

On the closure of the Lizorkin space in spaces of Beppo Levi type

by

TAKAHIDE KUROKAWA (Kagoshima)

*Dedicated to Professor Masayuki Ito
on the occasion of his sixtieth birthday*

Abstract. The purpose of this paper is to give a characterization of the closure of the Lizorkin space in spaces of Beppo Levi type. As preparations for the proof, we establish the invariance of the Lizorkin space, and give local integral representations for smooth functions.

1. Introduction and preliminaries. The purpose of this paper is to give a characterization of the closure of the Lizorkin space in spaces of Beppo Levi type. Let \mathbb{R}^n be the n -dimensional Euclidean space. We denote the Schwartz space on \mathbb{R}^n by \mathcal{S} . That is, \mathcal{S} is the class of all C^∞ -functions φ in \mathbb{R}^n such that

$$\sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \varphi(x)| < \infty$$

for all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ where $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $D^\beta = D_1^{\beta_1} \dots D_n^{\beta_n}$ ($D_j = \partial/\partial x_j$). Unless otherwise specified, all functions are assumed to be complex-valued. The Schwartz space \mathcal{S} contains the space \mathcal{D} of all C^∞ -functions with compact support. The *Lizorkin space* Φ is defined by

$$\Phi = \left\{ \varphi \in \mathcal{S} : \int_{\mathbb{R}^n} \varphi(x) x^\alpha dx = 0 \text{ for all } \alpha \right\}$$

([SKM: §25 in Chap. 5]). Further, we introduce the space Ψ as follows:

$$\Psi = \{ \psi \in \mathcal{S} : D^\alpha \psi(0) = 0 \text{ for all } \alpha \}.$$

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The Fourier transform $\mathcal{F}u$ and the inverse Fourier transform $\bar{\mathcal{F}}u$ of an integrable function u are defined by

$$\mathcal{F}u(x) = \int e^{-ix \cdot y} u(y) dy, \quad \bar{\mathcal{F}}u(x) = \int e^{ix \cdot y} u(y) dy = \mathcal{F}u(-x)$$

where $x \cdot y = x_1 y_1 + \dots + x_n y_n$. By the Fourier inversion theorem, for $u \in \mathcal{S}$ we have the equality

$$(1.1) \quad \bar{\mathcal{F}}\mathcal{F}u = \mathcal{F}\bar{\mathcal{F}}u = (2\pi)^n u.$$

Noting that

$$D^\alpha(\bar{\mathcal{F}}\varphi)(0) = \int \varphi(y)(iy)^\alpha dy$$

and

$$\int \mathcal{F}\psi(y)(iy)^\alpha dy = (2\pi)^n D^\alpha\psi(0)$$

for $\varphi, \psi \in \mathcal{S}$, we see that

$$(1.2) \quad \Phi = \mathcal{F}(\Psi), \quad \Psi = \mathcal{F}(\Phi)$$

by (1.1). Therefore, since the function $e^{-a|x|^2 - b/|x|^2}$ ($a > 0, b > 0$) belongs to Ψ , the function $\mathcal{F}(e^{-a|x|^2 - b/|x|^2})$ is an example of a function in Φ . We note that

$$\Phi \cap \mathcal{D} = \{0\}.$$

Throughout this paper, let $1 < p < \infty$. For real numbers r and s , we define the weighted Lebesgue spaces $L^{p,r,(\log)^s}$ as follows:

$$L^{p,r,(\log)^s} = \left\{ u : \|u\|_{p,r,(\log)^s} = \left(\int |u(x)|^p (1 + |x|)^{rp} (\log(e + |x|))^{sp} dx \right)^{1/p} < \infty \right\}$$

where e is the base of the natural logarithm. We simply write $L^{p,0,(\log)^0} = L^p$ and $\|u\|_{p,0,(\log)^0} = \|u\|_p$. Further, L^1 denotes the space of all integrable functions, and L^1_{loc} is the set of all locally integrable functions. For a positive integer m , the Sobolev space W^p_m and the space \mathcal{L}^p_m of Beppo Levi type are defined by

$$W^p_m = \left\{ u \in L^1_{\text{loc}} : \|u\|_{W^p_m} = \sum_{|\gamma| \leq m} \|D^\gamma u\|_p < \infty \right\}$$

and

$$\mathcal{L}^p_m = \left\{ u \in L^1_{\text{loc}} : |u|_{m,p} = \sum_{|\gamma|=m} \|D^\gamma u\|_p < \infty \right\}$$

where $|\gamma| = \gamma_1 + \dots + \gamma_n$ for a multi-index $\gamma = (\gamma_1, \dots, \gamma_n)$. The norm $\|u\|_{\mathcal{L}^p_m}$ in \mathcal{L}^p_m is given by

$$\|u\|_{\mathcal{L}^p_m} = \int_{|x| < 1} |u(x)| dx + |u|_{m,p}.$$

The denseness of Φ in L^p was established in P. I. Lizorkin [Li1] and S. G. Samko [Sa]. The denseness of Φ in the Bessel potential spaces is proved in P. I. Lizorkin [Li2], and hence Φ is dense in the Sobolev spaces W_m^p . In the final Section 4, we give a characterization of the closure of Φ in spaces \mathcal{L}_m^p of Beppo Levi type (Theorem 4.8). As a corollary we show that if $p \geq n$, then Φ is dense in \mathcal{L}_m^p , and if $p < n$, then Φ is not dense in \mathcal{L}_m^p (Corollary 4.9). Sections 2 and 3 are preparations for Section 4. In Section 2 we establish the invariance of Φ . Section 3 is devoted to local integral representations of smooth functions and interpolation inequalities. Throughout this paper we use the symbol C for a generic positive constant whose value may be different at each occurrence.

2. The invariance of Φ . In this section we establish the invariance of Φ relative to a class of operators which contains differential operators, the Riesz transforms and the Riesz potential operators. Let \mathbb{N} denote the set of natural numbers including zero; $2\mathbb{N}$ stands for the set of nonnegative even numbers.

LEMMA 2.1. *Let $m \in \mathbb{N}$. If*

$$\lim_{|x| \rightarrow 0} \frac{1}{|x|^m} \sum_{|\gamma|=m} a_\gamma x^\gamma = 0,$$

then $a_\gamma = 0$ for all γ with $|\gamma| = m$.

Proof. We put $P(x) = \sum_{|\gamma|=m} a_\gamma x^\gamma$. By assumption, for Θ with $|\Theta| = 1$ we have

$$0 = \lim_{t \rightarrow 0+0} \frac{P(t\Theta)}{|t\Theta|^m} = \lim_{t \rightarrow 0+0} \frac{t^m P(\Theta)}{t^m} = P(\Theta).$$

For $x \neq 0$, by putting $t = |x|$ and $\Theta = x/|x|$, we get

$$P(x) = P(t\Theta) = t^m P(\Theta) = 0.$$

Since it is clear that $P(0) = 0$, we conclude that $P(x)$ is identically 0, and hence $a_\gamma = 0$ for all γ with $|\gamma| = m$.

We define

$$\mathcal{H} = \{f \in C^\infty(\mathbb{R}^n) : D^\alpha f(0) = 0 \text{ for all } \alpha\}.$$

LEMMA 2.2. *Let $f \in C^\infty(\mathbb{R}^n)$. Then the following three conditions are equivalent:*

- (I) $f \in \mathcal{H}$.
- (II) For any multi-index α and any $l \in \mathbb{N}$,

$$\lim_{|x| \rightarrow 0} \frac{D^\alpha f(x)}{|x|^l} = 0.$$

(III) For any $l \in \mathbb{N}$,

$$\lim_{|x| \rightarrow 0} \frac{f(x)}{|x|^l} = 0.$$

Proof. First, we show (I) \Rightarrow (II). Let $f \in \mathcal{H}$. For any $l \in \mathbb{N}$, by the condition $f \in \mathcal{H}$ and Taylor's formula, we have

$$f(x) = \sum_{|\gamma|=l} \frac{D^\gamma f(\theta x)}{\gamma!} x^\gamma, \quad 0 < \theta < 1.$$

Hence

$$\lim_{|x| \rightarrow 0} \frac{f(x)}{|x|^l} = \sum_{|\gamma|=l} \lim_{|x| \rightarrow 0} \frac{D^\gamma f(\theta x)}{\gamma!} \cdot \frac{x^\gamma}{|x|^l} = 0,$$

because $\lim_{|x| \rightarrow 0} D^\gamma f(\theta x) = D^\gamma f(0) = 0$. Moreover, since $D^\alpha f \in \mathcal{H}$ for any multi-index α , by applying the above argument to $D^\alpha f$, we get

$$\lim_{|x| \rightarrow 0} \frac{D^\alpha f(x)}{|x|^l} = 0$$

for any $l \in \mathbb{N}$. Thus we obtain (II). The implication (II) \Rightarrow (III) is trivial.

We show (III) \Rightarrow (I). First, by taking $l = 0$, we get $f(0) = \lim_{|x| \rightarrow 0} f(x) = 0$. We assume that $D^\gamma f(0) = 0$ for any γ with $|\gamma| \leq m$. Then by Taylor's formula we have

$$\frac{f(x)}{|x|^{m+1}} = \frac{1}{|x|^{m+1}} \sum_{|\gamma|=m+1} \frac{D^\gamma f(0)}{\gamma!} x^\gamma + \sum_{|\gamma|=m+2} \frac{D^\gamma f(\theta x)}{\gamma!} \cdot \frac{x^\gamma}{|x|^{m+1}}, \quad 0 < \theta < 1.$$

By assumption we have $\lim_{|x| \rightarrow 0} f(x)/|x|^{m+1} = 0$, and moreover for γ with $|\gamma| = m + 2$, $\lim_{|x| \rightarrow 0} x^\gamma/|x|^{m+1} = 0$. Therefore

$$\lim_{|x| \rightarrow 0} \frac{1}{|x|^{m+1}} \sum_{|\gamma|=m+1} \frac{D^\gamma f(0)}{\gamma!} x^\gamma = 0.$$

Hence, it follows from Lemma 2.1 that $D^\gamma f(0) = 0$ for any γ with $|\gamma| = m + 1$. Consequently, by induction we obtain $D^\gamma f(0) = 0$ for any γ , and hence $f \in \mathcal{H}$. This proves (III) \Rightarrow (I).

Let $\Omega \subset \mathbb{R}^n$ be an open set and $m \in \mathbb{N}$. $C^m(\Omega)$ is the space of functions such that all partial derivatives $D^\alpha f$, $|\alpha| \leq m$, are continuous on Ω , and $C^\infty(\Omega) = \bigcap_{m=0}^\infty C^m(\Omega)$. The following lemma is a consequence of the mean value theorem of calculus.

LEMMA 2.3. *Let $f \in C^0(\mathbb{R}^n)$ and $f \in C^\infty(\mathbb{R}^n - \{0\})$. If $\lim_{|x| \rightarrow 0} D^\alpha f(x)$ exists for any multi-index α , then $f \in C^\infty(\mathbb{R}^n)$.*

LEMMA 2.4. *Let $\psi \in \Psi$ and $m \in C^\infty(\mathbb{R}^n - \{0\})$. If for any multi-index γ , there exist $l_\gamma \in \mathbb{N}$ and $C_\gamma > 0$ such that*

$$(2.1) \quad |D^\gamma m(x)| \leq C_\gamma \begin{cases} |x|^{l_\gamma}, & |x| \geq 1, \\ |x|^{-l_\gamma}, & 0 < |x| < 1, \end{cases}$$

then the function

$$f(x) = \begin{cases} m(x)\psi(x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is in Ψ .

Proof. First, it is clear that $f \in C^\infty(\mathbb{R}^n - \{0\})$. By (2.1), the assumption $\psi \in \Psi$ and Lemma 2.2 we see that

$$\lim_{|x| \rightarrow 0} |f(x)| = \lim_{|x| \rightarrow 0} |m(x)\psi(x)| \leq C_0 \lim_{|x| \rightarrow 0} \frac{|\psi(x)|}{|x|^{l_0}} = 0,$$

and hence $f \in C^0(\mathbb{R}^n)$. We show

$$(2.2) \quad \lim_{|x| \rightarrow 0} D^\alpha f(x) = 0 \quad \text{for any } \alpha.$$

Since

$$D^\alpha f(x) = \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} D^\gamma m(x) D^{\alpha-\gamma} \psi(x)$$

by the Leibniz formula, it is sufficient to show

$$D^\gamma m(x) D^{\alpha-\gamma} \psi(x) \rightarrow 0 \quad (|x| \rightarrow 0)$$

for $\gamma \leq \alpha$, which follows from (2.1), $\psi \in \Psi$ and Lemma 2.2 again. Hence we obtain (2.2), and hence $f \in C^\infty(\mathbb{R}^n)$ by Lemma 2.3. Further, (2.2) also implies $D^\alpha f(0) = 0$ for any multi-index α , and hence $f \in \mathcal{H}$. Similarly, the condition (2.1) and $\psi \in \Psi$ imply $f \in \mathcal{S}$. Thus we obtain $f \in \mathcal{H} \cap \mathcal{S} = \Psi$.

The dual space of \mathcal{S} is denoted by \mathcal{S}' .

PROPOSITION 2.5. *Suppose $T : \Phi \rightarrow \mathcal{S}'$ is a linear operator, and for $\varphi \in \Phi$,*

$$\mathcal{F}(T\varphi)(x) = m(x)\mathcal{F}\varphi(x) \quad \text{in } \mathcal{S}',$$

where $m \in C^\infty(\mathbb{R}^n - \{0\})$ and for any multi-index γ there exist $l_\gamma \in \mathbb{N}$ and $C_\gamma > 0$ such that

$$|D^\gamma m(x)| \leq C_\gamma \begin{cases} |x|^{l_\gamma}, & |x| \geq 1, \\ |x|^{-l_\gamma}, & 0 < |x| < 1. \end{cases}$$

Then the operator T carries Φ into Φ .

Proof. For $\varphi \in \Phi$, by Lemma 2.4 the function

$$\psi(x) = \begin{cases} m(x)\mathcal{F}\varphi(x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is in Ψ since $\mathcal{F}\varphi \in \Psi$. Therefore we have

$$T\varphi = \bar{\mathcal{F}}\mathcal{F}(T\varphi) = \bar{\mathcal{F}}\psi \in \Phi.$$

For a positive integer m , the *Riesz kernel* $\kappa_m(x)$ of order m is given by

$$\kappa_m(x) = \frac{1}{\gamma_{m,n}} \begin{cases} |x|^{m-n}, & m-n \notin 2\mathbb{N}, \\ (\delta_{m,n} - \log|x|)|x|^{m-n}, & m-n \in 2\mathbb{N}, \end{cases}$$

with

$$\gamma_{m,n} = \begin{cases} \pi^{n/2} 2^m \Gamma(m/2) / \Gamma((n-m)/2), & m-n \notin 2\mathbb{N}, \\ (-1)^{(m-n)/2} 2^{m-1} \pi^{n/2} \Gamma(m/2) ((m-n)/2)!, & m-n \in 2\mathbb{N}, \end{cases}$$

and

$$\delta_{m,n} = \frac{\Gamma'(m/2)}{2\Gamma(m)} + \frac{1}{2} \left(1 + \frac{1}{2} + \dots + \frac{1}{(m-n)/2} - \mathcal{C} \right) - \log \pi$$

where \mathcal{C} is Euler's constant. For $f \in \mathcal{S}$, we define the *Riesz potential* κ_m^f of f as follows:

$$\kappa_m^f(x) = \int \kappa_m(x-y) f(y) dy.$$

The Fourier transform of the Riesz kernel κ_m is given by

$$\mathcal{F}\kappa_m(x) = \text{Pf.}|x|^{-m} \quad ([\text{Sc: Sect. 7 in Chap. VII}])$$

where Pf. stands for the pseudo function ([Sc: Sect. 4 in Chap. II]). Since

$$\text{Pf.}|x|^{-m}\psi(x) = |x|^{-m}\psi(x)$$

for $\psi \in \Psi$, we have

$$(2.3) \quad \mathcal{F}\kappa_m^f(x) = |x|^{-m} \mathcal{F}f(x)$$

for $f \in \Phi$. Further, we define the *Riesz transforms* R_j ($j = 1, \dots, n$) as follows:

$$R_j f(x) = \lim_{\varepsilon \rightarrow 0} \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \int_{|x-y| \geq \varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) dy.$$

The Riesz transforms are bounded operators on L^p :

$$(2.4) \quad \|R_j f\|_p \leq C \|f\|_p, \quad j = 1, \dots, n,$$

and for $f \in L^2$, the Fourier transform of $R_j f$ is given by

$$(2.5) \quad \mathcal{F}(R_j f)(x) = -\frac{ix_j}{|x|} \mathcal{F}f(x).$$

For a multi-index $\gamma = (\gamma_1, \dots, \gamma_n)$, we set

$$R^\gamma = R_1^{\gamma_1} \dots R_n^{\gamma_n}.$$

For $f \in L^2$, it follows from (2.5) that

$$(2.6) \quad \mathcal{F}(R^\gamma f)(x) = \frac{(-i)^{|\gamma|} x^\gamma}{|x|^{|\gamma|}} \mathcal{F}f(x).$$

COROLLARY 2.6. *The differential operator D^γ , the Riesz potential operator κ_m^f and the Riesz transforms R_j ($j = 1, \dots, n$) carry Φ into Φ .*

Proof. This follows from the formula $\mathcal{F}(D^\gamma f)(x) = (ix)^\gamma \mathcal{F}f(x)$, (2.3), (2.5) and Proposition 2.5.

REMARK 2.7. The invariance of Φ relative to the Riesz potential operator was proved in [He] and [SKM].

For $f \in \mathcal{L}_m^p$ we put

$$D^m f = \sum_{|\gamma|=m} \frac{m!}{\gamma!} R^\gamma D^\gamma f.$$

COROLLARY 2.8. *If $u \in \Phi$, then $f = D^m u \in \Phi$ and $u = \kappa_m^f$.*

Proof. That $D^m u \in \Phi$ follows from Corollary 2.6. Since

$$\mathcal{F}(D^m u)(x) = |x|^m \mathcal{F}u(x)$$

for $u \in \Phi$ by (2.6), we see that

$$\begin{aligned} \mathcal{F}\kappa_m^f(x) &= \mathcal{F}\kappa_m(x) \mathcal{F}f(x) = \mathcal{F}\kappa_m(x) \mathcal{F}(D^m u)(x) \\ &= \text{Pf.}|x|^{-m} |x|^m \mathcal{F}u(x) = \mathcal{F}u(x). \end{aligned}$$

3. Local integral representations of smooth functions. If u is a C^m -function with compact support, then it can be represented by its partial derivatives of order m as follows:

$$(3.1) \quad u(x) = \sum_{|\alpha|=m} \frac{m}{\alpha! \sigma_n} \int \frac{(x-y)^\alpha}{|x-y|^n} D^\alpha u(y) dy$$

([Re]) where σ_n is the surface area of the unit sphere, and

$$(3.2) \quad u(x) = \sum_{|\alpha|=m} \frac{(-1)^m m!}{\alpha!} \int D^\alpha \kappa_{2m}(x-y) D^\alpha u(y) dy$$

([Wa]). In this section we give two kinds of integral representations of C^m -functions, which correspond to (3.1) and (3.2). One is based on Taylor's formula, and the other is deduced from the fact that the Riesz kernel κ_{2m} is a fundamental solution for the iterated Laplace operator Δ^m , namely

$$(3.3) \quad \Delta^m \kappa_{2m} = (-1)^m \delta$$

where δ is the point mass at the origin.

Let $0 < \varepsilon_1 < \varepsilon_2$. We take a function $\eta \in C^\infty(\mathbb{R})$ such that $\text{supp } \eta \subset \{\varepsilon_1 \leq t \leq \varepsilon_2\}$ and

$$\int_0^\infty \eta(t) t^{n-1} dt = \frac{1}{\sigma_n},$$

and set

$$\tau(t) = \int_t^\infty \eta(s) s^{n-1} ds.$$

Moreover we put

$$(3.4) \quad \omega(x) = \eta(|x|),$$

$$(3.5) \quad \chi(x) = \sigma_n \int_{|x|}^{\infty} \omega(tx/|x|) t^{n-1} dt.$$

Then $\int \omega(x) dx = 1$ and $\chi(x) = \sigma_n \tau(|x|)$. Since $\tau \in C^\infty([0, \infty))$ and

$$\tau(t) = \begin{cases} 1/\sigma_n & \text{for } 0 \leq t \leq \varepsilon_1, \\ 0 & \text{for } t \geq \varepsilon_2, \end{cases}$$

we have $\chi \in \mathcal{D}$ and

$$\chi(x) = \begin{cases} 1 & \text{for } |x| \leq \varepsilon_1, \\ 0 & \text{for } |x| \geq \varepsilon_2. \end{cases}$$

PROPOSITION 3.1. *Let $0 < \varepsilon_1 < \varepsilon_2$. Then there exist functions $\mu, \chi \in \mathcal{D}$ such that $\text{supp } \mu, \text{supp } \chi \subset \{|x| \leq \varepsilon_2\}$, $\mu(x) = 0$ on $\{|x| \leq \varepsilon_1\}$, $\chi(x) = 1$ on $\{|x| \leq \varepsilon_1\}$, and if $u \in C^m(\mathbb{R}^n)$, then*

$$(3.6) \quad u(x) = \int \mu(x-y)u(y) dy + \sum_{|\alpha|=m} \frac{m}{\alpha! \sigma_n} \int \frac{(x-y)^\alpha}{|x-y|^n} \chi(x-y) D^\alpha u(y) dy.$$

Proof. By Taylor's formula we have

$$(3.7) \quad u(x) = \sum_{|\gamma| < m} \frac{D^\gamma u(y)}{\gamma!} (x-y)^\gamma \\ + m \sum_{|\alpha|=m} \frac{(x-y)^\alpha}{\alpha!} \int_0^1 (1-t)^{m-1} D^\alpha u(y+t(x-y)) dt.$$

We take functions ω and χ defined by (3.4) and (3.5). Multiplying (3.7) by $\omega(x-y)$ and integrating with respect to y , we get

$$u(x) = \sum_{|\gamma| < m} \frac{1}{\gamma!} \int D^\gamma u(y) (x-y)^\gamma \omega(x-y) dy \\ + \sum_{|\alpha|=m} \frac{m}{\alpha!} \int (x-y)^\alpha \omega(x-y) \int_0^1 (1-t)^{m-1} D^\alpha u(y+t(x-y)) dt dy \\ = \sum_{|\gamma| < m} \frac{1}{\gamma!} \int D^\gamma u(y) (x-y)^\gamma \omega(x-y) dy \\ + \sum_{|\alpha|=m} \frac{m}{\alpha!} \int_0^1 (1-t)^{m-1} \left(\int (x-y)^\alpha \omega(x-y) D^\alpha u(y+t(x-y)) dy \right) dt \\ = I_1(x) + I_2(x).$$

By integration by parts we have

$$I_1(x) = \int \mu(x - y)u(y) dy \quad \text{where} \quad \mu(x) = \sum_{|\gamma| < m} \frac{1}{\gamma!} D^\gamma(x^\gamma \omega(x)).$$

Since $\omega \in \mathcal{D}$ and $\text{supp } \omega \subset \{\varepsilon_1 \leq |x| \leq \varepsilon_2\}$ by (3.4), μ also has the same properties. Further by the change of variables $y + t(x - y) = z$, we obtain

$$\begin{aligned} I_2(x) &= \sum_{|\alpha|=m} \frac{m}{\alpha!} \int_0^1 (1-t)^{m-1} \left(\int \frac{(x-z)^\alpha}{(1-t)^m} \omega\left(\frac{x-z}{1-t}\right) D^\alpha u(z) \frac{dz}{(1-t)^n} \right) dt \\ &= \sum_{|\alpha|=m} \frac{m}{\alpha!} \int D^\alpha u(z) (x-z)^\alpha \left(\int_0^1 \omega\left(\frac{x-z}{1-t}\right) \frac{dt}{(1-t)^{n+1}} \right) dz \end{aligned}$$

because $x - y = (x - z)/(1 - t)$. By the change of variable $|x - z|/(1 - t) = s$, we get

$$\begin{aligned} I_2(x) &= \sum_{|\alpha|=m} \frac{m}{\alpha!} \int D^\alpha u(z) (x-z)^\alpha \left(\int_{|x-z|}^\infty \omega\left(s \frac{x-z}{|x-z|}\right) \frac{s^{n-1}}{|x-z|^n} ds \right) dz \\ &= \sum_{|\alpha|=m} \frac{m}{\alpha! \sigma_n} \int D^\alpha u(z) \frac{(x-z)^\alpha}{|x-z|^n} \chi(x-z) dz \end{aligned}$$

because $(x - z)/(1 - t) = s(x - z)/|x - z|$. Thus we obtain (3.6).

PROPOSITION 3.2. *Let $0 < \varepsilon_1 < \varepsilon_2$. Then there exist functions $\zeta, \xi \in \mathcal{D}$ such that $\text{supp } \zeta, \text{supp } \xi \subset \{|x| \leq \varepsilon_2\}$, $\zeta(x) = 0$ on $\{|x| \leq \varepsilon_1\}$, $\xi(x) = 1$ on $\{|x| \leq \varepsilon_1\}$, and if $u \in C^m(\mathbb{R}^n)$, then*

$$(3.8) \quad u(x) = \int \zeta(x-y)u(y) dy + \sum_{|\alpha|=m} \frac{(-1)^m m!}{\alpha!} \int D^\alpha (\xi \kappa_{2m})(x-y) D^\alpha u(y) dy.$$

Proof. First we assume that $u \in \mathcal{D}$. Since $u(x - y)$ belongs to \mathcal{D} as a function of y , the formula (3.3) gives

$$\begin{aligned} u(x) &= \langle \delta(y), u(x - y) \rangle = \langle (-1)^m \Delta^m \kappa_{2m}(y), u(x - y) \rangle \\ &= \langle (-1)^m \kappa_{2m}(y), \Delta^m u(x - y) \rangle = \int (-1)^m \kappa_{2m}(y) \Delta^m u(x - y) dy \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ stands for the pairing between distributions and test functions. We take a function $\xi \in \mathcal{D}$ such that

$$(3.9) \quad \xi(x) = \begin{cases} 1, & |x| \leq \varepsilon_1, \\ 0, & |x| \geq \varepsilon_2. \end{cases}$$

If we set $\zeta(x) = (-1)^m \Delta^m ((1 - \xi) \kappa_{2m})(x)$, then by integration by parts we

have

$$\begin{aligned}
 u(x) &= \int (1 - \xi(y))(-1)^m \kappa_{2m}(y) \Delta^m u(x - y) dy \\
 &\quad + \int \xi(y)(-1)^m \kappa_{2m}(y) \Delta^m u(x - y) dy \\
 &= \int \zeta(y)u(x - y) dy + \sum_{|\alpha|=m} \frac{(-1)^m m!}{\alpha!} \int D^\alpha(\xi \kappa_{2m})(y) D^\alpha u(x - y) dy \\
 &= \int \zeta(x - y)u(y) dy + \sum_{|\alpha|=m} \frac{(-1)^m m!}{\alpha!} \int D^\alpha(\xi \kappa_{2m})(x - y) D^\alpha u(y) dy.
 \end{aligned}$$

By (3.3) and (3.9) we see that $\zeta(x) = 0$ for $|x| \leq \varepsilon_1$ and $|x| \geq \varepsilon_2$, and hence $\zeta \in \mathcal{D}$. Therefore we obtain the proposition for $u \in \mathcal{D}$. In case $u \in C^m(\mathbb{R}^n)$, the proposition is obtained by approximating u by a sequence $\{u_j\} \subset \mathcal{D}$ such that $D^\alpha u_j$ converges to $D^\alpha u$ locally uniformly as $j \rightarrow \infty$ for $|\alpha| \leq m$. This completes the proof.

By taking differentiation under the integral sign in (3.6) and (3.8), we obtain the following corollary.

COROLLARY 3.3. *Let $0 < \varepsilon_1 < \varepsilon_2$. Then there exist functions $\mu, \chi, \zeta, \xi \in \mathcal{D}$ such that $\text{supp } \mu, \text{supp } \chi, \text{supp } \zeta, \text{supp } \xi \subset \{|x| \leq \varepsilon_2\}$, $\mu(x) = \zeta(x) = 0$ on $\{|x| \leq \varepsilon_1\}$, $\chi(x) = \xi(x) = 1$ on $\{|x| \leq \varepsilon_1\}$, and if $u \in C^m(\mathbb{R}^n)$, then for $|\gamma| \leq m - 1$,*

$$D^\gamma u(x) = \int D^\gamma \mu(x - y)u(y) dy + \sum_{|\alpha|=m} \frac{m}{\alpha! \sigma_n} \int D^\gamma \chi_\alpha(x - y) D^\alpha u(y) dy$$

where $\chi_\alpha(x) = x^\alpha \chi(x) / |x|^n$, and

$$\begin{aligned}
 D^\gamma u(x) &= \int D^\gamma \zeta(x - y)u(y) dy \\
 &\quad + \sum_{|\alpha|=m} \frac{(-1)^m m!}{\alpha!} \int D^{\gamma+\alpha}(\xi \kappa_{2m})(x - y) D^\alpha u(y) dy.
 \end{aligned}$$

We set $B_1 = \{|x| < 1\}$. By cutting off functions belonging to $C^m(B_1)$, we obtain integral representations for $u \in C^m(B_1)$.

COROLLARY 3.4. *Let $r > 0$, $0 < \varepsilon_1 < \varepsilon_2$ and $r + \varepsilon_2 < 1$. Then there exist functions $\mu, \chi, \zeta, \xi \in \mathcal{D}$ such that $\text{supp } \mu, \text{supp } \chi, \text{supp } \zeta, \text{supp } \xi \subset \{|x| \leq \varepsilon_2\}$, $\mu(x) = \zeta(x) = 0$ on $\{|x| \leq \varepsilon_1\}$, $\chi(x) = \xi(x) = 1$ on $\{|x| \leq \varepsilon_1\}$, and if $u \in C^m(B_1)$, then for $|\gamma| \leq m - 1$ and $|x| \leq r$,*

$$\begin{aligned}
 D^\gamma u(x) &= \int D^\gamma \mu(x - y)u(y) dy \\
 &\quad + \sum_{|\alpha|=m} \frac{m}{\alpha! \sigma_n} \int D^\gamma \chi_\alpha(x - y) D^\alpha u(y) dy
 \end{aligned}$$

and

$$D^\gamma u(x) = \int D^\gamma \zeta(x-y)u(y) dy + \sum_{|\alpha|=m} \frac{(-1)^m m!}{\alpha!} \int D^{\gamma+\alpha}(\xi_{\kappa_{2m}})(x-y)D^\alpha u(y) dy.$$

By Corollaries 3.3 and 3.4 we get interpolation inequalities.

COROLLARY 3.5. *Let $r \geq 0$ and $\varepsilon > 0$. Then for $u \in C^m(\mathbb{R}^n)$ and $|\gamma| < m - n/p$,*

$$\max_{|x| \leq r} |D^\gamma u(x)| \leq C_\varepsilon^1 \left(\int_{|y| \leq r+\varepsilon} |u(y)| dy + \sum_{|\alpha|=m} \left(\int_{|y| \leq r+\varepsilon} |D^\alpha u(y)|^p dy \right)^{1/p} \right)$$

where C_ε^1 is independent of r .

Proof. Let $|x| \leq r$ and $|\gamma| < m - n/p$. By applying Corollary 3.3 for $\varepsilon_1 = \varepsilon/2$ and $\varepsilon_2 = \varepsilon$, we have

$$\begin{aligned} & |D^\gamma u(x)| \\ & \leq \int_{|x-y| \leq \varepsilon} |D^\gamma \mu(x-y)u(y)| dy \\ & \quad + \sum_{|\alpha|=m} \frac{m}{\alpha! \sigma_n} \int_{|x-y| \leq \varepsilon} |D^\gamma \chi_\alpha(x-y)D^\alpha u(y)| dy \\ & \leq \max_{|y| \leq \varepsilon} |D^\gamma \mu(y)| \int_{|y| \leq r+\varepsilon} |u(y)| dy \\ & \quad + \sum_{|\alpha|=m} \frac{m}{\alpha! \sigma_n} \left(\int_{|x-y| \leq \varepsilon} |D^\gamma \chi_\alpha(x-y)|^{p'} dy \right)^{1/p'} \left(\int_{|y| \leq r+\varepsilon} |D^\alpha u(y)|^p dy \right)^{1/p} \\ & \leq C_\varepsilon^1 \left(\int_{|y| \leq r+\varepsilon} |u(y)| dy + \sum_{|\alpha|=m} \left(\int_{|y| \leq r+\varepsilon} |D^\alpha u(y)|^p dy \right)^{1/p} \right) \end{aligned}$$

where

$$C_\varepsilon^1 = \max_{|\gamma| < m-n/p} \left(\max_{|y| \leq \varepsilon} |D^\gamma u(y)| + \max_{|\alpha|=m} \frac{m}{\alpha! \sigma_n} \left(\int_{|y| \leq \varepsilon} |D^\gamma \chi_\alpha(y)|^{p'} dy \right)^{1/p'} \right) < \infty.$$

COROLLARY 3.6. *Let $0 < \varepsilon < 1$ and $0 \leq r < 1-\varepsilon$. Then for $u \in C^m(B_1)$ and $|\gamma| \leq m - 1$,*

$$\max_{|x| \leq r} |D^\gamma u(x)| \leq C_\varepsilon^2 \left(\max_{|y| \leq r+\varepsilon} |u(y)| + \sum_{|\alpha|=m} \max_{|y| \leq r+\varepsilon} |D^\alpha u(y)| \right)$$

where C_ε^2 is independent of r .

Proof. Let $|x| \leq r$ and $|\gamma| \leq m - 1$. By applying Corollary 3.4 for $\varepsilon_1 = \varepsilon/2$ and $\varepsilon_2 = \varepsilon$, we have

$$\begin{aligned} |D^\gamma u(x)| &\leq \int_{|x-y| \leq \varepsilon} |D^\gamma \zeta(x-y)u(y)| dy \\ &\quad + \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{|x-y| \leq \varepsilon} |D^{\alpha+\gamma}(\xi\kappa_{2m})(x-y)D^\alpha u(y)| dy \\ &\leq \max_{|y| \leq r+\varepsilon} |u(y)| \int |D^\gamma \zeta(y)| dy \\ &\quad + \sum_{|\alpha|=m} \frac{m!}{\alpha!} \max_{|y| \leq r+\varepsilon} |D^\alpha u(y)| \int |D^{\alpha+\gamma}(\xi\kappa_{2m})(y)| dy \\ &\leq C_\varepsilon^2 \left(\max_{|y| \leq r+\varepsilon} |u(y)| + \sum_{|\alpha|=m} \max_{|y| \leq r+\varepsilon} |D^\alpha u(y)| \right) \end{aligned}$$

where

$$C_\varepsilon^2 = \max_{|\gamma| \leq m-1} \left(\int |D^\gamma \zeta(y)| dy + \max_{|\alpha|=m} \frac{m!}{\alpha!} \int |D^{\alpha+\gamma}(\xi\kappa_{2m})(y)| dy \right) < \infty.$$

4. The closure of Φ in spaces of Beppo Levi type. In order to define the Riesz potential of an L^p -function, we introduce the modified Riesz kernel $\kappa_{m,k}(x, y)$. For an integer $k < m$, we set

$$\kappa_{m,k}(x, y) = \begin{cases} \kappa_m(x-y) - \sum_{|\gamma| \leq k} \frac{D^\gamma \kappa_m(-y)}{\gamma!} x^\gamma, & 0 \leq k < m, \\ \kappa_m(x-y), & k < 0. \end{cases}$$

Moreover, for a locally integrable function f we define

$$U_{m,k}^f(x) = \int \kappa_{m,k}(x, y) f(y) dy$$

if it exists. Concerning the existence of $U_{m,k}^f$, we have the following.

LEMMA 4.1 (cf. [Ku: Corollary 5.9 and Proposition 5.15]). *Let $f \in L^p$ and $k = [m - n/p]$ be the integral part of $m - n/p$.*

(i) *In case $m - n/p \notin \mathbb{N}$, $U_{m,k}^f$ exists and*

$$\left(\int |U_{m,k}^f(x)|^p |x|^{-mp} dx \right)^{1/p} \leq C \|f\|_p.$$

(ii) *In case $m - n/p \in \mathbb{N}$, if we denote by f_1 and f_2 the restriction of f*

to $\{|x| < 1\}$ and to $\{|x| \geq 1\}$ respectively, then $U_{m,k-1}^{f_1}$ and $U_{m,k}^{f_2}$ exist and

$$\left(\int |U_{m,k-1}^{f_1}(x)|^p |x|^{-mp} (1 + |\log |x||)^{-p} dx\right)^{1/p} \leq C \|f_1\|_p,$$

$$\left(\int |U_{m,k}^{f_2}(x)|^p |x|^{-mp} (1 + |\log |x||)^{-p} dx\right)^{1/p} \leq C \|f_2\|_p.$$

For a set $E \subset \mathbb{R}^n$, we denote by $L^p(E)$ the set of L^p -functions f such that $f(x) = 0$ on the complement of E .

LEMMA 4.2 (cf. [Ku: Theorem 4.4]). *Let $k = [m - n/p]$ and $|\alpha| = m$.*

(i) *In case $m - n/p \notin \mathbb{N}$, for $f \in L^p$,*

$$D^\alpha U_{m,k}^f = (-1)^m R^\alpha f.$$

(ii) *In case $m - n/p \in \mathbb{N}$, for $f \in L^p(B_1)$ and $g \in L^p(B_1^c)$,*

$$D^\alpha U_{m,k-1}^f = (-1)^m R^\alpha f, \quad D^\alpha U_{m,k}^g = (-1)^m R^\alpha g.$$

where B_1^c is the complement of B_1 .

The Riesz potential space R_m^p is defined by

$$R_m^p = \begin{cases} \{U_{m,k}^f : f \in L^p\}, & m - n/p \notin \mathbb{N}, \\ \{U_{m,k-1}^{f_1} + U_{m,k}^{f_2} : f \in L^p, f_1 = f|_{B_1}, f_2 = f|_{B_1^c}\}, & m - n/p \in \mathbb{N}, \end{cases}$$

with $k = [m - n/p]$ where $f|_{B_1}$ is the restriction of f to B_1 . Concerning the relation between the Riesz potential spaces and the spaces of Beppo Levi type, we have

LEMMA 4.3 (cf. [Ku: Theorem 6.7]). *Let $k = [m - n/p]$. Then*

$$R_m^p + \mathcal{P}_k = \begin{cases} \mathcal{L}_m^p \cap L^{p,-m,(\log)^0}, & m - n/p \notin \mathbb{N}, \\ \mathcal{L}_m^p \cap L^{p,-m,(\log)^{-1}}, & m - n/p \in \mathbb{N}, \end{cases}$$

where \mathcal{P}_k is the set of polynomials of degree k with complex coefficients.

We introduce the following notation:

$$\mathcal{B} = \{f \in C^\infty : D^\alpha f \text{ is bounded for each } \alpha\}.$$

The following lemma holds.

LEMMA 4.4. (i) *If $u \in \mathcal{S}$ and $v \in \mathcal{H} \cap \mathcal{B}$, then $uv \in \Psi$.*

(ii) *If $u \in \Psi$ and $v \in \mathcal{B}$, then $uv \in \Psi$.*

We take a function $\lambda \in \mathcal{D}$ such that $\lambda(x) = 1$ on $|x| \leq 1$. We put

$$h(x) = \lambda(x) + e^{-|x|^2 - 1/|x|^2},$$

$$g(x) = 1 - \lambda(x) - e^{-|x|^2 - 1/|x|^2} = 1 - h(x).$$

We note that $g \in \mathcal{H} \cap \mathcal{B}$. For a function f and a positive number ε , we set

$$f_{(\varepsilon)}(x) = f(\varepsilon x).$$

We can easily check

- LEMMA 4.5. (i) For a multi-index α , $D^\alpha f_{(\varepsilon)}(x) = \varepsilon^{|\alpha|} D^\alpha f(\varepsilon x)$.
(ii) If $f \in L^1$, then $\mathcal{F}f_{(\varepsilon)}(x) = (1/\varepsilon^n) \mathcal{F}f(x/\varepsilon)$.

We put

$$\nu(x) = \frac{1}{(2\pi)^n} e^{-|x|^2}, \quad \varrho_\varepsilon(x) = \nu_{(\varepsilon)}(x) g_{(1/\varepsilon)}(x) = \nu(\varepsilon x) g(x/\varepsilon).$$

Since $\nu_{(\varepsilon)} \in \mathcal{S}$ and $g_{(1/\varepsilon)} \in \mathcal{H} \cap \mathcal{B}$, it follows from Lemma 2.1(i) that

$$(4.1) \quad \varrho_\varepsilon \in \Psi.$$

Further, we have

$$\varrho_\varepsilon(x) = \nu_{(\varepsilon)}(x) - \nu_{(\varepsilon)}(x) h_{(1/\varepsilon)}(x).$$

We set

$$\mathcal{K}_\varepsilon(x) = \mathcal{F}\varrho_\varepsilon(x), \quad \mathcal{K}_\varepsilon^1(x) = \mathcal{F}\nu_{(\varepsilon)}(x), \quad H_\varepsilon(x) = \mathcal{F}h_{(\varepsilon)}(x)$$

and

$$\mathcal{K}_\varepsilon^2(x) = \mathcal{F}(\nu_{(\varepsilon)} h_{(1/\varepsilon)})(x) = \mathcal{K}_\varepsilon^1 * H_{1/\varepsilon}(x)$$

where the symbol $*$ means convolution. Since $\varrho_\varepsilon \in \Psi$ by (4.1), we have $\mathcal{K}_\varepsilon \in \Phi$ on account of (1.2). By elementary calculations we have $\mathcal{K}_1^1(x) = 2^{-n} \pi^{-n/2} e^{-|x|^2/4}$, and hence

$$(4.2) \quad \int \mathcal{K}_1^1(x) dx = 1.$$

Moreover, by Lemma 2.2(ii) we have

$$(4.3) \quad \mathcal{K}_\varepsilon^1(x) = \frac{1}{\varepsilon^n} \mathcal{K}_1^1\left(\frac{x}{\varepsilon}\right), \quad H_{1/\varepsilon}(x) = \varepsilon^n H_1(\varepsilon x).$$

LEMMA 4.6. If $(1 + |x|)^m u(x) \in L^1$ for every nonnegative integer m , then $\mathcal{K}_\varepsilon * u \in \Phi$.

Proof. By (1.2) it suffices to show that $\bar{\mathcal{F}}(\mathcal{K}_\varepsilon * u) \in \Psi$. Since $\mathcal{K}_\varepsilon \in \mathcal{S}$ and $u \in L^1$, we see that

$$\bar{\mathcal{F}}(\mathcal{K}_\varepsilon * u)(x) = \bar{\mathcal{F}}\mathcal{K}_\varepsilon(x) \bar{\mathcal{F}}u(x) = (2\pi)^n \varrho_\varepsilon(x) \bar{\mathcal{F}}u(x)$$

by (1.1). The condition $(1 + |x|)^m u(x) \in L^1$ ($m = 0, 1, \dots$) implies $\bar{\mathcal{F}}u \in \mathcal{B}$. Therefore (4.1) and Lemma 4.4(ii) give the assertion.

We denote by Φ_m^p the closure of Φ in \mathcal{L}_m^p .

LEMMA 4.7. Let $k = [m - n/p]$. Then $\Phi_m^p \supset \mathcal{P}_k$.

Proof. Let $Q \in \mathcal{P}_k$. We take a function $\phi \in C^\infty(\mathbb{R})$ such that $\phi(t) = 1$ on $t \leq 1$ and $\phi(t) = 0$ on $t \geq 2$. If we set

$$\varphi_j(x) = \begin{cases} \phi\left(\frac{\log|x|}{\log j}\right), & x \neq 0, \\ 1, & x = 0, \end{cases}$$

then $\varphi_j \in \mathcal{D}$ and $\varphi_j(x) = 1$ on $|x| \leq j$ and $\varphi_j(x) = 0$ on $|x| \geq j^2$. We put $h_j = \varphi_j Q \in \mathcal{D}$. We shall show that h_j converges to Q as $j \rightarrow \infty$ in \mathcal{L}_m^p . It is clear that

$$\int_{|x|<1} |h_j(x) - Q(x)| dx \rightarrow 0 \quad (j \rightarrow \infty).$$

Let $|\alpha| = m$. Since $Q \in \mathcal{P}_k$ and $k < m$, by the Leibniz formula we have

$$\begin{aligned} D^\alpha(h_j - Q)(x) &= D^\alpha h_j(x) = \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} D^{\alpha-\gamma} \varphi_j(x) D^\gamma Q(x) \\ &= \sum_{\gamma \leq \alpha, |\gamma| \leq k} \binom{\alpha}{\gamma} D^{\alpha-\gamma} \varphi_j(x) D^\gamma Q(x). \end{aligned}$$

Since $\alpha - \gamma \neq 0$ for $|\gamma| \leq k$, we see that

$$\begin{aligned} &\left(\int |D^\alpha(h_j - Q)(x)|^p dx\right)^{1/p} \\ &\leq \sum_{\gamma \leq \alpha, |\gamma| \leq k} \binom{\alpha}{\gamma} \left(\int_{j \leq |x| \leq j^2} |D^{\alpha-\gamma} \varphi_j(x) D^\gamma Q(x)|^p dx\right)^{1/p} \\ &\leq C \sum_{\gamma \leq \alpha, |\gamma| \leq k} \left(\int_{j \leq |x| \leq j^2} |D^{\alpha-\gamma} \varphi_j(x)|^p |x|^{(k-|\gamma|)p} dx\right)^{1/p}. \end{aligned}$$

Since

$$|D^{\alpha-\gamma} \varphi_j(x)| \leq \frac{C}{(\log j)|x|^{|\alpha-\gamma|}}$$

on $j \leq |x| \leq j^2$, we obtain

$$\begin{aligned} &\int_{j \leq |x| \leq j^2} |D^{\alpha-\gamma} \varphi_j(x)|^p |x|^{(k-|\gamma|)p} dx \\ &\leq \frac{C}{(\log j)^p} \begin{cases} \int_{j \leq |x|} |x|^{(k-m)p} dx, & k < m - n/p, \\ \int_{j \leq |x| \leq j^2} |x|^{-n} dx, & k = m - n/p, \end{cases} \\ &= \frac{C}{(\log j)^p} \begin{cases} j^{(k-m)p+n}, & k < m - n/p, \\ \log j, & k = m - n/p, \end{cases} \\ &\rightarrow 0 \quad (j \rightarrow \infty) \end{aligned}$$

since $(k - m)p + n < 0$ for $k < m - n/p$. Thus we get

$$(4.4) \quad \|h_j - Q\|_{\mathcal{L}_m^p} \rightarrow 0 \quad (j \rightarrow \infty).$$

Further, we put $h_{j,\varepsilon} = \mathcal{K}_\varepsilon * h_j = \mathcal{K}_\varepsilon^1 * h_j - \mathcal{K}_\varepsilon^2 * h_j$. Since $h_j \in \mathcal{D}$, (4.2) and (4.3) imply

$$(4.5) \quad \|\mathcal{K}_\varepsilon^1 * h_j - h_j\|_{\mathcal{L}_m^p} \rightarrow 0 \quad (\varepsilon \rightarrow 0).$$

Since $\mathcal{K}_\varepsilon^2 * h_j = \mathcal{K}_\varepsilon^1 * H_{1/\varepsilon} * h_j$, we obtain

$$(4.6) \quad \|\mathcal{K}_\varepsilon^2 * h_j\|_{\mathcal{L}_m^p} \rightarrow 0 \quad (\varepsilon \rightarrow 0)$$

because $\|\mathcal{K}_\varepsilon^1\|_1 = 1$ and $\|H_{1/\varepsilon}\|_p \rightarrow 0$ ($\varepsilon \rightarrow 0$). Since $h_{j,\varepsilon} \in \Phi$ by Lemma 4.6, (4.4)–(4.6) give the lemma.

Now we are in a position to prove our main theorem.

THEOREM 4.8.

$$\Phi_m^p = \begin{cases} \mathcal{L}_m^p \cap L^{p,-m,(\log)^0}, & m - n/p \notin \mathbb{N}, \\ \mathcal{L}_m^p \cap L^{p,-m,(\log)^{-1}}, & m - n/p \in \mathbb{N}. \end{cases}$$

Proof. We begin with the proof that the left-hand side is contained in the right-hand side. Let $u \in \Phi_m^p$. By the definition there exists a sequence $\{u_j\} \subset \Phi$ which converges to u as $j \rightarrow \infty$ in \mathcal{L}_m^p . We put $f = D^m u$ and $f_j = D^m u_j$ ($j = 1, 2, \dots$). It follows from $u \in \mathcal{L}_m^p$ that $f \in L^p$, and by Corollary 2.8, $f_j \in \Phi$ and $u_j = \kappa_m^{f_j}$. Moreover, we see that

$$(4.7) \quad \begin{aligned} \|f_j - f\|_p &\leq \sum_{|\alpha|=m} \frac{m!}{\alpha!} \|R^\alpha D^\alpha (u_j - u)\|_p \\ &\leq C|u_j - u|_{m,p} \rightarrow 0 \quad (j \rightarrow \infty). \end{aligned}$$

Let $k = [m - n/p]$. We first consider the case $m - n/p \notin \mathbb{N}$. We have

$$u_j(x) = \int \kappa_m(x - y) f_j(y) dy = U_{m,k}^{f_j}(x) + \sum_{|\gamma| \leq k} a_{\gamma,j} x^\gamma$$

with

$$a_{\gamma,j} = \frac{1}{\gamma!} \int D^\gamma \kappa_m(-y) f_j(y) dy = \frac{D^\gamma u_j(0)}{\gamma!}.$$

Lemma 4.1(i) and (4.7) give

$$\int_{|x| < 1} |U_{m,k}^{f_j}(x) - U_{m,k}^f(x)| dx \rightarrow 0 \quad (j \rightarrow \infty).$$

Further, for $|\alpha| = m$, by Lemma 4.2(i), (2.4) and (4.7) we see that

$$\|D^\alpha (U_{m,k}^{f_j} - U_{m,k}^f)\|_p = C \|R^\alpha f_j - R^\alpha f\|_p \leq C \|f_j - f\|_p \rightarrow 0 \quad (j \rightarrow \infty).$$

Thus we conclude that $U_{m,k}^{f_j}$ converges to $U_{m,k}^f$ as $j \rightarrow \infty$ in \mathcal{L}_m^p . For $|\gamma| \leq k$, by applying Corollary 3.5 with $r = 0$ and $\varepsilon = 1$, we have

$$\begin{aligned} |a_{\gamma,j} - a_{\gamma,l}| &= \frac{1}{\gamma!} |D^\gamma u_j(0) - D^\gamma u_l(0)| \\ &\leq C \left(\int_{|y| \leq 1} |u_j(y) - u_l(y)| dy + \sum_{|\alpha|=m} \left(\int_{|y| \leq 1} |D^\alpha u_j(y) - D^\alpha u_l(y)|^p dy \right)^{1/p} \right) \\ &\leq C \|u_j - u_l\|_{\mathcal{L}_m^p} \rightarrow 0 \quad (j, l \rightarrow \infty). \end{aligned}$$

Thus $\{a_{\gamma,j}\}_{j=1}^\infty$ is a Cauchy sequence for each $|\gamma| \leq k$, and hence there exist a_γ ($|\gamma| \leq k$) such that $a_{\gamma,j}$ tends to a_γ as $j \rightarrow \infty$. This implies that $\sum_{|\gamma| \leq k} a_{\gamma,j} x^\gamma$ converges to $\sum_{|\gamma| \leq k} a_\gamma x^\gamma$ as $j \rightarrow \infty$ in \mathcal{L}_m^p . Consequently, we have

$$u = U_{m,k}^f + \sum_{|\gamma| \leq k} a_\gamma x^\gamma \in R_m^p + \mathcal{P}_k.$$

This implies that $u \in \mathcal{L}_m^p \cap L^{p,-m,(\log)^0}$ by Lemma 4.3.

We next consider the case $m - n/p \in \mathbb{N}$. In this case, putting $f_{j,1} = f_j|_{B_1}$ and $f_{j,2} = f_j|_{B_1^c}$, we have

$$u_j(x) = \int \kappa_m(x - y) f_j(y) dy = U_{m,k-1}^{f_{j,1}}(x) + U_{m,k}^{f_{j,2}}(x) + \sum_{|\gamma| \leq k} a_{\gamma,j} x^\gamma$$

with

$$a_{\gamma,j} = \begin{cases} \frac{1}{\gamma!} \int D^\gamma \kappa_m(-y) f_j(y) dy, & |\gamma| \leq k - 1, \\ \frac{1}{\gamma!} \int_{|y| \geq 1} D^\gamma \kappa_m(-y) f_j(y) dy, & |\gamma| = k. \end{cases}$$

Lemma 4.1(ii) and (4.7) give

$$\int_{|x| < 1} |U_{m,k-1}^{f_{j,1}}(x) - U_{m,k-1}^{f_1}(x)| dx \rightarrow 0 \quad (j \rightarrow \infty),$$

and

$$\int_{|x| < 1} |U_{m,k}^{f_{j,2}}(x) - U_{m,k}^{f_2}(x)| dx \rightarrow 0 \quad (j \rightarrow \infty)$$

where $f_1 = f|_{B_1}$ and $f_2 = f|_{B_1^c}$. Further, Lemma 4.2(ii), (2.4) and (4.7) give

$$|U_{m,k-1}^{f_{j,1}} - U_{m,k-1}^{f_1}|_{m,p} \rightarrow 0, \quad |U_{m,k}^{f_{j,2}} - U_{m,k}^{f_2}|_{m,p} \rightarrow 0 \quad (j \rightarrow \infty).$$

Thus $U_{m,k-1}^{f_{j,1}} + U_{m,k}^{f_{j,2}}$ converges to $U_{m,k-1}^{f_1} + U_{m,k}^{f_2}$ as $j \rightarrow \infty$ in \mathcal{L}_m^p . For $|\gamma| \leq k$, we shall show that $\{a_{\gamma,j}\}_{j=1}^\infty$ is a Cauchy sequence. First let $m - n/p \in \mathbb{N}$

and $m - n/p \neq 0$. For $|\gamma| \leq k - 1$, by using Corollary 3.5 we have

$$|a_{\gamma,j} - a_{\gamma,l}| = \frac{1}{\gamma!} |D^\gamma u_j(0) - D^\gamma u_l(0)| \leq C \|u_j - u_l\|_{\mathcal{L}_m^p}$$

because $|\gamma| < m - n/p$. This shows that $\{a_{\gamma,j}\}_{j=1}^\infty$ is a Cauchy sequence for $|\gamma| \leq k - 1$. Let $|\gamma| = k$. We put

$$v_j(x) = \int_{|y|<1} \kappa_m(x-y) f_j(y) dy, \quad w_j(x) = \int_{|y|\geq 1} \kappa_m(x-y) f_j(y) dy.$$

We show that $\{v_j\}_{j=1}^\infty$ and $\{w_j\}_{j=1}^\infty$ are Cauchy sequences in \mathcal{L}_m^p . Since

$$v_j(x) = U_{m,k-1}^{f_{j,1}}(x) + \sum_{|\beta|\leq k-1} \frac{x^\beta}{\beta!} \int_{|y|<1} D^\beta \kappa_m(-y) f_j(y) dy,$$

for $|\alpha| = m$ by Lemma 4.2(ii) we have

$$D^\alpha v_j = D^\alpha U_{m,k-1}^{f_{j,1}} = (-1)^m R^\alpha f_{j,1}.$$

Hence we see that

$$\begin{aligned} |v_j - v_l|_{m,p} &= \sum_{|\alpha|=m} \|R^\alpha f_{j,1} - R^\alpha f_{l,1}\|_p \leq C \|f_{j,1} - f_{l,1}\|_p \\ &\leq C \|f_j - f_l\|_p \rightarrow 0 \quad (j, l \rightarrow \infty). \end{aligned}$$

Further we obtain

$$\begin{aligned} \int_{|x|<1} |v_j(x) - v_l(x)| dx &= \int_{|x|<1} \left| \int_{|y|<1} \kappa_m(x-y) (f_j(y) - f_l(y)) dy \right| dx \\ &\leq \int_{|x|<2} |\kappa_m(x)| dx \int_{|y|<1} |f_j(y) - f_l(y)| dy \\ &\leq C \|f_j - f_l\|_p \rightarrow 0 \quad (j, l \rightarrow \infty). \end{aligned}$$

Thus $\{v_j\}_{j=1}^\infty$ is a Cauchy sequence in \mathcal{L}_m^p . Since $u_j = v_j + w_j$ and $\{u_j\}_{j=1}^\infty, \{v_j\}_{j=1}^\infty$ are Cauchy sequences in \mathcal{L}_m^p , so is $\{w_j\}_{j=1}^\infty$. Furthermore we shall show

$$(4.8) \quad \max_{|x|\leq 1/2} |w_j(x) - w_l(x)| \leq C \|w_j - w_l\|_{\mathcal{L}_m^p},$$

$$(4.9) \quad \sum_{|\alpha|=m} \max_{|x|\leq 1/2} |D^\alpha w_j(x) - D^\alpha w_l(x)| \leq C \|w_j - w_l\|_{\mathcal{L}_m^p}.$$

Since $w_j - w_l \in C^\infty(B_1)$, by applying Corollary 3.4 with $r = 1/2, 0 < \varepsilon_1 < \varepsilon_2 < 1/2$ and $\gamma = 0$, for $|x| \leq 1/2$ we have

$$\begin{aligned}
 & |w_j(x) - w_l(x)| \\
 & \leq \left| \int_{|x-y| \leq \varepsilon_2} \zeta(x-y)(w_j(y) - w_l(y)) dy \right| \\
 & \quad + \sum_{|\alpha|=m} \frac{m!}{\alpha!} \left| \int_{|x-y| \leq \varepsilon_2} D^\alpha(\xi\kappa_{2m})(x-y)D^\alpha(w_j - w_l)(y) dy \right| \\
 & \leq \|\zeta\|_\infty \int_{|y|<1} |w_j(y) - w_l(y)| dy \\
 & \quad + \sum_{|\alpha|=m} \frac{m!}{\alpha!} \left(\int_{|y|<1} |D^\alpha(\xi\kappa_{2m})(y)|^{p'} dy \right)^{1/p'} \left(\int_{|y|<1} |D^\alpha(w_j - w_l)(y)|^p dy \right)^{1/p} \\
 & \leq C\|w_j - w_l\|_{\mathcal{L}_m^p}
 \end{aligned}$$

with $\|\zeta\|_\infty = \sup_x |\zeta(x)|$, because $m - n/p > 0$ implies

$$\left(\int_{|y|<1} |D^\alpha(\xi\kappa_{2m})(y)|^{p'} dy \right)^{1/p'} < \infty.$$

Thus we obtain (4.8). Further, for $|\alpha| = m$ and $|x| \leq 1/2$ we have

$$\begin{aligned}
 |D^\alpha w_j(x) - D^\alpha w_l(x)| & = \left| \int_{|y| \geq 1} D^\alpha \kappa_m(x-y)(f_j(y) - f_l(y)) dy \right| \\
 & \leq \left(\int_{|y| \geq 1} |D^\alpha \kappa_m(x-y)|^{p'} dy \right)^{1/p'} \left(\int_{|y| \geq 1} |f_j(y) - f_l(y)|^p dy \right)^{1/p} \\
 & \leq \left(\int_{|y| \geq 1/2} |D^\alpha \kappa_m(y)|^{p'} dy \right)^{1/p'} \left(\int_{|y| \geq 1} |f_j(y) - f_l(y)|^p dy \right)^{1/p} \\
 & \leq C\|f_{j,2} - f_{l,2}\|_p = C \left\| \sum_{|\gamma|=m} \frac{m!}{\gamma!} R^\gamma D^\gamma(w_j - w_l) \right\|_p \\
 & \leq C|w_j - w_l|_{m,p} \leq C\|w_j - w_l\|_{\mathcal{L}_m^p}
 \end{aligned}$$

because

$$\left(\int_{|y| \geq 1/2} |D^\alpha \kappa_m(y)|^{p'} dy \right)^{1/p'} \leq C \left(\int_{|y| \geq 1/2} |y|^{-np'} dy \right)^{1/p'} < \infty.$$

Thus we obtain (4.9). Consequently, by taking $|\gamma| = k$, $r = 0$ and $\varepsilon = 1/2$ in Corollary 3.6, we obtain

$$|a_{\gamma,j} - a_{\gamma,l}| = \frac{1}{\gamma!} |D^\gamma w_j(0) - D^\gamma w_l(0)| \leq C\|w_j - w_l\|_{\mathcal{L}_m^p}.$$

This shows that $\{a_{\gamma,j}\}_{j=1}^\infty$ is a Cauchy sequence for $|\gamma| = k$. Next let $m - n/p = 0$. In this case we have

$$u_j(x) = \kappa_m^{f_j}(x) = U_{m,-1}^{f_{j,1}}(x) + U_{m,0}^{f_{j,2}}(x) + a_{0,j}.$$

By putting $\omega_n = \int_{|x|<1} dx$, we see that

$$\begin{aligned} |a_{0,j} - a_{0,l}| &= \frac{1}{\omega_n} \int_{|x|<1} |a_{0,j} - a_{0,l}| dx \\ &= \frac{1}{\omega_n} \int_{|x|<1} |u_j(x) - u_l(x) + U_{m,-1}^{f_{j,1}}(x) - U_{m,-1}^{f_{l,1}}(x) + U_{m,0}^{f_{j,2}}(x) - U_{m,0}^{f_{l,2}}(x)| dx \\ &\leq \frac{1}{\omega_n} \int_{|x|<1} |u_j(x) - u_l(x)| dx + \omega_n^{-1/p} \left(\int_{|x|<1} |U_{m,-1}^{f_{j,1}} - U_{m,-1}^{f_{l,1}}(x)|^p dx \right)^{1/p} \\ &\quad + \omega_n^{-1/p} \left(\int_{|x|<1} |U_{m,0}^{f_{j,2}} - U_{m,0}^{f_{l,2}}(x)|^p dx \right)^{1/p}. \end{aligned}$$

Therefore, Lemma 4.1(ii) gives

$$\begin{aligned} |a_{0,j} - a_{0,l}| &\leq C \|u_j - u_l\|_{\mathcal{L}_m^p} + C \|f_{l,1} - f_{j,1}\|_p + C \|f_{l,2} - f_{j,2}\|_p \\ &\leq C \|u_j - u_l\|_{\mathcal{L}_m^p} + C \|f_l - f_j\|_p \leq C \|u_j - u_l\|_{\mathcal{L}_m^p}. \end{aligned}$$

This shows that $\{a_{0,j}\}_{j=1}^\infty$ is a Cauchy sequence. Thus we conclude that $\{a_{\gamma,j}\}_{j=1}^\infty$ is a Cauchy sequence for $|\gamma| \leq k$. Hence there exist a_γ ($|\gamma| \leq k$) such that $a_{\gamma,j}$ tends to a_γ as $j \rightarrow \infty$. This implies that $\sum_{|\gamma| \leq k} a_{\gamma,j} x^\gamma$ converges to $\sum_{|\gamma| \leq k} a_\gamma x^\gamma$ as $j \rightarrow \infty$ in \mathcal{L}_m^p . Consequently, we have

$$u = U_{m,k-1}^{f_1} + U_{m,k}^{f_2} + \sum_{|\gamma| \leq k} a_\gamma x^\gamma \in R_m^p + \mathcal{P}_k.$$

This implies that $u \in \mathcal{L}_m^p \cap L^{p,-m,(\log)^{-1}}$ by Lemma 4.3.

Next, we show the converse inclusion. Let

$$u \in \begin{cases} \mathcal{L}_m^p \cap L^{p,-m,(\log)^0}, & m - n/p \notin \mathbb{N}, \\ \mathcal{L}_m^p \cap L^{p,-m,(\log)^{-1}}, & m - n/p \in \mathbb{N}. \end{cases}$$

By Lemma 4.3 there exists an L^p -function f such that

$$u = \begin{cases} U_{m,k}^f + \sum_{|\gamma| \leq k} a_\gamma x^\gamma, & m - n/p \notin \mathbb{N}, \\ U_{m,k-1}^{f_1} + U_{m,k}^{f_2} + \sum_{|\gamma| \leq k} a_\gamma x^\gamma, & m - n/p \in \mathbb{N}, \end{cases}$$

with $k = [m - n/p]$, $f_1 = f|_{B_1}$ and $f_2 = f|_{B_1^c}$. Since Φ is dense in L^p ([Li1]), there exists a sequence $\{f_j\} \subset \Phi$ such that f_j converges to f as $j \rightarrow \infty$ in L^p . We consider the case $m - n/p \notin \mathbb{N}$. Since f_j converges to f as $j \rightarrow \infty$ in L^p , $U_{m,k}^{f_j}$ converges to $U_{m,k}^f$ as $j \rightarrow \infty$ in \mathcal{L}_m^p by Lemmas 4.1(i) and 4.2(i).

Hence, for given $\varepsilon > 0$, there exists a function $f_l \in \Phi$ such that

$$(4.10) \quad \begin{aligned} \varepsilon/2 &> \|U_{m,k}^f - U_{m,k}^{f_l}\|_{\mathcal{L}_m^p} \\ &= \left\| U_{m,k}^f - \kappa_m^{f_l} + \sum_{|\gamma| \leq k} \frac{x^\gamma}{\gamma!} \int D^\gamma \kappa_m(-y) f_l(y) dy \right\|_{\mathcal{L}_m^p}. \end{aligned}$$

We put

$$P(x) = \sum_{|\gamma| \leq k} a_\gamma x^\gamma, \quad P_l(x) = \sum_{|\gamma| \leq k} \frac{x^\gamma}{\gamma!} \int D^\gamma \kappa_m(-y) f_l(y) dy.$$

Since $P - P_l \in \mathcal{P}_k$, by Lemma 4.7 there exists a function $\varphi \in \Phi$ such that

$$(4.11) \quad \|P - P_l - \varphi\|_{\mathcal{L}_m^p} < \varepsilon/2.$$

By (4.10) and (4.11) we have

$$\begin{aligned} \|u - \kappa_m^{f_l} - \varphi\|_{\mathcal{L}_m^p} &= \|U_{m,k}^f + P + P_l - P_l - \kappa_m^{f_l} - \varphi\|_{\mathcal{L}_m^p} \\ &\leq \|U_{m,k}^f - \kappa_m^{f_l} + P_l\|_{\mathcal{L}_m^p} + \|P - P_l - \varphi\|_{\mathcal{L}_m^p} \\ &= \|U_{m,k}^f - U_{m,k}^{f_l}\|_{\mathcal{L}_m^p} + \|P - P_l - \varphi\|_{\mathcal{L}_m^p} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\kappa_m^{f_l} + \varphi \in \Phi$ by Corollary 2.6, this implies that $u \in \Phi_m^p$. In case $m - n/p \in \mathbb{N}$, we can prove that $u \in \Phi_m^p$ in the same way. Thus we have completed the proof of the theorem.

REMARK 4.9. Since $\mathcal{L}_m^p = R_m^p + \mathcal{P}_{m-1}$ ([Ku: Theorems 6.1 and 6.3]), by Lemma 4.3 and Theorem 4.8 we conclude that if $p \geq n$, then Φ is dense in \mathcal{L}_m^p , and if $p < n$, then Φ is not dense in \mathcal{L}_m^p .

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Department of Mathematics
and Computer Science
Kagoshima University
890-0065 Kagoshima, Japan
E-mail: kurokawa@sci.kagoshima-u.ac.jp

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