \in \widehat{G}_{\mathrm{DIS}}$ is such that there exists a non-zero γ-scalar vector in X.*

Proof. Assume that (ii) holds, and let $|\cdot|$ be an invariant norm on X. Then we can apply Corollary 4 to infer that the identity map from $(X,\|\cdot\|)$ onto $(X,|\cdot|)$ is continuous. This clearly implies that both norms are equivalent.

We now assume that $x_0 \in X$ is a non-zero γ-scalar vector and that ϕ is a discontinuous γ-covariant linear functional on X. Then we take $\alpha \in \mathbb{C}\setminus\{0,\phi(x_0)\}$. It is easily seen that the map $x \mapsto \alpha x - \phi(x)x_0$ is a discontinuous invertible linear operator from X onto itself, and therefore we can define a complete norm $|\cdot|$ on X which is not equivalent to $\|\cdot\|$ by $|x| = \|\alpha x - \phi(x)x_0\|$ for each $x \in X$. Since $\tau(t)x_0 = t\cdot x_0 = \gamma(t)x_0$ for each $t \in G$, we have

$$\begin{aligned}|\tau(t)x| &= \|\alpha\tau(t)x - \phi(\tau(t)x)x_0\| = \|\alpha\tau(t)x - \phi(x)\gamma(t)x_0\|\\ &= \|\tau(t)(\alpha x - \phi(x)x_0)\| \le K\|\alpha x - \phi(x)x_0\| = K|x|\end{aligned}$$

for all $x \in X$ and $t \in G$, which shows that $|\cdot|$ is an invariant norm on X. ∎

Remark 2. Under the assumptions of Theorem 3 and Corollaries 4–7 a mild continuity hypothesis on the map $t \mapsto t\cdot y$ $(y \in Y)$ would imply that scalar submodules of Y correspond to continuous group homomorphisms, and then all of those results would apply with $\widehat{G}_{\mathrm{DIS}}$ replaced by $\widehat{G}$.

Remark 3. It is worth pointing out that the automatic continuity of the invariant linear functionals on X is far from being sufficient to characterize either the continuity of linear operators from X which commute with translations or the uniqueness of invariant norm on X for all Banach G-modules X. Indeed, let G be a locally compact abelian group, let $\gamma \in \widehat{G}\setminus\{1\}$, and consider an infinite-dimensional Banach space X. We endow X with the representation τ given by $\tau(t)x = \gamma(t)x$ for all $t \in G$ and $x \in X$. It is easily checked that the element 0 is the only invariant element of X, and that the functional 0 is the only invariant linear functional on X. Nevertheless it is clear that every function of X is γ-scalar and that every linear functional on X is γ-covariant. Since X is infinite-dimensional there is a discontinuous linear functional on X, and so Corollary 7 shows that X does not carry a unique invariant norm. In particular the identity map from X into itself, when endowed with some suitable norm, is a discontinuous linear operator which commutes with translations.

3.2. *Dynamical systems.* We now apply the results in the preceding section to show the significance of the dynamical systems in automatic continuity theory. In the following, $(\mathfrak{M}, \langle\cdot,\cdot\rangle)$ denotes a finite-dimensional Riemannian manifold, ω stands for its associated Riemannian density, and μ stands for the induced measure on $\mathfrak{M}$.

Suppose that φ is a diffeomorphism from $\mathfrak{M}$ onto itself. Then there exists a unique smooth function $J\varphi$ on $\mathfrak{M}$ such that the pull-back $\varphi^*\omega$ of ω by φ is given by $\varphi^*\omega = (J\varphi)\omega$. The function $J\varphi$ is the so-called *Jacobian determinant* of φ. If $f \in L^1(\mathfrak{M})$, then $(f\circ\varphi)|J\varphi| \in L^1(\mathfrak{M})$ and

$$\int_{\mathfrak{M}} f(p)\, d\mu(p) = \int_{\mathfrak{M}} f(\varphi(p))|(J\varphi)(p)|\, d\mu(p).$$

The evolution of a physical system may be described by a *flow* on a Riemannian manifold $\mathfrak{M}$. This is a smooth map $\mathcal{F}\colon \mathbb{R}\times\mathfrak{M}\to\mathfrak{M}$ such that

$$\mathcal{F}(s,\mathcal{F}(t,p)) = \mathcal{F}(s+t,p) \quad \text{and} \quad \mathcal{F}(0,p) = p$$

for all $s,t\in\mathbb{R}$ and $p\in\mathfrak{M}$. We thus obtain a one-parameter group of diffeomorphisms from $\mathfrak{M}$ onto itself given by $(\mathcal{F}_t)_{t\in\mathbb{R}}$, where $\mathcal{F}_t$ is defined by $\mathcal{F}_t(p) = \mathcal{F}(t,p)$ for all $t\in\mathbb{R}$ and $p\in\mathfrak{M}$. It is usually not $\mathcal{F}$ that is given, but rather the law of motion. In other words, differential equations are given that we must solve to find the flow. These equations of motion have the form

$$\gamma'(t) = \mathfrak{X}(\gamma(t)), \qquad \gamma(0) = p,$$

where $\mathfrak{X}$ is a smooth vector field on $\mathfrak{M}$; $\mathfrak{X}$ is called the *infinitesimal generator* of $\mathcal{F}$. The curve $t\mapsto\mathcal{F}(t,p)$ is nothing else than the solution of the preceding equation for each $p\in\mathfrak{M}$. Let us also recall that the vector field $\mathfrak{X}$ is said to be *conservative* if there exists a smooth function $U\colon\mathfrak{M}\to\mathbb{R}$, a *potential*, such that

$$\langle u,\mathfrak{X}(p)\rangle_p = -dU(p)(u)$$

for all $p\in\mathfrak{M}$ and $u\in T_p\mathfrak{M}$. The vector field $\mathfrak{X}$ is said to be *incompressible* or *divergence free* if $\operatorname{div}\mathfrak{X} = 0$, where the divergence $\operatorname{div}\mathfrak{X}$ of $\mathfrak{X}$ is the only smooth function on $\mathfrak{M}$ with the property that the Lie derivative of ω along $\mathfrak{X}$ equals $(\operatorname{div}\mathfrak{X})\omega$. This is equivalent to the property that $J\mathcal{F}_t = 1$.

In preparation for the following results, we point out that $J\mathcal{F}_t > 0$ for each $t\in\mathbb{R}$. Indeed, since $\mathcal{F}_t$ is a diffeomorphism, $J\mathcal{F}_t$ is nowhere zero; since it is continuous in t and equals 1 at $t = 0$, it is positive for each $t \in \mathbb{R}$. The flow $\mathcal{F}$ on the manifold $\mathfrak{M}$ induces a representation τ of $\mathbb{R}$ on the Banach space $L^p(\mathfrak{M})$ given by

$$\tau(t)f = (f \circ \mathcal{F}_{-t})(J\mathcal{F}_{-t})^{1/p}$$

for each $1 \leq p \leq \infty$ (with the usual convention $1/\infty = 0$). If $\mathfrak{X}$ is incompressible, then $(\mathcal{F}_t)_{t\in\mathbb{R}}$ is a one-parameter group of measure-preserving diffeomorphisms.

We refer the reader to [1] for the details about integration and dynamical systems on manifolds.

LEMMA 8. *Let $(\mathfrak{M}, \langle\cdot,\cdot\rangle)$ be a finite-dimensional Riemannian manifold, let $\mathcal{F}$ be a flow on $\mathfrak{M}$ whose infinitesimal generator $\mathfrak{X}$ is incompressible, and let $1 \leq p \leq \infty$. Suppose that the measure of the set $\{p \in \mathfrak{M} \colon \mathfrak{X}(p) = 0\}$ of equilibrium points is different from zero. Then there are an invariant non-zero function in $L^p(\mathfrak{M})$ and a discontinuous invariant linear functional on $L^p(\mathfrak{M})$. Accordingly, there is a discontinuous invertible linear operator from $L^p(\mathfrak{M})$ onto itself which commutes with translations.*

Proof. We begin by showing the existence of the invariant function. In the case where $p = \infty$ it suffices to take the function $\mathbf{1}$ (note that neither the incompressibility nor the condition $\mu(\{p \in \mathfrak{M} \colon \mathfrak{X}(p) = 0\}) > 0$ are required at all in this case). If $1 \leq p < \infty$, then we can take $f \in L^p(\mathfrak{M}) \setminus \{0\}$ which vanishes on $\{p \in \mathfrak{M} \colon \mathfrak{X}(p) \neq 0\}$. Since $\mathcal{F}_t(p) = p$ for each $t \in \mathbb{R}$ and each equilibrium point p, it follows immediately that $f \circ \mathcal{F}_t = f$ for each $t \in \mathbb{R}$.

We now prove the existence of the functional. Consider the closed subset $\mathfrak{N}$ of $\mathfrak{M}$ defined by $\mathfrak{N} = \{p \in \mathfrak{M} \colon \mathfrak{X}(p) = 0\}$ equipped with the measure induced by $\mathfrak{M}$. Let $\psi \colon L^p(\mathfrak{N}) \to \mathbb{C}$ be a discontinuous linear functional. Then the linear functional $f \mapsto \psi(f_{|\mathfrak{N}})$ on $L^p(\mathfrak{M})$ is easily seen to be discontinuous and invariant.

By Remark 6, there is a discontinuous invertible linear operator from $L^p(\mu)$ onto itself which commutes with translations. ∎

We now turn to the case when the measure of the set of equilibrium points of the flow is zero.

LEMMA 9. *Let $(\mathfrak{M}, \langle\cdot,\cdot\rangle)$ be a finite-dimensional Riemannian manifold, and let $\mathcal{F}$ be a flow on $\mathfrak{M}$ whose infinitesimal generator $\mathfrak{X}$ is conservative. Suppose that the measure of the set $\{p \in \mathfrak{M} \colon \mathfrak{X}(p) = 0\}$ of equilibrium points is zero. Then the only invariant function in $L^1(\mathfrak{M})$ is zero.*

Proof. Let U be a potential function for the field $\mathfrak{X}$.

We begin by observing that, for every $p \in \mathfrak{M}$ with $\mathfrak{X}(p) \neq 0$, the function $\theta_p \colon \mathbb{R} \to \mathbb{R}$ defined by

$$\theta_p(t) = U(\mathcal{F}(t,p))$$

for each $t \in \mathbb{R}$ is strictly decreasing. To this end we compute the derivative of θ_p, which is obviously given by

$$\theta_p'(t) = dU(\mathcal{F}(t,p))(\mathfrak{X}(\mathcal{F}(t,p))) = -\langle \mathfrak{X}(\mathcal{F}(t,p)), \mathfrak{X}(\mathcal{F}(t,p))\rangle_{\mathcal{F}(t,p)} < 0$$

for each $t \in \mathbb{R}$, where the last inequality holds because $\mathfrak{X}(\mathcal{F}(t,p)) \neq 0$ for each $t \in \mathbb{R}$, as is easy to check.

For every real number c and every (non-empty) interval $I \subset \mathbb{R}$ we define the set

$$\begin{aligned}\mathfrak{M}(c,I) &= \{\mathcal{F}(t,p) \colon t \in I,\, p \in \mathfrak{M},\, \mathfrak{X}(p) \neq 0,\, U(p) = c\} \\ &= \{q \in \mathfrak{M} \colon \mathfrak{X}(q) \neq 0,\, U(\mathcal{F}(-t,q)) = c \text{ for some } t \in I\}.\end{aligned}$$

It should be pointed out that $\mathfrak{M}(c,I) = \emptyset$ if and only if c is such that there is no $p \in \mathfrak{M}$ with $\mathfrak{X}(p) \neq 0$ and $U(p) = c$.

We proceed to show that $\mathfrak{M}(c,I)$ is open in the case where I is an open interval. Let $t_0 \in I$ and $p_0 \in \mathfrak{M}$ with $\mathfrak{X}(p_0) \neq 0$ and $U(p_0) = c$. We choose $\delta > 0$ such that $[t_0 - \delta, t_0 + \delta] \subset I$, and we observe that

$$U(\mathcal{F}(-t_0 + \delta, \mathcal{F}(t_0,p_0))) = U(\mathcal{F}(\delta,p_0)) < U(\mathcal{F}(0,p_0)) = U(p_0) = c$$

and

$$U(\mathcal{F}(-t_0 - \delta, \mathcal{F}(t_0,p_0))) = U(\mathcal{F}(-\delta,p_0)) > U(\mathcal{F}(0,p_0)) = U(p_0) = c.$$

We now consider the set

$$\mathcal{V} = \{p \in \mathfrak{M} \colon \mathfrak{X}(p) \neq 0,\, U(\mathcal{F}(-t_0+\delta,p)) < c,\, U(\mathcal{F}(-t_0-\delta,p)) > c\},$$

which is clearly open and contains the point $\mathcal{F}(t_0,p_0)$. We next prove $\mathcal{V} \subset \mathfrak{M}(c,I)$. If $q \in \mathcal{V}$, then there is $\eta \in \,]-\delta,\delta[$ such that $U(\mathcal{F}(-t_0+\eta,q)) = c$. Since $t_0 - \eta \in I$ and $\mathcal{F}(t_0-\eta, \mathcal{F}(-t_0+\eta,q)) = \mathcal{F}(0,q) = q$, we conclude that $q \in \mathfrak{M}(c,I)$.

If $c, d \in \mathbb{R}$, then

$$\mathfrak{M}(c,\{d\}) = \mathcal{F}_d(\{p \in \mathfrak{M} \colon \mathfrak{X}(p) \neq 0\} \cap \{p \in \mathfrak{M} \colon U(p) = c\}),$$

which is a Borel set since $\{p \in \mathfrak{M} \colon \mathfrak{X}(p) \neq 0\}$ is open, $\{p \in \mathfrak{M} \colon U(p) = c\}$ is closed, and $\mathcal{F}_d$ is a diffeomorphism. From this fact together with what has previously been proved, it may be concluded that $\mathfrak{M}(c,I)$ is a Borel subset of $\mathfrak{M}$ for all $c \in \mathbb{R}$ and intervals I.

Choose $c \in \mathbb{R}$. The task is now to show that $\mathfrak{M}(c,I) \cap \mathfrak{M}(c,J) = \emptyset$ in the case where I and J are disjoint intervals. Indeed, if $p \in \mathfrak{M}(c,I) \cap \mathfrak{M}(c,J)$, then $U(\mathcal{F}(-a,p)) = U(\mathcal{F}(-b,p)) = c$ with $a \in I$ and $b \in J$. This contradicts the fact that $a \neq b$ (because $I \cap J = \emptyset$) and the map $t \mapsto U(\mathcal{F}(t,p))$ is strictly decreasing.

Our next objective is to prove that

$$\{p \in \mathfrak{M} \colon \mathfrak{X}(p) \neq 0\} = \bigcup_{r \in \mathbb{Q}} \mathfrak{M}(r,\mathbb{R}). \tag{1}$$

Let $p \in \mathfrak{M}$ with $\mathfrak{X}(p) \neq 0$. Then $\{U(\mathcal{F}(t,p))\colon t \in \mathbb{R}\}$ is an open interval and so $U(\mathcal{F}(t,p)) = r \in \mathbb{Q}$ for some $t \in \mathbb{R}$. Hence $p = \mathcal{F}(-t, \mathcal{F}(t,p)) \in \mathfrak{M}(r, \mathbb{R})$.

We now assume that $f \in L^1(\mathfrak{M})$ is an invariant function. This means that $(f \circ \mathcal{F}_t) J\mathcal{F}_t = f$ for each $t \in \mathbb{R}$. For every $c \in \mathbb{R}$, we have

$$\begin{aligned}\infty > \int_{\mathfrak{M}} |f(p)|\, d\mu(p) &\geq \int_{\mathfrak{M}(c,\mathbb{R})} |f(p)|\, d\mu(p) = \sum_{k\in\mathbb{Z}} \int_{\mathfrak{M}(c,]k,k+1])} |f(p)|\, d\mu(p)\\ &= \sum_{k\in\mathbb{Z}} \int_{\mathcal{F}_{-k}(\mathfrak{M}(c,]k,k+1]))} |f(\mathcal{F}_k(p))| (J\mathcal{F}_k)(p)\, d\mu(p)\\ &= \sum_{k\in\mathbb{Z}} \int_{\mathfrak{M}(c,]0,1])} |f(\mathcal{F}_k(p))(J\mathcal{F}_k)(p)|\, d\mu(p) = \sum_{k\in\mathbb{Z}} \int_{\mathfrak{M}(c,]0,1])} |f(q)|^p\, d\mu(q),\end{aligned}$$

which clearly implies that $\int_{\mathfrak{M}(c,]0,1])} |f|\, d\mu = 0$ and hence that $\int_{\mathfrak{M}(c,\mathbb{R})} |f|\, d\mu = 0$. From (1) we deduce that

$$\int_{\{p\in\mathfrak{M}\colon\, \mathfrak{X}(p)\neq 0\}} |f|\, d\mu = 0.$$

This implies that $f = 0$ almost everywhere on $\{p \in \mathfrak{M}\colon \mathfrak{X}(p) \neq 0\}$. Accordingly, since $\mu(\{p \in \mathfrak{M}\colon \mathfrak{X}(p) = 0\}) = 0$, we have $f = 0$ almost everywhere on $\mathfrak{M}$. ∎

COROLLARY 10. *Let $(\mathfrak{M}, \langle\cdot,\cdot\rangle)$ be a finite-dimensional Riemannian manifold, and let $\mathcal{F}$ be a flow on $\mathfrak{M}$ whose infinitesimal generator $\mathfrak{X}$ is conservative. Suppose that the set $\{p \in \mathfrak{M}\colon \mathfrak{X}(p) = 0\}$ of equilibrium points has measure zero. Then every linear operator $\Phi\colon X \to L^p(\mathfrak{M})$ such that $\Phi(\tau(t)x) = (\Phi(x) \circ \mathcal{F}_{-t})(J\mathcal{F}_{-t})^{1/p}$ for all $t \in \mathbb{R}$ and $x \in X$ is continuous for each bounded representation τ of $\mathbb{R}$ on a Banach space X and each $1 \leq p < \infty$.*

Proof. Assume that $f \in L^p(\mathfrak{M})$ is a γ-scalar function for some $1 \leq p < \infty$ and some $\gamma \in \widehat{\mathbb{R}}_{\mathrm{DIS}}$. We check at once that $|f|^p$ is an invariant function of $L^1(\mathfrak{M})$. Lemma 9 now shows that $f = 0$ and Corollary 6 completes the proof. ∎

COROLLARY 11. *Let $(\mathfrak{M}, \langle\cdot,\cdot\rangle)$ be a finite-dimensional Riemannian manifold, and let $\mathcal{F}$ be a flow on $\mathfrak{M}$ whose infinitesimal generator $\mathfrak{X}$ is incompressible and conservative. Then $L^p(\mathfrak{M})$ carries a unique invariant (under the flow) norm if and only if the set $\{p \in \mathfrak{M}\colon \mathfrak{X}(p) = 0\}$ has measure zero.* ∎

REMARK 4. It is worth pointing out that Lemma 9 and Corollaries 10 and 11 may fail in the case when the conservativity is removed. Consider in the 2-dimensional Euclidean space the flow $\mathcal{F}\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathcal{F}(t,x,y) = (x\cos t - y\sin t, x\sin t + y\cos t)$$

for each $(t,x,y) \in \mathbb{R} \times \mathbb{R}^2$. Note that the diffeomorphism $\mathcal{F}_t$ is the rotation of angle t for each $t \in \mathbb{R}$ and that the infinitesimal generator of $\mathcal{F}$ is given by $\mathfrak{X}(x,y) = (-y,x)$ for each $(x,y) \in \mathbb{R}^2$. The field $\mathfrak{X}$ is incompressible but it is not conservative since $\operatorname{curl}(\mathfrak{X}) = 2 \neq 0$. Clearly, the only equilibrium point is the origin. Nevertheless the function $(x,y) \mapsto e^{-(x^2+y^2)}$ lies in $L^p(\mathbb{R}^2)$ for each $1 \leq p \leq \infty$ and is invariant under the translations induced by the flow $\mathcal{F}$. In fact, every radial function in $L^p(\mathbb{R}^2)$ is invariant. Thus Lemma 9 fails and, on account of Corollary 6, Corollary 10 also fails. Furthermore, since the subspace of $L^p(\mathbb{R}^2)$ consisting of radial functions is infinite-dimensional and G_{DIS} is amenable, Corollary 19 below yields a discontinuous invariant linear functional on $L^p(\mathbb{R}^2)$ for each $1 < p \leq \infty$, and Corollary 7 now shows that $L^p(\mathbb{R}^2)$ does not carry a unique invariant norm. Thus Corollary 11 also fails.

4. The basic principle. For a topological group G acting on Banach spaces X and Y, it is probably possible to characterize the automatic continuity of the linear operators $\Phi\colon X \to Y$ which commute with translations in terms of the appropriate non-abelian versions of both the scalar vectors of Y and the covariant linear functionals on X. This section is intended as an attempt to find this basic principle that underlies the theory of automatic continuity of linear operators which commute with translations.

4.1. *A non-abelian version of the covariant linear functionals.* Let τ be a representation of a topological group G on a Banach space X. In most of the remainder of this paper we shall be concerned with the representations $\pi \otimes \tau$ of G on the Banach space $H_\pi \otimes X$ defined through

$$(\pi \otimes \tau)(t)(u \otimes x) = \pi(t)u \otimes \tau(t)x$$

for all $u \in H_\pi$, $x \in X$, and $t \in G$, where $[\pi]$ ranges through $\widehat{G}_{\mathrm{FIN}}$. We will also be dealing with the representation of G on $H_\pi \otimes X$ which, by abuse of notation, we continue to write τ, defined through

$$\tau(t)(u \otimes x) = u \otimes \tau(t)x$$

for all $u \in H_\pi$, $x \in X$, and $t \in G$. Note that $H_\pi \otimes X$ can be identified with the space X^n, where $n = \dim H_\pi$. The important point to note here is that $H_\pi \otimes X$ becomes a Banach space with respect to any cross norm and that all of these are equivalent. From now on, we shall consider $H_\pi \otimes X$ as a Banach space without specifying any concrete norm.

Remark 5. Let τ be a representation of a locally compact abelian group G on a Banach space X, and let $\gamma \in \widehat{G}$. It is a simple matter to check that the representation $\gamma \otimes \tau$ can be naturally identified with the representation τ_γ of G on X given by $\tau_\gamma(t)x = \overline{\gamma(t)}\,\tau(t)x$ for all $x \in X$ and $t \in G$ in such a way that the $\gamma \otimes \tau$-invariant elements and the $\gamma \otimes \tau$-invariant linear functionals can be thought of as the τ_γ-invariant elements and the τ_γ-invariant linear functionals, respectively, which are just the γ-scalar elements and the γ-covariant linear functionals, respectively. Thus, the $\pi \otimes \tau$-invariant linear forms with $[\pi] \in \widehat{G}_{\mathrm{FIN}}$ may be thought of as a non-abelian version of the covariant linear forms.

Our task is now to prove that, for most Banach spaces of Radon measures on a locally compact group G, including spaces such as $M(G)$, $L^p(G)$ with $1 \le p \le \infty$, $C_{\mathrm{b}}(G)$, and $C_0(G)$, the $\pi \otimes \tau$-invariance is equivalent in some way to the τ-invariance.

Let G be a locally compact group acting on the left on a locally compact Hausdorff space Ω. This means that there is a continuous map $(t,\omega) \mapsto t\omega$ from $G \times \Omega$ into Ω such that $e\omega = \omega$ and $s(t\omega) = (st)\omega$ for all $s,t \in G$ and $\omega \in \Omega$. For every function $f\colon \Omega \to \mathbb{C}$ the translates are defined by

$$(\tau(t)f)(\omega) = f(t^{-1}\omega)$$

for all $t \in G$ and $\omega \in \Omega$, while for every Radon measure μ on Ω the translates are defined by

$$\int_\Omega f\, d(\tau(t)\mu) = \int_\Omega \tau(t^{-1})f\, d\mu$$

for all $t \in G$ and $f \in C_{\mathrm{c}}(\Omega)$, where $C_{\mathrm{c}}(\Omega)$ is the space of continuous functions on G with compact support. The case where $\Omega = G$ is of special interest; in that case we obtain the so-called *left regular representations* of G on a number of Banach spaces such as $C_0(G)$, $L^p(G)$ with $1 \le p \le \infty$, and $M(G)$.

Following [27], a *Banach space on a locally compact group G* is a translation-invariant linear space $\mathfrak{F}(G)$ of Radon measures on G which is a Banach space whose topology is stronger than the weak topology $\sigma(\mathfrak{F}(G), C_{\mathrm{c}}(G))$.

LEMMA 12. *Let τ be a representation of a topological group G on a Banach space X, and let $[\pi] \in \widehat{G}_{\mathrm{FIN}}$. Let $(e_i)_{i=1}^n$ be an orthonormal basis of H_π and, for every $i \in \{1,\dots,n\}$, let $P_i\colon H_\pi \otimes X \to X$ be the continuous linear operator defined through*

$$P_i(u \otimes x) = \langle u, e_i\rangle x$$

for all $u \in H_\pi$ and $x \in X$. Then

$$\zeta = \sum_{i=1}^n e_i \otimes P_i(\zeta),$$

$$P_i(\tau(t)\zeta) = \tau(t)P_i(\zeta) \quad (i = 1,\dots,n),$$

and

$$P_i((\pi \otimes \tau)(t)\zeta) = \sum_{j=1}^n \overline{\langle e_i, \pi(t)e_j\rangle}\, \tau(t)P_j(\zeta)$$

for all $\zeta \in H_\pi \otimes X$ and $t \in G$.

Proof. The first and second identities are obvious. On the other hand, for $u \in H_\pi$, $x \in X$, and $i \in \{1, \dots, n\}$, we have

$$\begin{aligned} P_i((\pi \otimes \tau)(t)(u \otimes x)) &= P_i(\pi(t)u \otimes \tau(t)x) = \langle \pi(t)u, e_i\rangle \tau(t)x \\ &= \langle u, \pi(t^{-1})e_i\rangle \tau(t)x = \Big\langle u, \sum_{j=1}^n \langle \pi(t^{-1})e_i, e_j\rangle e_j \Big\rangle \tau(t)x \\ &= \sum_{j=1}^n \overline{\langle e_i, \pi(t)e_j\rangle} \langle u, e_j\rangle \tau(t)x \\ &= \sum_{j=1}^n \overline{\langle e_i, \pi(t)e_j\rangle} \tau(t)(\langle u, e_j\rangle x) \\ &= \sum_{j=1}^n \overline{\langle e_i, \pi(t)e_j\rangle} \tau(t)P_j(u \otimes x), \end{aligned}$$

and so the third identity follows. ∎

LEMMA 13. *Let $\mathfrak{F}(G)$ be a Banach space on a locally compact group G, let $[\pi] \in \widehat{G}_{\mathrm{FIN}}$, let $(e_i)_{i=1}^n$ be an orthonormal basis of H_π, and let $(P_i)_{i=1}^n$ be the operators given in Lemma 12. Suppose that $\langle \pi(\cdot)u, v\rangle \mathfrak{F}(G) \subset \mathfrak{F}(G)$ for all $u, v \in H_\pi$, and define the map $\Phi_\pi \colon H_\pi \otimes \mathfrak{F}(G) \to H_\pi \otimes \mathfrak{F}(G)$ by*

$$\Phi_\pi(\zeta) = \sum_{i=1}^n \sum_{j=1}^n e_i \otimes \langle \pi(\cdot)e_j, e_i\rangle P_j(\zeta)$$

for all $\zeta \in H_\pi \otimes \mathfrak{F}(G)$, where $\langle \pi(\cdot)u, v\rangle$ stands for the function $t \mapsto \langle \pi(t)u, v\rangle$ on G for all $u, v \in H_\pi$. Then Φ_π is an invertible continuous linear operator such that

$$\Phi_\pi(\tau(t)\zeta) = (\pi \otimes \tau)(t)\Phi_\pi(\zeta)$$

for all $\zeta \in H_\pi \otimes \mathfrak{F}(G)$.

Proof. It is easily seen that the linear operator $\mu \mapsto \langle \pi(\cdot)u, v\rangle \mu$ from $\mathfrak{F}(G)$ into itself has a closed graph, and therefore is continuous, for all $u, v \in H_\pi$. This clearly forces the continuity of Φ_π.

For $t \in G$ and $\zeta \in H_\pi \otimes \mathfrak{F}(G)$, we have

$$\begin{aligned} (\pi \otimes \tau)(t)\Phi_\pi(\zeta) &= \sum_{i=1}^n \sum_{j=1}^n \pi(t)e_i \otimes \tau(t)(\langle \pi(\cdot)e_j, e_i\rangle P_j(\zeta)) \\ &= \sum_{i=1}^n \sum_{j=1}^n \pi(t)e_i \otimes \langle \pi(t^{-1}\cdot)e_j, e_i\rangle \tau(t)P_j(\zeta) \\ &= \sum_{i=1}^n \sum_{j=1}^n \pi(t)e_i \otimes \langle \pi(\cdot)e_j, \pi(t)e_i\rangle P_j(\tau(t)\zeta) \end{aligned}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \pi(t)e_i \otimes \Big\langle \pi(\cdot)e_j, \sum_{k=1}^{n}\langle \pi(t)e_i, e_k\rangle e_k \Big\rangle P_j(\tau(t)\zeta)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \langle e_k, \pi(t)e_i\rangle \pi(t)e_i \otimes \langle \pi(\cdot)e_j, e_k\rangle P_j(\tau(t)\zeta)$$
$$= \sum_{j=1}^{n}\sum_{k=1}^{n} e_k \otimes \langle \pi(\cdot)e_j, e_k\rangle P_j(\tau(t)\zeta) = \Phi_\pi(\tau(t)\zeta),$$

where $\sum_{i=1}^{n}\langle e_k, \pi(t)e_i\rangle \pi(t)e_i = e_k$ $(k = 1,\dots,n)$ because $(\pi(t)e_i)_{i=1}^{n}$ is an orthonormal basis of H_π.

We now define the linear operator $\Phi_{\pi^*} : H_\pi \otimes \mathfrak{F}(G) \to H_\pi \otimes \mathfrak{F}(G)$ by

$$\Phi_{\pi^*}(\zeta) = \sum_{i=1}^{n}\sum_{j=1}^{n} e_i \otimes \langle \pi(\cdot)^* e_j, e_i\rangle P_j(\zeta),$$

where $\langle \pi(\cdot)^* e_j, e_i\rangle$ stands for the function $t \mapsto \langle \pi(t)^* e_j, e_i\rangle$ on G for all $i, j = 1,\dots,n$. For every $\zeta \in H_\pi \otimes \mathfrak{F}(G)$, we have

$$(\Phi_\pi \circ \Phi_{\pi^*})(\zeta) = \sum_{i=1}^{n}\sum_{j=1}^{n} e_i \otimes \langle \pi(\cdot)e_j, e_i\rangle P_j\Big(\sum_{k=1}^{n}\sum_{l=1}^{n} e_k \otimes \langle \pi(\cdot)^* e_l, e_k\rangle P_l(\zeta)\Big)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n} e_i \otimes \langle \pi(\cdot)e_j, e_i\rangle\langle \pi(\cdot)^* e_l, e_j\rangle P_l(\zeta)$$
$$= \sum_{i=1}^{n}\sum_{l=1}^{n} e_i \otimes \Big(\sum_{j=1}^{n}\langle e_l, \pi(\cdot)e_j\rangle \overline{\langle e_i, \pi(\cdot)e_j\rangle}\Big) P_l(\zeta)$$
$$= \sum_{i=1}^{n}\sum_{l=1}^{n} e_i \otimes \langle e_l, e_i\rangle P_l(\zeta) = \sum_{i=1}^{n} e_i \otimes P_i(\zeta) = \zeta,$$

and likewise we see that $(\Phi_{\pi^*} \circ \Phi_\pi)(\zeta) = \zeta$. ∎

COROLLARY 14. *Let $\mathfrak{F}(G)$ be a Banach space on a locally compact group G and let $[\pi] \in \widehat{G}_{\mathrm{FIN}}$. Suppose that $\langle \pi(\cdot)u, v\rangle \mathfrak{F}(G) \subset \mathfrak{F}(G)$ for all $u, v \in H_\pi$. Then the following assertions hold*:

(i) *There is a non-zero τ-invariant vector in $\mathfrak{F}(G)$ if and only if there is a non-zero $\pi \otimes \tau$-invariant vector in $H_\pi \otimes \mathfrak{F}(G)$.*
(ii) *There is a discontinuous τ-invariant linear functional on $\mathfrak{F}(G)$ if and only if there is a discontinuous $\pi \otimes \tau$-invariant linear functional on $H_\pi \otimes \mathfrak{F}(G)$.*

Proof. Consider an orthonormal basis $(e_i)_{i=1}^{n}$ of H_π, the operators $(P_i)_{i=1}^{n}$ given in Lemma 12, and the operator Φ_π given in Lemma 13.

(i) Let μ be a non-zero τ-invariant vector of $\mathfrak{F}(G)$, and choose a vector $u \in H_\pi \setminus \{0\}$. Lemma 13 shows that

$$\Phi_\pi(u \otimes \mu) = \Phi_\pi(u \otimes \tau(t)\mu) = (\pi \otimes \tau)(t)\Phi_\pi(u \otimes \mu)$$

for each $t \in G$, which implies that $\Phi_\pi(u \otimes \mu)$ is a non-zero $\pi \otimes \tau$-invariant vector of $H_\pi \otimes \mathfrak{F}(G)$. Conversely, let ζ be a non-zero $\pi \otimes \tau$-invariant vector of $H_\pi \otimes \mathfrak{F}(G)$. Then Lemma 13 gives

$$\Phi_\pi(\tau(t)\Phi_\pi^{-1}(\zeta)) = (\pi \otimes \tau)(t)\zeta = \zeta,$$

which shows that $\Phi_\pi^{-1}(\zeta)$ is τ-invariant and therefore $P_i(\Phi_\pi^{-1}(\zeta))$ is a τ-invariant vector of $\mathfrak{F}(G)$ for each $i = 1, \ldots, n$. Clearly one of them is different from zero.

(ii) We now assume that ψ is a discontinuous τ-invariant linear functional on $\mathfrak{F}(G)$. Then we lift ψ to a discontinuous τ-invariant linear functional ψ_π on $H_\pi \otimes \mathfrak{F}(G)$ by $\psi_\pi(\zeta) = \sum_{i=1}^n \psi(P_i(\zeta))$ for each $\zeta \in H_\pi \otimes \mathfrak{F}(G)$. Lemma 13 now shows that $\psi_\pi \circ \Phi_\pi^{-1}$ is a discontinuous $\pi \otimes \tau$-invariant linear functional on $H_\pi \otimes \mathfrak{F}(G)$.

We finally assume that ϕ is a discontinuous $\pi \otimes \tau$-invariant linear functional on $H_\pi \otimes \mathfrak{F}(G)$. Then Lemma 13 clearly forces that $\phi \circ \Phi_\pi$ is a discontinuous τ-invariant linear functional on $H_\pi \otimes \mathfrak{F}(G)$. For every $i \in \{1, \ldots, n\}$, define $Q_i \colon \mathfrak{F}(G) \to H_\pi \otimes \mathfrak{F}(G)$ by $Q_i(\mu) = e_i \otimes \mu$ for each $\mu \in \mathfrak{F}(G)$. It is clear that

$$\phi \circ \Phi_\pi = \sum_{i=1}^n \phi \circ \Phi_\pi \circ Q_i \circ P_i,$$

and therefore $\phi \circ \Phi_\pi \circ Q_i$ is discontinuous for some $i \in \{1, \ldots, n\}$. On the other hand, it is obvious that $\phi \circ \Phi_\pi \circ Q_i$ is a τ-invariant linear functional on $\mathfrak{F}(G)$. ∎

In order to investigate the uniqueness-of-invariant-norm problem for a Banach space X on which a compact group G acts, we introduced in [32] the auxiliary spaces $L(H_\pi) \otimes X$, where π ranges through the irreducible unitary representations of G. We also introduced the auxiliary representation τ^π of G on $L(H_\pi) \otimes X$ through

$$\tau^\pi(t)(T \otimes x) = (\pi(t) \circ T) \otimes (\tau(t)x)$$

for all $T \in L(H_\pi)$, $x \in X$, and $t \in G$. It is obvious that the representation τ^π makes sense for an arbitrary topological group G and each $[\pi] \in \widehat{G}_{\mathrm{FIN}}$.

Our next goal is to relate the notions of τ^π-invariant vectors and functionals to those considered in the present paper of $\pi \otimes \tau$-invariant vectors and functionals, which seems to be more elementary. To this end, for every

$v \in H_\pi$ we define continuous linear operators

$$(2) \qquad \Psi_v\colon L(H_\pi) \otimes X \to H_\pi \otimes X, \quad \Psi_v(T \otimes x) = (Tv) \otimes x \qquad (T \in L(H_\pi),\ x \in X),$$

and

$$(3) \qquad \Psi^v\colon H_\pi \otimes X \to L(H_\pi) \otimes X, \quad \Psi^v(u \otimes x) = (u \otimes v) \otimes x \qquad (T \in L(H_\pi),\ x \in X),$$

where, as usual, $u \otimes v \in L(H_\pi)$ is defined by $(u \otimes v)w = \langle w, v\rangle u$ for each $w \in H_\pi$. We also recall that $L(H_\pi) \otimes X$ becomes a Banach $L(H_\pi)$-module with the operations given through

$$S \cdot (T \otimes x) = (S \circ T) \otimes x = (S \otimes x) \cdot T$$

for all $S, T \in L(H_\pi)$ and $x \in X$.

The proof of the following result is immediate, and it is left to the reader.

LEMMA 15. *Let G be a topological group, and let τ be a representation of G on a Banach space X. Let $[\pi] \in \widehat{G}_{\mathrm{FIN}}$, and let $u, v \in H_\pi$. Then the following assertions hold*:

(i) *Both Ψ_u and Ψ^u commute with translations.*
(ii) *$(\Psi_u \circ \Psi^v)(\zeta) = \langle u, v\rangle \zeta$ and $(\Psi^v \circ \Psi_u)(\vartheta) = \vartheta \cdot (u \otimes v)$ for all $\zeta \in H_\pi \otimes X$ and $\vartheta \in L(H_\pi) \otimes X$.* ∎

LEMMA 16. *Let G be a topological group, and let τ be a representation of G on a Banach space X. If $[\pi] \in \widehat{G}_{\mathrm{FIN}}$, then the following assertions are equivalent*:

(i) *Every $\pi \otimes \tau$-invariant linear functional on $H_\pi \otimes X$ is continuous.*
(ii) *Every τ^π-invariant linear functional on $L(H_\pi) \otimes X$ is continuous.*

Proof. Assume that (i) holds, and let $(e_i)_{i=1}^n$ be an orthonormal basis of H_π. If $\psi\colon L(H_\pi) \otimes X \to \mathbb{C}$ is τ^π-invariant, then by Lemma 15(i), $\psi \circ \Psi^{e_i}$ is a $\pi \otimes \tau$-invariant linear functional for each $i = 1, \ldots, n$. On account of Lemma 15(ii), we have $\psi = \sum_{i=1}^n \psi \circ \Psi^{e_i} \circ \Psi_{e_i}$, which is continuous.

We now assume that (ii) holds, and let $\phi\colon H_\pi \otimes X \to \mathbb{C}$ be a $\pi \otimes \tau$-invariant linear functional. Set $u \in H_\pi$ with $\|u\| = 1$. Then the linear functional $\phi \circ \Psi_u$ is τ^π-invariant (Lemma 15(i)) and therefore it is continuous, and hence so is $\phi \circ \Psi_u \circ \Psi^u = \phi$ (Lemma 15(ii)). ∎

4.2. *Discontinuous operators arising from invariant vectors and invariant functionals.* The following result provides some evidence for the existence of a principle as suggested at the beginning of this section.

THEOREM 17. *Let G be a topological group, and let τ_X and τ_Y be representations of G on Banach spaces X and Y, respectively. Suppose that there exist a discontinuous $\pi \otimes \tau_X$-invariant linear functional and a non-zero*

$\pi\otimes\tau_Y$-invariant vector for some $[\pi]\in\widehat{G}_{\mathrm{FIN}}$. Then there exists a discontinuous linear operator Φ from X into Y such that Φ commutes with translations.

Proof. Let $(e_i)_{i=1}^n$ be an orthonormal basis of H_π, and let $(P_i)_{i=1}^n$ be the operators given in Lemma 12.

Choose a non-zero $\pi\otimes\tau_Y$-invariant vector $\zeta\in H_\pi\otimes Y$ and a discontinuous $\pi\otimes\tau_X$-invariant linear functional $\phi\colon H_\pi\otimes X\to\mathbb{C}$. We define a linear operator $\Phi\colon X\to Y$ by

$$\Phi(x)=\sum_{i=1}^n\phi(e_i\otimes x)P_i(\zeta)$$

for each $x\in X$.

We first prove that Φ commutes with translations. Let $x\in X$ and $t\in G$. We have

$$\begin{aligned}\Phi(\tau_X(t)x)&=\sum_{i=1}^n\phi(e_i\otimes\tau_X(t)x)P_i(\zeta)\\&=\sum_{i=1}^n\phi((\pi\otimes\tau_X)(t)(\pi(t^{-1})e_i)\otimes x)P_i(\zeta)\\&=\sum_{i=1}^n\phi((\pi(t^{-1})e_i)\otimes x)P_i(\zeta)\\&=\sum_{i=1}^n\phi\Big(\sum_{j=1}^n\langle\pi(t^{-1})e_i,e_j\rangle e_j\otimes x\Big)P_i(\zeta)\\&=\sum_{j=1}^n\phi(e_j\otimes x)\sum_{i=1}^n\langle e_i,\pi(t)e_j\rangle P_i(\zeta).\end{aligned}\tag{4}$$

Since ζ is $\pi\otimes\tau_Y$-invariant, on account of Lemma 12, we have

$$P_i(\zeta)=P_i((\pi\otimes\tau_Y)(t)\zeta)=\sum_{j=1}^n\overline{\langle e_i,\pi(t)e_j\rangle}\,\tau_Y(t)P_j(\zeta)\tag{5}$$

for each $t\in G$. Identity (4) now becomes

$$\begin{aligned}\Phi(\tau_X(t)x)&=\sum_{j=1}^n\phi(e_j\otimes x)\sum_{i=1}^n\langle e_i,\pi(t)e_j\rangle\sum_{k=1}^n\overline{\langle e_i,\pi(t)e_k\rangle}\,\tau_Y(t)P_k(\zeta)\\&=\tau_Y(t)\Big(\sum_{j=1}^n\phi(e_j\otimes x)\sum_{k=1}^n\Big(\sum_{i=1}^n\langle e_i,\pi(t)e_j\rangle\,\overline{\langle e_i,\pi(t)e_k\rangle}\Big)P_k(\zeta)\Big)\\&=\tau_Y(t)\Big(\sum_{j=1}^n\phi(e_j\otimes x)\sum_{k=1}^n\langle\pi(t)e_k,\pi(t)e_j\rangle P_k(\zeta)\Big)\end{aligned}$$

$$= \tau_Y(t)\Big(\sum_{j=1}^{n}\phi(e_j\otimes x)\sum_{k=1}^{n}\langle e_k,e_j\rangle P_k(\zeta)\Big)$$
$$= \tau_Y(t)\Big(\sum_{j=1}^{n}\phi(e_j\otimes x)P_j(\zeta)\Big) = \tau_Y(t)\Phi(x).$$

We now proceed to show that Φ is discontinuous. Since ϕ is discontinuous, there exists $i_0\in\{1,\dots,n\}$ such that the linear functional $x\mapsto\phi(e_{i_0}\otimes x)$ on X is discontinuous. Choose $l\in\{1,\dots,n\}$. Since $\widetilde{\pi}(M_{\mathrm{f}}(G))=L(H_\pi)$, there exist $K\in\mathbb{N}$, $t_1,\dots,t_K\in G$, and $\alpha_1,\dots,\alpha_K\in\mathbb{C}$ such that

$$\Big(\sum_{k=1}^{K}\alpha_k\pi(t_k)\Big)e_i = \begin{cases} 0 & \text{if } i\neq l,\\ e_{i_0} & \text{if } i=l.\end{cases}$$

From identity (4), we deduce that

$$\begin{aligned}(6)\qquad \Phi\Big(\sum_{k=1}^{K}\alpha_k\tau_X(t_k^{-1})x\Big) &= \sum_{k=1}^{K}\alpha_k\tau_Y(t_k^{-1})\Phi(x)\\ &= \sum_{k=1}^{K}\alpha_k\Big(\sum_{j=1}^{n}\phi(e_j\otimes x)\sum_{i=1}^{n}\langle\pi(t_k)e_i,e_j\rangle P_i(\zeta)\Big)\\ &= \sum_{j=1}^{n}\sum_{i=1}^{n}\phi(e_j\otimes x)\Big\langle\Big(\sum_{k=1}^{K}\alpha_k\pi(t_k)\Big)e_i,e_j\Big\rangle P_i(\zeta)\\ &= \sum_{j=1}^{n}\phi(e_j\otimes x)\langle e_{i_0},e_j\rangle P_l(\zeta) = \phi(e_{i_0}\otimes x)P_l(\zeta)\end{aligned}$$

for each $x\in X$. From the discontinuity of the functional $x\mapsto\phi(e_{i_0}\otimes x)$ we deduce that there exists a sequence (x_n) in X with $\lim x_n=0$ and $\lim\phi(e_{i_0}\otimes x_n)=1$. Since $\lim\sum_{k=1}^{K}\alpha_k\tau_X(t_k^{-1})x_n=0$, (6) now clearly forces $P_l(\zeta)\in\mathfrak{S}(\Phi)$. We thus obtain

$$(7)\qquad P_j(\zeta)\in\mathfrak{S}(\Phi)\qquad (j=1,\dots,n).$$

As $\zeta\neq 0$, we have $P_{j_0}(\zeta)\neq 0$ for some some $j_0\in\{1,\dots,n\}$, and this entails that $\mathfrak{S}(\Phi)\neq\{0\}$, and therefore that Φ is discontinuous. ∎

Remark 6. It is worth pointing out that, under the assumptions of Theorem 17 with $(X,\tau_X)=(Y,\tau_Y)$, there exists a discontinuous linear invertible operator from X onto itself which commutes with translations. Indeed, let Φ be the linear operator given in that theorem, and take $\alpha\in\mathbb{C}\setminus(\mathrm{sp}(\Phi_{|\Phi(X)})\cup\{0\})$, where $\mathrm{sp}(\Phi_{|\Phi(X)})$ stands for the spectrum of the restriction of Φ to $\Phi(X)$ (it should be pointed out that $\dim\Phi(X)<\infty$). In the same manner as at the end of the proof of [32, Theorem 3.1] we can check that $\alpha I-\Phi$ is an invertible linear operator from X onto itself, and $\alpha I-\Phi$ is obviously discontinuous and commutes with translations. Consequently, the map $|\cdot|$ defined on X by $|x|=\|\alpha x-\Phi(x)\|$ for each $x\in X$ is an invariant norm on X which is not equivalent to $\|\cdot\|$. Note that this is similar to the reasoning in the proof of Corollary 7. We thus generalize [7, Theorem 5], [8, Theorem 6.1], and [32, Theorem 3.1].

4.3. *Amenable groups.* Our next objective is to investigate the size of the separating space of a linear operator which commute with translations corresponding to an amenable group in terms of an invariant mean. Until further notice we suppose that G is an amenable locally compact group. Recall that abelian groups as well as compact groups are examples of amenable groups.

THEOREM 18. *Let G be an amenable locally compact group, and let M be an invariant mean on $L^\infty(G)$. Let X be a Banach space, let X_* be a linear subspace of X^*, and let τ be a bounded $\sigma(X,X_*)$-continuous representation of G on X. Suppose that one of the following conditions holds*:

(a) $X_*=X^*$;
(b) *X is a dual Banach space and X_* is a predual of X.*

Then there exists a continuous linear operator $P_M\colon X\to(X_*)^*$ defined by

$$\xi(P_Mx)=M_t(\xi(\tau(t^{-1})x))$$

for all $x\in X$ and $\xi\in X_*$ (where M_t means that M is applied to the corresponding function of t), which satisfies the properties:

(i) $P_M\tau(t)x=P_Mx\ \ \forall x\in X,\ \forall t\in G$;
(ii) *$P_Mx=x$ whenever $x\in X$ is τ-invariant*;

and, *in case* (b),

(iii) $P_M^2=P_M$;
(iv) $P_MX=\{x\in X\colon \tau(t)x=x\ \forall t\in G\}$.

Proof. Let $x\in X$. For every $\xi\in X_*$, the function $t\mapsto\xi(\tau(t^{-1})x)$ is continuous and bounded. This shows that $M_t(\xi(\tau(t^{-1})x))$ makes sense. Moreover, it is easily checked that the linear functional $\xi\mapsto M_t(\xi(\tau(t^{-1})x))$ on X_* is continuous, and so P_Mx is well defined. It is a simple matter to show that P_M is a continuous linear operator from X into $(X_*)^*$.

The task is now to prove that $P_M\tau(s)x=P_Mx$ for all $x\in X$ and $s\in G$. Let $\xi\in X_*$. On account of the left invariance of M, we have

$$\begin{aligned}\xi(P_M\tau(s)x)&=M_t(\xi(\tau(t^{-1})\tau(s)x))=M_t(\xi(\tau((s^{-1}t)^{-1})x))\\&=M_t(\xi(\tau(t^{-1})x))=\xi(P_Mx).\end{aligned}$$

If $x\in X$ is invariant, then $\xi(P_Mx)=M_t(\xi(\tau(t^{-1})x))=M_t(\xi(x))=\xi(x)$ for each $\xi\in X_*$, since $M(\mathbf{1})=1$. Therefore $P_Mx=x$.

We now turn to the case (b). We first claim that $\tau(s)P_M x = P_M x$ for all $x \in X$ and $s \in G$. Indeed, on account of the right invariance of M, we have

$$\begin{aligned}\xi(\tau(s)P_M x) &= (\xi \circ \tau(s))(P_M x) = M_t((\xi \circ \tau(s))(\tau(t^{-1})x)) \\ &= M_t(\xi(\tau((ts^{-1})^{-1})x)) = M_t(\xi(\tau(t^{-1})x)) = \xi(P_M x).\end{aligned}$$

It is clear that the preceding property together with (i) and (ii) yields (iii) and (iv). ∎

COROLLARY 19. *Let G be an amenable locally compact group, let X be a dual Banach space, let X_* be a predual of X, and let τ be a bounded $\sigma(X, X_*)$-continuous representation of G on X. If the subspace of X consisting of the τ-invariant vectors is infinite-dimensional, then there exists a discontinuous τ-invariant functional.*

Proof. Let M be an invariant mean on $L^\infty(G)$, let P_M be the projection given in Theorem 18, and let Y be the space of τ-invariant vectors of X. Since $\dim Y = \infty$, it follows that there exists a discontinuous linear functional ψ on Y. The linear functional $\phi = \psi \circ P_M$ is easily seen to be discontinuous and, according to Theorem 18, we have

$$\phi(\tau(t)x) = \psi(P_M \tau(t)x) = \psi(P_M x) = \phi(x)$$

for all $t \in G$ and $x \in X$, so that ϕ is τ-invariant. ∎

COROLLARY 20. *Let G be an amenable locally compact group, let X be a dual Banach space and X_* be a predual of X, let Y be a Banach space, let τ_X be a bounded $\sigma(X, X_*)$-continuous representation of G on X, and let τ_Y be a representation of G on Y. If the subspace of $H_\pi \otimes X$ consisting of the $\pi \otimes \tau_X$-invariant vectors is infinite-dimensional and there is a non-zero $\pi \otimes \tau_Y$-invariant vector for some $[\pi] \in \widehat{G}_{\mathrm{FIN}}$, then there exists a discontinuous linear operator from X into Y which commutes with translations.*

Proof. It is clear that $H_\pi \otimes X$ is a dual Banach space, and Corollary 19 now shows that there exists a discontinuous $\pi \otimes \tau_X$-invariant linear functional. Theorem 17 completes the proof. ∎

LEMMA 21. *Let G be an amenable locally compact group, and let M be an invariant mean on $L^\infty(G)$. Let X be a Banach space, and let τ_X be a representation of G on X. Let τ_Y be a bounded $\sigma(Y, Y_*)$-continuous representation of G on a Banach space Y (where either $Y_* = Y^*$ or Y is a dual Banach space and Y_* is a predual of Y). Suppose that one of the following assertions holds*:

(i) *The only τ_Y-invariant vector of Y is zero.*
(ii) *Every τ_X-invariant functional on X is continuous.*

Then for every linear operator $\Phi\colon X \to Y$ which commutes with translations we have $P_M(\mathfrak{S}(\Phi)) = \{0\}$, where P_M is the operator given in Theorem 18 *for Y.*

Proof. First observe that in case (i) we have $P_M(Y) = \{0\}$.

We now consider the case where (ii) holds. On account of Theorem 18, for every $\phi \in (Y_*)^{**}$, the linear functional $\phi \circ P_M \circ \Phi$ is τ_X-invariant, and so it is continuous. This implies that $\phi(P_M(\mathfrak{S}(\Phi))) = \{0\}$ for each $\phi \in (Y_*)^{**}$ and hence $P_M(\mathfrak{S}(\Phi)) = \{0\}$. ∎

THEOREM 22. *Let G be an amenable locally compact group*, *let M be an invariant mean on $L^\infty(G)$, and let $[\pi] \in \widehat{G}_{\mathrm{FIN}}$. Let X be a Banach space, and let τ_X be a representation of G on X. Let τ_Y be a bounded $\sigma(Y, Y_*)$-continuous representation of G on a Banach space Y (where either $Y_* = Y^*$ or Y is a dual Banach space and Y_* is a predual of Y). Then the following assertions are equivalent*:

- (i) *For every linear operator $\Phi\colon X \to Y$ which commutes with translations*, *we have $M_t(\langle u, \pi(t)v\rangle \xi(\tau_Y(t^{-1})y)) = 0$ for all $u, v \in H_\pi$, $\xi \in Y_*$, and $y \in \mathfrak{S}(\Phi)$.*
- (ii) *Every $\pi \otimes \tau_X$-invariant linear functional on $H_\pi \otimes X$ is continuous whenever there exists a non-zero $\pi \otimes \tau_Y$-invariant vector in $H_\pi \otimes Y$.*

Proof. Assume that assertion (ii) holds. Let $\Phi\colon X \to Y$ be a linear operator which commutes with translations. We define the linear operator $\Phi_\pi\colon H_\pi \otimes X \to H_\pi \otimes Y$ by $\Phi_\pi(u \otimes x) = u \otimes \Phi(x)$ for all $u \in H_\pi$ and $x \in X$. We observe that

$$\begin{aligned}\Phi_\pi((\pi \otimes \tau_X)(t)(u \otimes x)) &= \Phi_\pi(\pi(t)u \otimes \tau_X(t)x) \\ &= \pi(t)u \otimes \Phi(\tau_X(t)x) = (\pi \otimes \tau_Y)(t)\Phi_\pi(u \otimes x)\end{aligned}$$

for all $u \in H_\pi$ and $x \in X$, which shows that Φ_π commutes with translations. On the other hand, we see at once that $\mathfrak{S}(\Phi_\pi) = H_\pi \otimes \mathfrak{S}(\Phi)$. On account of Lemma 21, we have $P_{M,\pi}(H_\pi \otimes \mathfrak{S}(\Phi)) = \{0\}$, where $P_{M,\pi}$ stands for the operator given by Theorem 18 for $H_\pi \otimes Y$. It should be pointed out that both $H_\pi \otimes Y$ and the representation $\pi \otimes \tau_Y$ of G on $H_\pi \otimes Y$ satisfy the requirements in both Theorem 18 and Lemma 21. This gives

$$0 = M_t(\zeta((\pi \otimes \tau_Y)(t^{-1})(u \otimes y))) = M_t(\zeta(\pi(t^{-1})u \otimes \tau_Y(t^{-1})y))$$

for all $\zeta \in H_\pi \otimes Y_*$, $u \in H_\pi$, and $y \in \mathfrak{S}(\Phi)$. By taking $\zeta = v \otimes \xi$ with $v \in H_\pi$ and $\xi \in Y_*$, this now becomes

$$M_t(\langle \pi(t^{-1})u, v\rangle \xi(\tau_Y(t^{-1})y)) = 0,$$

which yields assertion (i).

Our next concern is the case where there exist a discontinuous $\pi \otimes \tau_X$-invariant linear functional on $H_\pi \otimes X$ and a non-zero $\pi \otimes \tau_Y$-invariant vector $\zeta \in H_\pi \otimes Y$. We now consider the discontinuous linear operator $\Phi\colon X \to Y$

as defined in the proof of Theorem 17, and we retain the notation of that proof. Set $\xi \in Y_*$ with $\xi(P_{j_0}(\zeta)) \neq 0$. According to (5), we have

$$\tau_Y(t^{-1})P_i(\zeta) = \sum_{j=1}^{n} \overline{\langle e_i, \pi(t)e_j\rangle}\, P_j(\zeta)$$

for all $t \in G$ and $i = 1, \ldots, n$. We thus obtain

$$\begin{aligned}\sum_{i=1}^{n} M_t(\langle e_i, \pi(t)e_{j_0}\rangle \xi(\tau_Y(t^{-1})P_i(\zeta))) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} M_t(\langle e_i, \pi(t)e_{j_0}\rangle\, \overline{\langle e_i, \pi(t)e_j\rangle})\xi(P_j(\zeta)) \\ &= \sum_{j=1}^{n} M_t\Big(\sum_{i=1}^{n}\langle e_i, \pi(t)e_{j_0}\rangle\overline{\langle e_i, \pi(t)e_j\rangle}\Big)\xi(P_j(\zeta)) \\ &= \sum_{j=1}^{n} M_t(\langle \pi(t)e_j, \pi(t)e_{j_0}\rangle)\xi(P_j(\zeta)) \\ &= \sum_{j=1}^{n} M_t(\langle e_j, e_{j_0}\rangle)\xi(P_j(\zeta)) = \xi(P_{j_0}(\zeta)) \neq 0.\end{aligned}$$

Therefore there exists $i \in \{1, \ldots, n\}$ such that

$$M_t(\langle e_i, \pi(t)e_{j_0}\rangle \xi(\tau_Y(t^{-1})P_i(\zeta))) \neq 0.$$

On the other hand, on account of (7), we have $P_i(\zeta) \in \mathfrak{S}(\Phi)$ for each $i = 1, \ldots, n$. Consequently, the first assertion in the theorem does not hold. ∎

5. The rôle of compactness. It is important to note here that the automatic continuity of linear operators which commute with translations for a locally compact abelian group G strongly depends on whether G is compact or not. As a matter of fact it turns out from Corollary 6 that every linear operator $\Phi\colon X \to L^1(G)$ which commutes with translations is continuous for each bounded representation τ_X of G on a Banach space X if and only if G is non-compact. Indeed, since the function $\mathbf{1}$ is obviously translation-invariant and lies in $L^1(G)$ in the case where G is compact, Corollary 6 shows the existence of a discontinuous linear operator $\Phi\colon X \to L^1(G)$ which commutes with translations for some X. Conversely, if there exists such a Φ, then there is a non-zero γ-scalar function $f \in L^1(G)$ for some $\gamma \in \widehat{G}$. On the other hand, it is easily seen that $|f|$ is translation-invariant and therefore it is constant, which clearly entails that G is compact.

One of the purposes of this section is to examine whether such a strong dichotomy occurs in non-abelian context.

5.1. *Compact groups.* In the following, G stands for a compact group. It is well known that every irreducible unitary representation of G is finite-dimensional (see [11, Theorem 22.13] for example) and so $\widehat{G}_{\mathrm{FIN}} = \widehat{G}$. On the other hand, if τ is a $\sigma(X, X_*)$-continuous representation of G on a Banach space X, where either $X_* = X^*$ or X is a dual Banach space and X_* is a predual of X, then τ is bounded. Indeed, the continuity of the map $t \mapsto \xi(\tau(t)x)$ from G into X together with the compactness of G imply that the map is bounded for all $x \in X$ and $\xi \in X_*$. The uniform boundedness theorem now shows that the subset $\{\tau(t) \colon t \in G\}$ of $L(X)$ is bounded.

Theorem 23. *Let G be a compact group. Let X be a Banach space, and let τ_X be a representation of G on X. Let τ_Y be a $\sigma(Y, Y_*)$-continuous representation of G on a Banach space Y (where either $Y_* = Y^*$ or Y is a dual Banach space and Y_* is a predual of Y). Then the following assertions are equivalent*:

(i) *Every linear operator $\Phi \colon X \to Y$ which commutes with translations is continuous.*
(ii) *Every $\pi \otimes \tau_X$-invariant linear functional on $H_\pi \otimes X$ is continuous whenever $[\pi] \in \widehat{G}$ is such that there exists a non-zero $\pi \otimes \tau_Y$-invariant vector in $H_\pi \otimes Y$.*

Proof. First observe that integration with respect to the normalized Haar measure on G obviously defines an invariant mean on $L^\infty(G)$.

On account of Theorem 22 together with the preceding observation, assertion (ii) is equivalent to the property

$$\int_G \langle u, \pi(t)v\rangle \xi(\tau_Y(t^{-1})y)\, dt = 0 \qquad ([\pi] \in \widehat{G},\ u, v \in H_\pi,\ \xi \in Y_*,\ y \in \mathfrak{S}(\Phi)) \tag{8}$$

for every linear operator $\Phi \colon X \to Y$ which commutes with translations. We now note that (8) is equivalent to

$$\int_G p(t)\xi(\tau_Y(t^{-1})y)\, dt = 0 \qquad \forall \text{ trigonometric polynomial } p,\ \xi \in Y_*,\ y \in \mathfrak{S}(\Phi). \tag{9}$$

Since the trigonometric polynomials are dense in $C(G)$ with the uniform norm [12, Theorem 27.39], it may be concluded that (9) is equivalent to

$$\xi(\tau(t^{-1})y) = 0 \qquad (t \in G,\ \xi \in Y_*,\ y \in \mathfrak{S}(\Phi)). \tag{10}$$

By taking t to be the identity in G in (10) we see that (10) becomes $\xi(\mathfrak{S}(\Phi)) = \{0\}$ for each $\xi \in Y_*$. Of course, this latter condition is equivalent to $\mathfrak{S}(\Phi) = \{0\}$ and therefore to the continuity of Φ. ∎

Corollary 24. *Let G be a compact group, and let τ_X be a representation of G on a Banach space X. Then the following assertions are equivalent*:

(i) *Every linear operator* $\Phi\colon X \to Y$ *which commutes with translations is continuous for each* $\sigma(Y, Y_*)$*-continuous representation* τ_Y *of* G *on a Banach space* Y (*where either* $Y_* = Y^*$ *or* Y *is a dual Banach space and* Y_* *is a predual of* Y).

(ii) *Every linear* $\Phi\colon X \to \mathfrak{F}(G)$ *which commutes with translations is continuous for each Banach space* $\mathfrak{F}(G)$ *on* G.

(iii) *Every* $\pi \otimes \tau_X$*-invariant linear functional on* $H_\pi \otimes X$ *is continuous for each* $[\pi] \in \widehat{G}$.

Proof. Assume that (i) holds, and let $\Phi\colon X \to \mathfrak{F}(G)$ be a linear operator which commutes with translations for some Banach space $\mathfrak{F}(G)$ on G. The inclusion map $J\colon \mathfrak{F}(G) \to M(G)$ is a continuous linear operator which commutes with translations. Therefore $J \circ \Phi$ is continuous, and so $J(\mathfrak{S}(\Phi)) = \{0\}$, which clearly forces $\mathfrak{S}(\Phi) = \{0\}$. This shows that (i) implies (ii).

We now assume that there exists a discontinuous $\pi \otimes \tau_X$-invariant linear functional on $H_\pi \otimes X$ for some $[\pi] \in \widehat{G}$. Take $\mathfrak{F}(G) = L^1(G)$. Since G is compact, the function $\mathbf{1}$ is in $\mathfrak{F}(G)$ and obviously is translation-invariant. According to Corollary 14, there is a non-zero $\pi \otimes \tau$-invariant vector in $H_\pi \otimes \mathfrak{F}(G)$. By Theorem 17, there exists a discontinuous linear operator $\Phi\colon X \to \mathfrak{F}(G)$ which commutes with translations. Thus (ii) implies (iii).

Finally, by Theorem 23, it is clear that (iii) implies (i). ∎

Corollary 25. *Let* G *be a compact group, and let* τ_Y *be a* $\sigma(Y, Y_*)$*-continuous representation of* G *on a Banach space* Y (*where either* $Y_* = Y^*$ *or* Y *is a dual Banach space and* Y_* *is a predual of* Y). *Then the following assertions are equivalent*:

(i) *Every linear operator* $\Phi\colon X \to Y$ *which commutes with translations is continuous for each representation* τ_X *of* G *on a Banach space* X.

(ii) *The only* $\pi \otimes \tau_Y$*-invariant vector of* $H_\pi \otimes Y$ *is* $\{0\}$ *for each* $[\pi] \in \widehat{G}$.

Proof. From Theorem 23, we see that (ii) implies (i).

We now assume that there exists a non-zero $\pi \otimes \tau_Y$-invariant vector of $H_\pi \otimes Y$ for some $[\pi] \in \widehat{G}$. Set $K = G \times \mathbb{T}$ and $X = L^1(K)$ equipped with the left regular representation τ of K. Since K is an infinite compact group, [27, Theorem 1] yields a discontinuous τ-invariant linear functional on X. Corollary 14 now yields a discontinuous $\pi \otimes \tau$-invariant linear functional ϕ on $H_\pi \otimes X$. We now consider the representation τ_X of G on X defined by

$$\tau_X(t)x = \tau(t, 1)x$$

for all $x \in X$ and $t \in G$. It is obvious that ϕ is τ_X-invariant. Hence we can apply Theorem 23 to find a discontinuous linear operator $\Phi\colon X \to Y$ which commutes with translations. ∎

On account of Corollary 14, Theorem 23 and Corollary 24 have the following two straightforward consequences. In what follows, $\mathfrak{T}(G)$ denotes the algebra of all trigonometric polynomials on the compact group G.

COROLLARY 26. *Let G be a compact group. Let $\mathfrak{F}(G)$ be a Banach space on G such that $\mathfrak{T}(G)\mathfrak{F}(G) \subset \mathfrak{F}(G)$ and let τ_Y be a $\sigma(Y, Y_*)$-continuous representation of G on a Banach space Y (where either $Y_* = Y^*$ or Y is a dual Banach space and Y_* is a predual of Y). Then the following assertions are equivalent*:

(i) *Every linear operator $\Phi\colon \mathfrak{F}(G) \to Y$ which commutes with translations is continuous.*
(ii) *Every translation-invariant linear functional on $\mathfrak{F}(G)$ is continuous whenever there exists a non-zero $\pi \otimes \tau_Y$-invariant vector in $H_\pi \otimes Y$ for some $[\pi] \in \widehat{G}$.* ■

COROLLARY 27. *Let G be a compact group, and let $\mathfrak{F}(G)$ be a Banach space on G such that $\mathfrak{T}(G)\mathfrak{F}(G) \subset \mathfrak{F}(G)$. Then the following assertions are equivalent*:

(i) *Every linear operator $\Phi\colon \mathfrak{F}(G) \to Y$ which commutes with translations is continuous for each $\sigma(Y, Y_*)$-continuous representation τ_Y of G on a Banach space Y (where either $Y_* = Y^*$ or Y is a dual Banach space and Y_* is a predual of Y).*
(ii) *Every linear operator $\Phi\colon \mathfrak{F}(G) \to \mathfrak{F}'(G)$ which commutes with translations is continuous for each Banach space $\mathfrak{F}'(G)$ on G.*
(iii) *Every translation-invariant linear functional on $\mathfrak{F}(G)$ is continuous.* ■

THEOREM 28. *Let G be a compact group and let $1 < p < \infty$. Then the following assertions are equivalent*:

(i) *Every linear operator $\Phi\colon L^p(\Omega) \to Y$ which commutes with translations is continuous for each compact Hausdorff space Ω equipped with a positive Radon measure on which G acts by measure-preserving transformations with Ω/G finite and for each $\sigma(Y, Y_*)$-continuous representation τ_Y of G on a Banach space Y (where either $Y_* = Y^*$ or Y is a dual Banach space and Y_* is a predual of Y).*
(ii) *Every linear operator $\Phi\colon L^p(\Omega) \to \mathfrak{F}(G)$ which commutes with translations is continuous for each compact Hausdorff space Ω equipped with a positive Radon measure on which G acts by measure-preserving transformations with Ω/G finite and for each Banach space $\mathfrak{F}(G)$ on G.*
(iii) *$L^p(\Omega)$ carries a unique invariant norm for each compact Hausdorff space Ω equipped with a positive Radon measure on which G acts by measure-preserving transformations with Ω/G finite.*

(iv) *$L^p(G)$ carries a unique invariant norm.*
(v) *Every invariant linear functional on $L^p(G)$ is continuous.*
(vi) *Every invariant linear functional on $L^p(G)$ is a multiple of the Haar integral.*

Proof. On account of [32, Theorem 5.3], assertions (iii)–(vi) are already known to be equivalent.

We can prove that (i) implies (ii) in just the same way as we proved that (i) implies (ii) in Corollary 24.

Assume that there exists a discontinuous invariant linear functional on $L^p(G)$. Since the function $\mathbf{1}$ is obviously an invariant function in $L^p(G)$, Theorem 17 now shows that there is a discontinuous linear operator from $L^p(G)$ into itself which commutes with translations. This proves that (ii) implies (v).

We finally proceed to show that (v) implies (i). Let Ω be a compact Hausdorff space equipped with a positive Radon measure on which G acts as a group of measure-preserving transformations with Ω/G finite, and let τ_Y be a $\sigma(Y, Y_*)$-continuous representation of G on a Banach space Y (where either $Y_* = Y^*$ or Y is a dual Banach space and Y_* is a predual of Y). On account of Corollary 24, we only need to show that every $\pi \otimes \tau$-invariant linear functional on $H_\pi \otimes L^p(\Omega)$ is continuous for each $[\pi] \in \widehat{G}$. In the proof of [32, Theorem 5.3] it is shown that every τ^π-invariant linear functional on $L(H_\pi) \otimes L^p(\Omega)$ is continuous for each $[\pi] \in \widehat{G}$. From Lemma 16 we conclude that every $\pi \otimes \tau$-invariant linear functional on $H_\pi \otimes L^p(\Omega)$ is continuous for each $[\pi] \in \widehat{G}$. ∎

Remark 7. Of course, Theorem 28 completes the information given in [32, Theorem 5.3]. Clearly Banach spaces $\mathfrak{F}(G)$ with the property that $\mathfrak{T}(G)\mathfrak{F}(G) \subset \mathfrak{F}(G)$ include the spaces $M(G)$, $L^p(G)$ with $1 \le p \le \infty$, and $C(G)$. It should also be pointed out that translation-invariant linear functionals have been shown to be automatically continuous on some spaces $\mathfrak{F}(G)$ for a wide variety of groups G. As a matter of fact, translation-invariant linear functionals are automatically continuous on the following spaces: $L^2(G)$, where G is a compact abelian group with a finite number of connected components [22]; $L^2(G)$ for compact abelian groups G such that G/C is polythetic (that is, contains a finitely generated dense subgroup), where C is the connected component of the identity [17]; and $L^p(G)$, $1 < p < \infty$, when G is a connected metrizable compact abelian group [3]. Thus Theorem 28 generalizes [7, Theorem 10(iii)].

It turned out in [24, 25, 32, 35] that the automatic continuity of the translation-invariant linear functionals is closely related to the so-called *strong Kazhdan's property*. Recall that for a compact group G this property can be rephrased as follows: G has the strong Kazhdan's property (T) if, and only if, the restriction τ of the left regular representation of G to the invariant subspace $L_0^2(G) = \{f \in L^2(G)\colon \int_G f(t)\,dt = 0\}$ does not have almost invariant vectors when G is viewed as a discrete group. Such a group is described in [24] as not satisfying the *mean-zero weak containment property*. This property is related to the so-called *Banach–Ruziewicz problem*. This problem, for the spheres, asks whether the Lebesgue measure on the N-dimensional Euclidean sphere $\mathbb{S}^N$ is the unique normalized, finitely additive rotation invariant measure on all Lebesgue measurable subsets of $\mathbb{S}^N$. When solving this problem it was shown that the group $SO(N+1)$ has the strong Kazhdan's property (T) for $N \geq 2$ ([6] in the cases where $N = 2, 3$ and [18, 30] in the case where $N \geq 4$). Moreover, every simple compact connected Lie group has the strong Kazhdan's property (T) (see [19, Chapter III, 5.7] or [28, Theorem 5.17]). The unitary group $U(H)$ of an infinite-dimensional separable Hilbert space when equipped with the strong operator topology has the strong Kazhdan's property [2]. The compact unitary groups $U(N)$ also have the strong Kazhdan's property for $N \geq 2$.

Corollary 29. *Let G be a compact group which has the strong Kazhdan's property* (T). *Then every linear operator $\Phi\colon L^p(\Omega) \to Y$ which commutes with translations is continuous for each compact Hausdorff space Ω equipped with a positive Radon measure on which G acts as a group of measure-preserving transformations with Ω/G finite*, *each $1 < p < \infty$, and for each bounded $\sigma(Y, Y_*)$-continuous representation τ_Y of G on a Banach space Y (where either $Y_* = Y^*$ or Y is a dual Banach space and Y_* is a predual of Y).*

Proof. By [24], every invariant linear functional on $L^p(G)$ is continuous and so Theorem 28 now implies our corollary. ∎

Corollary 30. *Let $N \geq 2$, let $1 < p < \infty$, and let τ_Y be a bounded $\sigma(Y, Y_*)$-continuous representation of $SO(N+1)$ on a Banach space Y (where either $Y_* = Y^*$ or Y is a dual Banach space and Y_* is a predual of Y). Then every linear operator $\Phi\colon L^p(\mathbb{S}^N) \to Y$ which commutes with rotations is continuous.* ∎

5.2. *Non-compact* [MAP] *groups.* In the remainder of this section we assume G to be a [MAP] group, that is, a locally compact group with the property that the set of its finite-dimensional irreducible unitary representations separates the points of G.

Theorem 31. *Let G be a non-compact* [MAP] *group*, *let X be a Banach space*, *and let τ be a representation of G on X. Then every linear operator $\Phi\colon X \to M(G)$ which commutes with translations is continuous.*

Proof. Assume towards a contradiction that $\mathfrak{S}(\Phi) \neq \{0\}$. Let

$$\Delta = \{[\pi] \in \widehat{G}_{\mathrm{FIN}} : \widetilde{\pi}(\mathfrak{S}(\Phi)) \neq \{0\}\}.$$

This definition makes sense because the correspondence $\pi \leftrightarrow \widetilde{\pi}$ respects unitary equivalence.

We can now apply Lemma 2 by taking Σ in such a way that

$$\Delta = \{[\pi] : \pi \in \Sigma\}.$$

Let $\mu \in M_{\mathrm{f}}(G)$ satisfy the first assertion in that lemma, and let

$$\{\pi_1, \ldots, \pi_N\} = \{\pi \in \Sigma : \widetilde{\pi}(\mu) \neq 0\}.$$

We define $I = \overline{L^1(G) \star \mu \star \mathfrak{S}(\Phi) \star L^1(G)}$. It is clear that I is a closed two-sided ideal of $L^1(G)$ and that $\widetilde{\pi}(I) = \{0\}$ for each $[\pi] \in \widehat{G}_{\mathrm{FIN}} \setminus \{[\pi_1], \ldots, [\pi_N]\}$. Consequently, if $f \in I$ is such that $\widetilde{\pi}_k(f) = 0$ for each $k \in \{1, \ldots, N\}$, then $\widetilde{\pi}(f) = 0$ for each $[\pi] \in \widehat{G}_{\mathrm{FIN}}$. Since $\bigcap_{\pi \in \widehat{G}_{\mathrm{FIN}}} \ker \widetilde{\pi} = \{0\}$ [10, Theorem 3.2], it may be concluded that the map $f \mapsto (\widetilde{\pi}_1(f), \ldots, \widetilde{\pi}_N(f))$ from I into $L(H_{\pi_1}) \oplus \cdots \oplus L(H_{\pi_N})$ is injective. Since $\dim L(H_{\pi_k}) < \infty$ for each $k \in \{1, \ldots, N\}$, it follows that $\dim I < \infty$. From [9, Lemma 1] we see that $I = \{0\}$. Hence

$$\{0\} = \widetilde{\pi}_1(L^1(G) \star \mu \star \mathfrak{S}(\Phi) \star L^1(G)) = \widetilde{\pi}_1(L^1(G))\widetilde{\pi}_1(\mu)\widetilde{\pi}_1(\mathfrak{S}(\Phi))\widetilde{\pi}_1(L^1(G)).$$

Since $\widetilde{\pi}_1$ is an algebraically irreducible representation of $M(G)$ on H_{π_1}, it follows that $\widetilde{\pi}_1(\mu)\widetilde{\pi}_1(\mathfrak{S}(\Phi)) = \{0\}$. On the other hand, $\mathfrak{S}(\Phi)$ is easily checked to be left translation-invariant, and so the linear subspace of H_{π_1} generated by the vectors $\widetilde{\pi}_1(\mathfrak{S}(\Phi))x$ with $x \in H_{\pi_1}$ is invariant for π_1 and it is contained in $\ker \widetilde{\pi}_1(\mu) \neq H_{\pi_1}$. This entails that $\widetilde{\pi}_1(\mathfrak{S}(\Phi))x = 0$ for each $x \in H_{\pi_1}$, which contradicts the fact that $\pi_1 \in \Sigma$.

Finally, we are concerned with the case in which the second assertion of Lemma 2 holds. It should be pointed out that $\widetilde{\pi}_n(\mu_1^* \star \cdots \star \mu_n^*) \neq 0$ and $\widetilde{\pi}_n(\mu_1^* \star \cdots \star \mu_{n+1}^*) = 0$ for each $n \in \mathbb{N}$. Furthermore, for every $n \in \mathbb{N}$, μ_n^* can be expressed in the form

$$\mu_n^* = \sum_{k=1}^{N_n} \alpha_{n,k} \delta_{t_{n,k}}$$

with $N_n \in \mathbb{N}$, $\alpha_{n,1}, \ldots, \alpha_{n,N_n} \in \mathbb{C}$, and $t_{n,1}, \ldots, t_{n,N_n} \in G$, and we define a continuous linear operator S_n from X into itself by

$$S_n(x) = \sum_{k=1}^{N_n} \alpha_{n,k} \tau(t_{n,k}) x$$

for each $x \in X$. We now define $T_n : M(G) \to M(G)$ by $T_n(\mu) = \mu_n^* \star \mu$ for each $\mu \in M(G)$. It is clear that T_n is a continuous linear operator and that $\Phi S_n = T_n \Phi$ for each $n \in \mathbb{N}$. Thus we are now in a position to apply the stability lemma. Therefore there exists $N \in \mathbb{N}$ such that

$$\overline{(\mu_1^* \star \cdots \star \mu_N^*)(\mathfrak{S}(\Phi))} = \overline{(\mu_1^* \star \cdots \star \mu_{N+1}^*)(\mathfrak{S}(\Phi))}.$$

In particular

$$(\mu_1^* \star \cdots \star \mu_N^*)(\mathfrak{S}(\Phi)) \subset \overline{(\mu_1^* \star \cdots \star \mu_{N+1}^*)(\mathfrak{S}(\Phi))}.$$

Hence

$$\widetilde{\pi}_N((\mu_1^* \star \cdots \star \mu_N^*)(\mathfrak{S}(\Phi))) \subset \overline{\widetilde{\pi}_N((\mu_1^* \star \cdots \star \mu_{N+1}^*)(\mathfrak{S}(\Phi)))} = \{0\},$$

which implies that $\widetilde{\pi}_N(\mu_1^* \star \cdots \star \mu_N^*)\widetilde{\pi}_N(\mathfrak{S}(\Phi)) = \{0\}$. The linear subspace of H_π generated by the vectors $\widetilde{\pi}_N(\mathfrak{S}(\Phi))x$ with $x \in H_{\pi_N}$ is invariant for π_N, and it is contained in $\ker \widetilde{\pi}_N(\mu_1^* \star \cdots \star \mu_N^*) \neq H_{\pi_N}$. This entails that $\widetilde{\pi}_N(\mathfrak{S}(\Phi))x = 0$ for each $x \in H_{\pi_N}$, contrary to $\pi_N \in \Sigma$. ∎

Corollary 32. *Let G be a non-compact* [MAP] *group, let τ be a representation of G on a Banach space X, and let $\mathfrak{F}(G)$ be a Banach space on G. Then every linear operator $\Phi\colon X \to \mathfrak{F}(G)$ which commutes with translations is continuous.*

Proof. The inclusion map $J\colon \mathfrak{F}(G) \to M(G)$ is a continuous linear operator which commutes with translations. According to Theorem 31, $J \circ \Phi$ is continuous, and so $J(\mathfrak{S}(\Phi)) = \{0\}$, which clearly forces $\mathfrak{S}(\Phi) = \{0\}$. ∎

Corollary 33. *Let G be an infinite* [MAP] *group. Then the following assertions are equivalent*:

(i) *Every linear operator $\Phi\colon X \to L^1(G)$ which commutes with translations is continuous for each representation of G on a Banach space X.*
(ii) *Every linear operator $\Phi\colon X \to L^1(G, E)$ which commutes with translations is continuous for each representation of G on a Banach space X and for each Banach space E.*
(iii) *There is a non-zero Banach space E with the property that every linear operator $\Phi\colon X \to L^1(G, E)$ which commutes with translations is continuous for each representation of G on a Banach space X.*
(iv) *$L^1(G)$ carries a unique topologically invariant norm.*
(v) *$L^1(G, E)$ carries a unique topologically invariant norm for each Banach space E.*
(vi) *There is a non-zero Banach space E with the property that $L^1(G, E)$ carries a unique topologically invariant norm.*
(vii) *G is non-compact.*

Proof. We first assume that (i) holds. Let $\|\cdot\|$ be a complete norm on $L^1(G)$ that makes all the left translations from $(L^1(G), \|\cdot\|)$ into itself continuous. The identity map Φ from $(L^1(G), \|\cdot\|)$ onto $(L^1(G), \|\cdot\|_1)$ commutes with translations, and therefore it is continuous. On account of the open mapping theorem, $\|\cdot\|$ is equivalent to $\|\cdot\|_1$, and this proves (iv). In the same manner we can see that (ii) implies (v) and that (iii) implies (vi). Let X be a Banach space on which G acts, let E be a Banach space, and let $\Phi\colon X \to L^1(G,E)$ be a linear operator which commutes with translations. Assume towards a contradiction that $\mathfrak{S}(\Phi) \neq \{0\}$. Set $F_0 \in \mathfrak{S}(\Phi)$ with $F_0 \neq 0$. On account of [5, Corollary II.2.7], there exists a continuous linear functional ψ on E such that $\psi \circ F_0 \neq 0$. We define $\Psi\colon L^1(G,E) \to L^1(G)$ by $\Psi(F) = \psi \circ F$ for each $F \in L^1(G,E)$. It is easily checked that Ψ is a continuous linear operator which commutes with translations. Consequently, $\Psi \circ \Phi$ commutes with translations, and therefore it is continuous. This implies that $\Psi(\mathfrak{S}(\Phi)) = \{0\}$, and hence that $\psi \circ F_0 = 0$, a contradiction. We thus get (ii).

It is obvious that (ii) implies both (i) and (iii), that (iv) implies (vi), and that (v) implies both (iv) and (vi).

It is shown in [27, Theorem 1] that for every infinite compact group G, there exists a discontinuous translation invariant functional ϕ on $L^1(G)$. Let E be a non-zero Banach space and let ψ be a non-zero continuous linear functional on E. Then the functional $F \mapsto \phi(\psi \circ F)$ on $L^1(G,E)$ is easily seen to be translation-invariant and discontinuous. On the other hand, let $u \in E \setminus \{0\}$ and consider the function $F \in L^1(G,E)$ defined by $F(t) = u$ for each $t \in G$. It is clear that F is invariant and [32, Theorem 3.1] now shows that $L^1(G,E)$ does not carry a unique topologically invariant norm. Consequently, (vi) implies (vii).

Finally, the fact that (vii) implies (i) follows immediately from Theorem 31. ∎

REMARK 8. It should be noted that Corollary 32 generalizes [7, Theorem 10(i)] and that Corollary 33 generalizes [7, Theorem 3 and Corollary 4].

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