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Uniqueness of the topology on $L^1(G)$

by

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Abstract. Let G be a locally compact abelian group and let X be a translation invariant linear subspace of $L^1(G)$. If G is noncompact, then there is at most one Banach space topology on X that makes translations on X continuous. In fact, the Banach space topology on X is determined just by a single nontrivial translation in the case where the dual group $\widehat{G}$ is connected. For G compact we show that the problem of determining a Banach space topology on X by considering translation operators on X is closely related to the classical problem of determining whether or not there is a discontinuous translation invariant linear functional on X. As a matter of fact $L^1(G)$ does not carry a unique Banach space topology that makes translations continuous, but translations almost determine the Banach space topology on X. Moreover, if G is connected and compact and $1 < p < \infty$, then $L^p(G)$ carries a unique Banach space topology that makes translations continuous.

1. Introduction. It is easily seen that no infinite-dimensional Banach space carries a unique Banach space topology. However, if we restrict attention to those Banach space topologies that make some meaningful operators continuous, this situation becomes different. As a matter of fact if G is a locally compact group then it is well known that $L^1(G)$ is a semisimple Banach algebra. There is of course Johnson's famous theorem [4] that a semisimple Banach algebra carries a unique Banach algebra topology. Accordingly, $L^1(G)$ carries a unique Banach space topology that makes multiplication operators on $L^1(G)$ continuous. Recently K. Jarosz [3] has considered the question whether $L^1(G)$ carries a unique Banach space topology that makes translation operators on $L^1(G)$ continuous for the groups $\mathbb{R}$ and $\mathbb{T}$. In this paper we extend these results to function spaces on arbitrary locally compact abelian groups by proving the results given in the abstract.

It is worth pointing out that our arguments also apply to the study of the automatic continuity of translation invariant linear operators.

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2. Preliminaries. From now on, G denotes a locally compact abelian group, $\widehat{G}$ denotes its dual group, and $\widehat{f}$ the Fourier transform of f for each $f \in L^1(G)$. The Fourier transform on $L^1(G)$ is an injective continuous linear map from $L^1(G)$ into $C_0(\widehat{G})$.

For every $t \in G$ let T_t denote the operator of translation by t which is the map from $L^1(G)$ onto itself given by

$$(T_t f)(s) = f(s+t)$$

for all $f \in L^1(G)$ and $s \in G$. It is easy to check that $\widehat{T_t(f)}(\gamma) = \gamma(t)\widehat{f}(\gamma)$ for all $t \in G$, $f \in L^1(G)$ and $\gamma \in \widehat{G}$.

Suppose that X is a linear subspace of $L^1(G)$. We shall denote by I_X the identity map from X onto itself, and by i_X the inclusion map from X into $L^1(G)$. To shorten notation, if $t \in G$ is such that $T_t(X) \subset X$, we continue to write T_t for the operator of translation by t from X into itself. We say that a Banach space topology $\mathcal{T}$ on X *makes* T_t *continuous* if the map $T_t : (X, \mathcal{T}) \to (X, \mathcal{T})$ is continuous. X is said to be *translation invariant* when $T_t(X) \subset X$ for each $t \in G$. In such a case we say that $\mathcal{T}$ *makes translations continuous* if it makes all the translation operators on X continuous.

A key notion to study the continuity of a linear map Φ from a Banach space X into a Banach space Y is that of the *separating space* $\mathfrak{S}(\Phi)$ of Φ, which is defined as follows:

$$\mathfrak{S}(\Phi) = \{y \in Y : \text{there exists } (x_n) \to 0 \text{ in } X \text{ with } (\Phi(x_n)) \to y\}.$$

The separating space measures the closability of Φ and the closed graph theorem shows that Φ is continuous if and only if $\mathfrak{S}(\Phi) = \{0\}$. For a thorough discussion of the separating space we refer the reader to [8].

It should be noted that a linear subspace X of $L^1(G)$ has at most one Banach space topology not weaker than the topology of convergence in mean. Indeed, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are Banach space topologies on X which are not weaker than the topology of convergence in mean then the identity map from $(X, \mathcal{T}_1)$ onto $(X, \mathcal{T}_2)$ is easily seen to have a closed graph, which shows that it is continuous and therefore that $\mathcal{T}_2 \subset \mathcal{T}_1$.

In the case where X is a linear subspace of $L^1(G)$ which is endowed with a Banach space topology, for abbreviation, we write $\mathfrak{S}_X$ instead of $\mathfrak{S}(i_X)$. Furthermore we set

$$\Delta_X = \{\gamma \in \widehat{G} : \widehat{f}(\gamma) \neq 0 \text{ for some } f \in \mathfrak{S}_X\}.$$

In the following result we state some important properties of Δ_X. We omit the proofs because they are straightforward.

LEMMA 1. *Let X be a linear subspace of $L^1(G)$ which is endowed with a Banach space topology. Then the following assertions hold.*

(i) *Δ_X is an open subset of $\widehat{G}$.*

(ii) *The topology of X is not weaker than the topology of convergence in mean if and only if $\Delta_X = \emptyset$.*

(iii) $\dim(\mathfrak{S}_X) < \infty$ *if and only if Δ_X is finite.*

When considering a Banach space topology on a linear subspace X of $L^1(G)$ making a number of translations on X continuous we become involved with the classical stability lemma [8, Lemma 1.6]. The following result illustrates this technique and it yields information about such a topology.

LEMMA 2. *Let X be a linear subspace of $L^1(G)$, let (t_n) be a sequence in G such that $T_{t_n}(X) \subset X$ for each $n \in \mathbb{N}$, and let (γ_n) be a sequence in $\widehat{G}$. Suppose that X is endowed with a Banach space topology making the translation T_{t_n} on X continuous for each $n \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ with the property that*

$$(\gamma_1(t_1) - \gamma_{n+1}(t_1)) \ldots (\gamma_n(t_n) - \gamma_{n+1}(t_n))\widehat{f}(\gamma_{n+1}) = 0$$

for each $f \in \mathfrak{S}_X$.

Proof. For every $n \in \mathbb{N}$, let R_n and S_n be the continuous linear operators on X and $L^1(G)$ respectively, given by $R_n = \gamma_n(t_n)I_X - T_{t_n}$ and $S_n = \gamma_n(t_n)I_{L^1(G)} - T_{t_n}$. Since $i_X R_n = S_n i_X$ for each $n \in \mathbb{N}$, the classical stability lemma shows that there is $n \in \mathbb{N}$ such that

$$\overline{(S_1 \ldots S_n)(\mathfrak{S}_X)} = \overline{(S_1 \ldots S_{n+1})(\mathfrak{S}_X)}.$$

We thus get

$$(S_1 \ldots S_n)(\mathfrak{S}_X) \subset \overline{(S_1 \ldots S_{n+1})(\mathfrak{S}_X)}.$$

We now observe that the Fourier transform of every function of the latter set vanishes at γ_{n+1}. Indeed, we have

$$\begin{aligned}&[(S_1 \ldots S_{n+1})(f)]^\wedge(\gamma_{n+1}) \\ &\qquad = (\gamma_1(t_1) - \gamma_{n+1}(t_1)) \ldots (\gamma_{n+1}(t_{n+1}) - \gamma_{n+1}(t_{n+1}))\widehat{f}(\gamma_{n+1}) = 0\end{aligned}$$

for each $f \in \mathfrak{S}_X$. Consequently, for every $f \in \mathfrak{S}_X$,

$$\begin{aligned}&(\gamma_1(t_1) - \gamma_{n+1}(t_1)) \ldots (\gamma_n(t_n) - \gamma_{n+1}(t_n))\widehat{f}(\gamma_{n+1}) \\ &\qquad = [(S_1 \ldots S_n)(f)]^\wedge(\gamma_{n+1}) = 0. \ \blacksquare\end{aligned}$$

3. Noncompact groups. In [3] it was shown that a single nontrivial translation determines the Banach space topology on $L^1(\mathbb{R})$. We have found out that this property is just a consequence of the connectedness of $\widehat{\mathbb{R}}$.

THEOREM 1. *Let G be a locally compact abelian group such that $\widehat{G}$ is connected, let $t \in G \setminus \{0\}$, and let X be a linear subspace of $L^1(G)$ such that $T_t(X) \subset X$. Then every Banach space topology on X making the translation operator T_t on X continuous is not weaker than the topology of convergence* *in mean. Accordingly, there is at most one Banach space topology on X making the translation operator T_t on X continuous.*

Proof. We begin by proving that the set $\{\gamma(t) : \gamma \in \Delta_X\}$ is finite. On the contrary, suppose that it is infinite. Let (γ_n) be a sequence in Δ_X such that $\gamma_m(t) \neq \gamma_n(t)$ for $n \neq m$. For every $n \in \mathbb{N}$ there exists $f_n \in \mathfrak{S}_X$ such that $\widehat{f}_n(\gamma_{n+1}) \neq 0$ and so

$$(\gamma_1(t) - \gamma_{n+1}(t)) \ldots (\gamma_n(t) - \gamma_{n+1}(t)) \widehat{f}_n(\gamma_{n+1}) \neq 0,$$

which contradicts Lemma 2.

We can now proceed to show that Δ_X is empty. To obtain a contradiction suppose that Δ_X is nonempty. As $\{\gamma(t) : \gamma \in \Delta_X\}$ is finite, there exist $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$ such that

$$\Delta_X = \bigcup_{k=1}^{N} \{\gamma \in \Delta_X : \gamma(t) = \lambda_k\}.$$

The sets $\{\gamma \in \Delta_X : \gamma(t) = \lambda_k\}$ $(k = 1, \ldots, N)$ are pairwise disjoint closed subsets in Δ_X. Hence each of them is open in Δ_X and so it is open in $\widehat{G}$. Thus there exists $k \in \{1, \ldots, N\}$ such that $\{\gamma \in \Delta_X : \gamma(t) = \lambda_k\}$ is a nonempty open subset in $\widehat{G}$. Let $\gamma_0 \in \{\gamma \in \Delta_X : \gamma(t) = \lambda_k\}$ and let V be a symmetric open neighborhood of the identity e in $\widehat{G}$ such that

$$\gamma_0 V \subset \{\gamma \in \Delta_X : \gamma(t) = \lambda_k\}.$$

For every $\gamma \in V$ we have $\gamma_0(t)\gamma(t) = \lambda_k$ and therefore $\gamma(t) = 1$. Clearly the set $\bigcup_{n=1}^{\infty} V^n$ is an open subgroup of $\widehat{G}$ and [2, Proposition 2.1.d] shows that it is closed in $\widehat{G}$. By the connectedness of $\widehat{G}$ we conclude that $\bigcup_{n=1}^{\infty} V^n = \widehat{G}$. Therefore $\gamma(t) = 1$ for each $\gamma \in \widehat{G}$ and this leads to $t = 0$, a contradiction. ∎

COROLLARY 2. *Let G be a locally compact abelian group such that $\widehat{G}$ is connected and let $t \in G \setminus \{0\}$. Then $L^1(G)$ carries a unique Banach space topology making the translation operator T_t on $L^1(G)$ continuous.*

Clearly Corollary 2 generalizes [3, Theorem 3.3].

If we remove the connectedness assumption, we are still able to determine the Banach space topology if we consider all the translations on the space.

THEOREM 3. *Let G be a noncompact locally compact abelian group and let X be a translation invariant linear subspace of $L^1(G)$. Then every Banach space topology on X making translations on X continuous is not weaker than the topology of convergence in mean on X. Accordingly, there is at most one Banach space topology on X making translations on X continuous.*

Proof. Suppose $\varDelta_X$ were nonempty. Set $\varGamma_0 = \varDelta_X$ and $\gamma_1 \in \varGamma_0$. Since G is noncompact, $\widehat{G}$ is nondiscrete. Therefore $\varGamma_0$ is infinite and there exists $t_1 \in G$ such that the open set $\varGamma_1 = \{\gamma \in \varGamma_0 : \gamma(t_1) \neq \gamma_1(t_1)\}$ is infinite. We can successively choose $\varGamma_n \subset G$, $\gamma_n \in \widehat{G}$, and $t_n \in G$ such that $\gamma_n \in \varGamma_{n-1}$ and the open set $\varGamma_n = \{\gamma \in \varGamma_{n-1} : \gamma(t_n) \neq \gamma_n(t_n)\}$ is infinite for each $n \in \mathbb{N}$. Consequently, we have $\gamma_n(t_n) \neq \gamma_k(t_n)$ for all $k, n \in \mathbb{N}$ with $k > n$. For every $n \in \mathbb{N}$ let $f_n \in \mathfrak{S}_X$ be such that $\widehat{f_n}(\gamma_{n+1}) \neq 0$. We have

$$(\gamma_1(t_1) - \gamma_{n+1}(t_1)) \ldots (\gamma_n(t_n) - \gamma_{n+1}(t_n))\widehat{f_n}(\gamma_{n+1}) \neq 0$$

for each $n \in \mathbb{N}$, which contradicts Lemma 2. ∎

COROLLARY 4. *Let G be a noncompact locally compact abelian group. Then $L^1(G)$ carries a unique Banach space topology making translations on $L^1(G)$ continuous.*

REMARK 1. It is clear that the same proofs as in all the preceding results still work when we replace the group algebra $L^1(G)$ by the measure algebra $M(G)$. Accordingly, Theorems 1 and 3 and Corollaries 2 and 4 are still true with $L^1(G)$ replaced by $M(G)$.

REMARK 2. It is easy to see that Theorems 1 and 3 and Corollaries 2 and 4 still hold with $L^1(G)$ replaced by $L^1(G, E)$, where E is any Banach space. Similar results (just for $X = L^1(G, E)$) have recently been obtained in [9] by using much more sophisticated arguments.

4. Compact groups. Throughout this section, G is a compact abelian group and we assume that the Haar measure λ on G is normalized so that $\lambda(G) = 1$.

In this context the situation becomes completely different to that in the preceding section. In fact, the uniqueness of the Banach space topology making translations continuous on a translation invariant linear subspace X of $L^1(G)$ is closely related to the classical problem of whether there exists a discontinuous translation invariant linear functional on X. For an excellent survey about the last question we refer the reader to [6] and the references given there.

THEOREM 5. *Let G be a compact abelian group and let X be a translation invariant linear subspace of $L^1(G)$ such that $1 \in X$. Suppose that X is endowed with a Banach space topology making translations on X continuous and that X has a discontinuous translation invariant linear functional. Then X does not carry a unique Banach space topology making translations on X continuous.*

Proof. Let ϕ be a discontinuous invariant linear functional on X. We claim that there is a discontinuous invariant linear functional ψ on X such that $\psi(1) = 1$. Indeed, if $\phi(1) = \alpha \neq 0$ then we take $\psi = \alpha^{-1}\phi$. If $\phi(1) = 0$, then we define $\psi(f) = \phi(f) + \int_G f(t)\,dt$ for each $f \in X$.

Since the map $f \mapsto 2f - \psi(f)1$ is a linear bijection from X onto itself, it may be concluded that $|f| = \|2f - \psi(f)1\|$ is a complete norm on X that is not equivalent to $\|\cdot\|$. Let $t \in G$. For every $f \in X$ we have

$$\begin{aligned}|T_t f| &= \|2T_t(f) - \psi(T_t(f))1\| = \|T_t(2f) - \psi(f)1\| = \|T_t(2f - \psi(f)1)\| \\ &\le \|T_t\| \cdot \|2f - \psi(f)1\| = \|T_t\| \cdot |f|,\end{aligned}$$

which shows that T_t from $(X, |\cdot|)$ into itself is continuous. ∎

By using [7, Theorem 1] together with the preceding result we obtain the following.

Corollary 6. *If G is an infinite compact abelian group then $L^1(G)$ does not carry a unique Banach space topology making translations on $L^1(G)$ continuous.*

Despite the preceding facts we are going to prove that translations almost determine the Banach space topology of the translation invariant linear subspaces of $L^1(G)$.

Lemma 3. *Let K be a subset of G with nonempty interior and let Γ be an infinite subset of $\widehat{G}$. Then the set $\{\gamma_{|K} : \gamma \in \Gamma\}$ is infinite, where $\gamma_{|K}$ stands for the restriction of γ to K.*

Proof. Suppose the lemma were false. Then we could find $\xi_1, \ldots, \xi_N \in \Gamma$ such that $\Gamma = \bigcup_{k=1}^N \{\gamma \in \Gamma : \gamma = \xi_k \text{ on } K\}$. Let $t \in K$ and let U be a symmetric open neighborhood of 0 such that $t + U \subset K$. It is easily checked that the set $H = \bigcup_{n=1}^\infty (U + \overset{n}{\ldots} + U)$ is an open subgroup of G and that $\Gamma \subset \bigcup_{k=1}^N \{\gamma \in \Gamma : \gamma = \xi_k \text{ on } H\}$. [2, Proposition 2.1.d] shows that H is closed and from the compactness of G it follows that G/H is finite. [2, Corollary 4.7 and Theorem 4.39] now shows that $G/H = \widehat{G/H} = H^\perp$, where we write $H^\perp$ for the set $\{\gamma \in \widehat{G} : \gamma(H) = 1\}$. Therefore the set $\xi_k H^\perp$ is finite for each $k = 1, \ldots, N$. Since

$$\Gamma \subset \bigcup_{k=1}^N \{\gamma \in \Gamma : \gamma = \xi_k \text{ on } H\} = \bigcup_{k=1}^N \{\gamma \in \Gamma : \gamma \in \xi_k H^\perp\},$$

it may be concluded that Γ is finite. This contradicts our assumption. ∎

Lemma 4. *Let Γ be an infinite subset of $\widehat{G}$ and let $\gamma_1 \in \Gamma$. Then there are $t_1 \in G$ and $\Gamma_1 \subset \Gamma$ infinite such that $\gamma_1(t_1) \neq \gamma(t_1)$ for each $\gamma \in \Gamma_1$.*

Proof. We claim that there exist a sequence (γ_n) in Γ and a decreasing sequence (K_n) of compact subsets of G with nonempty interior such that $\gamma_1(t) \neq \gamma_n(t)$ for all $t \in K_n$ and $n \in \mathbb{N}$ with $n > 1$. We choose $\gamma_2 \in \Gamma$ with $\gamma_2 \neq \gamma_1$. By the continuity of γ_1 and γ_2 we can choose a compact subset

K_1 of G with nonempty interior such that $\gamma_1(t) \neq \gamma_2(t)$ for all $t \in K_1$. Assume that $\gamma_2, \ldots, \gamma_{n+1}$ and $K_1, \ldots, K_n$ have been chosen satisfying our requirements. According to the preceding lemma, the set $\{\gamma_{|K_n} : \gamma \in \Gamma\}$ is infinite. Hence there exists $\gamma_{n+2} \in \Gamma$ such that $\gamma_{1|K_n} \neq \gamma_{n+2|K_n}$. Let $K_{n+1} \subset G$ be a compact subset of G with nonempty interior such that $K_{n+1} \subset K_n$ and such that $\gamma_1(t) \neq \gamma_{n+2}(t)$ for all $t \in K_{n+1}$.

Since $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$ we can take $t_1 \in \bigcap_{n=1}^{\infty} K_n$. Of course $\gamma_1(t_1) \neq \gamma_n(t_1)$ for each $n > 1$. Therefore the element t_1 and the set $\Gamma_1 = \{\gamma_n : n > 1\}$ satisfy the requirement of the lemma. ∎

THEOREM 7. *Let G be a compact abelian group and let X be a translation invariant linear subspace of $L^1(G)$. Then for every Banach space topology on X making translations on X continuous the inclusion map from X into $L^1(G)$ has a finite-dimensional separating space.*

Proof. It suffices to prove that Δ_X is finite. On account of Lemma 4, if Δ_X were infinite, we could successively choose $\gamma_n \in \Gamma_{n-1}$, $\Gamma_n \subset \Gamma_{n-1} \subset \Delta_X$, and $t_n \in G$ such that $\gamma_n(t_n) \neq \gamma(t_n)$ for all $\gamma \in \Gamma_n$ and $n \in \mathbb{N}$. Here we write $\Gamma_0 = \Delta_X$. Thus $\gamma_n(t_n) \neq \gamma_k(t_n)$ for all $k, n \in \mathbb{N}$ with $k > n$. For every $n \in \mathbb{N}$ let $f_n \in \mathfrak{S}_X$ be such that $\widehat{f_n}(\gamma_{n+1}) \neq 0$. We have

$$(\gamma_1(t_1) - \gamma_{n+1}(t_1)) \ldots (\gamma_n(t_n) - \gamma_{n+1}(t_n))\widehat{f_n}(\gamma_{n+1}) \neq 0,$$

which contradicts Lemma 2. ∎

COROLLARY 8. *Let G be a compact abelian group and let $|\cdot|$ be a complete norm on $L^1(G)$ making translations on $L^1(G)$ continuous. Then there exists a finite-dimensional ideal $\mathfrak{S}$ of $L^1(G)$ such that the quotient norm of $|\cdot|$ and $\|\cdot\|_1$ are equivalent on the quotient linear space $L^1(G)/\mathfrak{S}$.*

Proof. Let $\mathfrak{S}$ be the separating space of the identity map from $(L^1(G), |\cdot|)$ onto $(L^1(G), \|\cdot\|_1)$. By Theorem 7, $\mathfrak{S}$ is a finite-dimensional subspace of $L^1(G)$. On the other hand, it follows immediately that $\mathfrak{S}$ is translation invariant. Therefore $\mathfrak{S}$ is an ideal of $L^1(G)$.

Since the quotient map from $(L^1(G), |\cdot|)$ onto $(L^1(G)/\mathfrak{S}, \|\cdot\|_1)$ is continuous it follows that the identity map from $(L^1(G)/\mathfrak{S}, |\cdot|)$ onto $(L^1(G)/\mathfrak{S}, \|\cdot\|_1)$ is continuous, as required. ∎

REMARK 3. Of course, Theorem 7 and Corollary 8 still hold if we replace $L^1(G)$ by $M(G)$.

THEOREM 9. *Let G be a connected compact abelian group and let $1 < p < \infty$. Then $L^p(G)$ carries a unique Banach space topology making translations on $L^p(G)$ continuous.*

Proof. Suppose that $|\cdot|$ is a complete norm on $L^p(G)$ making translations on $L^p(G)$ continuous. In order to show that $|\cdot|$ and $\|\cdot\|_p$ are equivalent it suffices to show that the identity map ϕ from $(L^p(G), \|\cdot\|_p)$ onto $(L^p(G), |\cdot|)$ is continuous. We claim that $\dim \mathfrak{S}(\phi) < \infty$. Indeed, let $f \in \mathfrak{S}(\phi)$ and let $(f_n) \to 0$ in $(L^p(G), \|\cdot\|_p)$ with $(f_n) \to f$ in $(L^p(G), |\cdot|)$. Then $(f - f_n) \to 0$ in $(L^p(G), |\cdot|)$, $(f - f_n) \to f$ in $(L^p(G), \|\cdot\|_p)$, and so $(f - f_n) \to f$ in $(L^1(G), \|\cdot\|_1)$. This shows that $\mathfrak{S}(\phi)$ is contained in the separating space of the inclusion map from $(L^p(G), |\cdot|)$ into $(L^1(G), \|\cdot\|_1)$, which is finite-dimensional on account of Theorem 7.

To obtain a contradiction, suppose that $\mathfrak{S}(\phi) \neq \{0\}$. Then we can find $\gamma \in \widehat{G}$ and $f_0 \in \mathfrak{S}(\phi)$ such that $\widehat{f_0}(\gamma) \neq 0$. Let $\mathcal{R}$ be the linear subspace of $\mathfrak{S}(\phi)$ given by

$$\mathcal{R} = \{f \in \mathfrak{S}(\phi) : \widehat{f}(\gamma) = 0\}$$

and let π denote the quotient map from $(L^p(G), |\cdot|)$ onto $(L^p(G)/\mathcal{R}, |\cdot|_{\mathcal{R}})$, where $|\cdot|_{\mathcal{R}}$ stands for the quotient norm of $|\cdot|$ on $L^p(G)/\mathcal{R}$. Since $\mathfrak{S}(\phi) \not\subset \mathcal{R}$, it follows that $\pi \circ \phi$ is discontinuous. We claim that for each $t \in G$ the map S_t from $(L^p(G), \|\cdot\|_p)$ into $(L^p(G)/\mathcal{R}, |\cdot|_{\mathcal{R}})$ given by

$$S_t(f) = \gamma(t)\pi(f) - \pi(T_t f)$$

is continuous. We have $S_t = \pi \circ R_t \circ \phi$, where R_t stands for the map from $(L^p(G), |\cdot|)$ into itself given by $R_t = \gamma(t) I_{L^p(G)} - T_t$. On account of [8, Lemma 1.3], we have $\mathfrak{S}(S_t) = \overline{\pi(R_t(\mathfrak{S}(\phi)))}$. [8, Lemma 1.2.iii] shows that $R_t(\mathfrak{S}(\phi)) \subset \mathfrak{S}(\phi)$. On the other hand, $\widehat{R_t(f)}(\gamma) = (\gamma(t) - \gamma(t))\widehat{f}(\gamma) = 0$ for each $f \in \mathfrak{S}(\phi)$, and so $R_t(\mathfrak{S}(\phi)) \subset \mathcal{R}$. Hence $\pi \circ R_t \circ \phi$ is continuous, as claimed.

According to [1, Proposition 3 and Corollary 4] together with the observation following [1, Proposition 3], there exist a constant C and $J \in \mathbb{N}$ with the property that, for every $f \in L^p(G)$ with $\int_G f(t)\,dt = 0$, there is a set $E_f \subset G^J$ with $\lambda^J(E_f) = 1$ (where λ^J stands for the normalized Haar measure on G^J) such that for every $(t_1, \ldots, t_J) \in E_f$ there are $f_1^{(t_1,\ldots,t_J)}, \ldots, f_J^{(t_1,\ldots,t_J)} \in L^p(G)$ such that

$$f = \sum_{j=1}^{J} (f_j^{(t_1,\ldots,t_J)} - T_{t_j}(f_j^{(t_1,\ldots,t_J)}))$$

and

$$\int_{G^J} \sum_{j=1}^{J} \|f_j^{(t_1,\ldots,t_J)}\|_p \, d(t_1, \ldots, t_J) \le C\|f\|_p.$$

For every $n \in \mathbb{N}$ let

$$H_n = \{(t_1, \ldots, t_J) \in G^J : \|S_{t_j}\| \le n,\ j = 1, \ldots, J\}.$$

Clearly $G^J = \bigcup_{n=1}^{\infty} H_n$. We claim that there exist $m, n \in \mathbb{N}$ with the property that every measurable set $E \subset G^J$ with $\lambda^J(E) \geq 1 - 1/m$ has a nonempty intersection with H_n. Suppose, contrary to our claim, that for all $m, n \in \mathbb{N}$ there exists $C_{m,n} \subset G^J$ such that $\lambda^J(C_{m,n}) \geq 1 - 1/m$ and $H_n \subset G^J \setminus C_{m,n}$. Then

$$G^J = \bigcup_{n=1}^{\infty} H_n \subset \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} (G^J \setminus C_{m,n}).$$

Since $\lambda^J(G^J \setminus C_{m,n}) \leq 1/m$ it follows that $\lambda^J(\bigcap_{m=1}^{\infty}(G^J \setminus C_{m,n})) = 0$ and hence that $\lambda^J(G^J) = 0$, a contradiction.

Fix $f \in L^p(G)$ and write $g = \overline{\gamma} f - \widehat{f}(\gamma)$. Then $g \in L^p(G)$ and $\int_G g(t)\,dt = 0$. Let

$$K = \Big\{(t_1, \ldots, t_J) \in E_g : \sum_{j=1}^{J} \|g_j^{(t_1,\ldots,t_J)}\|_p \leq Cm\|g\|_p\Big\}.$$

We have

$$\begin{aligned} Cm\|g\|_p \lambda^J(G^J \setminus K) &\leq \int_{G^J\setminus K} \sum_{j=1}^{J} \|g_j^{(t_1,\ldots,t_J)}\|_p \, d(t_1, \ldots, t_J) \\ &\leq \int_{G^J} \sum_{j=1}^{J} \|g_j^{(t_1,\ldots,t_J)}\|_p \, d(t_1, \ldots, t_J) \leq C\|g\|_p. \end{aligned}$$

Consequently, $\lambda^J(G^J \setminus K) \leq 1/m$ and so $\lambda^J(K) \geq 1 - 1/m$. Hence $H_n \cap K \neq \emptyset$. Choose $(t_1, \ldots, t_J) \in H_n \cap K$ and write $g_j = g_j^{(t_1,\ldots,t_J)}$ for $j = 1, \ldots, J$. Then g can be written in the form

$$g = \sum_{j=1}^{J} (g_j - T_{t_j} g_j),$$

where $g_1, \ldots, g_J \in L^p(G)$ are such that $\|g_j\|_p \leq Cm\|g\|_p \ \ (j = 1, \ldots, J)$. We thus get

$$f = \widehat{f}(\gamma)\gamma + \sum_{j=1}^{J} (\gamma(t_j) f_j - T_{t_j} f_j),$$

where $f_j = \overline{\gamma(t_j)} \gamma g_j$ and so $\|f_j\|_p = \|g_j\|_p$ for $j = 1, \ldots, J$. Hence

$$(\pi \circ \phi)(f) = \widehat{f}(\gamma)\pi(\gamma) + \sum_{j=1}^{J} S_{t_j}(f_j)$$

and so

$$
\begin{aligned}
|(\pi\circ\phi)(f)|_{\mathcal{R}} &\le |\widehat{f}(\gamma)|\cdot|\pi(\gamma)|_{\mathcal{R}}+\sum_{j=1}^{J}|S_{t_j}(f_j)|_{\mathcal{R}}\\
&\le \|f\|_p|\pi(\gamma)|_{\mathcal{R}}+\sum_{j=1}^{J}\|S_{t_j}\|\cdot\|f_j\|_p\\
&\le \|f\|_p|\pi(\gamma)|_{\mathcal{R}}+\sum_{j=1}^{J}2nCm\|f\|_p\\
&= (|\pi(\gamma)|_{\mathcal{R}}+2JnCm)\|f\|_p.
\end{aligned}
$$

Consequently, $\pi\circ\phi$ is continuous, a contradiction. ∎

The preceding generalizes [3, Theorem 3.5].

Remark 4. It should be noted that if the group G has finitely many components, then a similar analysis to that in the proof of the preceding theorem shows that this theorem still holds true in this case. If the group has infinitely many components, then the situation becomes different. In [5] it was shown that there exist discontinuous translation invariant linear functionals on $L^2(G)$ in the case where G is the totally disconnected infinite compact abelian group usually referred to as the Cantor discontinuum (for a more detailed discussion of this topic we refer the reader to [6]). On account of Theorem 5, $L^2(G)$ does not carry a unique Banach space topology making translations continuous.

5. Translation invariant operators. Here by a *group space* on G we mean a translation invariant linear subspace X of $L^1(G)$ which is endowed with a Banach space topology that makes translations on X continuous. The uniqueness of the group space topology on X is closely related to the automatic continuity of translation invariant linear operators from X. In fact, a careful analysis of the proofs given in the preceding sections leads to the following result.

Theorem 10. *Let X and Y be group spaces on a locally compact abelian group G and let Φ be a translation invariant linear operator from X into Y. Then the following assertions hold.*

(i) *If G is noncompact, then Φ is continuous.*

(ii) *If G is compact, then* $\dim\mathfrak{S}(\Phi)<\infty$.

(iii) *If G is connected and compact and $X=L^p(G)$ for some* $1<p<\infty$, *then Φ is continuous.*

Proof. We begin by observing that assertions (i) and (ii) hold in the case where $Y=(L^1(G),\|\cdot\|_1)$. Indeed, we can apply the same arguments as in the preceding sections, with i_X replaced by Φ. The details are left to the reader. For the general case, we first apply what has previously been observed to the operator $i_Y \circ \Phi$. If G is noncompact, then i_Y and $i_Y \circ \Phi$ are continuous and [8, Lemma 1.3] shows that

$$\{0\} = \mathfrak{S}(i_Y \circ \Phi) = \overline{i_Y(\mathfrak{S}(\Phi))},$$

which shows that $\mathfrak{S}(\Phi) = \{0\}$. If G is compact, then we consider the quotient map from $L^1(G)$ onto $L^1(G)/\mathfrak{S}_Y$. Since π and $\pi \circ i_Y$ are continuous, [8, Lemma 1.3] now shows that

$$\overline{\pi(\mathfrak{S}(i_Y \circ \Phi))} = \mathfrak{S}(\pi \circ (i_Y \circ \Phi)) = \mathfrak{S}((\pi \circ i_Y) \circ \Phi) = \overline{\pi(i_Y(\mathfrak{S}(\Phi)))}.$$

Since $\dim \mathfrak{S}_Y < \infty$ and $\dim \mathfrak{S}(i_Y \circ \Phi) < \infty$ it follows that $\dim \mathfrak{S}(\Phi) < \infty$.

Finally the third assertion can be proved by the same method as in the proof of Theorem 9 with ϕ replaced by Φ. ∎

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