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The Shirali–Ford theorem as a consequence of Pták theory for hermitian Banach algebras

by

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Dedicated to the memory of Professor Vlastimil Pták

Abstract. A simple application of Pták theory for hermitian Banach algebras, combined with a result on normed Q-algebras, gives a non-technical new proof of the Shirali–Ford theorem. A version of this theorem in the setting of non-normed topological algebras is also provided.

0. Introduction. The aim of this paper is, on the one hand, to give a new proof of the celebrated Shirali–Ford theorem [17] by using the powerful results of V. Pták [13; Section 5] for hermitian Banach algebras (see Theorem 3.3), and on the other hand, to provide a generalization of the same theorem in the more general framework of (non-normed) topological algebras (cf. Theorem 4.7). The idea for the afore-mentioned new proof originates from an "algebraic analogue" of the Shirali–Ford theorem due to D. Birbas [4; Theorem 3.2]; see also Theorem 4.1 in Section 4.

A generalization of the Shirali–Ford theorem to involutive Arens–Michael algebras (inverse limits of Banach algebras) appears in a 1985 paper by D. Štěrbová [19; Theorem 2.5]. But the proof of this result depends upon Lemma 2.1 of [19], in the proof of which relation (2) is unclear. A proof of the Shirali–Ford theorem in the class of involutive Arens–Michael Q-algebras has been given by this author (see, e.g., [7; Theorem 7.2]) by applying standard techniques. The corresponding result presented here (Theorem 4.7) contains the previous one and it is obtained as a corollary of the afore-mentioned "algebraic analogue" of the Shirali–Ford theorem by D. Birbas [4].

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1. Preliminaries. Throughout this paper we deal with complex algebras and Hausdorff topological spaces.

A *topological algebra* is an algebra, which is, in addition, a topological vector space such that the ring multiplication is separately continuous. An *lmc* (locally m-convex) *algebra* is a topological algebra A whose topology is defined by a saturated family, say $\Gamma = \{p\}$, of algebra seminorms; i.e., each seminorm $p \in \Gamma$ is *submultiplicative*, in the sense that $p(xy) \leq p(x)p(y)$ for $x, y \in A$. A complete lmc algebra is called an *Arens–Michael algebra* [8; p. 65, Definition (I.2.4)]. Let $(A, \Gamma = \{p\})$ be an Arens–Michael algebra and $N_p \equiv \ker(p)$, $p \in \Gamma$. The Banach algebra completion of the normed algebra $(A/N_p, \|\cdot\|_p)$, with $\|x_p\|_p := p(x)$ for $x_p \equiv x + N_p \in A/N_p$, is denoted by A_p, $p \in \Gamma$. It is known (cf., e.g., [8, 9]) that

$$A = \varprojlim A_p,$$

up to a topological algebraic isomorphism. A topological algebra A is called a *Q-algebra* if the set G_A^{q} of its quasi-invertible elements is open. An element $x \in A$ is called *quasi-invertible* if there is $y \in A$ with $x \circ y = 0 = y \circ x$, where $x \circ y := x + y - xy$. In the case of a unital algebra A, with unit e, G_A stands for the invertible elements of A. In this case, *$x \in G_A^{\mathrm{q}}$ with quasi-inverse $y \in A$ iff $x - e \in G_A$ with inverse $y - e$.* Given an algebra A and an element $x \in A$ denote by $\mathrm{sp}_A(x)$ (resp. $r_A(x)$) the *spectrum* (resp. *spectral radius*) of x. Y. Tsertos has proved in [20; Corollary 4.1] that *an lmc algebra $(A, \Gamma = \{p\})$ is a Q-algebra iff there is $p_0 \in \Gamma$ such that*

$$r_A(x) \leq p_0(x), \quad \forall x \in A; \tag{1.1}$$

in fact, this characterization is shown for any topological algebra A, with the gauge function of a balanced 0-neighborhood in place of p_0 (ibid., Theorem 4.1). The corresponding result for a normed algebra is due to B. Yood [22; Lemma 2.1]. Now a topological algebra A is called *advertibly complete* (Warner; see [21] and [9; p. 45, Definition 6.4]) whenever every Cauchy net $(x_\lambda)_{\lambda \in \Lambda}$ in A with the property

$$x_\lambda \circ x \to 0 \leftarrow x \circ x_\lambda, \quad \text{for some } x \in A, \tag{1.2}$$

converges in A. *Every Q-algebra A is advertibly complete, and when A is a normed algebra the two notions coincide* [21; Theorem 7]. The algebra $\mathcal{D}(\mathbb{R})$ of compactly supported C^∞-functions on $\mathbb{R}$, endowed with the C^∞-topology from $C^\infty(\mathbb{R})$ is an advertibly complete, non-complete, non-Q-algebra. But with its usual inductive limit topology, $\mathcal{D}(\mathbb{R})$ is a Q-algebra (see, e.g., [6; p. 86]).

A *spectral algebra* (Palmer, [11; Definition 2.4.1]) is an algebra A that can be equipped with an algebra seminorm q such that $r_A(x) \leq q(x)$ for all $x \in A$. If A is an involutive algebra and q a *$*$-preserving* (i.e., $q(x^*) = q(x)$ for all $x \in A$) algebra seminorm on A satisfying the preceding inequality, then

A is said to be a *spectral $*$-algebra*. Such algebras are called by Palmer [12] *S^*-algebras*. The algebra seminorm q is called a *spectral seminorm* in the first case and *spectral $*$-seminorm* in the second case. Furthermore, if A is an involutive algebra and q a *C^*-seminorm* on A (i.e., $q(x^*x) = q(x)^2$ for all $x \in A$) that dominates the spectral radius r_A, then A is called a *C^*-spectral algebra* and q a *spectral C^*-seminorm*. Such algebras were introduced and studied by S. J. Bhatt, A. Inoue and H. Ogi [3]. Later, in a series of papers by the same authors and sometimes jointly with K.-D. Kürsten (cf., e.g., [2]), these algebras were used, very effectively and in a smart way, for the construction of unbounded $*$-representations. Algebras of this kind have also been considered in [12; Section 10.4]. In the category of involutive Banach algebras, C^*-spectral algebras are exhausted by the hermitian ones [3; Corollary 2.7]. Some further information on C^*-spectral algebras, fitting to the present environment, is given in Remark 3.5.

Note that *every spectral algebra is a* (not necessarily Hausdorff) *lmc Q-algebra and every lmc Q-algebra is a spectral algebra* (see (1.1)). But *a spectral lmc algebra $(A, \Gamma = \{p\})$ is not a Q-algebra unless the spectral seminorm q belongs to Γ.*

Let us now fix some further notation. Given an algebra A, denote by J_A the *Jacobson radical* of A. For an involutive algebra A, set

$$H(A) := \{x \in A : x^* = x\}, \qquad N(A) := \{x \in A : x^*x = xx^*\};$$

the elements of $H(A)$ are called *self-adjoint*, while those of $N(A)$ are named *normal*. A subset S of A is called *s*elf-adjoint if $x \in S$ implies $x^* \in S$. It is easily seen (apply, e.g., [6; Lemma 8.11]) that in an involutive algebra A, *the Jacobson radical J_A is a self-adjoint ideal.* An involutive algebra A is called *hermitian* (resp. *symmetric*) if $\mathrm{sp}_A(x) \subseteq \mathbb{R}$ for all $x \in H(A)$ (resp. $-x^*x \in G_A^{\mathrm{q}}$ for all $x \in A$; or $e + x^*x \in G_A$ for all $x \in A$, in the case where A is unital with unit e). In the notions of *hermitian* (resp. *symmetric*) *topological algebra* no continuity of the involution is assumed. On the contrary, the terms *hermitian* (resp. *symmetric*) *topological $*$-algebra* always postulate continuity of the involution. A useful geometrical characterization of a symmetric algebra A is given by the positivity of the elements x^*x ($x \in A$), in the sense that these elements have positive spectra. As a direct consequence, *every C^*-algebra is symmetric* (see, e.g., [5; Theorem (12.6)]). For a more general result of this kind, cf. [6; Corollary 6.2].

2. Some general results. The following can be found implicitly in [9; p. 95, Remark], for lmc algebras.

2.1. Proposition. *Let A be an advertibly complete topological algebra whose completion $\widetilde{A}$ is also a topological algebra* (take, for instance, A to have continuous multiplication). *Let $x \in A$. Then $x \in G_A^{\mathrm{q}} \Leftrightarrow x \in G_{\widetilde{A}}^{\mathrm{q}}$.*

Proof. Clearly $x \in A$ with $x \in G_A^{\mathrm{q}}$ yields $x \in G_{\widetilde{A}}^{\mathrm{q}}$. So let $x \in A$ with $x \in G_{\widetilde{A}}^{\mathrm{q}}$. Then there is $y \in \widetilde{A}$ such that

$$x \circ y = 0 = y \circ x. \tag{2.1}$$

In addition, there is a net $(y_\lambda)_{\lambda \in \Lambda}$ in A with $y = \lim_\lambda y_\lambda$. Using continuity of addition and separate continuity of multiplication in A we get

$$x \circ y_\lambda \to 0 \leftarrow y_\lambda \circ x, \tag{2.2}$$

where $(y_\lambda)_{\lambda \in \Lambda}$ is moreover a Cauchy net in A. Since A is advertibly complete we deduce from (2.2) (see also (1.2)) that $(y_\lambda)_{\lambda \in \Lambda}$ converges in A, say to $z \in A$. Then (2.2) implies

$$x \circ z = 0 = z \circ x. \tag{2.3}$$

From (2.1), (2.3) we clearly have $y = z \in A$, so that $x \in G_A^{\mathrm{q}}$. ■

As we noticed in Section 1, advertible completeness coincides with property Q in normed algebras. Hence one has the following (see also [6; Corollary 2.3]).

2.2. Corollary. *Let A be a normed Q-algebra and $\widetilde{A}$ the Banach algebra completion of A. Let $x \in A$. Then $x \in G_A^{\mathrm{q}} \Leftrightarrow x \in G_{\widetilde{A}}^{\mathrm{q}}$.* ■

In 1966 B. A. Barnes proved in [1; Lemma 1.2] that *a pre-C^*-algebra A* (i.e., an involutive algebra A equipped with an algebra norm satisfying the C^*-property) *is a Q-algebra iff $r_A(x) \leq \|x\|$ for all $x \in H(A)$.* In fact, he proved the following more general result.

2.3. Proposition. *Let A be an involutive algebra. The following are equivalent*:

(1) *A is a spectral $*$-algebra.*

(2) *$r_A(x) \leq q(x)$ for all $x \in H(A)$, for some $*$-preserving algebra seminorm q on A.*

Proof. (1)⇒(2). This is immediate from the definition of a spectral $*$-algebra (see Section 1).

(2)⇒(1). Repeat the corresponding proof of [1; Lemma 1.2] with q in place of the C^*-norm. ■

3. A new proof of the Shirali–Ford theorem. Let A be an involutive algebra and

$$p_A(x) := r_A(x^*x)^{1/2}, \qquad \forall x \in A; \tag{3.1}$$

we call p_A the *Pták function* of A. T. W. Palmer names the preceding function the Raĭkov–Pták functional (see [12; Definition 10.2.5 and comments on p. 1095], while he uses the term *Raĭkov's inequality* (cf. [10; p. 524]) for what we call *Pták inequality* (see Theorem 3.2(3) below).

The following is a direct consequence of (3.1).

3.1. Proposition. *The Pták function p_A of an involutive algebra A has the following properties*:

(1) $p_A(\lambda x) = |\lambda| p_A(x)$ *for all* $\lambda \in \mathbb{C}$, $x \in A$.
(2) $p_A(x^*) = p_A(x)$ *for all* $x \in A$.
(3) $p_A(x^* x) = p_A(x)^2$ *for all* $x \in A$.
(4) $p_A(x) = r_A(x)$ *for all* $x \in H(A)$.
(5) $J_A \subseteq \{x \in A : p_A(x) = 0\}$. ∎

V. Pták proved in 1972 that hermiticity of a Banach algebra is equivalent to subadditivity of the Pták function [13; Theorem (5,10)]; so that for each hermitian Banach algebra A, the real-valued function p_A is a C^*-seminorm (cf. Proposition 3.1). In fact, the Pták function is an algebra C^*-seminorm for any hermitian Banach algebra A and its subadditivity is a consequence of its submultiplicativity; the latter is not a general rule, since according to a result of Z. Sebestyén [16] (see also [5; Theorem (38.1)]) *any C^*-senimorm on an involutive algebra A is automatically submultiplicative* (and $*$-preserving). It is worth mentioning that *p_A is the largest C^*-seminorm on a* (unital) *hermitian Banach $*$-algebra A*. This follows easily from D. A. Raĭkov's criterion for symmetry (see [14] and/or [15; Theorem (4.7.21)]), which, in fact, can be reformulated (referee's remark) in the following way: *a* (unital) *Banach $*$-algebra A is symmetric if and only if p_A is the largest C^*-seminorm on A.*

In the next theorem we list some well-known properties of hermitian (resp. symmetric) algebras, which we use in the proof of Theorem 3.3. Complete proofs of these results can be found in the book of Doran–Belfi [5; Proposition (32.9), Theorems (33.1), (33.7) and Proposition (B.5.14)(a)].

For more general classes of hermitian algebras than that of hermitian Banach algebras, the reader should consult the second volume of T. W. Palmer's book [12, Section 10.4], where apart from the interesting material he will find important comments and historical notes.

3.2. Theorem. *Let A be an involutive algebra.*

(1) (Wichmann) *If I is a self-adjoint ideal of A, then A is hermitian* (*resp. symmetric*) *iff I and A/I are hermitian* (*resp. symmetric*).

(2) *J_A is a self-adjoint ideal of A, which* (consisting entirely of quasi-invertible elements) *is symmetric*, *hence hermitian.*

If A is moreover Banach, one has:

(3) (Pták) *A is hermitian iff $r_A(x) \leq p_A(x)$ for all $x \in A$* (Pták inequality).

(4) (Pták) *A is hermitian iff $p_A(x+y) \leq p_A(x) + p_A(y)$ for all $x \in A$; in other words* (see also Proposition 3.1), *A is hermitian iff p_A is a C^*-seminorm.*

(5) (Pták) *A hermitian implies $J_A = \ker(p_A)$.* ■

A direct consequence of Theorem 3.2(1), (2) is that: *The problem of proving hermiticity or symmetry of an involutive algebra A always reduces to the semisimple case through the involutive algebra $B \equiv A/J_A$.*

The proofs of the Shirali–Ford theorem one usually meets in the literature are technical (see, for instance, [5; Theorem (33.2) and comments before it], as well as [13; Theorem (5,9)]), based on: (i) Gel'fand representation theory applied to a suitable commutative $*$-subalgebra of the given hermitian Banach algebra, say A; and (ii) the fact that the positive elements of A form a convex cone. A different (less technical) proof, involving properties of maximal modular left ideals, has been given by T. W. Palmer [10]; in the same paper, hermiticity of an involutive Banach algebra is characterized (among other conditions) by the "property Q" of the Gel'fand–Naimark pseudo-norm (see e.g. (1.1) with the Gel'fand–Naimark pseudo-norm in place of p_0).

In this section, thanks to Pták's smart theory for hermitian Banach algebras (see Theorem 3.2) and to a suitable use of the "property Q" (cf. Corollary 2.2), we present a new proof of the Shirali–Ford theorem that provides a more conceptual argument that frees us from calculations.

3.3. Theorem (Shirali–Ford). *Every hermitian Banach algebra A is symmetric.*

Proof. According to the above, it suffices to show that the semisimple hermitian Banach algebra $B \equiv A/J_A$ is symmetric. Hermiticity of B implies

$$r_B(x + J_A) \leq p_B(x + J_A), \qquad \forall x \in A, \tag{3.2}$$

with p_B a C^*-seminorm and $J_B = \ker(p_B)$. Semisimplicity of B makes p_B a C^*-norm. Thus the completion $\widetilde{B}$ of (B, p_B) is a C^*-algebra, hence symmetric, while from (3.2) (see also (1.1)) (B, p_B) is a (normed) Q-algebra. Applying now Corollary 2.2, we clearly get symmetry of B. ■

3.4. Corollary. *An involutive Banach algebra is hermitian iff it is symmetric.* ■

We now give some extra information about C^*-spectral algebras that we promised in Section 1.

3.5. Remark. (1) *Every spectral C^*-seminorm is unique and coincides with the Pták function* (cf. e.g., [2; Lemma 4.5(1)] and [12; Proposition 9.5.3]).

(2) *Every C^*-spectral algebra is symmetric.*

Proof. (1) If (A, q) is a C^*-spectral algebra, one has

$$q(x)^2 = q(x^*x) = r_A(x^*x) = p_A(x)^2, \quad \forall x \in A,$$

where the middle equality follows from the formula $r_A(x) = \lim_n q(x^n)^{1/n}$, $x \in A$ (cf. [11; Theorem 2.2.5]), due to the spectrality of A.

(2) Let (A, q) be a C^*-spectral algebra. The result follows directly from Theorem 4.1 below, by using (1). It is also easily derived from [6; Corollary 6.2], if we endow A with the topology induced by the C^*-seminorm q.

Nevertheless, one can give a self-reliant proof based on the spirit of the proof of Theorem 3.3. Indeed, since $J_A = \ker(q) \equiv N_q$ (see e.g., [6; Lemma 8.11]), one has

$$r_{A/N_q}(x_q) \leq r_A(x) \leq q(x) =: \|x_q\|_q, \quad \forall x_q \in A/N_q;$$

therefore $(A/N_q, \|\cdot\|_q)$ is a Q-algebra whose completion is symmetric as a C^*-algebra. So A/N_q (hence A too) is symmetric by Proposition 2.1. ∎

A consequence of (2) is that *every C^*-spectral algebra is hermitian.* This property can also be proved independently, but symmetry cannot be derived from hermiticity, since an arbitrary C^*-spectral algebra is not necessarily complete, so Theorem 4.7 e.g. (cf. Section 4) cannot be applied.

4. A generalization of the Shirali–Ford theorem. In 1998 D. Birbas [4; Theorem 3.2(i)] proved an "algebraic analogue" of the Shirali–Ford theorem. More precisely, using the result of B. A. Barnes mentioned in Section 2 (cf., e.g., Proposition 2.3), D. Birbas [4; Lemma 3.1] showed that an involutive algebra A with subadditive real-valued Pták function satisfies the statements (3) and (5) of Theorem 3.2, i.e.,

$$r_A(x) \leq p_A(x), \quad \forall x \in A; \quad J_A = \ker(p_A).$$

Using the preceding results, as well as two algebraic facts: Theorem 3.2(1) and the identification of the spectral radii r_A, r_{A/J_A}, for any algebra A [5; Proposition (B.5.16)], he applied arguments similar to those of Theorem 3.3, to obtain the following.

4.1. Theorem (Birbas). *Let A be an involutive algebra having a subadditive real-valued Pták function. Then A is symmetric.* ∎

In this section we prove that a certain class of hermitian Arens–Michael algebras, containing all hermitian Arens–Michael Q-algebras, have a subadditive real-valued Pták function (see Proposition 4.6); so that one has from

Theorem 4.1 a non-normed version of the Shirali–Ford theorem (cf. Theorem 4.7). The technique we use is that of [13] combined with the general theory of non-normed topological algebras (see, e.g., [9]). Although some of these results have been exposed in [6; Section 8] for symmetric Arens–Michael (occasionally Q-) algebras, we shall outline their proofs for clarity's sake.

4.2. THEOREM. *Let* $(A, \Gamma = \{p\})$ *be an involutive Arens–Michael algebra. Consider the following conditions*:

(1) *A is hermitian.*
(2) $r_A(x) \le p_A(x)$ *for all* $x \in A$.
(3) $r_A(x)^2 = r_A(x^*x) \Leftrightarrow r_A(x) = p_A(x)$, *for all* $x \in N(A)$.

Then (1)⇒(2)⇒(3) *and if moreover* $r_A(x) < \infty$ *for all* $x \in H(A)$, *one also has* (3)⇒(1).

Proof. (1)⇒(2). The unitization A_1 of A endowed with the product topology is a hermitian (cf. [5; Proposition (32.8)]) Arens–Michael algebra. So without loss of generality we may suppose that A is unital with unit e. Suppose that (2) is not true. Then there are $x \in A$ and $\lambda \in \mathrm{sp}_A(x)$ such that

$$|\lambda| > p_A(x) \;\Leftrightarrow\; |\lambda|^2 > r_A(x^*x).$$

Thus if $z \equiv \lambda^{-1}x$, we have

$$r_A(e - (e - z^*z)) < 1 \quad \text{with} \quad e - z^*z \in H(A),$$

whence (see [18; Theorem 3.9]) there is a unique $y \in H(A)$ with

$$y^2 = e - z^*z \quad \text{and} \quad r_A(e - y) < 1.$$

From [9; p. 101, Proposition 6.1] we now deduce that $y \in G_A$. On the other hand, denoting by i the imaginary unit we have

$$(e + z^*)(e - z) = y^2 - (z - z^*) = -iy(ie - iy^{-1}(z - z^*)y^{-1})y, \tag{4.1}$$

where $w \equiv iy^{-1}(z - z^*)y^{-1} \in H(A)$, therefore $\mathrm{sp}_A(w) \subseteq \mathbb{R}$ by (1). Hence $i \notin \mathrm{sp}_A(w) \Leftrightarrow ie - w \in G_A$, consequently (4.1) implies that $(e + z^*)(e - z) \in G_A$. So $e - z$ has a left inverse. Now since $r_A(xx^*) = r_A(x^*x) < |\lambda|^2$, we can repeat the preceding argument for the element $e - zz^* \in H(A)$ to deduce that $e - z$ has a right inverse. Thus

$$e - z \in G_A \;\Leftrightarrow\; \lambda e - x \in G_A \;\Leftrightarrow\; \lambda \notin \mathrm{sp}_A(x),$$

which is a contradiction. Therefore $r_A(x) \le p_A(x)$ for all $x \in A$.

(2)⇒(3). Let $x \in N(A)$. Then (cf. [9; p. 100, Corollary 6.1(5)]) $r_A(x^*x) \le r_A(x^*)r_A(x) = r_A(x)^2$, whence $p_A(x) \le r_A(x)$.

(3)⇒(1). Suppose $r_A(x) < \infty$ for all $x \in H(A)$ and let $x \in H(A)$ with $\alpha + i\beta \in \mathrm{sp}_A(x)$, $\alpha, \beta \in \mathbb{R}$, $\beta \ne 0$. Then

$$y \equiv \beta^{-1}(x - \alpha e) \in H(A) \quad \text{and} \quad i \in \mathrm{sp}_A(y).$$

Let n be an arbitrary natural number and $z \equiv y + ine \in A$. Then

$$z^*z = zz^* = y^2 + n^2 e \quad \text{and} \quad (n+1)i \in \mathrm{sp}_A(z);$$

so that from (3) and [9; p. 100, Corollary 6.1(4)] one obtains

$$|(n+1)i|^2 \le r_A(z)^2 = r_A(z^*z) = r_A(y^2 + n^2 e) \le r_A(y)^2 + n^2.$$

This yields $2n + 1 \le r_A(y^2)$, where $y^2 \in H(A)$, therefore $r_A(y^2) < \infty$. For $n \to \infty$ we are led to a contradiction. Thus $\beta = 0$, and this proves (1). ∎

4.3. Proposition. *For a hermitian spectral Arens–Michael algebra* $(A, \Gamma = \{p\})$, *we have*:

(1) $r_A(xy) \le r_A(x) r_A(y)$ *for all* $x, y \in H(A)$.
(2) $p_A(xy) \le p_A(x) p_A(y)$ *for all* $x, y \in A$.

That is, the spectral radius r_A *is submultiplicative on the self-adjoint elements of* A, *while* p_A *is submultiplicative everywhere on* A.

Proof. (1) Let $x, y \in H(A)$. Using Theorem 4.2 and standard properties of the spectral radius, we have

$$r_A(xy)^2 \le r_A((xy)^*(xy)) = r_A(yxxy) = r_A(x^2y^2).$$

Inductively one gets

$$r_A(xy) \le r_A(x^{2^n} y^{2^n})^{1/2^n}, \quad \forall x, y \in H(A),\ n \in \mathbb{N}. \tag{4.2}$$

Since A is spectral, there is a spectral seminorm q on A such that $r_A(x) \le q(x)$ for all $x \in A$, so that (4.2) implies

$$r_A(xy) \le q(x^{2^n})^{1/2^n} q(y^{2^n})^{1/2^n}, \quad \forall x, y \in H(A),\ n \in \mathbb{N}.$$

But [11; p. 210, Theorem 2.2.2] $\lim_n q(x^n)^{1/n} \le r_A(x)$ for all $x \in A$, therefore taking limits for all $n \to \infty$, we deduce (1).

(2) Using standard properties of the spectral radius and (1) we have

$$\begin{aligned} p_A(xy)^2 &= r_A((xy)^*(xy)) = r_A(x^*xyy^*) \le r_A(x^*x) r_A(yy^*) \\ &= p_A(x)^2 p_A(y)^2, \quad \forall x, y \in A. \end{aligned}$$ ∎

Let A be an involutive algebra. An element $x \in A$ is called *positive*, resp. *strictly positive* (in symbols $x \ge 0$, resp. $x > 0$) if $x \in H(A)$ and $\mathrm{sp}_A(x) \subseteq [0, \infty)$, resp. $\mathrm{sp}_A(x) \subseteq (0, \infty)$.

4.4. Proposition. *For a hermitian spectral Arens–Michael algebra* $(A, \Gamma = \{p\})$, *we have*:

(1) $x + y \ge 0$ *for any positive elements* $x, y \in A$.
(2) $r_A(x + y) \le r_A(x) + r_A(y)$ *for all* $x, y \in H(A)$.

Proof. We may suppose that A is unital with unit e (see proof of Theorem 4.2).

(1) If either of x, y or both are zero, the assertion is clear. So let $x, y \in A$ with $x > 0$ and $y > 0$. Observe that the elements $e + x$, $e + y$ are invertible and

$$x + y > 0 \;\Leftrightarrow\; -1 \notin \mathrm{sp}_A(x + y) \;\Leftrightarrow\; e + x + y \in G_A.$$

On the other hand,

$$e + x + y = (e + x)(e + y) - xy = (e + x)(e - zw)(e + y), \tag{4.3}$$

with $z = (e + x)^{-1}x$ and $w = y(e + y)^{-1}$. Additionally [9; p. 93, (4.3)]

$$\mathrm{sp}_A(z) = \bigcup_{p\in\Gamma} \mathrm{sp}_{A_p}((e_p + x_p)^{-1}x_p) = \{(1 + \lambda)^{-1}\lambda : \lambda \in \mathrm{sp}_A(x)\},$$

where $\mathrm{sp}_A(x) \subseteq (0, \infty)$. Hence $r_A(z) < 1$ and similarly $r_A(w) < 1$. On the other hand, since the inverse of a self-adjoint element is also self-adjoint and $x(e + x)^{-1} = (e + x)^{-1}x$, we conclude that $z \in H(A)$. Analogously, $w \in H(A)$. Hence (see Proposition 4.3(1) and [9; p. 101, Proposition 6.1])

$$r_A(zw) \le r_A(z)r_A(w) < 1 \;\Rightarrow\; e - zw \in G_A,$$

which according to (4.3) completes the proof of (1).

(2) Let $x \in H(A)$. Then $r_A(x)e \pm x \in H(A)$ and

$$\mathrm{sp}_A(r_A(x)e \pm x) = \{r_A(x) \pm \lambda : \lambda \in \mathrm{sp}_A(x)\} \ge 0.$$

Thus taking a second element $y \in H(A)$, we get, by (1),

$$(r_A(x) + r_A(y))e \pm (x + y) \ge 0, \quad \forall x, y \in H(A),$$

whence (2) follows. ∎

4.5. Proposition. *Let $(A, \Gamma = \{p\})$ be a hermitian spectral Arens–Michael algebra. Then $r_A(x + x^*) \le 2p_A(x)$ for all $x \in A$.*

Proof. We again suppose that A is unital with unit e. Let $x \in A$. Then there are unique $y, z \in H(A)$ with $x = y + iz$. Thus

$$xx^* + x^*x = 2(y^2 + z^2) \in H(A), \tag{4.4}$$

where $y^2 \ge 0$ and $z^2 \ge 0$. Also $r_A(y^2 + z^2)e - (y^2 + z^2) \ge 0$, so that (Proposition 4.4(1)) $r_A(y^2 + z^2)e - y^2 \ge 0$. From the latter inequality we get

$$r_A(y)^2 = r_A(y^2) \le r_A(y^2 + z^2). \tag{4.5}$$

Using now (4.4), (4.5) and Proposition 4.4(2), we obtain

$$\begin{aligned} r_A(x + x^*)^2 &= 4r_A(y^2) \le 2r_A(xx^* + x^*x) \\ &\le 4r_A(x^*x) = (2p_A(x))^2, \quad \forall x \in A. \end{aligned}$$ ∎

4.6. Proposition. *Let $(A, \Gamma = \{p\})$ be a hermitian spectral Arens–Michael algebra. Then $p_A(x + y) \le p_A(x) + p_A(y)$ for all $x, y \in A$.*

Proof. Applying Propositions 4.3–4.5 and 3.1(2), we have

$$\begin{aligned} p_A(x+y)^2 &= r_A((x+y)^*(x+y)) = r_A(x^*x + y^*y + (x^*y + y^*x)) \\ &\le r_A(x^*x) + r_A(y^*y) + r_A(x^*y + y^*x) \le p_A(x)^2 + p_A(y)^2 + 2p_A(x^*y) \\ &\le p_A(x)^2 + p_A(y)^2 + 2p_A(x)p_A(y) = (p_A(x) + p_A(y))^2, \quad \forall x, y \in A. \ \blacksquare \end{aligned}$$

We are now in a position to state a version of the Shirali–Ford theorem in the context of (non-normed) topological algebras.

2.3. Theorem. *Every hermitian spectral Arens–Michael algebra A is symmetric.*

Proof. Since A is an Arens–Michael algebra, $\mathrm{sp}_A(x) \neq \emptyset$ for all $x \in A$ [9; p. 58, Corollary 4.2]. On the other hand, $r_A(x) < \infty$ for all $x \in A$, since A is spectral. Hence p_A is a real-valued function. Additionally, p_A is subadditive from Proposition 4.6, so that the assertion follows from Theorem 4.1. ■

The next corollary has been proved in [7; Theorem 7.2] by using classical techniques.

4.8. Corollary. *Every hermitian Arens–Michael Q-algebra is symmetric.*

Proof. This follows from Theorem 4.7, since every Arens–Michael Q-algebra is spectral (see (1.1)). ■

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