Almost periodicity of \(C\)-semigroups, integrated semigroups and \(C\)-cosine functions

by

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Abstract. We investigate the characterization of almost periodic \(C\)-semigroups, via the Hille–Yosida space \(Z_0\), in case of \(R(C)\) being non-dense. Analogous results are obtained for \(C\)-cosine functions. We also discuss the almost periodicity of integrated semigroups.

0. Introduction. Characterizations of almost periodic semigroups and groups of class \(C_0\) were studied by Bart and Goldberg [1] in 1978. Later, Cioranescu [3], Piskarev [14, 15] and others discussed the almost periodicity of strongly continuous cosine functions. Recently, Zheng and Liu [21] studied the almost periodicity of \(C\)-semigroups and \(C\)-cosine functions under the assumption that \(R(C)\) is dense.

In this paper, we investigate the situation where \(R(C)\) is allowed to be non-dense. We characterize the generator of an almost periodic \(C\)-semigroup, \(A\), via the Hille–Yosida space, \(Z_0\), which is a maximal continuously imbedded subspace of \(X\) on which \(A\) generates a strongly continuous semigroup. Kantorovitz [13] first introduced the Hille–Yosida space for a closed operator \(A\) with \((0, \infty) \subset \rho(A)\), on which the restriction of \(A\) generates a semigroup of class \(C_0\). R. deLaubenfels [8] extended it to more general cases that \(A\) has no eigenvalues in \((0, \infty)\), and used it to connect \(C\)-semigroups with semigroups of class \(C_0\). Similarly, Cioranescu [2] constructed the Hille–Yosida space of cosine functions. For the extensive literature on this subject, we refer to [19].

Let \(\mathcal{I} := \text{span}\{x \in D(A) : Ax = irx \text{ for some } r \in \mathbb{R}\}\). We show in Theorem 2.4 that if \(A\) has no eigenvalues in \((0, \infty)\) and \(C^{-1}AC = A\), then \(A\) generates an almost periodic \(C\)-semigroup if and only if the image of \(C\) is contained in \((Z_0)_a\), the closure of \(\mathcal{I}\) in \(Z_0\), the Hille–Yosida space for \(A\); and \((Z_0)_a\) is proved to be a maximal continuously imbedded subspace of \(X\) on

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which \( A \) generates an almost periodic semigroup of class \( C_0 \) of contractions (Theorem 2.6). The key fact here is that a solution of the abstract Cauchy problem is almost periodic in \( Z_0 \) if and only if it is almost periodic in \( X \). The same method applies to the case of asymptotic almost periodicity of \( C \)-semigroups; but this is the subject of another paper ([18]). Theorem 4.2 gives the analogous result for \( C \)-cosine functions. We also consider the periodicity (Theorems 2.8 and 4.3). Our results generalize the corresponding ones in [21].

If \( (\mathcal{A}) \cap i\mathbb{R} \) is at most countable, then a \( C \)-semigroup \( T(t) \) is almost periodic if and only if \( e^{-\lambda t}T(t)x \) has uniformly convergent means for \( \lambda \in \sigma(A) \cap i\mathbb{R}, x \in X \). This is proved in Theorem 2.9.

In Section 3 the almost periodicity of integrated semigroups is discussed. Theorem 3.3 asserts that, if \( A \) generates a bounded \( (rA)^{-1} \)-semigroup \( T(t) \) and a bounded integrated semigroup \( S(t) \), then \( T(t) \) is almost periodic if and only if \( S(t) \) is almost periodic. Theorem 3.3 relates almost periodicity of bounded \( (rA)^{-1} \)-groups and bounded integrated groups to uniformly convergent means.

Throughout this paper, \( X \) will be a Banach space, the dual space will be denoted by \( X^* \). All operators are linear. The space of all bounded linear operators on \( X \) will be denoted by \( B(X) \). \( C \in B(X) \) will be injective. For an operator \( A \), we will write \( D(A) \) for its domain, \( R(A) \) for its range. Finally, \( J = \mathbb{R} \) or \( \mathbb{R}^+ \), where \( \mathbb{R}^+ = [0, \infty) \).

1. Preliminaries. First, we recall the definition and basic properties of \( C \)-semigroups or groups.

**Definition 1.1.** A strongly continuous family \( T(t) (t \in J) \subset B(X) \) is called a \( C \)-semigroup \((J = \mathbb{R}^+)\) or a \( C \)-group \((J = \mathbb{R})\) if \( T(t+s)C = T(t)T(s) \) for \( t, s \in J \) and \( T(0) = C \). The generator \( A \) is defined by

\[
D(A) = \{ x \in X : \lim_{J \ni t \to 0} t^{-1}(T(t)x - Cx) \text{ exists and belongs to } R(C) \}
\]

with

\[
Ax = C^{-1} \left( \lim_{J \ni t \to 0} t^{-1}(T(t)x - Cx) \right) \quad \text{for } x \in D(A).
\]

The complex number \( \lambda \) is in \( \varrho_C(A) \), the \( C \)-resolvent set of \( A \), if \( \lambda - A \) is injective and \( R(C) \subset R(\lambda - A) \); we set \( \sigma_C(A) := C \setminus \varrho_C(A) \).

**Lemma 1.2 ([8]).** Let \( T(t) (t \in J) \) be a \( C \)-semigroup or \( C \)-group with generator \( A \). Then

(a) \( A \) is closed and \( R(C) \subset \overline{D(A)} \);

(b) \( \int_0^t T(s)x \, ds \in D(A) \) with \( AT_0^t T(s)x \, ds = T(t)x - Cx \) for all \( x \in X \) and \( t \in J \);
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(c) \(T(t)x \in D(A)\) with \(AT(t)x = T(t)Ax\), and \(\int_0^t T(s)Ax \, ds = T(t)x - Cx\) for all \(x \in D(A)\) and \(t \in J\);

(d) if \(T(t)\) is uniformly bounded, then \(\{\lambda \in \mathbb{C} : \text{Re}\, \lambda \in J \setminus \{0\}\} \subset \rho(C(A))\) and \((\lambda - A)^{-1}Cx = \int_0^\infty e^{-\lambda s}T(t)x \, dt\) for all \(x \in X\) and \(\text{Re}\, \lambda > 0\).

Next, we need to introduce the Hille–Yosida space for an operator; for the details we refer to [8].

**Definition 1.3.** Suppose \(A\) has no eigenvalues in \((0, \infty)\) and is a closed linear operator. The Hille–Yosida space for \(A\), \(Z_0\), is defined by

\[Z_0 = \{x \in X : \text{the Cauchy problem } u'(t) = Au(t), u(0) = x \text{ has a bounded uniformly continuous mild solution } u(\cdot, x)\}\]

with

\[\|x\|_{Z_0} = \sup \{\|u(t, x)\| : t \geq 0\} \quad \text{for } x \in Z_0.\]

**Lemma 1.4 ([8]).** Let \(A\) generate a bounded strongly uniformly continuous C-semigroup \(T(t)\). Then \(R(C) \subset Z_0\) and \(A|_{Z_0}\) generates a contraction semigroup of class \(C_0\) given by \(S(t) = C^{-1}T(t)\) and

\[Z_0 = \{x : t \rightarrow C^{-1}T(t)x \text{ is bounded and uniformly continuous}\}\]

with

\[\|x\|_{Z_0} = \sup_{t \geq 0} \|C^{-1}T(t)x\|.\]

Now we introduce the notion of a mild C-existence family, which is more general than C-semigroup.

**Definition 1.5.** The family of operators \(\{T(t)\}_{t \geq 0} \subseteq B(X)\) is a mild C-existence family for \(A\) if

(a) the map \(t \mapsto T(t)x\), from \([0, \infty)\) into \(X\), is continuous, for all \(x \in X\);

(b) for all \(x \in X\) and \(t > 0\), \(\int_0^t T(s)x \, ds \in D(A)\) with \(A(\int_0^t T(s)x \, ds) = T(t)x - Cx\).

**Definition 1.6.** (a) A function \(f \in C(J, X)\) is almost periodic, written \(f \in \text{AP}(J, X)\), if for every \(\varepsilon > 0\), there exists \(l > 0\) such that every subinterval of \(J\) of length \(l\) contains at least one \(\tau\) satisfying \(\|f(t + \tau) - f(t)\| \leq \varepsilon\) for all \(t \in J\).

(b) Let \(F(t) \in B(X) (t \in J)\) be a strongly continuous operator family. Then \(F(t)\) is almost periodic if for every \(x \in X\), \(F(\cdot)x\) is almost periodic; \(F(t)\) is periodic with period \(p\) if \(F(t + p) = F(t)\) for all \(t \in J\).

We collect some basic results on vector-valued almost periodic functions in the following lemma (see [21]).
Lemma 1.7. Let $f \in \text{AP}(\mathbb{R}, X)$. Then

(a) $f(t)$ is bounded, i.e., $\sup_{t \in \mathbb{R}} \|f(t)\| < \infty$;
(b) if $g \in \text{AP}(\mathbb{R}, X)$, $h \in \text{AP}(\mathbb{R}, C)$, then $f + g, hf \in \text{AP}(\mathbb{R}, X)$;
(c) $a_r(f) := \lim_{t \to \infty} t^{-1} \int_0^t e^{-i\alpha s} f(s) \, ds$ exists and
\[
a_r(f) = \lim_{t \to \infty} \frac{1}{t} \int_{\alpha}^{\alpha+t} e^{-i\alpha s} f(s) \, ds \quad \text{for all } r, \alpha \in \mathbb{R};
\]
(d) if $a_r(f) = 0$ for all $r \in \mathbb{R}$, then $f(t) = 0$ for all $t \in \mathbb{R}$;
(e) $\sigma(f) := \{ r \in \mathbb{R} : a_r(f) \neq 0 \}$ is at most countable;
(f) if $X \not\cong c_0$ (that is, $X$ does not contain an isomorphic copy of $c_0$, where $c_0$ is the space of all numerical sequences converging to 0), and $g(t) = \int_0^t f(s) \, ds$ $(t \in \mathbb{R})$ is bounded, then $g \in \text{AP}(\mathbb{R}, X)$;
(g) if $\{f_n\}_{n \in \mathbb{N}} \subset \text{AP}(\mathbb{R}, X)$ and $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly to $f$, then $f \in \text{AP}(\mathbb{R}, X)$;
(h) if $f'(t)$ exists and is uniformly continuous, then $f' \in \text{AP}(\mathbb{R}, X)$.

The following lemma follows immediately from Lemmas 1.4 and 1.7.

Lemma 1.8. Suppose $T(t)$ is an almost periodic $C$-semigroup with generator $A$. Then

(a) $T(t)$ is bounded and strongly uniformly continuous;
(b) $R(C) \subset Z_0$, the Hille–Yosida space for $A$, and $T(t) = e^{tA}|_{Z_0} C$.

2. Almost periodic $C$-semigroups and $C$-groups. In this section, we discuss the almost periodicity of $C$-semigroups and $C$-groups. The following is the main result of this section.

Theorem 2.1. Let $T(t)$ be a $C$-semigroup on $X$ with generator $A$. Then $T(t)$ is almost periodic if and only if $R(C) \subset (Z_0)_a$, the closure of $I$ in $Z_0$.

Proof. Sufficiency. Since $R(C) \subset (Z_0)_a$, for fixed $x \in X$ and $\varepsilon > 0$, there exist finitely many points $r_k \in \mathbb{R}$ and $x_k \in \ker(ir_k - A)$ such that $\|Cx - \sum \alpha_k x_k\|_{Z_0} \leq \varepsilon$. Thus $\|e^{tA}|_{Z_0} Cx - \sum \alpha_k e^{tA}|_{Z_0} x_k\|_{Z_0} \leq \varepsilon$. But $Ax_k = ir_k x_k$, so $e^{tA}|_{Z_0} x_k = e^{ir_k t} x_k \in \text{AP}(\mathbb{R}^+, X)$, i.e.,
\[
\left\| e^{tA}|_{Z_0} Cx - \sum \alpha_k e^{ir_k t} x_k \right\|_{Z_0} \leq \varepsilon.
\]
So we have
\[
\left\| e^{tA}|_{Z_0} Cx - \sum \alpha_k e^{ir_k t} x_k \right\|_{Z_0} \leq \left\| e^{tA}|_{Z_0} Cx - \sum \alpha_k e^{ir_k t} x_k \right\|_{Z_0} \leq \varepsilon \quad \text{for } t \geq 0.
\]
Hence $T(t)x = e^{tA}|_{Z_0} Cx \in \text{AP}(\mathbb{R}^+, X)$, and so $T(t)$ is almost periodic.

Necessity. Define $P_r x = \lim_{t \to \infty} t^{-1} \int_0^t e^{-ir s} T(s)x \, ds$ for each $r \in \mathbb{R}$ and $x \in X$. Then by Lemma 1.7(c) and from the proof of [21, Theorem 2.1], we
know that $P_r x$ exists and belongs to $D(A)$ with $A P_r x = i r P_r x$. Thus,

$$T(t) P_r x = \lim_{s \to \infty} \frac{1}{s} \int_0^s e^{-i r \tau} T(t + \tau) C x \, d\tau = C \lim_{s \to \infty} \frac{1}{s} \int_t^{t+s} e^{-i r (\tau-t)} T(\tau) x \, d\tau$$

$$= C e^{i r t} \lim_{s \to \infty} \frac{1}{s} \int_t^{t+s} e^{-i r \tau} T(\tau) x \, d\tau = e^{i r t} C P_r x.$$ 

Hence, $T(t) P_r x \in R(C)$ and $C^{-1} T(t) P_r x = e^{i r t} P_r x$ is bounded, and uniformly continuous. This implies $P_r x \in Z_0$ and $\{P_r x : r \in \mathbb{R}, x \in X\} \subset D(A|Z_0)$ with $A|Z_0 P_r x = i r P_r x$.

For every $x \in X$, since $t \mapsto T(t) x$ is bounded and uniformly continuous, we see that $T(t) x \in Z_0$ for $t \geq 0$. Next, we show $T(t)x \in AP(\mathbb{R}^+, Z_0)$. Since $T(t)x \in AP(\mathbb{R}^+, X)$, for every $\varepsilon > 0$, there exists $l > 0$ such that every subinterval of $\mathbb{R}^+$ of length $l$ contains at least one $\tau$ satisfying $\sup_{t \in \mathbb{R}^+} \|T(t+\tau)x - T(t)x\| \leq \varepsilon$. Then

$$\sup_{t \geq 0} \|T(t+\tau)x - T(t)x\|_{Z_0} = \sup_{t,s \geq 0} \|C^{-1} T(s)T(t+\tau)x - C^{-1} T(s)T(t)x\|$$

$$= \sup_{t,s \geq 0} \|T(t+s+\tau)x - T(t+s)x\|$$

$$\leq \sup_{t \geq 0} \|T(t+\tau)x - T(t)x\| \leq \varepsilon,$$

i.e., $T(t)x \in AP(\mathbb{R}^+, Z_0)$. If $f \in Z_0^\perp$ is such that $f(P_r x) \equiv 0$ for all $x \in X$ and $r \in \mathbb{R}$, then $\lim_{t \to \infty} t^{-1} \int_0^t e^{-i r s} f(T(s)x) \, ds = f(P_r x) \equiv 0$. But $f(T(t)x) \in AP(\mathbb{R}^+, C)$. Thus by Lemma 1.7(d), we get $f(T(t)x) \equiv 0$ for all $t \in \mathbb{R}^+$ and $x \in X$. In particular, $f(Cx) \equiv 0$. Therefore, $\{P_r x : r \in \mathbb{R}, x \in X\}^\perp \subset R(C)^\perp$, i.e.,

$$R(C) \subset \perp(R(C)^\perp) \subset \perp(\{P_r x : r \in \mathbb{R}, x \in X\}^\perp) = \text{span}\{P_r x : r \in \mathbb{R}, x \in X\} \subset (Z_0)\!a = \overline{\mathcal{I}},$$

where all the closures are taken in $Z_0$. ■

Now we have the following result ([21, Theorem 2.1]) as a corollary.

**COROLLARY 2.2.** If $\overline{R(C)} = X$, then $T(t)$ is an almost periodic $C$-semigroup with generator $A$ if and only if $T(t)$ is bounded and $X = X_a$, where $X_a$ is the closure of $\mathcal{I}$ in $X$.

**Proof.** The sufficiency is obvious. For the converse, since $Z_0 \hookrightarrow X$, a Cauchy sequence in $Z_0$ is also a Cauchy sequence in $X$, so that $(Z_0)\!a \subseteq X_a$. By Theorem 2.1, $R(C) \subset (Z_0)\!a$, hence $R(C) \subset X_a$; taking closure on both sides yields $X = X_a$. ■

By Definition 1.5 and combining Theorem 2.1 with [8, Theorem 5.16], we have
Theorem 2.3. Suppose $A$ has no eigenvalues in $(0, \infty)$. Then there exists an almost periodic mild $C$-existence family for $A$ if and only if $R(C) \subset (Z_0)_a$.

Moreover, combining Theorem 2.1 with [8, Theorem 5.17] and [10, Corollary 3.14] gives

Theorem 2.4. Suppose $A$ is closed and has no eigenvalues in $(0, \infty)$, and $C^{-1}AC = A$. Then $A$ generates an almost periodic $C$-semigroup if and only if $R(C) \subset (Z_0)_a$.

Now we investigate a special case.

Corollary 2.5. If $C = (r - A)^{-n}$ for some $n \in \mathbb{N}$, and $T(t)$ is a bounded strongly uniformly continuous $C$-semigroup generated by $A$, then $T(t)$ is almost periodic if and only if $S(t) := e^{t|A|Z_0}$ is almost periodic.

Proof. From the proof of Theorem 2.1, we see that $T(t)$ almost periodic on $X$ implies $T(t) = S(t)(r - A)^{-n}$ is almost periodic on $Z_0$. Applying Lemma 1.7(h) $n$ times, we deduce that $S(t)$ is almost periodic. The converse holds since $T(t) = S(t)C$ and $Z_0 \hookrightarrow X$.

The following theorem clarifies the relations between almost periodic $C$-semigroups and semigroups of class $C_0$.

Theorem 2.6. Let $T(t)$ be an almost periodic $C$-semigroup with generator $A$. Then there exists a maximal continuously imbedded subspace $W$ of $X$ such that $A|_W$ generates a contraction almost periodic semigroup of class $C_0$ on $W$ and $R(C) \subset W$; $W$ is maximal-unique in the sense that if $Y \hookrightarrow X$ and $A|_Y$ generates a contraction almost periodic semigroup of class $C_0$ on $Y$, then $Y \hookrightarrow W$.

Proof. Let $S(t)$ be the semigroup of class $C_0$ generated by $A|_{Z_0}$. Since $S(t)x = e^{it}x$, for $Ax = irx$, $S(t)$ clearly takes $I$ to itself, therefore, since $S(t)$ is continuous, it takes the closure of $I$ to itself, that is to say, $S(t)(Z_0)_a \subset (Z_0)_a$. Set $W = (Z_0)_a$; the first half of the result follows.

Now suppose $Y \hookrightarrow X$ and $A|_Y$ generates a contraction almost periodic semigroup of class $C_0$. Then $Y \hookrightarrow Z_0$, since $Z_0$ is maximal (cf. [8, Theorem 5.5]). It follows that $(Z_0)_a$ contains the closure of span $\{x \in D(A|_Y) : Ax = irx$ for some $r \in \mathbb{R}\}$ in $Y$, which is exactly $Y$, so that $Y \hookrightarrow W = (Z_0)_a$.

Remark 2.7. We can consider $(Z_0)_a$ for any closed operator $A$ with $s - A$ injective for $s > 0$. The results of Theorem 2.5 are also true, and it is not hard to see that $(Z_0)_a$ equals the set of all almost periodic orbits.

Theorem 2.8. Assume that $A$ generates a $C$-group $T(t)$. Then $T(t)$ is a periodic $C$-group with period $p$ if and only if $\sigma_C(A) \subset (2\pi i/p)\mathbb{Z}$ and $R(C) \subset (Z_0)_a$. 

Proof. Necessity. By Lemma 1.2(b) and the fact that $T(p) = C$, 
\[ (\lambda - A) \int_0^p e^{-\lambda s} T(s)x \, ds = (1 - e^{-\lambda p}) Cx \quad \text{for all } x \in X. \]
Combining this with $T(s)Ax = AT(s)x$ for every $x \in D(A)$, we get $\sigma_C(A) \subset (2\pi i/p)\mathbb{Z}$, while $R(C) \subset (Z_0)_a$ follows from Theorem 2.1.

Sufficiency. If $x \in \ker((2\pi ik/p - A)$ for some $k \in \mathbb{Z}$, then $T(t)x = e^{2\pi ikt/p} Cx$, which implies $T(t + p)x = T(t)x$ for $t \in \mathbb{R}$; the same holds for every $x \in \mathcal{I}$. Since $T(t)$ is continuous in $Z_0$, we have $T(t + p)x = T(t)x$ for all $x \in (Z_0)_a$; in particular, $T(t + p)Cx = T(t)Cx$ for all $x \in X$ by our assumption $R(C) \subset (Z_0)_a$, therefore, since $C$ is injective, we obtain $T(t + p) = T(t)$. ■

It is shown in [1] that every almost periodic semigroup of class $C_0$ can be extended to an almost periodic group; from [21, Theorem 3.1], we know that every almost periodic $C$-semigroup can also be extended to an almost periodic $C$-group. So we can assume that $A$ generates an almost periodic $C$-group.

Applying [17, Theorem 4.4] and the Hille–Yosida space, we obtain the following result, where we say that a function $u$ has uniformly convergent means if
\[ \lim_{R \to \infty} \frac{1}{R} \int_{-R}^R u(s) \, ds \]
exists, uniformly in $a \in \mathbb{R}$.

**Theorem 2.9.** Suppose $T(t)$ is a bounded strongly uniformly continuous $C$-group with generator $A$ such that $\sigma(A) \cap \mathbb{R}$ is at most countable. Then the following assertions are equivalent.

(a) $T(t)$ is almost periodic.

(b) For $\lambda \in \sigma(A) \cap \mathbb{R}$, $x \in X$, $e^{-\lambda t} T(t)x$ has uniformly convergent means.

Proof. By [21, Theorem 3.1], $A$ and $-A$ generate $C$-semigroups $T(t)$ and $T(-t)$ ($t \geq 0$), respectively, so that the Cauchy problem $u'(t) = Au(t)$ has a bounded uniformly continuous mild solution $T(t)x$ on $\mathbb{R}$.

Suppose $S(t)$ is the semigroup of class $C_0$ generated by $A|_{Z_0}$. From the proof of Theorem 2.1, we know $T(t)x$ is almost periodic if and only if $S(t)Cx$ is almost periodic in $Z_0$. To see that (b) implies (a), by [17, Theorem 4.4], we only need to show that $e^{-\lambda t} S(t)x$ has uniformly convergent means in $Z_0$ for $\lambda \in \sigma(A|_{Z_0}) \cap \mathbb{R}$. This can be achieved by a small modification of [9, Theorem 4].

(a)$\Rightarrow$(b) is trivial, since $T(t)x$ and $e^{-\lambda t} T(t)x$ ($\lambda \in \mathbb{R}$) are almost periodic. ■
3. Almost periodicity of integrated semigroups. An integrated semigroup is a strongly continuous family \( S(t) \) such that \( S(0) = 0 \) and

\[
S(t)S(s) = \int_{t}^{s+t} S(r) dr - \int_{0}^{s} S(r) dr
\]

for all \( s, t \geq 0 \).

Let \( r \in \rho(A) \neq \emptyset \). From [8, Theorem 18.3], we know that \( A \) generates an \((r-A)^{-1}\)-semigroup \( T(t) \) if and only if \( A \) generates an integrated semigroup \( S(t) \), and \( T(t)x = \frac{d}{dt} S(t)(r - A)^{-1}x \).

Suppose \( T(t) \) and \( S(t) \) are bounded, and strongly uniformly continuous. If \( S(t) \) is almost periodic, then \( S(t)(r - A)^{-1}x \) is almost periodic, and \( T(t)x = \frac{d}{dt} S(t)(r - A)^{-1}x \) is uniformly continuous, so that \( T(t)x \) is almost periodic.

Conversely, suppose \( T(t) \) is almost periodic, and \( X \) does not contain an isomorphic copy of \( c_0 \). Since

\[
S(t)x = (r - A) \int_{0}^{t} T(s)x ds = r \int_{0}^{t} T(s)x ds - T(t)x + (r - A)^{-1}x
\]

is bounded, we conclude that \( \int_{0}^{t} T(s)x ds \) is bounded; by Lemma 1.7(f), \( \int_{0}^{t} T(s)x ds \) is almost periodic, therefore so is \( S(t)x \).

Combining the above with Theorem 2.1, we have

**Theorem 3.1.** Suppose \( r \in \rho(A) \neq \emptyset \), \( A \) generates a bounded strongly uniformly continuous \((r - A)^{-1}\)-semigroup \( T(t) \) and a bounded integrated semigroup \( S(t) \), and suppose \( X \) does not contain an isomorphic copy of \( c_0 \). Then the following statements are equivalent.

(a) \( T(t) \) is almost periodic.

(b) \( S(t) \) is almost periodic.

(c) \( D(A) \subset (Z_0)_a \).

**Remark 3.2.** (a) If \( A \) generates an almost periodic \((r - A)^{-1}\)-semigroup \( T(t) \), then \( A \) also generates an integrated semigroup \( S(t) \). However, the almost periodicity of \( T(t) \) does not guarantee the almost periodicity of \( S(t) \).

In fact, if \( T(t) \) is periodic with period \( p \), and \( \int_{0}^{p} T(t)x dt \neq 0 \), then \( \int_{0}^{t} T(s)x ds \) is not bounded, so that \( S(t)x \) is not bounded. So the assumption that \( S(t) \) is bounded in Theorem 3.1 is necessary.

(b) The assumption that \( X \not\supset c_0 \) is not needed for the implication (b)\( \Rightarrow \) (a) of Theorem 3.1; the same holds for (b)\( \Rightarrow \) (a) of Theorem 3.3.

Let \( C = (r - A)^{-1} \). Suppose \( A \) generates a \( C \)-group \( T(t) \). Then \( A \) and \( -A \) generate \( C \)-semigroups \( T(t) \) and \( T(-t) \) \((t \geq 0)\), respectively. Hence \( A \) and \( -A \) also generate integrated semigroups \( S(t) \) and \( S(-t) \) \((t \geq 0)\) such that \( T(t) = \frac{d}{dt} S(t)(r - A)^{-1} \), \( T(-t) = \frac{d}{dt} S(-t)(r - A)^{-1} \), respectively. It is
easy to verify that (1) holds for all $t, s \in \mathbb{R}$. So we call $S(t)$ ($t \in \mathbb{R}$) an integrated group.

**Theorem 3.3.** Let $r \in \rho(A) \neq \emptyset$. Suppose $A$ generates a bounded strongly uniformly continuous $(r - A)^{-1}$-group $T(t)$ and a bounded integrated group $S(t)$ such that $\sigma(A) \cap i\mathbb{R}$ is at most countable, and $X$ does not contain an isomorphic copy of $c_0$. Then the following statements are equivalent.

(a) $T(t)$ is almost periodic.
(b) $S(t)$ is almost periodic.
(c) For $\lambda \in \sigma(A) \cap i\mathbb{R}$ and $x \in X$, $e^{-\lambda t}T(t)x$ has uniformly convergent means.
(d) For $\lambda \in \sigma(A) \cap i\mathbb{R}$ and $x \in X$, $e^{-\lambda t}S(t)x$ has uniformly convergent means.

**Proof.** We only need to show (c) $\Leftrightarrow$ (d).

(c) $\Rightarrow$ (d). By (c) and Theorem 3.1, $S(t)$ is almost periodic, thus $S(t)$ has uniformly convergent means, i.e., (d) holds for $\lambda = 0$.

Now suppose $\lambda \in \sigma(A) \cap i\mathbb{R}\setminus\{0\}$. Fix $\varepsilon > 0$. Then by the assumption of (c), there exists $T_\varepsilon$ such that

$$\left\| \frac{1}{T} \int_{h-T}^{h+T} e^{-\lambda t}T(t)x \, dt - \frac{1}{S} \int_{h-S}^{h+S} e^{-\lambda t}T(t)x \, dt \right\| < \varepsilon$$

for all $T, S > T_\varepsilon$ and $h \in \mathbb{R}$.

To prove $e^{-\lambda t}S(t)x$ has uniformly convergent means, by (2), it suffices to show $e^{-\lambda t}\int_0^T x \, ds$ has uniformly convergent means. Suppose $\| \int_0^T x \, ds \| \leq M$ and $T, S > 1/|\lambda\varepsilon|$. Then

$$\left\| \frac{1}{T} \int_{h-T}^{h+T} e^{-\lambda t}T(t)x \, dt - \frac{1}{S} \int_{h-S}^{h+S} e^{-\lambda t}T(t)x \, dt \right\|$$

$$= \left\| \frac{1}{\lambda T} \int_{h-T}^{h+T} e^{-\lambda t}T(t)x \, dt - \frac{1}{\lambda S} \int_{h-S}^{h+S} e^{-\lambda t}T(t)x \, dt \right\|$$

$$- \frac{1}{\lambda T} e^{-\lambda (h+T)} \int_0^{h+T} T(t)x \, dt + \frac{1}{\lambda T} e^{-\lambda (h-T)} \int_0^{h-T} T(t)x \, dt$$

$$+ \frac{1}{\lambda S} e^{-\lambda (h+S)} \int_0^{h+S} T(t)x \, dt - \frac{1}{\lambda S} e^{-\lambda (h-S)} \int_0^{h-S} T(t)x \, dt$$

$$< \varepsilon + 4M\varepsilon;$$

the result then follows.
(d)⇒(c). Given $\varepsilon > 0$ and $x \in X$, there exists $T_\varepsilon$ such that
\[
\left\| \frac{1}{K} \int_{h-K}^{h+K} e^{-\lambda t} S(t)(r-A)^{-1} x \, dt - \frac{1}{L} \int_{h-L}^{h+L} e^{-\lambda t} S(t)(r-A)^{-1} x \, dt \right\| < \varepsilon
\]
for all $K, L > T_\varepsilon$ and $h \in \mathbb{R}$. Suppose $\|S(t)(r-A)^{-1} x\| \leq M$ and $K, L > 1/\varepsilon$. Then
\[
\left\| \frac{1}{K} \int_{h-K}^{h+K} e^{-\lambda t} T(t) x \, dt - \frac{1}{L} \int_{h-L}^{h+L} e^{-\lambda t} T(t) x \, dt \right\|
\]
\[
= \left\| \frac{1}{K} \int_{h-K}^{h+K} e^{-\lambda t} \frac{d}{dt} S(t)(r-A)^{-1} x \, dt - \frac{1}{L} \int_{h-L}^{h+L} e^{-\lambda t} \frac{d}{dt} S(t)(r-A)^{-1} x \, dt \right\|
\]
\[
= \left\| \frac{\lambda}{K} \int_{h-K}^{h+K} e^{-\lambda t} S(t)(r-A)^{-1} x \, dt - \frac{\lambda}{L} \int_{h-L}^{h+L} e^{-\lambda t} S(t)(r-A)^{-1} x \, dt \right\|
\]
\[
+ \frac{1}{K} e^{-\lambda(h+K)} S(h+K)(r-A)^{-1} x - \frac{1}{K} e^{-\lambda(h-K)} S(h-K)(r-A)^{-1} x
\]
\[
- \frac{1}{L} e^{-\lambda(h+L)} S(h+L)(r-A)^{-1} x + \frac{1}{L} e^{-\lambda(h-L)} S(h-L)(r-A)^{-1} x
\]
\[
< \lambda \varepsilon + 4M \varepsilon;
\]
thus we get (c). \[\fbox{\textbf{4. Almost periodic $C$-cosine functions.}}\] A $C$-cosine function $C(t)$ is a strongly continuous operator family such that $C(0) = C$ and $2C(t)C(s) = C(t+s)C + C(s-t)C$ for all $t, s \in \mathbb{R}$. The corresponding $C$-sine function, $S(t)$, is defined by $S(t) = \int_{0}^{t} C(s) \, ds$. The generator $A$ of $C(t)$ is defined by
\[
\text{D}(A) = \left\{ x \in X : \lim_{t \to 0} \frac{2}{t^2} (C(t)x - Cx) \right\},
\]
\[
Ax = C^{-1} \left( \lim_{t \to 0} \frac{2}{t^2} (C(t)x - Cx) \right) \quad \text{for } x \in \text{D}(A).
\]
For more details on cosine and $C$-cosine functions, we refer to [11, 19, 21].

First we introduce the interpolation space for $C$-cosine functions (cf. [19, Theorem 1.2.5]).

**Lemma 4.1**. Suppose $A$ generates a strongly uniformly continuous and uniformly bounded $C$-cosine function. Then there exists a Banach space $Y$ such that
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(1) $A|_Y$ generates a bounded strongly continuous cosine function $G(t)$, with corresponding sine function $H(t);
(2) $R(C) \subseteq Y \hookrightarrow X$;
(3) $C(t) = G(t)C$, $S(t) = H(t)C$, $Y$ may be chosen as $Y = \{ x \in X : t \rightarrow C^{-1}C(t)x \text{ is bounded and uniformly continuous} \}$ and
$$\|x\|_Y = \sup_{t \in \mathbb{R}} \|C^{-1}C(t)x\|.$$ Using the above results and arguments similar to those in Section 2, we can prove the following theorem on the almost periodicity of $C$-cosine functions.

**Theorem 4.2.** (a) A $C$-cosine function $C(t)$ is almost periodic if and only if $C(t)$ is bounded and $R(C) \subseteq Y_b := \text{span}\{ x \in D(A|_Y) : Ax = -r^2x \text{ for some } r \in \mathbb{R} \}$, the closure taken in $Y$, where $Y$ is as in Lemma 4.1.
(b) $S(t)$ is almost periodic if and only if $S(t)$ is bounded, $0 \notin P_\sigma(A)$ and $R(C) \subseteq Y_b$.

We can also derive [21, Theorem 4.1] from Theorem 4.2, as in the proof of Corollary 2.2.

Finally, we characterize the periodicity of $C$-cosine functions.

**Theorem 4.3.** A $C$-cosine function $C(t)$ is periodic with period $p$ if and only if $C(t)$ is bounded, $\sigma_C(A) \subseteq \{-4\pi^2k^2/p^2 : k \in \mathbb{N} \}$ and $R(C) \subseteq Y_b$.

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