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Affine bijections of $\mathcal{C}(\mathcal{X}, I)$

by

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Abstract. Let $\mathcal{X}$ be a compact Hausdorff space which satisfies the first axiom of countability, $I = [0, 1]$ and $\mathcal{C}(\mathcal{X}, I)$ the set of all continuous functions from $\mathcal{X}$ to I. If $\varphi : \mathcal{C}(\mathcal{X}, I) \to \mathcal{C}(\mathcal{X}, I)$ is a bijective affine map then there exists a homeomorphism $\mu : \mathcal{X} \to \mathcal{X}$ such that for every component C in $\mathcal{X}$ we have either $\varphi(f)(x) = f(\mu(x))$, $f \in \mathcal{C}(\mathcal{X}, I)$, $x \in C$, or $\varphi(f)(x) = 1 - f(\mu(x))$, $f \in \mathcal{C}(\mathcal{X}, I)$, $x \in C$.

1. Introduction and statement of the result. The problem we consider in this paper has been motivated by results of L. Molnár in [15] and [17]. Molnár and several other authors studied preservers of various operations, relations and quantities on Hilbert space effect algebras (see [2, 5, 9, 10, 12, 15–17]). Let $\mathcal{A}$ be a unital C^*-algebra. The *effects* in $\mathcal{A}$ are the positive elements of $\mathcal{A}$ which are less than or equal to the unit of $\mathcal{A}$. The set of all effects in $\mathcal{A}$ is denoted by $E(\mathcal{A})$. If $\mathcal{A}$ equals the algebra $B(\mathcal{H})$ of all bounded linear operators on the complex Hilbert space $\mathcal{H}$, then the corresponding effects are called *Hilbert space effects*. The concept of effects plays an important role in certain parts of quantum mechanics, for instance, in the quantum theory of measurement (see [1, 4, 11]).

The set $E(\mathcal{H})$ of Hilbert space effects can be equipped with several algebraic operations and relations, each having a physical content. In [6] Gudder and Nagy defined the *sequential product* between effects (see also [7]) by

$$A \circ B = A^{1/2} B A^{1/2}, \qquad A, B \in E(\mathcal{A}).$$

If $\mathcal{A}$, $\mathcal{B}$ are unital C^*-algebras, then a bijective map $\varphi : E(\mathcal{A}) \to E(\mathcal{B})$ is called a *sequential isomorphism* if

$$\varphi(A \circ B) = \varphi(A) \circ \varphi(B), \qquad A, B \in E(\mathcal{A}).$$

Gudder and Greechie described in [5] the general form of sequential automorphisms of the set of all Hilbert space effects assuming that the underlying Hilbert space is at least three-dimensional: every such automorphism

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φ is implemented either by a unitary or an antiunitary operator U of the underlying Hilbert space $\mathcal{H}$, via

$$\varphi(A) = UAU^*, \quad A \in E(\mathcal{H}).$$

This result was generalized by Molnár [17] to the case of effects in general von Neumann algebras. Namely, if $\mathcal{A}$, $\mathcal{B}$ are von Neumann algebras and $\varphi : E(\mathcal{A}) \to E(\mathcal{B})$ is a sequential isomorphism, then there are direct decompositions

$$\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3 \quad \text{and} \quad \mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \mathcal{B}_3$$

within the category of von Neumann algebras and there are bijective maps

$$\varphi_1 : E(\mathcal{A}_1) \to E(\mathcal{B}_1), \quad \Phi_2 : \mathcal{A}_2 \to \mathcal{B}_2, \quad \Phi_3 : \mathcal{A}_3 \to \mathcal{B}_3$$

such that

(i) $\mathcal{A}_1$, $\mathcal{B}_1$ are commutative and $\mathcal{A}_2 \oplus \mathcal{A}_3$, $\mathcal{B}_2 \oplus \mathcal{B}_3$ have no commutative direct summands;
(ii) φ_1 is a multiplicative bijection, Φ_2 is an algebra *-isomorphism, Φ_3 is an algebra *-antiisomorphism and $\varphi = \varphi_1 \oplus \Phi_2 \oplus \Phi_3$ on $E(\mathcal{A})$.

In our recent paper [13] we studied the first factor in the above decomposition, i.e., the bijective multiplicative maps between the sets of effects in commutative von Neumann algebras or, more generally, in commutative unital C^*-algebras. It is well known that every commutative C^*-algebra is isomorphic to the algebra of all continuous complex-valued functions on a compact Hausdorff space $\mathcal{X}$. Therefore, it is enough to consider the set $\mathcal{C}(\mathcal{X}, I)$ of all continuous functions from $\mathcal{X}$ to the unit interval I. The main result in [13] describes the general form of bijective multiplicative maps of $\mathcal{C}(\mathcal{X}, I)$ under the technical condition that $\mathcal{X}$ satisfies the first axiom of countability.

The set $E(\mathcal{H})$ may also be equipped with a partial order $\leq$, which comes from the usual order between self-adjoint operators on $\mathcal{H}$, and one can also define the operation of *orthocomplementation* by

$$A \mapsto I - A, \quad A \in E(\mathcal{H}).$$

It turns out that automorphisms of $E(\mathcal{H})$, $\dim \mathcal{H} > 1$, with respect to order and orthocomplementation are again implemented by a unitary or an antiunitary operator U of $\mathcal{H}$ as in the case of sequential automorphisms (see [2, 12, 18]). It seems that a necessary step in understanding the structure of preservers of different types on general von Neumann algebra effects is to investigate the transformations of $\mathcal{C}(\mathcal{X}, I)$. We presented in [14] a structural result describing the bijective transformations of $\mathcal{C}(\mathcal{X}, I)$ which preserve the order $\leq$ in both directions, i.e., $f \leq g$ if and only if $\varphi(f) \leq \varphi(g)$ for all $f, g \in \mathcal{C}(\mathcal{X}, I)$. Again, we assumed that $\mathcal{X}$ satisfies the first axiom of countability.

The Hilbert space effect algebra is clearly a convex set. So, it is natural to equip it with the operation of convex combinations (called mixture in physics). Automorphisms with respect to this operation were studied, for example, in [8]. These automorphisms, called *affine*, were determined in [15, Corollary 2], stating that the bijective maps $\varphi : E(\mathcal{H}) \to E(\mathcal{H})$ which satisfy

$$\varphi(\lambda A + (1-\lambda)B) = \lambda\varphi(A) + (1-\lambda)\varphi(B)$$

for all $A, B \in E(\mathcal{H})$ and $\lambda \in I$ are exactly the transformations which are either of the form

$$\varphi(A) = UAU^*, \qquad A \in E(\mathcal{H}),$$

or of the form

$$\varphi(A) = U(I-A)U^*, \qquad A \in E(\mathcal{H}),$$

where U is either a unitary or an antiunitary operator on $\mathcal{H}$.

In this paper we will study the bijective affine transformations of $\mathcal{C}(\mathcal{X}, I)$, i.e., the bijective maps $\varphi : \mathcal{C}(\mathcal{X}, I) \to \mathcal{C}(\mathcal{X}, I)$ which satisfy

$$\varphi(\lambda f + (1-\lambda)g) = \lambda\varphi(f) + (1-\lambda)\varphi(g)$$

for all $f, g \in \mathcal{C}(\mathcal{X}, I)$ and $\lambda \in I$, under the technical condition that $\mathcal{X}$ satisfies the first axiom of countability.

THEOREM 1.1. *Let $\mathcal{X}$ be a compact Hausdorff space which satisfies the first axiom of countability and $I = [0,1]$. If $\varphi : \mathcal{C}(\mathcal{X}, I) \to \mathcal{C}(\mathcal{X}, I)$ is a bijective affine map then there exists a homeomorphism $\mu : \mathcal{X} \to \mathcal{X}$ such that for every component C in $\mathcal{X}$, either*

$$\varphi(f)(x) = f(\mu(x)) \quad \text{for every } f \in \mathcal{C}(\mathcal{X}, I) \text{ and } x \in C$$

or

$$\varphi(f)(x) = 1 - f(\mu(x)) \quad \text{for every } f \in \mathcal{C}(\mathcal{X}, I) \text{ and } x \in C.$$

We believe that the same result holds without the first countability assumption.

2. Proof of Theorem 1.1. Let us first recall some well known facts. Let $\mathcal{A}$ be a C^*-algebra. An element $p \in \mathcal{A}$ is a *projection* if $p = p^* = p^2$. An element $x \in \mathcal{A}$ is *positive* if there exists $y \in \mathcal{A}$ such that $x = y^*y$. A point x of a convex set C in a linear space $\mathcal{X}$ is an *extreme point* of C if the condition $x = ty + (1-t)z$, where $y, z \in C$ and $0 < t < 1$, implies that $x = y = z$. Recall also (see, for example, [3]) that the extreme points of the set of all positive elements from the unit ball in $\mathcal{A}$ are exactly the projections. Clearly, the set $\mathcal{C}(\mathcal{X})$ of all continuous complex-valued functions on $\mathcal{X}$ is a C^*-algebra with the pointwise operations, the supremum norm and $f^* = \bar{f}$, $f \in \mathcal{C}(\mathcal{X})$. The set of all positive elements in the unit ball of $\mathcal{C}(\mathcal{X})$ is $\mathcal{C}(\mathcal{X}, I)$.

LEMMA 2.1. *Let $\mathcal{X}$ be a compact Hausdorff space. If $\varphi : \mathcal{C}(\mathcal{X}, I) \to \mathcal{C}(\mathcal{X}, I)$ is a bijective affine map then φ preserves the extreme points of $\mathcal{C}(\mathcal{X}, I)$.*

The trivial proof of Lemma 2.1 is omitted.

From now on, let $\varphi : \mathcal{C}(\mathcal{X}, I) \to \mathcal{C}(\mathcal{X}, I)$ be a bijective affine map. By Lemma 2.1, φ preserves the projections. A function $f \in \mathcal{C}(\mathcal{X})$ is a projection if $\bar{f} = f = f^2$. So, $f \in \mathcal{C}(\mathcal{X}, I)$ is a projection if $f^2 = f$ and therefore $f(x)(f(x) - 1) = 0$. For a projection $f \in \mathcal{C}(\mathcal{X}, I)$ and $x \in \mathcal{X}$ we thus obtain

$$f(x) = 1 \quad \text{or} \quad f(x) = 0.$$

For $c \in I$, let $c_{\mathcal{X}}(x) = c$ for every $x \in \mathcal{X}$. Since $0_{\mathcal{X}}$ and $1_{\mathcal{X}}$ are projections we see that for every $x \in \mathcal{X}$, $\varphi(1_{\mathcal{X}})(x) = 0$ or 1 and $\varphi(0_{\mathcal{X}})(x) = 0$ or 1. Also,

$$\varphi(c_{\mathcal{X}}) = \varphi(c1_{\mathcal{X}} + (1-c)0_{\mathcal{X}}) = c\varphi(1_{\mathcal{X}}) + (1-c)\varphi(0_{\mathcal{X}}).$$

So, if for an $x \in \mathcal{X}$ we have $\varphi(1_{\mathcal{X}})(x) = 0$ and $\varphi(0_{\mathcal{X}})(x) = 0$, then $\varphi(c_{\mathcal{X}})(x) = 0$ for all $c \in I$. Let X_1 be the set of all $x \in \mathcal{X}$ which satisfy this condition. If $\varphi(1_{\mathcal{X}})(x) = 1$ and $\varphi(0_{\mathcal{X}})(x) = 1$ for an $x \in \mathcal{X}$, then $\varphi(c_{\mathcal{X}})(x) = 1$ for all $c \in I$. Let X_2 be the set of all $x \in \mathcal{X}$ which satisfy this condition. If $\varphi(1_{\mathcal{X}})(x) = 1$ and $\varphi(0_{\mathcal{X}})(x) = 0$ for an $x \in \mathcal{X}$, we obtain $\varphi(c_{\mathcal{X}})(x) = c$ for all $c \in I$. Again, let X_3 be the set of all $x \in \mathcal{X}$ which satisfy this condition. Finally, if $\varphi(1_{\mathcal{X}})(x) = 0$ and $\varphi(0_{\mathcal{X}})(x) = 1$ for an $x \in \mathcal{X}$, we obtain $\varphi(c_{\mathcal{X}})(x) = 1 - c$ for all $c \in I$. Let X_4 be the set of all $x \in \mathcal{X}$ which satisfy this condition.

Assume that $X_1 \neq \emptyset$ and let $x_0 \in X_1$. Let $f \in \mathcal{C}(\mathcal{X}, I)$ and $g = 1_{\mathcal{X}} - f$. Then $g \in \mathcal{C}(\mathcal{X}, I)$. So, $\frac{1}{2}g + \frac{1}{2}f = \left(\frac{1}{2}\right)_{\mathcal{X}}$ and hence

$$0 = \varphi\left(\left(\frac{1}{2}\right)_{\mathcal{X}}\right)(x_0) = \frac{1}{2}\,\varphi(g)(x_0) + \frac{1}{2}\,\varphi(f)(x_0).$$

But then $\varphi(f)(x_0) = 0$ for every $f \in \mathcal{C}(\mathcal{X}, I)$, which is a contradiction since φ is surjective. So, $X_1 = \emptyset$. Similarly we prove that $X_2 = \emptyset$. It follows that

$$\varphi(c_{\mathcal{X}})(x) = c \quad \text{or} \quad \varphi(c_{\mathcal{X}})(x) = 1 - c$$

for all $x \in \mathcal{X}$ and $c \in I$. Notice that then $\varphi\big(\left(\frac{1}{2}\right)_{\mathcal{X}}\big) = \left(\frac{1}{2}\right)_{\mathcal{X}}$. Since $\left(\frac{1}{2}\right)_{\mathcal{X}} = \frac{1}{2}f + \frac{1}{2}(1_{\mathcal{X}} - f)$ and hence $\left(\frac{1}{2}\right)_{\mathcal{X}} = \frac{1}{2}\varphi(f) + \frac{1}{2}\varphi(1_{\mathcal{X}} - f)$, we get

$$\varphi(1_{\mathcal{X}} - f) = 1_{\mathcal{X}} - \varphi(f). \tag{2.1}$$

Notice also that

$$X_3 = \varphi(1_{\mathcal{X}})^{-1}(1), \qquad X_4 = \varphi(1_{\mathcal{X}})^{-1}(0), \qquad X_3 \cup X_4 = \mathcal{X}.$$

LEMMA 2.2. *Let $f \in \mathcal{C}(\mathcal{X}, I)$. Then $0 < f(x) < 1$ for every $x \in \mathcal{X}$ if and only if $0 < \varphi(f)(x) < 1$ for every $x \in \mathcal{X}$.*

Proof. Let $f \in \mathcal{C}(\mathcal{X}, I)$. For $\lambda \in I$ and $x_0 \in X_3$ we obtain

$$\varphi(\lambda f)(x_0) = \lambda\varphi(f)(x_0) + (1-\lambda)\varphi(0_{\mathcal{X}})(x_0) = \lambda\varphi(f)(x_0) \tag{2.2}$$

and for $x_0 \in X_4$ we get

$$\varphi(\lambda f)(x_0) = \lambda\varphi(f)(x_0) + (1-\lambda). \tag{2.3}$$

Suppose that $f(x) \in (0,1)$ for every $x \in \mathcal{X}$. Since $\mathcal{X}$ is compact there exists $a = \max f$. Then $a \in (0,1)$. Let $1 < \lambda_0 < 1/a$ and $g = \lambda_0 f$. Since $\lambda_0 f(x) \le \lambda_0 a < 1$ for every $x \in \mathcal{X}$, we conclude that $g \in \mathcal{C}(\mathcal{X}, I)$ and $g(x) < 1$ for every $x \in \mathcal{X}$. Let $\lambda_1 = 1/\lambda_0$. Also, suppose first that $x_0 \in X_3$. Since $\lambda_1 \in (0,1)$ we obtain, by (2.2),

$$\varphi(f)(x_0) = \varphi(\lambda_1 g)(x_0) = \lambda_1\varphi(g)(x_0) < 1.$$

If $x_0 \in X_4$ then by (2.3),

$$\varphi(f)(x_0) = \varphi(\lambda_1 g)(x_0) = \lambda_1\varphi(g)(x_0) + 1 - \lambda_1 > 0.$$

Let now $b = \min f$ and let $\lambda_0 > 1-b$ be such that $\lambda_0 \in (0,1)$. Let $\lambda_1 = 1/\lambda_0$. Also, let $h(x) = \lambda_1(f(x)-1)+1$ for every $x \in \mathcal{X}$. Then, since $f(x) < 1$ for every $x \in \mathcal{X}$, we get $h(x) < 1$ for every $x \in \mathcal{X}$. Also,

$$h(x) = \lambda_1(f(x)-1)+1 \ge \lambda_1(b-1)+1 > \frac{1}{1-b}(b-1)+1 = 0.$$

We conclude that $h \in \mathcal{C}(\mathcal{X}, I)$. Notice that $f = \lambda_0 h + 1 - \lambda_0$. Again, let first $x_0 \in X_3$. Then

$$\varphi(f)(x_0) = \lambda_0\varphi(h)(x_0) + (1-\lambda_0)\varphi(1_{\mathcal{X}})(x_0) = \lambda_0\varphi(h)(x_0) + 1 - \lambda_0 > 0.$$

If $x_0 \in X_4$ then

$$\varphi(f)(x_0) = \lambda_0\varphi(h)(x_0) + (1-\lambda_0)\varphi(1_{\mathcal{X}})(x_0) = \lambda_0\varphi(h)(x_0) < 1.$$

So, $\varphi(f)(x) \in (0,1)$ for every $x \in \mathcal{X}$. Conversely, if $\varphi(f)(x) \in (0,1)$ for every $x \in \mathcal{X}$ then, since φ^{-1} has the same properties as φ, we conclude that $f(x) \in (0,1)$ for every $x \in \mathcal{X}$. ∎

Throughout the proof we will need the notions of 0-proper and 1-proper functions in $\mathcal{C}(\mathcal{X}, I)$. Let $f \in \mathcal{C}(\mathcal{X}, I)$. If $f^{-1}(0) \ne \mathcal{X}$ and $\operatorname{Int} f^{-1}(0) \ne \emptyset$ then f is called 0-*proper* and we write $\operatorname{Int} f^{-1}(0) = Z_f$. Similarly, if $f^{-1}(1) \ne \mathcal{X}$ and $\operatorname{Int} f^{-1}(1) \ne \emptyset$ then f is called 1-*proper* and we set $\operatorname{Int} f^{-1}(1) = O_f$.

LEMMA 2.3. *Let U be an open nonempty subset of $\mathcal{X}$ with $\overline{U} \ne \mathcal{X}$. Then there exists $f \in \mathcal{C}(\mathcal{X}, I)$ such that $f \ne 1_{\mathcal{X}}$, $f(\overline{U}) = \{1\}$ and $f(x) \ne 0$ for every $x \in \mathcal{X}$. Furthermore, for every such f the function $\varphi(f)$ is either* 1*-proper or* 0*-proper.*

Proof. By Urysohn's lemma there exists $g \in C(\mathcal{X}, I)$ such that $g \equiv 1$ on $\overline{U}$ and $g \ne 1_{\mathcal{X}}$. Let $c \in (0,1)$ and $f = \max\{g, c_{\mathcal{X}}\}$. Then $f \in \mathcal{C}(\mathcal{X}, I)$, $f \equiv 1$ on $\overline{U}$, $f \ne 1_{\mathcal{X}}$ and $f(x) \ne 0$ for every $x \in \mathcal{X}$.

Let $f_1 \in \mathcal{C}(\mathcal{X}, I)$, $f_1 \neq 1_{\mathcal{X}}$, $f_1(\overline{U}) = \{1\}$ and $f_1(x) \neq 0$ for every $x \in \mathcal{X}$. By Urysohn's lemma there exists $f_2 \in \mathcal{C}(\mathcal{X}, I)$ such that $f_2 \equiv 1$ on U^{c}, $f_2 \neq 1_{\mathcal{X}}$ and $f_2(x) \neq 0$ for every $x \in \mathcal{X}$. Similarly, there exist f_3, $f_4 \in \mathcal{C}(\mathcal{X}, I)$ such that $f_3 \equiv 0$ on $\overline{U}$, $f_3 \neq 0_{\mathcal{X}}$, $f_3(x) \neq 1$ for every $x \in \mathcal{X}$, and $f_4 \equiv 0$ on U^{c}, $f_4 \neq 0_{\mathcal{X}}$, $f_4(x) \neq 1$ for every $x \in \mathcal{X}$. Let $h \in \mathcal{C}(\mathcal{X}, I)$ be such that $h(x_0) = 1$ for some $x_0 \in \mathcal{X}$ and let $\lambda \in (0,1)$. Then there exists $i \in \{1,2\}$ such that $f_i(x_0) = 1$ and hence

$$\lambda f_i(x_0) + (1-\lambda)h(x_0) = 1.$$

By Lemma 2.2 there exists $x_1 \in \mathcal{X}$ such that either $\varphi(\lambda f_i + (1-\lambda)h)(x_1) = 1$ or $\varphi(\lambda f_i + (1-\lambda)h)(x_1) = 0$. Therefore either

$$\lambda\varphi(f_i)(x_1) + (1-\lambda)\varphi(h)(x_1) = 1 \quad \text{or} \quad \lambda\varphi(f_i)(x_1) + (1-\lambda)\varphi(h)(x_1) = 0$$

and hence either $\varphi(f_i)(x_1) = \varphi(h)(x_1) = 1$ or $\varphi(f_i)(x_1) = \varphi(h)(x_1) = 0$. Similarly, if $h(x_0) = 0$ then there exist $i \in \{3,4\}$ and $x_2 \in \mathcal{X}$ such that either $\varphi(f_i)(x_2) = \varphi(h)(x_2) = 1$ or $\varphi(f_i)(x_2) = \varphi(h)(x_2) = 0$.

Suppose now that there exists $x_\lambda \in \mathcal{X}$ such that $\varphi(f_i)(x_\lambda) \neq 1$ for all $i \in \{1,2,3,4\}$. By the continuity of $\varphi(f_i)$ there exists an open neighbourhood V of x_λ such that $\varphi(f_i)(x) \neq 1$ for all $x \in V$ and $i \in \{1,2,3,4\}$. By Urysohn's lemma and the surjectivity of φ there exists $f_\lambda \in \mathcal{C}(\mathcal{X}, I)$ such that $\varphi(f_\lambda)(x_\lambda) = 1$, $\varphi(f_\lambda)(x) \neq 1$ for every $x \in V^{\mathrm{c}}$ and $\varphi(f_\lambda)(x) \neq 0$ for every $x \in \mathcal{X}$. On the one hand, by Lemma 2.2 there exists $x \in \mathcal{X}$ such that $f_\lambda(x) = 1$ or $f_\lambda(x) = 0$. Since $\varphi(f_\lambda)^{-1}(1) \cap \varphi(f_i)^{-1}(1) = \emptyset$ for every $i \in \{1,2,3,4\}$ and $\varphi(f_\lambda)(x) \neq 0$ for every $x \in \mathcal{X}$, we obtain

$$0 < \lambda\varphi(f_\lambda)(x) + (1-\lambda)\varphi(f_i)(x) < 1$$

for all $i \in \{1,2,3,4\}$ and $x \in \mathcal{X}$, and therefore by Lemma 2.2,

$$0 < \lambda f_\lambda(x) + (1-\lambda)f_i(x) < 1$$

for all $i \in \{1,2,3,4\}$ and $x \in \mathcal{X}$. So, on the other hand, $f_\lambda^{-1}(1) \cap f_i^{-1}(1) = \emptyset$ and $f_\lambda^{-1}(0) \cap f_i^{-1}(0) = \emptyset$ for every $i \in \{1,2,3,4\}$ and therefore $f_\lambda(x) \in (0,1)$ for every $x \in \mathcal{X}$, which is a contradiction.

Thus for every $x \in \mathcal{X}$ there exists $i \in \{1,2,3,4\}$ such that $\varphi(f_i)(x) = 1$. Similarly, for every $x \in \mathcal{X}$ there exists $j \in \{1,2,3,4\}$ such that $\varphi(f_j)(x) = 0$. Also for $i \in \{1,2\}$ and $j \in \{3,4\}$ we get $0 < \lambda f_i(x) + (1-\lambda)f_j(x) < 1$ for all $x \in \mathcal{X}$ and $\lambda \in (0,1)$. Therefore, by Lemma 2.2 we obtain

$$0 < \lambda\varphi(f_i)(x) + (1-\lambda)\varphi(f_j)(x) < 1$$

for all $x \in \mathcal{X}$ and $\lambda \in (0,1)$. So, if $\varphi(f_i)(x) = 0$ then $\varphi(f_j)(x) \neq 0$ and if $\varphi(f_i)(x) = 1$ then $\varphi(f_j)(x) \neq 1$, $i \in \{1,2\}$, $j \in \{3,4\}$.

Assume that there does not exist a nonempty open set $V_1 \subset \mathcal{X}$ such that $\varphi(f_1) \equiv 1$ on V_1 or $\varphi(f_1) \equiv 0$ on V_1. If $\varphi(f_1)(x_0) = 1$ for some $x_0 \in \mathcal{X}$ then for each open neighbourhood V of x_0 there exists $x \in V$ such that $\varphi(f_1)(x) \neq 1$

and therefore $\varphi(f_i)(x) = 1$ for some $i \in \{2,3,4\}$. Since $\mathcal{X}$ is first countable we can construct a sequence $\{x_j : j \in \mathbb{N}\}$ such that $\lim_{j\to\infty} x_j = x_0$ and for some $i \in \{2,3,4\}$ we have $\varphi(f_i)(x_j) = 1$ for every $j \in \mathbb{N}$. Then

$$1 = \lim_{j\to\infty} \varphi(f_i)(x_j) = \varphi(f_i)(x_0).$$

If $i \in \{3,4\}$ then we get a contradiction by the argument above. So, if $\varphi(f_1)(x_0) = 1$ then $\varphi(f_2)(x_0) = 1$. Similarly, if $\varphi(f_1)(x_0) = 0$ then $\varphi(f_2)(x_0) = 0$.

By Urysohn's lemma and the continuity of f_2 there exist $h \in \mathcal{C}(\mathcal{X}, I)$ and a nonempty open set $U_1 \subset U$ such that $h(x) = 1$ for some $x \in U_1$, $h(x) \neq 1$ for every $x \in U_1^{\mathrm{c}}$ and $f_2(x) \neq 1$ for every $x \in U_1$. So, there does not exist $x \in \mathcal{X}$ such that $h(x) = f_2(x) = 1$. Since $f_2(x) \neq 0$ for every $x \in \mathcal{X}$ we see that if $\varphi(h)(x) = 1$ then $\varphi(f_2)(x) \neq 1$ and if $\varphi(h)(x) = 0$ then $\varphi(f_2)(x) \neq 0$. By Lemma 2.2 there exists $x \in \mathcal{X}$ such that either $\varphi(f_1)(x) = \varphi(h)(x) = 1$ or $\varphi(f_1)(x) = \varphi(h)(x) = 0$. But then $\varphi(f_2)(x) = 1$ in the former case and $\varphi(f_2)(x) = 0$ in the latter, which is a contradiction.

Hence there exists a nonempty open set V_1 such that either $\varphi(f_1) \equiv 1$ on V_1 or $\varphi(f_1) \equiv 0$ on V_1. Assume the former. Define $V = \mathrm{Int}(\varphi(f_1)^{-1}(1))$. We have proven that $V \neq \emptyset$. Observe that $\varphi(f_1)^{-1}(1) \neq \mathcal{X}$ since f_1 is not a projection and φ preserves the projections. This shows that $\varphi(f_1)$ is 1-proper. Similarly, if we assume the latter then $\varphi(f_1)$ is 0-proper. ∎

The first part of the next lemma can be proven in the same way as Lemma 2.3, and the second part is a direct consequence of Lemma 2.3 and (2.1).

Lemma 2.4. *Let U be an open nonempty subset of $\mathcal{X}$ with $\overline{U} \neq \mathcal{X}$. Then there exists $f \in \mathcal{C}(\mathcal{X}, I)$ such that $f \neq 0_{\mathcal{X}}$, $f(\overline{U}) = \{0\}$ and $f(x) \neq 1$ for every $x \in \mathcal{X}$. Furthermore, for every such f the function $\varphi(f)$ is either 0-proper or 1-proper.*

Let $c \in (0,1)$. By the surjectivity of φ there exists $g_1 \in \mathcal{C}(\mathcal{X}, I)$ such that

$$\varphi(g_1) = \max\{\varphi(1_{\mathcal{X}}), c_{\mathcal{X}}\}.$$

Then $\varphi(g_1)^{-1}(1) = X_3$. From now on let

$$A = g_1^{-1}(1).$$

Notice that if $\varphi(1_{\mathcal{X}}) = 1_{\mathcal{X}}$ then $A = \mathcal{X}$ since φ is injective, and if $\varphi(1_{\mathcal{X}}) = 0_{\mathcal{X}}$ then $A = \emptyset$ by Lemma 2.2.

Lemma 2.5. *Suppose that $\varphi(1_{\mathcal{X}}) \neq 1_{\mathcal{X}}$ and $\varphi(1_{\mathcal{X}}) \neq 0_{\mathcal{X}}$. Then A is an open and closed nonempty subset of $\mathcal{X}$, $A \neq \mathcal{X}$, and if $f_1 \in \mathcal{C}(\mathcal{X}, I)$ satisfies $f_1(x) \neq 0$ for every $x \in \mathcal{X}$ and $f_1(x) \neq 1$ for every $x \in A^{\mathrm{c}}$, then $\varphi(f_1)(x) \neq 0$ for every $x \in \mathcal{X}$. Similarly, if $f_2 \in \mathcal{C}(\mathcal{X}, I)$ satisfies $f_2(x) \neq 0$*

for every $x \in \mathcal{X}$ *and* $f_2(x) \neq 1$ *for every* $x \in A$, *then* $\varphi(f_2)(x) \neq 1$ *for every* $x \in \mathcal{X}$.

Proof. Notice first that $X_3 \neq \emptyset$ and $X_3 \neq \mathcal{X}$. Also, X_3 is open and closed since $\varphi(1_{\mathcal{X}})$ is continuous. By the surjectivity of φ there exists $g_2 \in \mathcal{C}(\mathcal{X}, I)$ such that

$$\varphi(g_2) = \min\{\varphi(1_{\mathcal{X}}), c_{\mathcal{X}}\}.$$

So, $\varphi(g_2)^{-1}(0) = X_4 = X_3^{\mathrm{c}}$ and $\varphi(g_2) \equiv c$ on X_3. Since $\varphi(g_1)^{-1}(1) = X_3$ and $\varphi(g_1) \equiv c$ on X_3^{c}, we obtain $\varphi(g_1)(x) \neq \varphi(g_2)(x)$ for every $x \in \mathcal{X}$. By (2.1) we obtain $\varphi(0_{\mathcal{X}}) = 1_{\mathcal{X}} - \varphi(1_{\mathcal{X}})$ and therefore $\varphi(0_{\mathcal{X}})(x) \neq \varphi(g_1)(x)$ and $\varphi(0_{\mathcal{X}})(x) \neq \varphi(g_2)(x)$ for every $x \in \mathcal{X}$. It then follows by Lemma 2.2 that $g_1(x) \neq 0$ and $g_2(x) \neq 0$ for every $x \in \mathcal{X}$. By Lemmas 2.3 and 2.4 and since φ^{-1} has the same properties as φ we establish that g_1 and g_2 are 1-proper. Also, for every $x \in \mathcal{X}$ there exist $h_1, h_2 \in \{\varphi(g_1), \varphi(g_2), \varphi(0_{\mathcal{X}})\}$ such that $h_1(x) = 0$ and $h_2(x) = 1$. Therefore, since $\varphi(g_1)(x) \neq \varphi(g_2)(x)$ for every $x \in \mathcal{X}$, we establish (by using similar arguments to the proof of Lemma 2.3) that $(g_1^{-1}(1))^{\mathrm{c}} = g_2^{-1}(1)$. Clearly, $A \neq \emptyset$ and $A \neq \mathcal{X}$. By the continuity of g_1 and g_2 we also conclude that A is open and closed.

Let now $f_1 \in \mathcal{C}(\mathcal{X}, I)$ be such that $f_1(x) \neq 0$ for every $x \in \mathcal{X}$ and $f_1(x) \neq 1$ for every $x \in A^{\mathrm{c}}$. Since $f_1(x) \neq 1$ if $g_2(x) = 1$, Lemma 2.2 shows that $\varphi(f_1)(x) \neq 0$ if $\varphi(g_2)(x) = 0$. Similarly, since $f_1(x) \neq 0$ for every $x \in \mathcal{X}$, we obtain $\varphi(f_1)(x) \neq 0$ if $\varphi(0_{\mathcal{X}})(x) = 0$. So, as $\varphi(0_{\mathcal{X}})^{-1}(0) = X_3$ and $\varphi(g_2)^{-1}(0) = X_3^{\mathrm{c}}$ we conclude that $\varphi(f_1)(x) \neq 0$ for every $x \in \mathcal{X}$. If $f_2 \in \mathcal{C}(\mathcal{X}, I)$ is such that $f_2(x) \neq 0$ for every $x \in \mathcal{X}$ and $f_2(x) \neq 1$ for every $x \in A$, then similar arguments yield $\varphi(f_2)(x) \neq 1$ for every $x \in \mathcal{X}$. ∎

The next remark follows from the argument already used before. Namely, if $f, g \in \mathcal{C}(\mathcal{X}, I)$ satisfy $f^{-1}(0) \cap g^{-1}(0) = \emptyset$ and $f^{-1}(1) \cap g^{-1}(1) = \emptyset$ then $\varphi(f)^{-1}(0) \cap \varphi(g)^{-1}(0) = \emptyset$ and $\varphi(f)^{-1}(1) \cap \varphi(g)^{-1}(1) = \emptyset$.

Remark 2.6. If $\varphi(1_{\mathcal{X}}) = 1_{\mathcal{X}}$ then $\varphi(0_{\mathcal{X}}) = 0_{\mathcal{X}}$ and therefore if $f_1 \in \mathcal{C}(\mathcal{X}, I)$ is never zero then $\varphi(f_1)(x)$ is never zero. Similarly, if $\varphi(1_{\mathcal{X}}) = 0_{\mathcal{X}}$ then $\varphi(0_{\mathcal{X}}) = 1_{\mathcal{X}}$ and therefore if $f_2 \in \mathcal{C}(\mathcal{X}, I)$ is never zero then $\varphi(f_2)(x)$ is never 1.

Lemma 2.7. *Suppose that* $\varphi(1_{\mathcal{X}}) \neq 0_{\mathcal{X}}$. *The functions* $f_1, \ldots, f_n$ *are* 1*-proper and* $f_i(x) \neq 0$ *for every* $x \in \mathcal{X}$ *and* $f_i(x) \neq 1$ *for every* $x \in A^{\mathrm{c}}$, $i \in \{1, \ldots, n\}$, *if and only if* $\varphi(f_1), \ldots, \varphi(f_n)$ *are* 1*-proper and* $\varphi(f_i)(x) \neq 0$ *for every* $x \in \mathcal{X}$ *and* $\varphi(f_i)(x) \neq 1$ *for every* $x \in X_3^{\mathrm{c}}$, $i \in \{1, \ldots, n\}$. *Furthermore, in this case*

$$O_{f_1} \cap \cdots \cap O_{f_n} \neq \emptyset \quad \textit{if and only if} \quad O_{\varphi(f_1)} \cap \cdots \cap O_{\varphi(f_n)} \neq \emptyset.$$

Proof. Let $f_1 \ldots, f_n$ be 1-proper, never zero on $\mathcal{X}$ and never 1 on A^{c}. By Lemma 2.3, each function $\varphi(f_i)$, $i \in \{1, \ldots, n\}$, is 1-proper or 0-proper.

If $\varphi(1_{\mathcal{X}}) \neq 1_{\mathcal{X}}$ then we conclude by Lemma 2.5 that $\varphi(f_i)(x) \neq 0$ for all $x \in \mathcal{X}$ and $i \in \{1, \ldots, n\}$. So, $\varphi(f_1), \ldots, \varphi(f_n)$ are 1-proper. Also, by Lemma 2.2 since $\varphi(0_{\mathcal{X}})^{-1}(1) = X_3^{\mathrm{c}}$, we see that $\varphi(f_i)(x) \neq 1$ for all $x \in X_3^{\mathrm{c}}$ and $i \in \{1, \ldots, n\}$. If $\varphi(1_{\mathcal{X}}) = 1_{\mathcal{X}}$ then $X_3 = \mathcal{X}$ and so we get the same conclusions using Remark 2.6. Since φ^{-1} has the same properties as φ we prove the converse implication in the same way.

Let $O_{f_1} \cap \cdots \cap O_{f_n} \neq \emptyset$. The set $(O_{f_1} \cap \cdots \cap O_{f_n})^{\mathrm{c}}$ is closed, so by Urysohn's lemma there exist $h_1 \in \mathcal{C}(\mathcal{X}, I)$ and a nonempty open set $U \subset O_{f_1} \cap \cdots \cap O_{f_n}$ such that $h_1 \equiv 1$ on $(O_{f_1} \cap \cdots \cap O_{f_n})^{\mathrm{c}}$ and $h_1(x) \neq 1$ for every $x \in U$. Also, there exists a 1-proper function h_2 such that $O_{h_2} \subset U$, $h_2(x) \neq 0$ for every $x \in \mathcal{X}$ and $h_2(x) \neq 1$ for every $x \in U^{\mathrm{c}}$. By Lemmas 2.3 and 2.5 or in case $\varphi(1_{\mathcal{X}}) = 1_{\mathcal{X}}$ by Lemma 2.3 and Remark 2.6 we conclude that $\varphi(h_2)$ is 1-proper and never zero. Since $h_2^{-1}(1) \cap h_1^{-1}(1) = \emptyset$ and h_2 is never zero, we obtain

$$\varphi(h_2)^{-1}(1) \cap \varphi(h_1)^{-1}(1) = \emptyset. \tag{2.4}$$

Again, since h_2 is never zero, we get

$$\varphi(h_2)^{-1}(1) \cap \varphi(0_{\mathcal{X}})^{-1}(1) = \emptyset. \tag{2.5}$$

Also, for every $x \in \mathcal{X}$ and $i \in \{1, \ldots, n\}$ we obtain $h_1(x) = 1$ or $f_i(x) = 1$. This implies that for every $i \in \{1, \ldots, n\}$ and $x \in \mathcal{X}$ there exist $g_{i_1}, g_{i_2} \in \{\varphi(0_{\mathcal{X}}), \varphi(h_1), \varphi(f_i)\}$ such that $g_{i_1}(x) = 0$ and $g_{i_2}(x) = 1$. We then conclude by (2.4) and (2.5) that $\varphi(h_2)^{-1}(1) \subset \varphi(f_i)^{-1}(1)$ for every $i \in \{1, \ldots, n\}$. Therefore

$$\emptyset \neq O_{\varphi(h_2)} \subset O_{\varphi(f_1)} \cap \cdots \cap O_{\varphi(f_n)}.$$

We prove the converse implication in a similar way since φ^{-1} has the same properties as φ. ■

We prove the following lemma in a similar way to Lemma 2.7 by additionally using Lemma 2.4.

LEMMA 2.8. *Suppose that* $\varphi(1_{\mathcal{X}}) \neq 1_{\mathcal{X}}$. *The functions* $f_1, \ldots, f_n$ *are* 1*-proper and* $f_i(x) \neq 0$ *for every* $x \in \mathcal{X}$ *and* $f_i(x) \neq 1$ *for every* $x \in A$, $i \in \{1, \ldots, n\}$, *if and only if* $\varphi(f_1), \ldots, \varphi(f_n)$ *are* 0*-proper and* $\varphi(f_i)(x) \neq 1$ *for every* $x \in \mathcal{X}$ *and* $\varphi(f_i)(x) \neq 0$ *for every* $x \in X_3^{\mathrm{c}}$, $i \in \{1, \ldots, n\}$. *Furthermore*, *in this case*

$$O_{f_1} \cap \cdots \cap O_{f_n} \neq \emptyset \quad \textit{if and only if} \quad Z_{\varphi(f_1)} \cap \cdots \cap Z_{\varphi(f_n)} \neq \emptyset.$$

From now on, let $|\mathcal{X}| > 1$. We will use this assumption nearly to the end of the proof. In the next step we will construct a homeomorphism $\mu : \mathcal{X} \to \mathcal{X}$. First assume that $A \neq \emptyset$ and let $x_0 \in A$. Since A is open there exists an open neighbourhood A_{x_0} of x_0 such that $A_{x_0} \subset A$ and $\overline{A}_{x_0} \neq \mathcal{X}$. By Urysohn's lemma there exists a 1-proper function f such that $x_0 \in O_f$, $\overline{O}_f \subset A_{x_0}$, $f(x) \neq 0$ for every $x \in \mathcal{X}$ and $f(x) \neq 1$ for every $x \in A^{\mathrm{c}}$. Let

$\mathcal{F}_{A_{x_0}}$ be the set all such 1-proper functions f. Then $x_0 \in \bigcap_{f\in\mathcal{F}_{A_{x_0}}} O_f$. Let $x_1 \in \mathcal{X}$, $x_1 \neq x_0$. Then there exist open sets A_1, A_2 such that $A_1 \cap A_2 = \emptyset$ and $x_0 \in A_1$ and $x_1 \in A_2$. Again by Urysohn's lemma there exists $f \in \mathcal{F}_{A_{x_0}}$ such that $\overline{O}_f \subset A_1 \cap A_{x_0}$. So, $O_f \cap A_2 = \emptyset$ and hence $x_1 \notin \bigcap_{f\in\mathcal{F}_{A_{x_0}}} O_f$. This gives us

$$\bigcap_{f\in\mathcal{F}_{A_{x_0}}} O_f = \{x_0\}.$$

The set A is nonempty, so $\varphi(1_\mathcal{X}) \neq 0_\mathcal{X}$. Let $f \in \mathcal{F}_{A_{x_0}}$. By Lemma 2.7 then $\varphi(f)$ is also 1-proper. We will next show that there exists $x_1 \in \mathcal{X}$ such that

$$\bigcap_{f\in\mathcal{F}_{A_{x_0}}} O_{\varphi(f)} = \{x_1\}.$$

First assume that $\bigcap_{f\in\mathcal{F}_{A_{x_0}}} \overline{O}_{\varphi(f)} = \emptyset$. Since $\mathcal{X}$ is compact there exist $f_1, \ldots, f_n \in \mathcal{F}_{A_{x_0}}$ such that $\overline{O}_{\varphi(f_1)} \cap \cdots \cap \overline{O}_{\varphi(f_n)} = \emptyset$. But since $O_{f_1} \cap \cdots \cap O_{f_n} \neq \emptyset$ Lemma 2.7 shows that $O_{\varphi(f_1)} \cap \cdots \cap O_{\varphi(f_n)} \neq \emptyset$, a contradiction. So,

$$\bigcap_{f\in\mathcal{F}_{A_{x_0}}} \overline{O}_{\varphi(f)} \neq \emptyset.$$

Next assume that

$$\bigcap_{f\in\mathcal{F}_{A_{x_0}}} O_{\varphi(f)} = \emptyset.$$

Then there exist $x_\lambda \in \bigcap_{f\in\mathcal{F}_{A_{x_0}}} \overline{O}_{\varphi(f)}$ and $f_\lambda \in \mathcal{F}_{A_{x_0}}$ such that $x_\lambda \in \overline{O}_{\varphi(f_\lambda)}$ and $x_\lambda \notin O_{\varphi(f_\lambda)}$. By Lemma 2.7 we infer that $\overline{O}_{\varphi(f_\lambda)} \subset X_3$. By Urysohn's lemma there exists $g_\lambda \in \mathcal{C}(\mathcal{X}, I)$ such that $g_\lambda(x_0) \neq 1$, $g_\lambda(x) \neq 0$ for every $x \in \mathcal{X}$ and $g_\lambda \equiv 1$ on $O^{\mathrm{c}}_{f_\lambda}$. So, $g_\lambda^{-1}(1) \cup f_\lambda^{-1}(1) = \mathcal{X}$ and therefore for every $x \in \mathcal{X}$ there exist $h_1, h_2 \in \{\varphi(0_\mathcal{X}), \varphi(f_\lambda), \varphi(g_\lambda)\}$ such that $h_1(x) = 0$ and $h_2(x) = 1$. Let C_λ be any open neighbourhood of x_λ such that $C_\lambda \subset X_3$. Since $x_\lambda \in \overline{O}_{\varphi(f_\lambda)} \backslash O_{\varphi(f_\lambda)}$ there exists $x \in C_\lambda$ such that $\varphi(f_\lambda)(x) \neq 1$. Since $\mathcal{X}$ is first countable we can construct a sequence $\{x_i : i \in \mathbb{N}\} \subset X_3 \backslash \overline{O}_{\varphi(f_\lambda)}$ such that $\lim_{i\to\infty} x_i = x_\lambda$ and $\varphi(f_\lambda)(x_i) \neq 1$, $i \in \mathbb{N}$. Since $\varphi(0_\mathcal{X})^{-1}(0) = X_3$, this shows that $\varphi(g_\lambda)(x_i) = 1$ for every $i \in \mathbb{N}$ and therefore by the continuity of $\varphi(g_\lambda)$,

$$\varphi(g_\lambda)(x_\lambda) = 1.$$

Since $g_\lambda(x_0) \neq 1$ and g_λ is continuous there exists an open neighbourhood U_1 of x_0 such that $g_\lambda(x) \neq 1$ for every $x \in U_1$ and $U_1 \subset U_\lambda$. By Urysohn's lemma there exists a 1-proper function h_λ such that $O_{h_\lambda} \subset U_1 \subset U_\lambda$, $x_0 \in O_{h_\lambda}$, $h_\lambda(x) \neq 0$ for every $x \in \mathcal{X}$ and $h_\lambda(x) \neq 1$ for every $x \in U_1^{\mathrm{c}}$. So, $h_\lambda \in \mathcal{F}_{A_{x_0}}$. Notice that $h_\lambda(x) = g_\lambda(x) = 1$ for no $x \in \mathcal{X}$. But then, since h_λ is never zero, Lemma 2.2 shows that $\varphi(h_\lambda)(x) = \varphi(g_\lambda)(x) = 1$ for no $x \in \mathcal{X}$. Hence

$$\varphi(h_\lambda)(x_\lambda) \neq 1.$$

This is a contradiction since $h_\lambda \in \mathcal{F}_{A_{x_0}}$ and therefore $x_\lambda \in \overline{O}_{\varphi(h_\lambda)}$. We have proven that

$$\bigcap_{f \in \mathcal{F}_{A_{x_0}}} O_{\varphi(f)} \neq \emptyset.$$

Now assume that there exist $x_1, x_2 \in \mathcal{X}$, $x_1 \neq x_2$, such that $\{x_1, x_2\} \subset \bigcap_{f \in \mathcal{F}_{A_{x_0}}} O_{\varphi(f)}$. So, $x_1, x_2 \in X_3$. Let $V', V'' \subset X_3$ be disjoint open neighbourhoods of x_1 and x_2 respectively. By Urysohn's lemma and the surjectivity of φ there exists a 1-proper function $\varphi(g_1)$ such that $\overline{O}_{\varphi(g_1)} \subset V'$, $x_1 \in O_{\varphi(g_1)}$, $\varphi(g_1)(x) \neq 0$ for every $x \in \mathcal{X}$, and $\varphi(g_1)(x) \neq 1$ for every $x \in V'^{\mathrm{c}}$. Similarly, there exists a 1-proper function $\varphi(g_2)$ such that $\overline{O}_{\varphi(g_2)} \subset V''$, $x_2 \in O_{\varphi(g_2)}$, $\varphi(g_2)(x) \neq 0$ for every $x \in \mathcal{X}$, $\varphi(g_2)(x) \neq 1$ for every $x \in V''^{\mathrm{c}}$. By Lemma 2.7 we conclude that g_1 and g_2 are also 1-proper. Furthermore, $O_{g_1} \cap O_{g_2} = \emptyset$, $g_i(x) \neq 0$ for every $x \in \mathcal{X}$ and $g_i(x) \neq 1$ for every $x \in A^{\mathrm{c}}$, $i \in \{1, 2\}$. Since $\varphi(g_1)^{-1}(1) \cap \varphi(g_2)^{-1}(1) = \emptyset$ and $\varphi(g_1)$ is never zero we obtain $g_1^{-1}(1) \cap g_2^{-1}(1) = \emptyset$. Hence, $\overline{O}_{g_1} \cap \overline{O}_{g_2} = \emptyset$. Without loss of generality we may assume that $x_0 \notin \overline{O}_{g_1}$. By Urysohn's lemma there exists $g_3 \in \mathcal{F}_{A_{x_0}}$ such that $\overline{O}_{g_1} \cap \overline{O}_{g_3} = \emptyset$. By Lemma 2.7 we obtain

$$O_{\varphi(g_1)} \cap O_{\varphi(g_3)} = \emptyset,$$

hence $x_1 \notin O_{\varphi(g_3)}$. But since $g_3 \in \mathcal{F}_{A_{x_0}}$ we get $x_1 \in O_{\varphi(g_3)}$, a contradiction. Therefore

$$\bigcap_{f \in \mathcal{F}_{A_{x_0}}} O_{\varphi(f)} = \{x_1\}.$$

It is easy to check that this intersection is independent of the selection of the neighbourhood A_{x_0}.

Now assume that $A^{\mathrm{c}} \neq \emptyset$ and let $x_0 \in A^{\mathrm{c}}$. As before, there exists an open neighbourhood A_{x_0} of x_0 such that $A_{x_0} \subset A^{\mathrm{c}}$ and $\overline{A}_{x_0} \neq \mathcal{X}$. There also exists a 1-proper function f such that $x_0 \in O_f$, $\overline{O}_f \subset A_{x_0}$, $f(x) \neq 0$ for every $x \in \mathcal{X}$ and $f(x) \neq 1$ for every $x \in A$. Let now $\mathcal{F}_{A_{x_0}}$ be the set of all such 1-proper functions f. Let $f \in \mathcal{F}_{A_{x_0}}$. By Lemma 2.8, $\varphi(f)$ is then 0-proper. By using Lemma 2.8 we can prove as before that there exists $y_1 \in \mathcal{X}$ such that

$$\bigcap_{f \in \mathcal{F}_{A_{x_0}}} Z_{\varphi(f)} = \{y_1\}.$$

Again we observe that this intersection is independent of the selection of the neighbourhood of x_0.

We now define $\psi : \mathcal{X} \to \mathcal{X}$ in the following way. If $x_0 \in A$ then ψ maps x_0 to x_1, and if $x_0 \in A^{\mathrm{c}}$ then ψ maps x_0 to y_1. Notice that $x_1 \in X_3$

and $y_1 \in X_3^{\mathrm{c}}$. We will prove that ψ is a homeomorphism. Let $x_a \neq x_b$, $x_a, x_b \in A$. There exist 1-proper functions f_1 and f_2 such that $f_i(x) \neq 0$ for every $x \in \mathcal{X}$, $f_i(x) \neq 1$ for every $x \in A^{\mathrm{c}}$, $i \in \{1,2\}$, and O_{f_1}, O_{f_2} are disjoint neighbourhoods of x_a, x_b respectively. Since $O_{f_1} \cap O_{f_2} = \emptyset$, by Lemma 2.7 we get $O_{\varphi(f_1)} \cap O_{\varphi(f_2)} = \emptyset$. This yields

$$\{\psi(x_a)\} = \bigcap_{f \in \mathcal{F}_{A_{x_a}}} O_{\varphi(f)} \neq \bigcap_{f \in \mathcal{F}_{A_{x_b}}} O_{\varphi(f)} = \{\psi(x_b)\}.$$

So, $\psi(x_a) \neq \psi(x_b)$. Similarly, if $y_a, y_b \in A^{\mathrm{c}}$, $y_a \neq y_b$, then $\psi(y_a) \neq \psi(y_b)$ by Lemma 2.8. If $x_a \in A$ and $y_b \in A^{\mathrm{c}}$ then $\psi(x_a) \in X_3$ and $\psi(y_b) \in X_3^{\mathrm{c}}$. So, ψ is injective. We prove that ψ is also surjective by using Lemmas 2.7 and 2.8 and the fact that φ^{-1} has the same properties as φ.

Assume now $\varphi(f)$ is 1-proper and $\varphi(f)(x) \neq 0$ for every $x \in \mathcal{X}$ and $\varphi(f)(x) \neq 1$ for every $x \in X_3^{\mathrm{c}}$. Let $x \in O_{\varphi(f)}$. The set O_f is then a neighbourhood of $\psi^{-1}(x)$. So, $\psi^{-1}(O_{\varphi(f)}) \subset O_f$. Similarly, $\psi(x) \in O_{\varphi(f)}$ for each $x \in O_f$, which yields $\psi(O_f) \subset O_{\varphi(f)}$ and therefore

$$\psi^{-1}(O_{\varphi(f)}) = O_f.$$

Hence $\psi^{-1}(O_{\varphi(f)})$ is an open set. Similarly, if $\varphi(f)$ is 0-proper and $\varphi(f)(x) \neq 1$ for every $x \in \mathcal{X}$ and $\varphi(f)(x) \neq 0$ for every $x \in X_3$, then as before we prove that $\psi^{-1}(Z_{\varphi(f)}) = O_f$.

Let now $C \subset \mathcal{X}$ be any nonempty open set. Set $A_1 = X_3 \cap C$ and $A_2 = X_3^{\mathrm{c}} \cap C$. Suppose $A_1 \neq \emptyset$. By Urysohn's lemma we may find for every $a \in A_1$ a 1-proper function f_a such that $a \in O_{f_a}$, $\overline{O}_{f_a} \subset A_1$, $f_a(x) \neq 0$ for every $x \in \mathcal{X}$ and $f_a(x) \neq 1$ for every $x \in X_3^{\mathrm{c}}$. Therefore $A_1 = \bigcup_{a \in A_1} O_{f_a}$, hence

$$\psi^{-1}(A_1) = \bigcup_{a \in A_1} \psi^{-1}(O_{f_a}).$$

Similarly, assuming that $A_2 \neq \emptyset$ we may find for every $b \in A_2$ a 0-proper function f_b such that $b \in Z_{f_b}$, $\overline{Z}_{f_b} \subset A_2$, $f_b(x) \neq 1$ for every $x \in \mathcal{X}$ and $f_b(x) \neq 0$ for every $x \in X_3$. Therefore $A_2 = \bigcup_{b \in A_2} Z_{f_b}$, hence

$$\psi^{-1}(A_2) = \bigcup_{b \in A_2} \psi^{-1}(Z_{f_b}).$$

It follows that $\psi^{-1}(C)$ is an open set. This shows that ψ is continuous. Since $\mathcal{X}$ is a compact Hausdorff space and ψ is a continuous bijection we conclude that ψ is a homeomorphism. Set

$$\mu = \psi^{-1}.$$

To conclude the proof we need the following auxiliary result.

LEMMA 2.9. *Let* $f \in \mathcal{C}(\mathcal{X}, I)$, $x_1 \in X_3$ *and* $y_1 \in X_3^{\rm c}$. *If* $\max f = f(\mu(x_1))$ *then* $\varphi(f)(x_1) = f(\mu(x_1))$, *and if* $\max f = f(\mu(y_1))$ *then* $\varphi(f)(y_1) = 1 - f(\mu(y_1))$.

Proof. Let $f \in \mathcal{C}(\mathcal{X}, I)$, $f \neq 0_{\mathcal{X}}$. Since $\mathcal{X}$ is compact and μ surjective there exists $x_1 \in \mathcal{X}$ such that $\max f = f(\mu(x_1)) = \lambda_0$. Suppose first $x_1 \in X_3$ and let $g = (1/\lambda_0)f$. Then $g \in \mathcal{C}(\mathcal{X}, I)$ and $g(\mu(x_1)) = 1$. Suppose $\varphi(g)(x_1) \neq 1$. Since $\varphi(g)$ is continuous there exists an open neighbourhood V of x_1, $V \subset X_3$, such that $\varphi(g)(x) \neq 1$ for every $x \in V$. By Urysohn's lemma and the surjectivity of φ there exists a 1-proper function $\varphi(h)$ such that $x_1 \in O_{\varphi(h)}$, $\varphi(h)(x) \neq 0$ for every $x \in \mathcal{X}$, and $\varphi(h)(x) \neq 1$ for every $x \in V^{\rm c}$. It follows that $\mu(x_1) \in O_h$. On the one hand, we obtain

$$h(\mu(x_1)) = g(\mu(x_1)) = 1,$$

but on the other hand, since $\varphi(h)^{-1}(1) \cap \varphi(g)^{-1}(1) = \emptyset$ and $\varphi(h)(x) \neq 0$ for every $x \in \mathcal{X}$, we conclude that $h(x) = g(x) = 1$ for no $x \in \mathcal{X}$, a contradiction. So, $\varphi(g)(x_1) = 1$. Since $x_1 \in X_3$ it follows that

$$\varphi(f)(x_1) = \varphi(\lambda_0 g + (1-\lambda_0)0_{\mathcal{X}})(x_1) = \lambda_0\varphi(g)(x_1) + (1-\lambda_0)\varphi(0_{\mathcal{X}})(x_1) = \lambda_0.$$

So, $\varphi(f)(x_1) = f(\mu(x_1))$. Suppose now that $y_1 \in X_3^{\rm c}$ and $\max f = f(\mu(y_1)) = \lambda_0$. Define g as before. Again, by using Lemma 2.8 we prove that $\varphi(g)(x_1) = 0$. So,

$$\varphi(f)(y_1) = \lambda_0\varphi(g)(y_1) + (1-\lambda_0)\varphi(0_{\mathcal{X}})(y_1) = 1 - \lambda_0$$

and hence $\varphi(f)(y_1) = 1 - f(\mu(y_1))$. ∎

Let $f \in \mathcal{C}(\mathcal{X}, I)$, $x_0 \in \mathcal{X}$ and suppose $f(\mu(x_0)) \neq \max f$. Let $D_{x_0} = \{x : f(x) \geq f(\mu(x_0))\}$. Then D_{x_0} is a nonempty closed set. Also, define $g : D_{x_0} \to I$ by $g(x) = 1 - f(x)$, $x \in D_{x_0}$. Then $\max g = g(\mu(x_0))$. Since g is continuous there exists by Tietze's theorem a continuous extension $g_1 : \mathcal{X} \to I$. Define $g_2 : \mathcal{X} \to I$ by

$$g_2(x) = \begin{cases} g_1(x), & x \in D_{x_0}, \\ \min\{g_1(x), g_1(\mu(x_0))\}, & x \in D_{x_0}^{\rm c}. \end{cases}$$

Then $g_2 \in \mathcal{C}(\mathcal{X}, I)$ and $\max g_2 = g_2(\mu(x_0))$. Let now $h = \frac{1}{2}\, g_2 + \frac{1}{2}\, f$. Clearly, $h \in \mathcal{C}(\mathcal{X}, I)$. Suppose first $x \in D_{x_0}^{\rm c}$. Then $f(x) < f(\mu(x_0))$ and hence

$$h(x) < \frac{1}{2}\, g_2(\mu(x_0)) + \frac{1}{2}\, f(\mu(x_0)) = h(\mu(x_0)).$$

Next, let $x \in D_{x_0}$. Then $g_2(x) = 1 - f(x)$ and hence

$$h(x) = \frac{1}{2}\,(1 - f(x)) + \frac{1}{2}\, f(x) = \frac{1}{2}.$$

Since $\mu(x_0) \in D_{x_0}$ and hence $h(\mu(x_0)) = 1/2$ it follows that $\max h = h(\mu(x_0)) = 1/2$. Suppose $x_0 \in X_3$. Since $\varphi(h) = \frac{1}{2}\varphi(g_2) + \frac{1}{2}\varphi(f)$ Lemma 2.9

yields

$$\varphi(f)(x_0) = 2h(\mu(x_0)) - g_2(\mu(x_0)) = 1 - g_2(\mu(x_0)) = f(\mu(x_0)).$$

Let now $x_0 \in X_3^c$. Again, by Lemma 2.9 we get

$$\varphi(f)(x_0) = 2(1 - h(\mu(x_0))) - (1 - g_2(\mu(x_0))) = g_2(\mu(x_0)) = 1 - f(\mu(x_0)).$$

Let C be a component in $\mathcal{X}$. Then $\varphi(1_{\mathcal{X}})(C)$ is connected. It follows that either $\varphi(1_{\mathcal{X}})(x) = 1$ for every $x \in C$ or $\varphi(1_{\mathcal{X}})(x) = 0$ for every $x \in C$. So, $C \subset X_3$ or $C \subset X_3^c$, which shows that either

$$\varphi(f)(x) = f(\mu(x)) \quad \text{for every } x \in C \text{ and } f \in \mathcal{C}(\mathcal{X}, I),$$

or

$$\varphi(f)(x) = 1 - f(\mu(x)) \quad \text{for every } x \in C \text{ and } f \in \mathcal{C}(\mathcal{X}, I).$$

Finally, let $|\mathcal{X}| = 1$. Clearly, then $\varphi(f)(x) = f(x)$ or $\varphi(f)(x) = 1 - f(x)$ for a unique $x \in \mathcal{X}$. This concludes the proof of Theorem 1.1. ∎

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