Compactness of the integration operator associated with a vector measure

by

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Abstract. A characterization is given of those Banach-space-valued vector measures $m$ with finite variation whose associated integration operator $I_m : f \mapsto \int f \, dm$ is compact as a linear map from $L^1(m)$ into the Banach space. Moreover, in every infinite-dimensional Banach space there exist nontrivial vector measures $m$ (with finite variation) such that $I_m$ is compact, and other $m$ (still with finite variation) such that $I_m$ is not compact. If $m$ has infinite variation, then $I_m$ is never compact.

1. Introduction and statement of results. Let $X$ be a (complex) Banach space with norm $\| \cdot \|$ and dual space $X'$. Let $\Sigma$ be a $\sigma$-algebra of subsets of a nonempty set $\Omega$ and $m : \Sigma \to X$ be a vector measure, i.e., a $\sigma$-additive set function. Associated with $m$ is the Banach space $L^1(m)$ of all (equivalence classes of) $m$-integrable functions $f : \Omega \to \mathbb{C}$ together with the integration operator $I_m : L^1(m) \to X$ given by $f \mapsto \int_\Omega f \, dm$. The operator $I_m$ is always linear and continuous. Although vector measures $m$ and the Banach spaces $L^1(m)$ have received considerable attention since their conception (see [1–4, 6, 7, 11, 14–17, 23] and the references therein, for example), the same is not true of the integration operator $I_m$. This is somewhat surprising since, for example, such an important operator as the Fourier transform map $f \mapsto \hat{f}$ from $L^1([-\pi, \pi])$ into $c_0(\mathbb{Z})$ is of the form $I_m$ for a suitable $c_0(\mathbb{Z})$-valued measure $m$ (see [18]). The same is also true for other kernel operators, such as those of Volterra type, for example [4, 20]. Or, the representation of cyclic Banach spaces is given via the integration operator with respect to a suitable vector measure [8] and so on. Whereas the weak compactness of integration operators $I_m$ has been systematically treated [18, 19], the same is not true of compactness, although some results

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for particular vector measures are known [4, 20]. The aim of this paper is to present a systematic investigation of the compactness properties of integration operators. It is time to be more precise.

The variation $|m|$ of a vector measure $m : \Sigma \to X$ is the smallest $\sigma$-additive, nonnegative scalar measure on $\Sigma$ satisfying $\|m(E)\| \leq |m|(E)$ for $E \in \Sigma$. This is equivalent to the usual definition via the “partition process” [6, pp. 2–3]. The variation $|m|$ is called finite (resp. $\sigma$-finite) if it is a finite (resp. $\sigma$-finite) measure. It turns out always to be the case that $L^1(|m|) \subseteq L^1(m)$; see Section 2.

For the definition of Bochner integrals we refer to [6, Ch. II]. Let $\lambda : \Sigma \to [0, \infty)$ be a finite measure and let $B(\lambda, X)$ denote the space of all $X$-valued, Bochner $\lambda$-integrable functions on $\Omega$. Given $G \in B(\lambda, X)$, the Bochner integral of $G$ over a set $E \in \Sigma$ (with respect to $\lambda$) is denoted by $\int_E Gd\lambda$ and is an element of $X$. The indefinite Bochner $\lambda$-integral of $G$ is defined to be the vector measure $G \cdot \lambda$ on $\Sigma$ given by $E \mapsto \int_E Gd\lambda$. We point out that the scalar function $\|G(\cdot)\|$ is always $\Sigma$-measurable and $\lambda$-integrable. Given a vector measure $m : \Sigma \to X$ with finite variation, if there exists $G \in B(\lambda, X)$, necessarily unique, such that $m$ equals the indefinite Bochner $\lambda$-integral $G \cdot \lambda$, then $G$ is called the Radon–Nikodym derivative of $m$ with respect to $\lambda$ and we write $G = dm/d\lambda$. A function $H : \Omega \to X$ is said to have $\lambda$-essentially relatively compact range if there exists a $\lambda$-null set $E \in \Sigma$ such that $H(\Omega \setminus E)$ is relatively compact in $X$ (i.e. its closure is compact).

Our first theorem characterizes compactness of $I_m$. Its proof (and of the other results of this section) is given in Section 3.

**Theorem 1.** Let $X$ be a Banach space and $m : \Sigma \to X$ be a vector measure with finite variation. Then the integration operator $I_m : L^1(m) \to X$ is compact if and only if both of the following conditions hold:

(i) the measure $m$ has a Radon–Nikodym derivative $G = dm/d|m| \in B(|m|, X)$ with respect to $|m|$ (i.e., $m = G \cdot |m|$),

(ii) the function $G$ has $|m|$-essentially relatively compact range in $X$.

In this case, the identity $L^1(m) = L^1(|m|)$ necessarily holds, and $I_mf = (B)-\int_{\Omega} f \cdot Gd|m|$ for every $f \in L^1(m)$.

As an immediate consequence of the proof of Theorem 1 (see Section 3) we have the following useful fact.

**Corollary 1.1.** Let $X$ be a Banach space and $m : \Sigma \to X$ be a vector measure with finite variation. If $J : L^1(|m|) \to L^1(m)$ denotes the natural injection, then the integration operator $I_m : L^1(m) \to X$ is compact if and only if $I_m \circ J : L^1(|m|) \to X$ is compact.
Condition (i) of Theorem 1 has received some attention in the literature and so it may be worthwhile to record some equivalent properties, namely:

(i)' Given any $E \in \Sigma$ with $|m|(E) > 0$ there exists a $\Sigma$-measurable set $F \subseteq E$ with $|m|(F) > 0$ such that

$$\text{Ave}_F(m) := \left\{ \frac{m(H)}{|m|(H)} : H \in \Sigma, H \subseteq F \text{ and } |m|(H) > 0 \right\}$$

is relatively compact in $X$.

(i)'' The integration operator $I_m$ restricted to the Banach space $L_\infty(|m|) = L_\infty(m)$, equipped with the essential sup-norm, is an $X$-valued nuclear operator.

The equivalence (i)$\Leftrightarrow$(i)' is due to M. A. Rieffel; see [21, Theorem 4.1] for the formulation given above. The equivalence (i)$\Leftrightarrow$(i)'' can be found in [21, Theorem 5.1], as a special case of [6, Theorem VI.4.4]. Further equivalences with condition (i) of Theorem 1 can be found in [21, Theorem 5.2], for example.

If $X$ is finite-dimensional and $m$ is any $X$-valued measure, then $I_m$ is necessarily compact. The existence of (nontrivial) compact integration operators in general Banach spaces $X$ is guaranteed by the following result.

**Theorem 2.** Let $X$ be an infinite-dimensional Banach space. Then there exists an $X$-valued vector measure $m$ such that $m$ has finite variation, the range of $m$ is not contained in any finite-dimensional subspace of $X$, and the integration operator $I_m$ is compact.

It is also the case, for vector measures of finite variation, that noncompact integration operators exist in every infinite-dimensional space.

**Theorem 3.** Let $X$ be an infinite-dimensional Banach space. Then there exists an $X$-valued vector measure $m$ with finite variation such that its integration operator $I_m : L^1(m) \to X$ is not compact. Furthermore, such an $m$ can be chosen to satisfy $L^1(|m|) = L^1(m)$ and have a Radon–Nikodým derivative $dm/d|m| \in \mathcal{B}(|m|, X)$.

Concerning arbitrary vector measures $m$, the determination of whether or not $I_m$ is compact reduces to the situation of finite variation.

**Theorem 4.** Let $X$ be a Banach space and $m$ be an $X$-valued vector measure. If the integration operator $I_m : L^1(m) \to X$ is compact, then $m$ must have finite variation.

If the integration operator $I_m$ is compact, then $m$ has finite variation (cf. Theorem 4), and hence, $L^1(m) = L^1(|m|)$ by Theorem 1. Moreover, if $L^1(m)$ is infinite-dimensional, then $L^1(|m|)$ contains a complemented subspace isomorphic to $\ell^1$. This can be seen by decomposing the finite measure
into the direct sum of its atomic part \(|m|_a\) and its (disjointly supported) nonatomic part \(|m|_{na}\) and then applying the argument (and discussion) from [5, pp. 201–202] to \(|m|_a\) if \(|m|\) has infinitely many atoms, and to \(|m|_{na}\) otherwise. So, Theorem 4 provides an alternative proof of the following (slightly more general) result due to G. Curbera [3, Claim, p. 3800].

**Corollary 4.1.** Let \(X\) be a Banach space and \(m\) be an \(X\)-valued vector measure such that its integration operator \(I_m : L^1(m) \to X\) is compact and \(L^1(m)\) is infinite-dimensional. Then \(m\) has finite variation and \(L^1(m)\) contains a complemented copy of \(\ell^1\).

By Theorem 4, if a vector measure has infinite variation, then its integration operator is not compact. There are many examples of such measures. To see this, let \(\lambda : \Sigma \to [0, \infty]\) be any infinite but \(\sigma\)-finite measure. For the definition of a function \(G : \Omega \to X\) being Pettis \(\lambda\)-integrable we refer to [6, Ch. II, §3]. In this case, the indefinite Pettis integral of \(G\) with respect to \(\lambda\), namely, \(E \mapsto (P)\int_E G d\lambda\) for \(E \in \Sigma\) (where the element \((P)\int_E G d\lambda\) of \(X\) denotes the Pettis integral of \(G\) over \(E\) with respect to \(\lambda\)) is an \(X\)-valued vector measure with \(\sigma\)-finite variation [10, Proposition 5.6(iv)]. So, let \(G : \Omega \to X\) be any strongly measurable function (see [6, p. 41] for the definition) which is Pettis \(\lambda\)-integrable but not Bochner \(\lambda\)-integrable. Then its indefinite Pettis \(\lambda\)-integral has infinite but \(\sigma\)-finite variation. Whenever \(X\) is infinite-dimensional and \(\lambda\) is Lebesgue measure on the half line \([0, \infty)\) such a function \(G : [0, \infty) \to X\) always exists; see the proof of [21, Theorem 3.3]. For a characterization of vector measures with \(\sigma\)-finite variation we refer to [22, Theorem 2.4]. In every infinite-dimensional Banach space there also exist vector measures with infinite but not \(\sigma\)-finite variation, and such measures can even be chosen to have relatively compact range, [23, p. 90]. See also [9, 12] for further information about such measures.

2. Preliminaries. Let \(\Sigma\) be a \(\sigma\)-algebra of subsets of a nonempty set \(\Omega\). Let \(X\) be a Banach space and \(m : \Sigma \to X\) be a vector measure. Given \(x' \in X'\), let \(\langle m, x' \rangle\) denote the complex measure \(E \mapsto \langle m(E), x' \rangle\); its variation \(|\langle m, x' \rangle|\) is then a finite measure. A \(\Sigma\)-measurable function \(f : \Omega \to \mathbb{C}\) is called \(m\)-integrable if it is \(\langle m, x' \rangle\)-integrable for all \(x' \in X'\), and if there is a set function \(f m : \Sigma \to X\), necessarily unique, satisfying \(\langle (f m)(E), x' \rangle = \int_E f d\langle m, x' \rangle\) for all \(x' \in X'\) and \(E \in \Sigma\). Then the Orlicz–Pettis theorem ensures that \(f m\) is also a vector measure. The classical notation \(\int_E f dm := (f m)(E)\), for \(E \in \Sigma\), will also be used. The vector space of all \(m\)-integrable functions is denoted by \(L^1(m)\). Define a seminorm on \(L^1(m)\) by

\[
\|f\|_m := \sup_{\Omega} \left\{ \left| \int_{\Omega} |f| d|\langle m, x' \rangle| : x' \in B[X'] \right\} \}, \quad f \in L^1(m),
\]
where for any Banach space $Y$ we define $B[Y] := \{ y \in Y : \|y\| \leq 1 \}$. The space $L^1(m)$ is then a complete seminormed space which contains the set of all $\mathbb{C}$-valued, $\Sigma$-simple functions as a dense subspace; see [14, Ch. IV] or [15, Theorem 2.4]. Associated with $m$ is its integration operator $I_m : L^1(m) \to X$ defined by

$$I_m f := \int f \, dm,$$

where $f \in L^1(m)$. It is clear that $I_m$ is linear. Moreover, since $\|I_m f\| \leq \|f\|_m$ for $f \in L^1(m)$, the operator $I_m$ is also continuous.

For each set $E \in \Sigma$, its characteristic function is denoted by $\chi_E$. The semivariation $\|m\|$ of $m$ is the set function $\|m\| : \Sigma \to [0, \infty)$ defined by $\|m\|(E) := \|\chi_E\|_m$ for $E \in \Sigma$. In particular, $\|m\|$ is always finite (unlike the variation $|m|$, in general). It follows that $\|m\|(E) \leq |m|(E)$ for every $E \in \Sigma$ (see [6, Proposition I.1.11(a)]). An element $f \in L^1(m)$ is called $m$-null if $fm$ is the zero vector measure. This is equivalent to $\|f\|_m = 0$. The quotient space $L^1(m)$ modulo the $m$-null functions and equipped with the quotient norm induced by $\|\cdot\|_m$ is a Banach space; since no confusion will occur, we denote this quotient Banach space again by $L^1(m)$ and identify it with the seminormed space from which it arises (in the usual manner).

Sets $E \in \Sigma$ satisfying $\|m\|(E) = 0$ are called $m$-null. The $m$-null and $|m|$-null sets coincide. This is immediate from the partition definition of $|m|$ (see [6, p. 2]) and the inequalities

$$\|m(E)\| \leq \|m\|(E) \leq |m|(E), \quad E \in \Sigma.$$

A property which holds outside an $m$-null set is said to hold $m$-almost everywhere (briefly, $m$-a.e.). A $\Sigma$-measurable function $f : \Omega \to \mathbb{C}$ is called $m$-essentially bounded if it is bounded $m$-a.e. The space of all such functions is denoted by $L^\infty(m)$ and is equipped with the essential sup-norm $\|\cdot\|_\infty$. It is known that $L^\infty(m) \subseteq L^1(m)$ and that, for each $f \in L^\infty(m)$,

$$\left\| \int_E f \, dm \right\| \leq \|f\|_\infty \cdot \|m\|(E), \quad E \in \Sigma;$$

see [14, Theorem II.3.1] and [6, p. 6]. The quotient Banach space of $L^\infty(m)$ modulo the $m$-null functions is also denoted by $L^\infty(m)$. In particular, $L^\infty(|m|) = L^\infty(m)$.

For the following fact we refer to [6, Theorem II.2.4].

**Lemma 2.1.** Let $\lambda : \Sigma \to [0, \infty)$ be a finite measure and $X$ be a Banach space. If $G : \Omega \to X$ is a Bochner $\lambda$-integrable function, then its indefinite Bochner $\lambda$-integral $G \cdot \lambda : \Sigma \to X$ is a vector measure with finite variation given by

$$|G \cdot \lambda|(E) = \int_E \|G(\omega)\| \, d\lambda(\omega), \quad E \in \Sigma.$$
The next result, [6, Theorem III.2.2 & p. 70], is one of the main tools of this paper.

**Lemma 2.2.** Let $\lambda : \Sigma \to [0, \infty)$ be a finite measure and $X$ be a Banach space. Then a continuous linear operator $T : L^1(\lambda) \to X$ is compact (i.e., the closure $T(B[L^1(\lambda)])$ is compact in $X$) if and only if there is a Bochner $\lambda$-integrable function $G \in \mathcal{B}(\lambda, X)$ with $\lambda$-essentially relatively compact range such that

$$Tf = (B)\int_{\Omega} f(\omega) G(\omega) \, d\lambda(\omega), \quad f \in L^1(\lambda).$$

In this case, on the complement of some $\lambda$-null set the function $G$ takes its values in the closure (in $X$) of $\{Tf : \|f\|_{L^1(\lambda)} = 1\}$.

Recall that the weak topology of a Banach space $X$ is determined by the saturated family of seminorms

$$q_F(x) := \sum_{x' \in F} |\langle x, x' \rangle|, \quad x \in X,$$

as $F$ varies through all finite subsets of $X'$. The following notion will play a crucial role. A subset $W \subseteq X$ is said to be $w$-seminorm dominated (the "$w$" denotes "weak") if there exists a finite set $F \subseteq X'$ such that

$$\|x\| \leq q_F(x), \quad x \in W,$$

where $q_F$ is given by (2.2).

**Example 2.3.** (i) Every finite-dimensional subspace $Y$ of a Banach space $X$ is $w$-seminorm dominated. In fact, let $\dim Y = n$. Take a basis $\{e_1, \ldots, e_n\}$ of $Y$ and elements $\{x'_1, \ldots, x'_n\} \subseteq X'$ such that $\langle e_j, x'_k \rangle = \delta_{jk}$ for all $j, k \in \{1, \ldots, n\}$. Since the norm on $Y$ induced by $X$ is equivalent to the norm $y \mapsto \sum_{j=1}^n |\langle y, x'_j \rangle|$ for $y \in Y$, it follows that $Y$ is $w$-seminorm dominated.

(ii) Let $\lambda : \Sigma \to [0, \infty]$ be any measure. Then the positive cone $W := \{f \in L^1(\lambda) : f \geq 0\}$ of $X := L^1(\lambda)$ is $w$-seminorm dominated. Indeed, the subset of $X'$ consisting of the single function $1$ (constantly equal to 1 on $\Omega$) satisfies

$$\|f\| = \int_{\Omega} |f| \, d\lambda = \int_{\Omega} f \, d\lambda = \langle f, 1 \rangle = |\langle f, 1 \rangle| = q_{\{1\}}(f), \quad f \in W.$$  

An important class of sets which are $w$-seminorm dominated is given by the following result.

**Lemma 2.4.** Let $X$ be a Banach space and $S[X] := \{x \in X : \|x\| = 1\}$ be its unit sphere. Then every relatively compact subset of $S[X]$ is $w$-seminorm dominated.
Proof. Let $W \subseteq S[X]$ be a relatively compact set and define $U(x) := \{y \in X : \|x - y\| < 1/2\}$ for every $x \in \overline{W}$. The compact set $\overline{W} \subseteq S[X]$ includes finitely many points $x_1, \ldots, x_n$ satisfying $\overline{W} \subseteq \bigcup_{j=1}^n U(x_j)$. Fix $j \in \{1, \ldots, n\}$ and let $C_j$ denote the closed, convex hull of $\overline{W} \cap U(x_j)$ in $X$. Then $\|y\| \leq 3/2$ for all $y \in C_j$. Since $0 \notin C_j$, there is $x'_j \in X'$ such that
$$\inf \{\|\langle y, x'_j \rangle\| : y \in C_j\} \geq 3/2;$$
x'_j is the complexification of $u_j \in (X_\mathbb{R})'$, with $u_j$ suitably chosen as in [13, Theorem 7.3.4] for $A = \{0\}$ and $B = C_j$. Thus $\|y\| \leq 3/2 \leq |\langle y, x'_j \rangle|$ for every $y \in C_j$. Consequently,
$$\|x\| \leq \sum_{j=1}^n |\langle x, x'_j \rangle|,$$
because $W \subseteq \overline{W} \subseteq \bigcup_{j=1}^n C_j$.

The converse of the previous result fails in general.

**Lemma 2.5.** Let $X$ be an infinite-dimensional Banach space. Then there exists a subset of $S[X]$ which is w-seminorm dominated but not relatively compact.

**Proof.** Choose any basic sequence $\{x_n\}_{n=1}^\infty$ of unit vectors in $X$. That is, it is a Schauder basis for its closed linear span $Y$ in $X$ and there is a constant $K > 0$ satisfying $\|\sum_{j=1}^n a_j x_j\| \leq K \|\sum_{j=1}^k a_j x_j\|$ for all choices of $n, k \in \mathbb{N}$ with $n < k$ and scalars $\{a_j\}_{j=1}^k \subseteq \mathbb{C}$ (see [5, Theorem V.1 & Corollary V.3]). Given $n \geq 2$ choose any $a_{n+1} > 0$ satisfying $\|\frac{1}{2} x_1 + \frac{1}{4} x_n + a_{n+1} x_{n+1}\| = 1$ and define $y_n := \frac{1}{2} x_1 + \frac{1}{4} x_n + a_{n+1} x_{n+1}$. Then, for all $n, k \in \mathbb{N}$ with $2 \leq n < k$,
$$\|y_n - y_k\| = \|\frac{1}{4} x_n + (a_{n+1} x_{n+1} - \frac{1}{4} x_k - a_{k+1} x_{k+1})\| \geq K^{-1} \|\frac{1}{4} x_n\| = (4K)^{-1}.$$
So, the subset $W := \{y_n : n \geq 2\}$ of $S[X]$ is not relatively compact. To see that $W$ is w-seminorm dominated, choose any $\xi \in Y'$ such that $\langle x_1, \xi \rangle = 1$ and $\langle x_j, \xi \rangle = 0$ for $j \geq 2$. By the Hahn–Banach theorem there is $x' \in X'$ which coincides with $\xi$ on $Y$. Since
$$|\langle y_n, 2x' \rangle| = 2|\langle y_n, x' \rangle| = 1 = \|y_n\|, \quad n \geq 2,$$
we see that $W$ is w-seminorm dominated.

Given a vector measure $m : \Sigma \to X$, we always have the inclusion $L^1(|m|) \subseteq L^1(m)$. Moreover, a function $f \in L^1(m)$ belongs to $L^1(|m|)$ if and only if its indefinite integral $fm : \Sigma \to X$ has finite variation, in which case $|fm|(E) = \int_E |f| dm$ (see [16, Theorem 4.2]). The natural inclusion $J : L^1(|m|) \to L^1(m)$ is continuous because
$$\|f\|_m \leq |fm|(\Omega) = \int_\Omega |f| dm = \|f\|_{L^1(|m|)}, \quad f \in L^1(|m|).$$
In the case when $L^1(m) = L^1(|m|)$ as vector spaces, the open mapping theorem implies that the Banach spaces $L^1(m)$ and $L^1(|m|)$ are isomorphic. If $|m|(\Omega) = \infty$, then the inclusion $L^1(|m|) \subseteq L^1(m)$ is always proper as $1 \in L^1(m) \setminus L^1(|m|)$. Even if $|m|(\Omega) < \infty$, this inclusion may be proper. Indeed, such measures $m$ exist in every infinite-dimensional Banach space [18, Remark 2(d)].

Given a function $G : \Omega \to X$ and $x' \in X'$, let $\langle G(\cdot), x' \rangle$ denote the scalar-valued function $\omega \mapsto \langle G(\omega), x' \rangle$ for $\omega \in \Omega$.

**Lemma 2.6.** Let $X$ be a Banach space and $m : \Sigma \to X$ be a vector measure with finite variation.

(i) If there exists a nonempty finite set $F \subseteq X'$ such that
\begin{equation}
|m|(E) \leq \sum_{x' \in F} |\langle m, x' \rangle|(E), \quad E \in \Sigma,
\end{equation}
then necessarily $L^1(m) = L^1(|m|)$.

(ii) Suppose that there exists a finite measure $\lambda : \Sigma \to [0, \infty)$ with the property that $m$ has a Radon–Nikodým derivative $G \in \mathcal{B}(\lambda, X)$ with respect to $\lambda$ (i.e., $m = G \cdot \lambda$). Then there exists an $m$-null set $E \in \Sigma$ such that $G(\Omega \setminus E) \subseteq X$ is $w$-seminorm dominated if and only if (2.3) holds for some nonempty finite set $F \subseteq X'$.

**Proof.** (i) This is an immediate consequence of the inclusion $L^1(|m|) \subseteq L^1(m)$ and (2.3) since
\[ L^1(|m|) \subseteq L^1(m) \subseteq \bigcap_{x' \in F} L^1(|\langle m, x' \rangle|) \subseteq L^1(|m|). \]

(ii) It follows from Lemma 2.1 that $|\langle m, x' \rangle| = |\langle G(\cdot), x' \rangle| \cdot \lambda$ on $\Sigma$ for all $x' \in F$, and $|m| = \|G(\cdot)\| \cdot \lambda$ on $\Sigma$. Therefore (2.3) holds if and only if
\[ \int_E \|G(\omega)\| \, d\lambda(\omega) \leq \sum_{E' \in F} \|\langle G(\omega), x' \rangle\| \, d\lambda(\omega), \quad E \in \Sigma, \]
which is the case if and only if $\|G(\omega)\| \leq \sum_{x' \in F} |\langle G(\omega), x' \rangle|$ for $m$-almost every $\omega \in \Omega$.

**Remark 2.7.** (i) If the range of a vector measure $m : \Sigma \to X$ is $w$-seminorm dominated, then $m$ has finite variation and satisfies (2.3) for some nonempty finite set $F \subseteq X'$. In particular, $L^1(m) = L^1(|m|)$.

The converse is false in general, i.e. there exists a vector measure $m$ (it can even be chosen with $I_m$ compact!) such that (2.3) holds for some nonempty finite set $F \subseteq X'$ but the range of $m$ is not $w$-seminorm dominated. Indeed, let $X := \ell^2$ and let $\Sigma$ denote the $\sigma$-algebra of all Borel subsets of $\Omega := [0, 2]$. Let $\alpha_n > 0$ for $n = 2, 3, \ldots$ be any sequence decreasing to 0 and, for each $n \geq 1$, let $A_n := [1/(n + 1), 1/n)$. Define $G : \Omega \to X$ by
$G(\omega) := -\chi_{[1, 2]}(\omega) \cdot e_1 + \sum_{n=1}^{\infty} \chi_{A_n}(\omega) \cdot (e_1 + \alpha_{n+1} \varepsilon_{n+1})$, \quad \omega \in \Omega,$

where $\{e_n\}_{n=1}^{\infty}$ is the standard orthonormal basis of $X$. Since $\langle G(\cdot), e_1 \rangle = \chi_{(0, 1)}(\cdot) - \chi_{[1, 2]}(\cdot)$ and $\|G(\omega)\| \leq (1 + \alpha_2^2)^{1/2}$ for every $\omega \in \Omega$, it follows that

$$\|G(\omega)\| \leq |\langle G(\omega), (1 + \alpha_2^2)^{1/2}e_1 \rangle|, \quad \omega \in \Omega,$$

i.e. the range of $G$ is $w$-seminorm dominated. But $G$ is easily seen to be Bochner integrable with respect to Lebesgue measure $\lambda$ on $\Sigma$. Hence, if we let $m := G \cdot \lambda$, then it follows from Lemma 2.6(ii) that $m$ satisfies (2.3) with $F := \{(1 + \alpha_2^2)^{1/2}e_1\}$.

For each $n \geq 1$, let $B_n := ((n + 2)/(n + 1), (n + 1)/n)$ and note that $m(A_n \cup B_n) = \lambda(A_n)\alpha_{n+1} \varepsilon_{n+1}$. Suppose that there exists a nonempty finite set $H \subseteq X'$ such that

$$\|m(E)\| \leq \sum_{x' \in H} |\langle m(E), x' \rangle|, \quad E \in \Sigma.$$

By choosing $E$ to be $A_n \cup B_n$ it would follow that $1 \leq \sum_{x' \in H} |\langle e_{n+1}, x' \rangle|$ for every $n \in \mathbb{N}$, which is impossible. So, $m(\Sigma)$ is not $w$-seminorm dominated.

Since $dm/d|m|$ is the function $\omega \mapsto G(\omega)/\|G(\omega)\|$ for $\omega \in \Omega$, and this function has $|m|$-essential range equal to the relatively compact subset $\{-e_1\} \cup \{(1 + \alpha_{n+1}^2)^{-1/2}(e_1 + \alpha_{n+1} \varepsilon_{n+1}) : n \in \mathbb{N}\}$ of $S[X]$, it follows from Theorem 1 that $I_m$ is compact.

(ii) The converse of Lemma 2.6(i) is not valid in general. To see this, let $X := c_0$ and consider the functions

$$g_n := r_n \chi_{A_n}, \quad n \in \mathbb{N},$$

where $A_n := [0, 1 - 1/(n + 1)]$ and $\{r_n\}_{n=1}^{\infty} \subseteq L^\infty([0, 1])$ is the sequence of Rademacher functions. Since $\{g_n\}_{n=1}^{\infty}$ is a weak-star null sequence in the dual space $L^\infty([0, 1])$ of $L^1([0, 1])$, it is clear that the set function $m : \Sigma \to X$ defined by

$$m(E) := \left(\int_E g_n \, d\lambda\right)_{n=1}^{\infty} \in c_0, \quad E \in \Sigma,$$

is a vector measure, where $\Sigma$ is the $\sigma$-algebra of all Borel subsets of $\Omega := [0, 1]$ and $\lambda$ is Lebesgue measure on $\Sigma$. From the fact that $|r_n(\omega)| = 1$ for all $\omega \in \Omega$ and $n \in \mathbb{N}$, it follows that $m$ has finite variation given by $|m| = \lambda$. Let $\{e'_n\}_{n=1}^{\infty} \subseteq X' = \ell^1$ be the standard basis of $\ell^1$. For each $f \in L^1(m)$ we have

$$\int_0^{1-1/(n+1)} |f| \, d\lambda = \int_0^1 |f| \, d|\langle m, e'_n \rangle| \leq \|f\|_m, \quad n \in \mathbb{N}.$$

The monotone convergence theorem implies that $\int_0^{1-1/(n+1)} |f| \, d\lambda \to \|f\|_{L^1(\lambda)}$ as $n \to \infty$, from which it is clear that $L^1(m) = L^1(|m|)$ with equality.
of norms. If \( x' = (x'_n)_{n=1}^\infty \in X' \), then \( d|\langle m, x' \rangle|/d\lambda \leq \sum_{n=1}^\infty |x'_n| \cdot |g_n(\cdot)| \) pointwise a.e. on \( \Omega \). So, if (2.3) holds for some nonempty finite set \( F \subseteq X' \), then there exists \( \xi = (\xi_n)_{n=1}^\infty \in X' \), with \( \xi_n > 0 \) for all \( n \in \mathbb{N} \), such that

\[
\sum_{x' \in F} |\langle m, x' \rangle|(E) \leq \sum_{n=1}^\infty \xi_n \int_E |g_n(\omega)| \, d\lambda(\omega), \quad E \in \Sigma.
\]

Choosing \( E :=[1-1/k, 1] \) gives

\[
\sum_{x' \in F} |\langle m, x' \rangle|([1-1/k, 1]) \leq \sum_{n \geq k} \xi_n \cdot \lambda([1-1/k, 1]), \quad k \in \mathbb{N}.
\]

It then follows from the identity \( |m| = \lambda \) and (2.3) that

\[
\lambda([1-1/k, 1]) = |m|([1-1/k, 1]) \leq \sum_{n \geq k} \xi_n \cdot \lambda([1-1/k, 1]), \quad k \in \mathbb{N},
\]

which is impossible as \( \xi \in \ell^1 \). So, there is no nonempty finite set \( F \subseteq X' \) such that (2.3) holds.

We conclude with the following result needed later.

**Lemma 2.8.** Let \( Y \) be a Banach space and \( Z \) be a closed subspace of \( Y \) having finite codimension. For each \( \varepsilon > 0 \) there exists a finite set \( F_\varepsilon \subseteq B[Y] \) such that

\[
B[Y] \subseteq (1 + \varepsilon)(\text{bco}(F_\varepsilon) + 2B[Z]),
\]

where \( \text{bco}(H) \) denotes the balanced convex hull of any set \( H \subseteq Y \).

**Proof.** Let \( \delta := \varepsilon/(2(1 + \varepsilon)) \). For \( y \in Y \), let \( \widetilde{y} \) denote its equivalence class in the quotient Banach space \( Y/Z \) which is equipped with the quotient norm \( \| \cdot \|_{Y/Z} \). Since \( 0 < \delta < 1/2 \) and \( \dim(Y/Z) < \infty \), there are finitely many (say \( N \)) open balls of radius \( \delta \), having their centres in the open unit ball of \( Y/Z \), which cover \( B[Y/Z] \). So, there exists a finite set \( F_\varepsilon := \{w_1, \ldots, w_N\} \subseteq B[Y] \) with the property that for every \( y \in B[Y] \) there is \( j(y) \in \{1, \ldots, N\} \) such that \( \|\widetilde{y} - (w_{j(y)})\|_{Y/Z} < \delta \), and hence, there is also \( z \in Z \) (depending on \( y \)) such that

\[
\|y - z - w_{j(y)}\| < \delta.
\]

Observe that \( z \) also satisfies \( \|z\| \leq 1 + \delta + \|y\| \). Define a closed, balanced, convex set \( G \subseteq Y \) by \( G := 2B[Z] + \text{bco}(F_\varepsilon) \). Then we have just established that for every \( y \in (1-\delta)B[Y] \subseteq B[Y] \) there exists \( x \in G \) (depending on \( y \)) with \( \|y - x\| < \delta \). Indeed, the choice \( x := z + w_{j(y)} \) as in (2.4) has the desired property since \( \|z\| \leq 1 + \delta + \|y\| \leq 1 + \delta + (1-\delta) = 2 \) shows that \( z \in 2B[Z] \).

Fix \( y' \in Y' \). Then we have

\[
\sup\{|\langle y, y' \rangle| : y \in (1 - 2\delta)B[Y]\} = \sup\{|\langle y, y' \rangle| : y \in (1 - \delta)B[Y]\} - \delta\|y'\|.
\]
But, for each $y \in (1 - \delta)B[Y]$, choose a vector $x_y \in G$ satisfying $\|y - x_y\| < \delta$. Then

$$|\langle y, y' \rangle| \leq |\langle y - x_y, y' \rangle| + |\langle x_y, y' \rangle| \leq \delta \|y'\| + \sup\{|\langle x, y' \rangle| : x \in G\},$$

from which it is clear that

$$\sup\{|\langle y, y' \rangle| : y \in (1 - \delta)B[Y]\} \leq \delta \|y'\| + \sup\{|\langle x, y' \rangle| : x \in G\}.$$  

Then (2.5) implies (2.6)

$$\sup\{|\langle y, y' \rangle| : y \in B[Y]\} \leq \sup\{|\langle x, y' \rangle| : x \in (1 - 2\delta)^{-1}G\}, \quad y' \in Y'.$$

It follows that

$$B[Y] \subseteq (1 - 2\delta)^{-1}G.$$  

In fact, assume the contrary. Then there would exist a vector $y_0 \in B[Y] \setminus (1 - 2\delta)^{-1}G$. Since $\{y_0\}$ is compact and convex and since $(1 - 2\delta)^{-1}G$ is balanced, convex and closed, by [13, Corollary 5, p. 131] there would exist $y'_0 \in Y'$ satisfying

$$\sup\{|\langle x, y'_0 \rangle| : x \in (1 - 2\delta)^{-1}G\} < |\langle y_0, y'_0 \rangle|.$$  

But this contradicts the inequality

$$|\langle y_0, y'_0 \rangle| \leq \sup\{|\langle x, y'_0 \rangle| : x \in (1 - 2\delta)^{-1}G\}$$

which follows from (2.6) because $y_0 \in B[Y]$. Thus (2.7) holds, and hence,

$$B[Y] \subseteq (1 - 2\delta)^{-1}G = (1 + \varepsilon)G = (1 + \varepsilon)(\text{bco}(F) + 2B[Z]).$$  

The previous result is optimal in the sense that there exist a Banach space $Y$ and a closed subspace $Z$ of $Y$ (with finite codimension) having the property that if

$$B[Y] \subseteq K + \beta B[Z]$$

for any compact set $K \subseteq Y$, then necessarily $\beta > 2$. Moreover, if (2.8) holds for some $K$ of the form $K = \alpha \text{bco}(F)$ with $F$ a finite subset of $B[Y]$, then necessarily $\alpha > 1$. Indeed, take $Y := \ell^1$ and $Z := \ker(\psi)$, where $\psi \in (\ell^1)'$ is the linear functional given by

$$\langle y, \psi \rangle := \sum_{n=1}^{\infty} (1 - n^{-1})y_n, \quad y = (y_n)_{n=1}^{\infty} \in Y.$$  

The point is that $\|\psi\| = 1$ but $|\langle y, \psi \rangle| < 1$ for every $y \in B[Y]$.

Suppose that there exist $\beta > 0$ and a compact set $K \subseteq Y$ such that (2.8) holds. It can be seen (argue by contradiction) that there exists $\varepsilon > 0$ with the property that $|\langle y, \psi \rangle| \leq 1 - \varepsilon$ for all $y \in K \cap (1 + \varepsilon)B[Y]$. Choose any $n_0 \in \mathbb{N}$ satisfying

$$|y_j| < \varepsilon/3 \quad \text{for all } y \in K \text{ and } j \geq n_0.$$  


Now choose an integer $n > n_0$ such that $n^{-1} < \varepsilon$ and let $e_n \in B[Y]$ be the corresponding $n$th standard basis vector of $Y$. By (2.8) there are $y \in K$ and $z \in B[Z]$ such that $e_n = y + \beta z$. Then

$$1 - \varepsilon < 1 - n^{-1} = \langle e_n, \psi \rangle = \langle y, \psi \rangle$$

and, by the above property of $\varepsilon$, we have $\|y\| > 1 + \varepsilon$. It follows that $\beta > 2$ since

$$\beta \geq \|\beta z\| = \|e_n - y\| = |1 - y_n| + \sum_{j \in \mathbb{N}\setminus\{n\}} |y_j|$$

$$= \|y\| - |y_n| + |1 - y_n| > 1 + \varepsilon + 1 - 2|y_n|$$

$$> 2 + \varepsilon - (2\varepsilon/3) > 2.$$
ensures that $L^1(|m|) = L^1(m)$. Since $J$ is then the identity operator and $I_m \circ J = I_m$, the compactness of $I_m$ follows from that of $I_m \circ J$. ■

Some additional comments concerning Theorem 1 are in order. If $X$ has the Radon–Nikodým property and $m : \Sigma \to X$ has finite variation, then condition (i) of Theorem 1 is automatically satisfied. Accordingly, the compactness of $I_m$ is then solely determined by whether or not $dm/d|m| \in B(|m|, X)$ has $|m|$-essentially relatively compact range in $X$. This is not always the case.

**Example 3.1.** Let $\Omega := [0, 1]$ and $\Sigma$ be the $\sigma$-algebra of all Borel subsets of $\Omega$. Fix $p \in (1, \infty)$, in which case the reflexive Banach space $X := L^p([0, 1])$ has the Radon–Nikodým property [6, p. 218]. Define $m : \Sigma \to X$ by

$$m(E) : t \mapsto \int_0^t \chi_E(s) \, ds, \quad t \in [0, 1],$$

for $E \in \Sigma$. Then $m$ is a vector measure with finite variation and $|m|(E) = \int_E (1 - s)^{1/p} \, ds$ for every $E \in \Sigma$. Moreover, $G = dm/d|m|$ is the function $G(s) = (1 - s)^{-1/p} \cdot \chi_{[s, 1]}(\cdot)$ for $s \in \Omega$. It is shown in the proof of [20, Proposition 5.2(ii)] that $G$ (called $h_p$ in [20]) does not have $|m|$-essentially relatively compact range, and hence $I_m$ is not compact. ■

A similar phenomenon to that in Example 3.1 can occur in spaces without the Radon–Nikodým property.

**Example 3.2.** Let $\Omega$ and $\Sigma$ be as in Example 3.1. Then $X := L^1([0, 1])$ does not have the Radon–Nikodým property [6, p. 219]. Define a vector measure $m : \Sigma \to X$ again by the formula (3.1), in which case $m$ has finite variation given by $|m|(E) = \int_E (1 - s) \, ds$ for $E \in \Sigma$. Moreover, $G := dm/d|m|$ exists and is the $X$-valued function $G(s) = (1 - s)^{-1} \chi_{[s, 1]}(\cdot)$ for $s \in \Omega$. Since $I_m$ is not even weakly compact [18, Proposition 2.7], it cannot be compact. By Theorem 1, $G$ does not have $|m|$-essentially relatively compact range. ■

For examples of vector measures of finite variation which have no Radon–Nikodým derivative with respect to their variation (i.e. condition (i) of Theorem 1 fails) we refer to the Volterra measures considered in [20] in the spaces $C([0, 1])$ and $L^\infty([0, 1])$. Neither of these spaces has the Radon–Nikodým property [6, p. 219].

**Remark 3.3.** (i) The vector measure $m$ of Example 3.2 satisfies $L^1(m) = L^1(|m|)$; see [18, Lemma 2.4], or use Example 2.3(ii) and Remark 2.7. This shows that the compactness of $I_m$ is not equivalent to the equality $L^1(m) = L^1(|m|)$; see the statement of Theorem 1.
(ii) The compactness of $I_m$ clearly implies the relative compactness of $m(\Sigma)$ in $X$. The converse is not true in general. To see this let $X := L^p([0, \infty])$ for any $1 \leq p \leq \infty$ and define $m$ as in Examples 3.1 and 3.2. It is known that the classical Volterra operator $V : X \to X$ defined by $Vf : t \mapsto \int_0^t f(s) \, ds$, for $t \in [0,1]$ and every $f \in X$, is a compact operator. Since $m(E) = V(\chi_E)$ for all $E \in \Sigma$, it follows that $m(\Sigma)$ is relatively compact in $X$. ■

Proof of Theorem 2. Let $\Omega := [0,1]$ and $\lambda$ be Lebesgue measure on the Borel $\sigma$-algebra $\Sigma$ of $\Omega$. Choose any basic sequence $\{x_n\}_{n=1}^\infty$ of unit vectors in $X$ (see [5, Corollary V.3]). By using the fact that the sequence $\{n^{-3}e^{in\omega}x_n\}_{n=1}^\infty$ is absolutely summable in $X$ for every $\omega \in \Omega$, we can define a function $H : \Omega \to X$ by $H(\omega) := \sum_{n=1}^\infty n^{-3}e^{in\omega}x_n$ for $\omega \in \Omega$. Then $H$ is continuous because $\|H(\omega) - H(\omega')\| \leq |\omega - \omega'|(\sum_{n=1}^\infty n^{-2})$ whenever $u, \omega \in \Omega$. Moreover, $H(\omega) \neq 0$ for each $\omega \in \Omega$. This is a consequence of the fact that $\{x_n\}_{n=1}^\infty$ is a Schauder basis for its closed linear span $Y$ and the fact that $H(\omega) = 0$ if and only if $n^{-3}e^{in\omega} = 0$ for every $n \in \mathbb{N}$, which never occurs. Clearly $\langle H(\cdot), x' \rangle$ is $\Sigma$-measurable for each $x' \in X'$. Since $H$ takes its values in the separable subspace $Y$ of $X$, it follows from the Pettis measurability theorem [6, p. 42] that $H$ is strongly measurable. Accordingly, $H \in B(\lambda, X)$. Let $m := H \cdot \lambda$ be the indefinite Bochner $\lambda$-integral of $H$ with respect to $\lambda$. By Lemma 2.1, $m$ has finite variation $|m| = \|H(\cdot)\| \cdot \lambda$. Since $\omega \mapsto 1/\|H(\omega)\|$ is continuous and strictly positive on $\Omega$, it follows that $G := H(\cdot)/\|H(\cdot)\|$ is continuous on $\Omega$ and so has compact range in $S[X]$. In particular, $G = dm/d|m|$ is Bochner $|m|$-integrable. Theorem 1 ensures that $I_m$ is compact. ■

Proof of Theorem 3. Let $W := \{y_n : n \geq 2\} \subseteq S[X]$ be a set which is not relatively compact but is w-seminorm dominated; see Lemma 2.5. Let $\Omega := \mathbb{N}$ and $\Sigma := 2^{\mathbb{N}}$, and define a vector measure $m : \Sigma \to X$ by $m(E) := \sum_{n \in E} 2^{-n}y_{n+1}$ for $E \in \Sigma$. Then $m$ has finite variation given by $|m|(E) := \sum_{n \in E} 2^{-n}$ for $E \in \Sigma$. Define $G : \Omega \to X$ by $G(n) := y_{n+1}$ for $n \in \Omega$. Since $\langle G(\cdot), x' \rangle$ is $\Sigma$-measurable for each $x' \in X'$ (as $\Sigma = 2^{\mathbb{N}}$) and $G$ takes its values in the separable subspace of $X$ generated by $W$, the Pettis measurability theorem ensures that $G$ is strongly measurable. Moreover, $\|G(\cdot)\| \, d|m| = |m|(\Omega) < \infty$ and so $G \in B(|m|, X)$. It is routine to verify that $m = G \cdot |m|$ and so $dm/d|m| = G \in B(|m|, X)$. Since $G(\Omega) = W$ is w-seminorm dominated, Lemma 2.6 implies that $L^1(m) = L^1(|m|)$. But $G$ does not have $|m|$-essentially relatively compact range in $X$ and so Theorem 1 shows that $I_m$ is not compact. ■

Proof of Theorem 4. Denote the domain of $m$ by $\Sigma$ and let $\lambda : \Sigma \to [0, \infty)$ be a control measure for $m$. That is, $\lambda$ is a finite measure such that $\lambda(E) \to 0$ implies $m(E) \to 0$, or equivalently $\lambda(E) = 0$ implies $m(E) = 0$;
see [6, p. 10 & p. 14]. By Rybakov’s theorem [6, Theorem IX.2.2], it is possible to choose (which we do) \( \lambda = |\langle m, x_0' \rangle| \) for a suitable \( x_0' \in X' \). It is then clear that

\[
L^\infty(m) = L^\infty(\lambda) \subseteq L^1(m) \subseteq L^1(\lambda).
\]

Moreover, if we define \( \psi_{x'} := d\langle m, x' \rangle / d\lambda \) for \( x' \in X' \), then \( \psi_{x'} \in L^1(\lambda) \) and

\[
\|f\|_m = \sup \left\{ \int_{\Omega} |f| \cdot |\psi_{x'}| \, d\lambda : x' \in B[X'] \right\}, \quad f \in L^1(m).
\]

Suppose that \( m : \Sigma \to X \) does not have finite variation. Then for every \( h \in L^1(\lambda) \) there exists \( A \in \Sigma \) (depending on \( h \)) such that \( \|m(A)\| > \int_A |h| \, d\lambda \). It is to be proved that \( I_m \) is not compact. To this end we construct, inductively, a sequence \( \{g_n\}_{n=1}^\infty \subseteq B[L^1(m)] \) such that \( \|I_m(g_n) - I_m(g_k)\| \geq 1/4 \) whenever \( n \neq k \).

Choose \( g_1 \in B[L^1(m)] \) arbitrarily and suppose that functions \( g_1, \ldots, g_n \in B[L^1(m)] \) have been constructed with the stated property. Let \( H \subseteq X \) be the finite-dimensional subspace spanned by \( \{I_m(g_1), \ldots, I_m(g_n)\} \). By Lemma 2.8 with \( \varepsilon := 1/4 \), \( Y := X' \) and \( Z := H^\perp = \{x' \in X' : \langle I_m(g_j), x' \rangle = 0 \) for \( 1 \leq j \leq n \} \), there is a finite set \( F = \{x'_1, \ldots, x'_N\} \) in \( B[X'] \) with \( N \in \mathbb{N} \) such that

\[
B[X'] \subseteq \frac{5}{4} (bco(F) + 2B[H^\perp]).
\]

Define \( \psi \) by \( \omega \mapsto \psi(\omega) := \max\{|\psi_{x'_j}(\omega)| : 1 \leq j \leq N\} \) for \( \omega \in \Omega \). Since \( 5\psi \in L^1(\lambda) \) and \( |m|(\Omega) = \infty \), we noted above that there is a set \( A \in \Sigma \) with \( \|m(A)\| > 5 \int_A \psi \, d\lambda \). Then \( \alpha := \|\chi_A\|_m \) satisfies

\[
\alpha \geq \|m(A)\| > 5 \int_A \psi \, d\lambda \geq 0.
\]

Moreover, \( \alpha = \sup \{\int_A |\psi_{u'}| \, d\lambda : u' \in B[X'] \} \) by (3.2), and so we can choose \( x' \in B[X'] \) such that \( \frac{7}{8} \alpha < \int_A |\psi_{x'}| \, d\lambda \). By (3.3) there exist complex numbers \( \alpha_j \), for \( 1 \leq j \leq N \), with \( \sum_{j=1}^N |\alpha_j| \leq 1 \) and \( z' \in B[H^\perp] \) such that \( x' = \frac{5}{4} \left( \sum_{j=1}^N \alpha_j x'_j + 2z' \right) \). Then \( \psi_{x'} = \frac{5}{4} \left( \sum_{j=1}^N \alpha_j \psi_{x'_j} + 2\psi_{z'} \right) \) satisfies \( |\psi_{x'}| \leq \frac{5}{4} (\psi + 2|\psi_{z'}|) \) and hence

\[
\frac{7}{8} \alpha < \frac{5}{4} \int_A \psi \, d\lambda + \frac{5}{2} \int_A |\psi_{z'}| \, d\lambda < \alpha + \frac{5}{2} \int_A |\psi_{z'}| \, d\lambda.
\]

It follows that \( \alpha/4 < \int_A |\psi_{z'}| \, d\lambda \). Now define

\[
g_{n+1} := \alpha^{-1} \chi_A \cdot |\psi_{z'}| / |\psi_{z'}|,
\]

with the understanding that \( 0/0 = 1 \), and note that \( g_{n+1} \in L^\infty(\lambda) = L^\infty(m) \subseteq L^1(m) \). Moreover, \( |g_{n+1}| = \alpha^{-1} \chi_A \), from which it follows that \( \|g_{n+1}\|_m = 1 \), i.e., \( g_{n+1} \in B[L^1(m)] \). Now fix \( k \in \{1, \ldots, n\} \). Then we have
\( \langle I_m(g_k), z' \rangle = 0 \), since \( z' \in H^\perp \). Accordingly, since also \( \|z'\| \leq 1 \), we see that

\[
\|I_m(g_{n+1}) - I_m(g_k)\| \geq |\langle I_m(g_{n+1}) - I_m(g_k), z' \rangle| = |\langle I_m(g_{n+1}), z' \rangle| = \left| \int_{\Omega} \psi_z' g_{n+1} d\lambda \right| = \alpha^{-1} \int_{\Lambda} |\psi_z'| d\lambda > \frac{1}{4}.
\]

This completes the construction of \( g_{n+1} \), and hence also the proof. ■

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