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The Bergman projection on weighted spaces: L^1 and Herz spaces

by

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Abstract. We find necessary and sufficient conditions on radial weights w on the unit disc so that the Bergman type projections of Forelli–Rudin are bounded on $L^1(w)$ and in the Herz spaces $K_p^q(w)$.

1. Introduction and preliminaries. The purpose of this paper is to study spaces of analytic functions on the unit disc $\mathbb{D}$ provided with a norm of a weighted Herz space. More precisely we consider the classical family of Bergman projections P_s, $s > -1$, and we give necessary and sufficient conditions on the weight making these projections continuous in the corresponding weighted Herz space. We also consider the continuity of these projections in the weighted L^1 space. The continuity of the projections P_s has been studied by many authors in several settings like weighted L^p continuity or weighted mixed norms (see for example [2, 5, 8, 12, 15, 16, 18] and [1, 3, 4, 17, 19] for related literature on Bergman type spaces).

Throughout the paper $dm(z)$ is the normalized area measure on the disc, that is, $dm(z) = \pi^{-1} r dr d\theta$. By a *weight* we understand a function w such that $0 < w(z) < \infty$. If f is a function on $\mathbb{D}$ and $s \geq -1$, we set $f_s(z) = (1-|z|^2)^s f(z)$. We write $r_n = 1 - 2^{-n}$, $I_n = \{r : r_n < r < r_{n+1}\}$ and $A_n = \{z \in \mathbb{D} : r_n < |z| < r_{n+1}\}$. We define

$$\|f\|_{L^p(w)} = \Big(\int_{\mathbb{D}} |f(z)|^p w(z)\, dm(z)\Big)^{1/p},$$

and

$$M_p(f,r) = \Big(\int_0^{2\pi} |f(re^{i\theta})|^p \,\frac{d\theta}{2\pi}\Big)^{1/p}.$$

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We write $w(A) = \int_A w(z)\,dm(z)$ for any measurable subset A of $\mathbb{D}$. Given a function ψ integrable on $[0,1)$ we denote by $M_n(\psi) = \int_0^1 \psi(r) r^n\,dr$ the moment of order n for $n \in \mathbb{N}$ or $n = 0$.

Define the spaces $K_q^{\alpha,p}$ to consist of all measurable functions f on $\mathbb{D}$ such that

$$\sum_{n=1}^{\infty} 2^{-\alpha q n}\Big(\int_{A_n} |f(z)|^p\,dm(z)\Big)^{q/p} < \infty.$$

These spaces are a variant of those introduced by C. Herz in [10]. In this paper we will consider a more general class of spaces, the weighted Herz spaces $K_q^p(w)$, $1 \le p, q \le \infty$, introduced by Lu and Yang in [14] (see also [13] for power weights). These spaces consist of all measurable functions f on the disc such that $(\|f\|_{L_w^p(A_n)}) \in \ell^q$. The norm in $K_q^p(w)$ is defined by

$$\|f\|_{K_q^p(w)} = \|(\|f\|_{L_w^p(A_n)})\|_{\ell^q}.$$

EXAMPLE 1.1. (a) If $f = \sum_{m=1}^{\infty} a_n \chi_{A_n}$ then $f \in K_q^p(w)$ if and only if

$$\sum_{m=1}^{\infty} |a_n|^q w(A_n)^{q/p} < \infty.$$

(b) Let w be a radial weight and $f(z) = \phi(r)\psi(\theta)$ for $z = re^{i\theta}$ where ϕ, ψ are measurable functions in $[0,1)$ and $[0,2\pi)$ respectively. Then

$$\|f\|_{K_q^p(w)} = \|\psi\|_{L^p([0,2\pi))}\Big(\sum_{n=1}^{\infty}\Big(\int_{I_n} |\phi(r)|^p w(r) r\,dr\Big)^{q/p}\Big)^{1/q}.$$

For $s > -1$ we consider the family of Bergman projections

$$P_s f(z) = \int_{\mathbb{D}} K_s(z,\xi) f(\xi)(1-|\xi|^2)^s\,dm(\xi),$$

where

$$K_s(z,\xi) = \frac{1}{(1-z\overline{\xi})^{2+s}} = \frac{1}{\Gamma(s+2)}\sum_{n=0}^{\infty}\frac{\Gamma(n+s+2)}{n!}\,z^n\overline{\xi}^n.$$

LEMMA 1.2. (a) *If $f(z) = \phi(r)\psi(\theta)$ for $z = re^{i\theta}$ then*

$$P_s f(z) = \frac{2}{\Gamma(s+2)}\sum_{n=0}^{\infty}\frac{\Gamma(n+s+2)}{n!}\,M_{n+1}(\phi_s)\widehat{\psi}(n) z^n.$$

(b) *Fix $s > -1$. Then $P_s\chi_{A_n}(z) = c_{n,s} \sim 2^{-n(s+1)}$ for all $z \in \mathbb{D}$.*

Proof. To prove (a), we use polar coordinates to get

$$P_s f(z) = 2\int_0^1\Big(\int_0^{2\pi}\frac{\psi(\theta)}{(1-re^{-i\theta}z)^{2+s}}\,\frac{d\theta}{2\pi}\Big)(1-r^2)^s\phi(r) r\,dr$$

$$= \frac{2}{\Gamma(s+2)} \int_0^1 \Big(\sum_{n=0}^{\infty} \frac{\Gamma(n+s+2)}{n!} \widehat{\psi}(n) r^n z^n \Big) (1-r^2)^s \phi(r) r\, dr$$
$$= \frac{2}{\Gamma(s+2)} \sum_{n=0}^{\infty} \frac{\Gamma(n+s+2)}{n!} M_{n+1}(\phi_s) \widehat{\psi}(n) z^n.$$

The proof of (b) easily follows from (a). ∎

Define now the spaces $H_{pq}(w)$ to consist of all functions f holomorphic on the disc $\mathbb{D}$ such that

$$\Big(\int_0^1 M_p(f,r)^q w(r) r\, dr \Big)^{1/q} < \infty.$$

For the weight $w(r) = (1-r^2)^{q\alpha-1}$ the spaces are sometimes denoted by $H(p,q,\alpha)$.

Using the fact that $M_p(f,r)$ is increasing for holomorphic functions one gets the following

PROPOSITION 1.3. *Let w be a weight such that $w(A_n) \le C w(A_{n+1})$, for instance $w(r) = (1-r^2)^{\beta}$ or $w = \sum a_n \chi_{A_n}$ with $a_n/a_{n+1} \le M$. Then*

1. *$f \in H_{pq}(w)$ if and only if $\sum_{n=1}^{\infty} M_p(f, r_n)^q w(A_n) < \infty$.*
2. *$f \in K_q^p(w) \cap \mathrm{Hol}(D)$ if and only if $\sum_{n=1}^{\infty} M_p(f,r_n)^q w(A_n)^{q/p} < \infty$.*

In particular, $K_q^p(w) \cap \mathrm{Hol}(D) = H_{pq}(w^{q/p})$.

1.1. *The class B_s^p.* In [2] Bekollé introduced the class B_s^p of weight functions. Let $1 < p < \infty$. A radial weight $w = w(r)$ belongs to B_s^p if

$$(1) \quad \Big(\int_{1-h}^1 w(r)(1-r^2)^s r\, dr \Big) \Big(\int_{1-h}^1 w(r)^{-p'/p} (1-r^2)^s r\, dr \Big)^{p/p'} \le C h^{(s+1)p}.$$

EXAMPLE 1.4. (a) If $w = \sum_{n=1}^{\infty} a_n \chi_{A_n}$ with $a_n > 0$, then $w \in B_s^p$ if and only if

$$\Big(\sum_{k=n}^{\infty} a_k 2^{-(s+1)k} \Big) \Big(\sum_{k=n}^{\infty} a_k^{-p'/p} 2^{-(s+1)k} \Big)^{p/p'} \le C 2^{-(s+1)np}.$$

(b) If $w(r) = (1-r^2)^{\alpha - s}$ then $w \in B_s^p$ if and only if

$$(2) \qquad 0 < \alpha + 1 < p(s+1).$$

In [2] it was proved that B_s^p is precisely the class of weight functions making P_s a continuous projection:

THEOREM 1.5. *Let $1 < p < \infty$. P_s is continuous in $L^p(w_s)$ if and only if $w \in B_s^p$.*

Notice in particular that P_s is continuous on $L^p((1-r^2)^{\alpha})$ if and only if the inequality (2) holds. Also for $p = 1$ a weak type continuity result was obtained in [2] and the B^p_s condition was shown to be equivalent to the boundedness in $L^p(w_s)$ of P^*_s where

$$P^*_s f(z) = \int_{\mathbb{D}} \frac{(1-|\xi|^2)^s f(\xi)}{|1-\overline{\xi}z|^{2+s}}\, dm(\xi).$$

2. Continuity on $L^1(w)$. If we write the condition B^p_s as the existence of a constant $C>0$ such that for all $0<h<1$,

$$\|w^{1/p}\|_{L^p([h,1),d\nu_{h,s})}\|w^{-1/p}\|_{L^{p'}([h,1),d\nu_{h,s})} \le C \tag{3}$$

with

$$d\nu_{h,s} = \frac{(1-r^2)^s r\,dr}{(1-h)^{s+1}},$$

then the natural substitute of (3) for $p=1$ is true:

PROPOSITION 2.1. *Let $w=w(r)$ and let P_s be bounded on $L^1(w_s)$. Then*

(a) $M_{n+1}(w_s)\sup_{0<r<1} r^n w(r)^{-1} \le C/(n+1)^{s+1}$.

(b) $\|w\|_{L^1([h,1),d\nu_{h,s})}\|w^{-1}\|_{L^\infty([h,1),d\nu_{h,s})} \le C$.

Proof. Let $f_n(re^{i\theta}) = \phi(r)e^{in\theta}$ with $\phi \ge 0$. Then

$$P_s f_n(z) = 2\,\frac{\Gamma(n+s+2)}{\Gamma(s+2)n!}\, M_{n+1}(\phi_s) z^n,$$

and

$$\|P_s f_n\|_{L^1(w_s)} = 2\,\frac{\Gamma(n+s+2)}{\Gamma(s+2)n!}\, M_{n+1}(w_s)\Big(\int_0^1 \phi(r)(1-r^2)^s r^{n+1}\,dr\Big).$$

Therefore, using the boundedness of P_s, one gets

$$2\,\frac{\Gamma(n+s+2)}{\Gamma(s+2)n!}\, M_{n+1}(w_s)\Big(\int_0^1 \phi(r)w(r)w(r)^{-1}(1-r^2)^s r^{n+1}\,dr\Big) \le C\int_0^1 \phi(r)w_s(r)r\,dr.$$

This, by duality, implies that for all $n \ge 0$,

$$\sup_{0<r<1} r^n w(r)^{-1} \le \frac{C_s n!}{\Gamma(n+s+2)M_n(w_s)} \le \frac{C_s}{(n+1)^{s+1}M_n(w_s)},$$

since by the Stirling formula we have

$$\Gamma(n+s+2)/n! \sim (n+1)^{s+1}.$$

Notice in particular that w_s is integrable on $\mathbb{D}$ and w^{-1} is bounded.

To see (b) observe that for each $0<h<1$ we can take $n\in\mathbb{N}$ such that $1-1/(n+1) < h \le 1-1/n$ and that for $r > 1-1/n$ we have

$r^n \ge (1-1/n)^n \ge C$, provided $n \ge 2$. Hence

$$\begin{aligned}
&\|w\|_{L^1([h,1),d\nu_{h,s})}\|w^{-1}\|_{L^\infty([h,1),d\nu_{h,s})} \\
&\quad = \left(\frac{1}{(1-h)^{s+1}}\int_h^1 w(r)(1-r^2)^s r\,dr\right)\sup_{h<r<1} w(r)^{-1} \\
&\quad \le C\left((n+1)^{s+1}\int_{1-1/n}^1 w(r)(1-r^2)^s r\,dr\right)\sup_{1-1/(n+1)<r<1} w(r)^{-1}r^n \\
&\quad \le C(n+1)^{s+1}M_n(w)\sup_{0<r<1} w(r)^{-1}r^n \le C. \ \blacksquare
\end{aligned}$$

Remark 2.2. If P_s is bounded on $L^1(w_s)$ then P_s is also bounded on $L^p(w_s)$ for all $1<p<\infty$. Indeed, part (b) in Proposition 2.1 implies Bekollé's condition as in (3).

Let us now get a necessary condition for the boundedness of P_s on $L^1(w)$ for a general weight w.

Theorem 2.3. *Let w be a radial weight. If P_s is bounded on $L^1(w)$ then there exists a constant $C>0$ so that*

$$\int_0^1 \frac{w(r)}{(1-rt)^{s+1}}\,r\,dr \le C\frac{w(t)}{(1-t)^s}\log\left(\frac{1}{1-t}\right),$$

and there exist $C_\alpha>0$ for all $\alpha>0$ such that

$$\int_0^1 \frac{w(r)}{(1-rt)^{s+\alpha+1}}\,dr \le C_\alpha\frac{w(t)}{(1-t)^{s+\alpha}}.$$

Proof. Take $f=\phi(r)\psi(\theta)$ where $\psi\in H^1(\mathbb{T})=\{\psi\in L^1([0,2\pi):\widehat{\psi}(n)=0$ for $n<0\}$. Recall that the Hardy inequality (see [7]) gives that, for all $0<r<1$,

$$\sum_{n=0}^\infty \frac{|\widehat{\psi}(n)|r^n}{n+1} \le CM_1(\psi,r).$$

Then

$$\begin{aligned}
\|P_s f\|_{L^1(w)} &= \int_0^1 w(r)M_1(P_s f,r)\,r\,dr \\
&\ge C_s\int_0^1 w(r)\left(\sum_{n=0}^\infty \frac{\Gamma(n+s+2)}{(n+1)!}\,M_{n+1}(\phi_s)|\widehat{\psi}(n)|r^n\right)r\,dr \\
&= C_s\int_0^1 G(t)(1-t^2)^s\phi(t)t\,dt,
\end{aligned}$$

where

$$G(t)=\int_0^1\Big(\sum_{n=0}^{\infty}\frac{\Gamma(n+s+2)}{(n+1)!}\,t^n r^n|\widehat{\psi}(n)|\Big)w(r)r\,dr.$$

Using the continuity of P_s we deduce by duality that

$$\sup_{0<t<1}(1-t^2)^s w(t)^{-1}G(t)\le C\|\psi\|_1.$$

If for each $\alpha\ge 0$ and $0<t<1$ we let $\psi(z)=1/(1-tz)^{\alpha+1}$, we have

$$\|\psi\|_1\sim\begin{cases}\dfrac{1}{(1-t)^\alpha}, & \alpha>0,\\ \log\Big(\dfrac{1}{1-t}\Big), & \alpha=0.\end{cases}$$

For this ψ we obtain

$$G(t)=\int_0^1\bigg(\sum_{n=0}^{\infty}\frac{\Gamma(n+s+2)}{(n+1)!}\cdot\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!}\,t^{2n}r^n\bigg)w(r)r\,dr.$$

Then from $\Gamma(n+\lambda)/n!\sim n^{\lambda-1}$ and the expansion

$$\frac{1}{(1-t)^\lambda}=\sum_{n=0}^{\infty}\frac{\Gamma(n+\lambda)}{\Gamma(\lambda)n!}\,t^n,$$

it follows that

$$G(t)\sim\int_0^1\frac{w(r)}{(1-rt^2)^{s+\alpha+1}}\,r\,dr\sim\int_0^1\frac{w(r)}{(1-rt)^{s+\alpha+1}}\,r\,dr,$$

and the proof is complete. ∎

We finish this section by showing that

$$\int_0^1\frac{w(r)}{(1-rt)^{s+1}}\,r\,dr\le C\frac{w(t)}{(1-t)^s}$$

implies the continuity of P_s on $L^1(w)$. Actually this will be equivalent to the boundedness of P_s^*.

LEMMA 2.4. *Let w be weight. Then P_s^* is bounded on $L^1(w)$ if and only if*

$$\int_{\mathbb{D}}\frac{w(z)}{|1-\overline{y}z|^{2+s}}\,dm(z)\le C\frac{w(y)}{(1-|y|^2)^s}\qquad a.e.$$

Proof. For any positive function f one has

$$\int_{\mathbb{D}}P_s^*f(z)w(z)\,dm(z)=\int_{\mathbb{D}}f(y)(1-|y|^2)^s\bigg(\int_{\mathbb{D}}\frac{w(z)}{|1-\overline{y}z|^{2+s}}\,dm(z)\bigg)dm(w).$$

Then the lemma follows by duality. ∎

PROPOSITION 2.5. *Let w be a radial weight. The following are equivalent*:

(a) P_s^* *is bounded on* $L^1(w)$.

(b) *There exists a constant* $C>0$ *so that*

$$\int_0^1 \frac{w(r)}{(1-rt)^{s+1}}\, r\,dr \le C\frac{w(t)}{(1-t)^s} \qquad a.e.$$

(c) $\displaystyle\int_0^t \frac{w(r)}{(1-r)^{s+1}}\, r\,dr \le C\frac{w(t)}{(1-t)^s} \quad a.e., \quad \frac{1}{1-t}\int_t^1 w(r)r\,dr \le Cw(t) \quad a.e.$

Proof. (a) is equivalent to (b) according to the previous lemma since

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|1-zre^{-i\theta}|^{2+s}} \sim \frac{C}{(1-|z|^2r^2)^{1+s}}.$$

To see that (b) is equivalent to (c) observe that

$$\begin{aligned}\int_0^1 \frac{w(r)}{(1-rt)^{s+1}}\, r\,dr &= \int_0^t \frac{w(r)}{(1-rt)^{s+1}}\, r\,dr + \int_t^1 \frac{w(r)}{(1-rt)^{s+1}}\, r\,dr \\ &\sim \int_0^t \frac{w(r)}{(1-r)^{s+1}}\, dr + \frac{1}{(1-t)^{s+1}}\int_t^1 w(r)r\,dr. \ \blacksquare\end{aligned}$$

Let us recall that a weight w is called *normal* (see [8] or [18]) if there exist a and b, $0<a<b$, such that

(i) $w(r)/(1-r)^a$ is nonincreasing with $\lim_{r\to 1} w(r)/(1-r)^a = 0$,

(ii) $w(r)/(1-r)^b$ is nondecreasing with $\lim_{r\to 1} w(r)/(1-r)^b = \infty$.

Set $b(w) = \inf\{b : b \text{ satisfies (ii)}\}$.

COROLLARY 2.6. *Let w be a normal weight. If $s>b(w)$ then P_s^* is bounded on $L^1(w)$.*

Proof. Let us check that (c) of Proposition 2.5 is satisfied. Set $b=b(w)$. Then

$$\int_0^t \frac{w(r)}{(1-r)^{s+1}}\, r\,dr = \int_0^t \frac{w(r)}{(1-r)^b}\cdot\frac{(1-r)^b}{(1-r)^{s+1}}\, r\,dr \le C\frac{w(t)}{(1-t)^s}$$

and

$$\frac{1}{1-t}\int_t^1 w(r)r\,dr = \frac{1}{1-t}\int_t^1 \frac{w(r)}{(1-r)^a}(1-r)^a r\,dr \le Cw(t). \ \blacksquare$$

3. Necessary conditions for the boundedness on Herz spaces

PROPOSITION 3.1. *Let $1 \le p, q < \infty$, and assume the constant functions belong to $K_q^p(w)$, that is, $\sum_{n=1}^{\infty} w(A_n)^{q/p} < \infty$. If P_s is bounded on $K_q^p(w)$ then the sequence $(2^{-n(s+1)} w(A_n)^{-1/p}) \in \ell_{q'}$.*

Proof. Fix N and take

$$f = \sum_{n=1}^{N} \frac{a_n}{w(A_n)^{1/p}} \, \chi_{A_n}.$$

From Lemma 1.2,

$$P_s f = \sum_{n=1}^{N} \frac{a_n}{w(A_n)^{1/p}} \, c_{n,s}.$$

Hence

$$\|P_s f(z)\|_{K_q^p(w)} = \Big| \sum_{n=1}^{N} \frac{a_n}{w(A_n)^{1/p}} \, c_{n,s} \Big| \Big(\sum_{n=1}^{\infty} w(A_n)^{q/p} \Big)^{1/q}$$

and

$$\|f\|_{K_q^p(w)} = \Big(\sum_{m=1}^{\infty} |a_n|^q \Big)^{1/q}.$$

Now the result follows by duality. ∎

COROLLARY 3.2. *Let $\alpha > -1$. If P_s is bounded on $K_q^p((1-r^2)^\alpha)$ then $\alpha + 1 < (s+1)p$.*

Proof. This follows from Proposition 3.1 and the fact that $w(A_n) \sim 2^{-n(\alpha+1)}$ in this case. ∎

Let us now give some more accurate necessary conditions for the boundedness of P_s on $K_q^p(w_s)$.

PROPOSITION 3.3. *Let w be a radial weight. If $1 < p, q < \infty$ and P_s is bounded on $K_q^p(w_s)$, then there exists a constant C such that for all $n \in \mathbb{N}$,*

$$\|r^n\|_{K_q^p(w_s)} \|r^n\|_{K_{q'}^{p'}((w^{-p'/p})_s)} \le \frac{C}{(n+1)^{s+1}}. \tag{4}$$

Proof. Applying the boundedness to the functions $f_n(z) = \phi(r) e^{in\theta}$ with $\phi \ge 0$ and $n \in \mathbb{Z}$ we have

$$P_s f_n(z) = 2 \, \frac{\Gamma(n+s+2)}{\Gamma(s+2) n!} \, M_{n+1}(\phi_s) z^n,$$

hence

$$\|P_s f_n\|_{K_q^p(w_s)} = 2 M_{n+1}(\phi_s) \, \frac{\Gamma(n+s+2)}{\Gamma(s+2) n!} \, \|r^n\|_{K_q^p(w_s)} \le C \|\phi\|_{K_q^p(w_s)},$$

which implies that for all $n \geq 0$,

$$\int_0^1 \phi(r) r^{n+1}(1-r^2)^s\, dr \leq \frac{C\Gamma(s+2)n!}{\Gamma(n+s+2)\|r^n\|_{K_q^p(w_s)}} \|\phi\|_{K_q^p(w_s)}. \tag{5}$$

Writing

$$\int_0^1 \phi(r) r^{n+1}(1-r^2)^s\, dr = \sum_{k=1}^{\infty} \int_{I_k} \phi(r) w(r)^{1/p} w(r)^{-1/p} r^n (1-r^2)^s r\, dr$$

and taking the supremum over all $\|\phi\|_{K_q^p(w_s)} \leq 1$ one deduces from the duality in Herz spaces (see [9, Th. 2.1]) that

$$\begin{aligned}\Big(\sum_{k=1}^{\infty}\Big(\int_{I_k} w(r)^{-p'/p} r^{np'}(1-r^2)^s r\, dr\Big)^{q'/p'}\Big)^{1/q'} &\leq \frac{C\Gamma(s+2)n!}{\Gamma(n+s+2)\|r^n\|_{K_q^p(w)}} \\ &\leq \frac{C_s}{(n+1)^{s+1}\|r^n\|_{K_q^p(w)}}. \ \blacksquare\end{aligned}$$

Corollary 3.4. *Let w be a radial weight. If $1 < p, q < \infty$ and P_s is bounded on $K_q^p(w_s)$ then there exists a constant C such that for all $n \in \mathbb{N}$,*

$$\|\chi_{[h,1)}\|_{K_q^p(w_s)} \|\chi_{[h,1)}\|_{K_{q'}^{p'}((w^{-p'/p})_s)} \leq C(1-h)^{s+1}. \tag{6}$$

Proof. Notice that there exists $C > 0$ such that $\chi_{[1-1/n,1)} \leq Cr^n$ for all $n \in \mathbb{N}$. Given $0 < h < 1$ take n such that $1 - 1/n < h \leq 1 - 1/(n+1)$. Then the lattice structure of these spaces gives

$$\|\chi_{[h,1)}\|_{K_q^p(w_s)} \|\chi_{[h,1)}\|_{K_{q'}^{p'}((w^{-p'/p})_s)} \leq C(1-h)^{s+1}. \ \blacksquare$$

Remark 3.5. (a) If $w(r) = v(r^2)$, then for $p = q = 2$, the condition (4) can be written as

$$M_n((1-r^2)^s v)^{1/2} M_n((1-r^2)^s v^{-1})^{1/2} \leq \frac{C}{(n+1)^{s+1}}.$$

(b) If $p = q$, then inequality (6) is precisely Bekollé's condition (1).

4. Sufficient conditions for the boundedness on Herz spaces. Let us start with some conditions on the weight to have $w \in B_t^p$ for $s - \varepsilon < t < s + \varepsilon$.

Lemma 4.1. *If there exists $\gamma > 1$ such that*

$$\int_0^1 \frac{w(r)^\gamma (1-r^2)^s}{(1-rt)^{s+1}} r\, dr \leq Cw(t)^\gamma$$

then $(1-r^2)^{\pm\varepsilon} w \in B_s^p$ for all $0 < \varepsilon < \min\{(s+1)/\gamma', (p/p')(s+1)\}$.

Proof. By Proposition 2.5, P_s^* is continuous in $L^1((w^\gamma)_s)$. Then Proposition 2.1 implies that

$$\Big(\int_{1-h}^{1} w(r)^\gamma(1-r^2)^s\,r\,dr\Big)\sup_{1-h<r<1} w(r)^{-\gamma}\le Ch^{s+1}.$$

Let $-(s+1)/\gamma'<\varepsilon<(p/p')(s+1)$. Then

$$\begin{aligned}
&\Big(\int_{1-h}^{1} w(r)(1-r^2)^{\varepsilon+s}r\,dr\Big)\Big(\int_{1-h}^{1} w(r)^{-p'/p}(1-r^2)^{-\varepsilon p'/p+s}r\,dr\Big)^{p/p'}\\
&\quad\le\Big(\int_{1-h}^{1} w(r)^\gamma(1-r^2)^{s}r\,dr\Big)^{1/\gamma}\Big(\int_{1-h}^{1}(1-r^2)^{\gamma'\varepsilon+s}r\,dr\Big)^{1/\gamma'}\\
&\quad\quad\times\sup_{1-h<r<1} w(r)^{-1}\Big(\int_{1-h}^{1}(1-r^2)^{-\varepsilon p'/p+s}r\,dr\Big)^{p/p'}\\
&\quad\le C\Big(\sup_{1-h<r<1} w(r)^{-\gamma}\int_{1-h}^{1} w(r)^\gamma(1-r^2)^{s}r\,dr\Big)^{1/\gamma}h^{\varepsilon+(s+1)/\gamma'}h^{-\varepsilon+(s+1)p/p'}\\
&\quad\le Ch^{(s+1)p}. \blacksquare
\end{aligned}$$

THEOREM 4.2. *If there exists $\gamma>1$ such that*

$$\int_0^1\frac{w(r)^\gamma(1-r^2)^s}{(1-rt)^{s+1}}\,r\,dr\le Cw(t)^\gamma,$$

then P_s is bounded on $K_q^p(w_s)$ for every $1<p<\infty$ and $1\le q<\infty$.

Proof. By Lemma 4.1, there exists $\varepsilon>0$ such that $(1-r^2)^{\pm\varepsilon}w\in B_s^p$, hence P_s is continuous in $L^p((1-r^2)^{\pm\varepsilon}w_s)$, that is, there exists $C>0$ such that

$$\int_D |P_sf(z)|^p(1-r^2)^{\pm\varepsilon}w_s(z)\,dm(z)\le C\int_D|f(z)|^p(1-r^2)^{\pm\varepsilon}w_s(z)\,dm(z).$$

In particular, given $n,m\in\mathbb{N}$, if $\operatorname{supp}(f)\subset A_n$ then

$$\int_{A_m}|P_sf(z)|^pw_s(z)\,dm(z)\le C2^{\pm\varepsilon(m-n)}\int_D|f(z)|^pw_s(z)\,dm(z).$$

Let f be any function in $K_q^p(w)$. Write $f=\sum f_n$ with $f_n=f\chi_{A_n}$. Assume that the series is a sum with only a finite number of terms. Then

$$\|P_sf\|_{L^p_{w_s}(A_m)}\le C\sum_n\|P_sf_n\|_{L^p_{w_s}(A_m)}=C\sum_n 2^{\pm\varepsilon(m-n)/p}\|f_n\|_{L^p_{w_s}(A_n)}$$

$$= C \sum_{n<m} 2^{\pm\varepsilon(m-n)/p} \|f_n\|_{L^p_{w_s}(A_n)} + C \sum_{n\ge m} 2^{\pm\varepsilon(m-n)/p} \|f_n\|_{L^p_{w_s}(A_n)}$$
$$= I_1 + I_2.$$

Consider the sequences $X = (x_n)$ and $Y = (y_n)$ where $x_n = 2^{-\varepsilon|n|/p}$ and

$$y_n = \begin{cases} \|f_n\|_{L^p_{w_s}(A_n)}, & n \ge 0, \\ 0, & n < 0. \end{cases}$$

Then

$$\|P_s f\|_{L^p_{w_s}(A_m)} \le C X * Y(m), \qquad m \in \mathbb{N}.$$

Finally from Young's inequality it follows that

$$\|P_s f\|_{K_{pq}(w_s)} \le C\|X\|_{\ell^1} \|f\|_{K^p_q(w_s)}. \ \blacksquare$$

We notice that the proof of Theorem 4.2 was based on the existence of a positive number ε such that $(1-r^2)^{\pm\varepsilon} w \in B^p_s$. With the same idea we have the following

THEOREM 4.3. *If $w \in B^p_t$ then P_s is bounded on $K^p_q(w_s)$ for every $s > t$.*

Proof. An easy calculation shows that for $\varepsilon > 0$ we have

$$(1-r^2)^{-\varepsilon} w \in B^p_{t+\varepsilon}, \qquad (1-r^2)^{\varepsilon} w \in B^p_{t+\varepsilon p'/p}.$$

If we take $\varepsilon > 0$ small enough so that $\max(t + \varepsilon p'/p, t + \varepsilon) < s$, we obtain $(1-r^2)^{\pm\varepsilon} w \in B^p_s$ since the class B^p_t increases in t. $\blacksquare$

COROLLARY 4.4. *Let $\alpha > -1$ and $w(r) = (1-r^2)^\alpha$. Then P_s is continuous in $K^p_q(w)$ if and only if $\alpha + 1 < p(s+1)$. In this case P_s maps $K^p_q(w)$ onto $H_{pq}(w^{q/p})$.*

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