STUDIA MATHEMATICA 162 (1) (2004)

Unitary Banach algebras

by

Julio Becerra Guerrero (Granada), Simon Cowell (Swansea),
Ángel Rodríguez Palacios (Granada) and
Geoffrey V. Wood (Swansea)

Abstract. In a Banach algebra an invertible element which has norm one and whose inverse has norm one is called unitary. The algebra is unitary if the closed convex hull of the unitary elements is the closed unit ball. The main examples are the C^*-algebras and the ℓ_1 group algebra of a group. In this paper, different characterizations of unitary algebras are obtained in terms of numerical ranges, dentability and holomorphy. In the process some new characterizations of C^*-algebras are given.

1. Introduction. By a *norm-unital normed algebra* we mean a normed (associative) algebra having a unit $\mathbf{1}$ such that $\|\mathbf{1}\| = 1$. *Unitary elements* of a norm-unital normed algebra A are defined as those invertible elements u of A satisfying $\|u\| = \|u^{-1}\| = 1$. By a *unitary normed algebra* we mean a norm-unital normed algebra (say A) such that the convex hull of the set of its unitary elements is norm-dense in the closed unit ball of A. Relevant examples of unitary Banach algebras are all unital C^*-algebras and the group algebras $\ell_1(G)$ for every group G.

Unitary Banach algebras were first considered by E. R. Cowie in her Ph.D. thesis [17], but most of her results were not published elsewhere. Indeed, in Cowie's paper [18] we only find some incidental references to unitary Banach algebras. More recently, unitary Banach algebras have been reconsidered by M. L. Hansen and R. V. Kadison [31]. One of the main goals in both [17] and [31] is to obtain characterizations of unital C^*-algebras among unitary Banach algebras by some extra conditions. An example in this line is the remark, in both Cowie's and Hansen–Kadison's works, that unitary closed subalgebras of C^*-algebras are in fact C^*-algebras. For a review of some of the more important results in [17], [18], and [31] the reader is referred to [47].

2000 *Mathematics Subject Classification*: Primary 46H05; Secondary 43A20, 46B04.

Partially supported by Junta de Andalucía grant FQM 0199 and Acción Integrada HB1999-0052.

A Banach $*$-algebra that is unitary is an example of a U^*-algebra as defined by T. W. Palmer in [41, Section 10.3]. This is a $*$-algebra which is the linear span of the "unitary" elements ($uu^* = u^*u = 1$). In a U^*-algebra A a seminorm is defined by

$$\nu(x) = \inf\Big\{\sum_{j=1}^{\infty}|\lambda_j| : x = \sum_{j=1}^{\infty}\lambda_j u_j,\ \lambda_j \in \mathbb{C},\ u_j \text{ unitary},\ \|u_j\| = 1\Big\}$$

and so A is unitary if ν is equal to the norm.

In the present paper we try to study unitary Banach algebras by themselves. In Section 2 we prove several stability conditions for the class of unitary Banach algebras. After the observation that quotients of unitary Banach algebras are unitary Banach algebras (Proposition 2.1), our main result in this line states that every unitary Banach algebra is a quotient of $\ell_1(G)$ for a suitable group G (Theorem 2.3). We also prove that both the complete projective tensor product and the ℓ_∞-sum of two unitary Banach algebras is a unitary Banach algebra (Propositions 2.4 and 2.8).

In Section 3, we extend to unitary Banach algebras some results previously known in the case of unital C^*-algebras (see Remark 3.9(c) and the comment before Corollary 3.13), and prove some other results for unitary Banach algebras, which could be new even in the C^*-algebra case. The main theorem in the section (Theorem 3.8) asserts that, in most cases, each one of the facts asserted in those results actually characterizes unitary Banach algebras among norm-unital Banach algebras. As an example, a norm-unital Banach algebra A is unitary if and only if the numerical range of each continuous linear operator F on A is the closed convex hull of the union of the numerical ranges relative to A of the elements of the form $F(u^{-1})u$, where u runs over the set of all unitary elements of A.

We prove Theorem 3.8 through a general discussion in the setting of Banach spaces, involving "big points" and "strong subdifferentiability of the norm", notions which have been introduced in [5] and [28], respectively. Motivation comes from the facts that, if A is a norm-unital Banach algebra, then the norm of A is strongly subdifferentiable at its unit, and that, if in addition A is unitary, then the unit of A is a big point of A. Among the consequences of such a discussion we emphasize that the normed space numerical index of every unitary Banach algebra coincides with that of its dual space (Corollary 3.13).

In Section 4 we study dentability of closed balls in unitary Banach algebras. Concerning this topic, the situation is very different in the two fundamental classes of examples. Indeed, closed unit balls of group algebras always have denting points in abundance, whereas closed unit balls of infinite-dimensional C^*-algebras have no denting point, nor even are dentable [4].

Our main result in this line asserts that the closed unit ball B_A of a unitary Banach algebra A is dentable (if and) only if the unit of A is a denting point of B_A, and moreover, if this is the case, then denting points of B_A, unitary elements of A, and big points of the Banach space of A coincide (Theorem 4.5).

In the concluding section of the paper (Section 5) we invoke some results of [3] and [34] on infinite-dimensional holomorphy to obtain several new characterizations of unital C^*-algebras among unitary Banach algebras (Theorem 5.2). One of them (namely, that unital C^*-algebras are precisely those unitary complex Banach algebras which satisfy the von Neumann inequality) is implicitly contained in [3] (see Remark 5.3(a)). Among the essentially new results collected in Theorem 5.2, we emphasize the one asserting that unital C^*-algebras are precisely those unitary complex Banach algebras A such that the distance from the unit of A to the "symmetric part" of the Banach space of A is less than one.

2. Stability. From now on, $\mathbb{K}$ will denote the field of real or complex numbers. All normed spaces will be over $\mathbb{K}$ unless otherwise stated. We denote by S_X and B_X the closed unit sphere and the closed unit ball, respectively, of X, and, for a norm-unital normed algebra A, we denote by U_A the group of all unitary elements of A.

Some stability conditions for the class of unitary normed algebras are almost trivial. Thus, for example, *the completion of every unitary normed algebra is a unitary normed algebra.* We will give our results mostly in terms of Banach algebras rather than normed algebras. Another stability condition is the following.

PROPOSITION 2.1. *Let A be a unitary Banach algebra, and let I be a proper closed* (*two-sided*) *ideal of A. Then the Banach algebra A/I is unitary.*

Proof. Let $\pi : A \mapsto A/I$ be the natural quotient homomorphism. It is obvious that $\pi(u)$ is a unitary element of A/I whenever u is a unitary element of A. Let z be in $\operatorname{int}(B_{A/I})$, and let $\varepsilon > 0$. Then there exists $a \in \operatorname{int}(B_A)$ with $\pi(a) = z$, and $\alpha_1, \dots, \alpha_n \in \mathbb{R}_0^+$ and $u_1, \dots, u_n \in U_A$ satisfying $\sum_{i=1}^n \alpha_i = 1$ and $\|a - \sum_{i=1}^n \alpha_i u_i\| < \varepsilon$. Therefore we have $\|z - \sum_{i=1}^n \alpha_i \pi(u_i)\| < \varepsilon$ with $\pi(u_1), \dots, \pi(u_n) \in U_{A/I}$. ∎

Given a set Γ, we denote by $\ell_1(\Gamma)$ the Banach space of all functions $\psi : \Gamma \to \mathbb{K}$ such that $\|\psi\| := \sum_{\gamma\in\Gamma} |\psi(\gamma)| < \infty$, and, for μ in Γ, we denote by δ_μ the element of $\ell_1(\Gamma)$ defined by $\delta_\mu(\gamma) = 1$ if $\gamma = \mu$, and $\delta_\mu(\gamma) = 0$ otherwise. Now, let X and Y be normed spaces. By a *metric surjection* from X to Y we mean a continuous linear mapping (say Ψ) from X onto Y such that the induced bijection $X/\ker(\Psi) \to Y$ is an isometry. It is easy to see that a linear mapping $\Psi : X \to Y$ is a metric surjection if and only if $\Psi(\mathrm{int}(B_X)) = \mathrm{int}(B_Y)$. The next lemma is of folklore type.

LEMMA 2.2. *Let X be a Banach space, and let Γ be a subset of B_X such that $\overline{\mathrm{co}}(\Gamma) = B_X$. Then, for ψ in $\ell_1(\Gamma)$, the family $\{\psi(\gamma)\gamma\}_{\gamma\in\Gamma}$ is summable in X, and the mapping $\psi \to \sum_{\gamma\in\Gamma}\psi(\gamma)\gamma$ becomes a metric surjection from $\ell_1(\Gamma)$ to X. As a consequence, given $x \in X$ and $\varepsilon > 0$, there are sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ in $\mathbb{K}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ in Γ satisfying $\sum_{n=1}^{\infty}|\lambda_n| < \|x\| + \varepsilon$ and $\sum_{n=1}^{\infty}\lambda_n\gamma_n = x$.*

Proof. For ψ in $\ell_1(\Gamma)$ we have $\sum_{\gamma\in\Gamma}\|\psi(\gamma)\gamma\| \le \sum_{\gamma\in\Gamma}|\psi(\gamma)| < \infty$, and therefore, since X is a Banach space, the family $\{\psi(\gamma)\gamma\}_{\gamma\in\Gamma}$ is summable in X [16, Proposition VII.9.18].

Now let Ψ denote the mapping from $\ell_1(\Gamma)$ to X defined by $\Psi(\psi) := \sum_{\gamma\in\Gamma}\psi(\gamma)\gamma$, which is clearly a linear contraction. For μ in Γ we have $\Psi(\delta_\mu) = \mu$, and hence $\Gamma \subseteq \Psi(B_{\ell_1(\Gamma)})$. Since $\overline{\mathrm{co}}(\Gamma) = B_X$, we have in fact $B_X \subseteq \overline{\Psi(B_{\ell_1(\Gamma)})}$. Now, from the main tool in the proof of Banach's open mapping theorem (see for example [11, Lemma 48.3]) we deduce $\mathrm{int}(B_X) \subseteq \Psi(B_{\ell_1(\Gamma)})$. Since Ψ is a linear contraction, the above shows that Ψ is in fact a metric surjection. ■

Let G be a group. Then the *group algebra* $\ell_1(G)$ is a norm-unital Banach algebra for the *convolution product* $\star$ defined by

$$(\psi \star \varphi)(t) := \sum_{s\in G}\psi(s^{-1}t)\varphi(s)$$

for every $t \in G$. Then $\ell_1(G)$ is clearly a unitary Banach algebra [18, p. 9].

Now that we know that quotients of unitary Banach algebras are unitary Banach algebras (Proposition 2.1), and that, for every group G, $\ell_1(G)$ is a unitary Banach algebra, we prove that every unitary Banach algebra is a quotient of $\ell_1(G)$ for a suitable group G.

THEOREM 2.3. *Let A be a unitary Banach algebra. Then there exists a closed ideal I of $\ell_1(U_A)$ such that A is isometrically algebra-isomorphic to $\ell_1(U_A)/I$.*

Proof. According to Lemma 2.2, the mapping $\Psi : \psi \mapsto \sum_{u\in U_A}\psi(u)u$, from $\ell_1(U_A)$ to A, is a metric surjection. On the other hand, it is easy to check that Ψ is an algebra homomorphism. Therefore $I := \ker(\Psi)$ is a closed ideal of $\ell_1(U_A)$, and the induced mapping $\ell_1(U_A)/I \to A$ is an isometric algebra isomorphism. ■

Let A and B be algebras. The vector space $A \otimes B$ becomes in a natural manner an algebra under the product determined on elementary tensors by $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1a_2) \otimes (b_1b_2)$. If A and B are Banach algebras, then the algebra $A \otimes B$ is a normed algebra under the projective tensor norm

$\|\cdot\|_\pi$ [11, Proposition 43.18]. The Banach algebra obtained by completing $A \otimes_\pi B$ is called the *projective tensor product* of A and B, and is denoted by $A \widehat{\otimes}_\pi B$.

PROPOSITION 2.4. *Let A and B be unitary Banach algebras. Then $A \widehat{\otimes}_\pi B$ is a unitary Banach algebra.*

Proof. The proof is straightforward. ∎

Note that any real Banach algebra A can be complexified by taking the projective tensor product $\mathbb{C} \otimes_\pi A$ (see [15, Proposition 13.3]). The next corollary follows from Proposition 2.4.

COROLLARY 2.5. *Let A be unitary real Banach algebra. Then the complexification of A is a unitary complex Banach algebra.* ∎

By an *involution* on an algebra A over $\mathbb{R}$ (respectively, $\mathbb{C}$) we mean a linear (respectively, conjugate-linear) involutive anti-automorphism of A. If G is a group, then $\ell_1(G)$ has a natural involution, namely $*$ defined by $\psi^*(s) := \overline{\psi(s^{-1})}$ for all $\psi \in \ell_1(G)$ and $s \in G$. Such an involution is isometric, and sends each unitary element to its inverse. The same occurs for the natural involution of every unital C^*-algebra.

COROLLARY 2.6. *Let A be a finite-dimensional unitary Banach algebra over $\mathbb{K}$. Then there exists a* (*unique*) *isometric involution $*$ on A satisfying $u^* = u^{-1}$ for every unitary element u of A. Moreover*, *if $\mathbb{K} = \mathbb{C}$*, *then $(A, *)$ is $*$-isomorphic to a C^*-algebra.*

Proof. In view of Corollary 2.5, we can assume that $\mathbb{K} = \mathbb{C}$. Then, by Theorem 2.3, we have $A = \ell_1(G)/M$ for some group G and some closed ideal M of $\ell_1(G)$. Let $\pi : \ell_1(G) \mapsto \ell_1(G)/M = A$ be the natural quotient homomorphism. We always have $\pi(U_{\ell_1(G)}) \subseteq U_A$, but, looking at the proof of Theorem 2.3, we can choose the couple (G, M) in such a way that actually the equality $\pi(U_{\ell_1(G)}) = U_A$ holds.

On the other hand, since M is of finite codimension in $\ell_1(G)$, it follows from [20, Corollary 3.3.27] that M is invariant under the natural involution $*$ of $\ell_1(G)$, and that $A = \ell_1(G)/M$ (endowed with the quotient involution, also denoted by $*$) is $*$-isomorphic to a C^*-algebra. Now, clearly, the involution $*$ on A is isometric, and π becomes a $*$-homomorphism. If u is in U_A, then there exists v in $U_{\ell_1(G)}$ such that $\pi(v) = u$, and hence $u^* = (\pi(v))^* = \pi(v^*) = \pi(v^{-1}) = (\pi(v))^{-1} = u^{-1}$. ∎

In relation to Corollary 2.6, it is worth mentioning that a finite-dimensional unitary complex Banach algebra need not be isometrically isomorphic to a C^*-algebra. This is the case, for example, of the algebra $\ell_1(G)$ for every finite group G not reduced to its unit. Nevertheless, an elegant isometric characterization of finite-dimensional C^*-algebras among finite-dimensional unitary complex Banach algebras is obtained in [31]. This characterization is rediscovered in Corollary 2.7 immediately below. Following [31, p. 536], we say that a Banach algebra A is *maximal unitary* if A is unitary and U_A is a maximal bounded subgroup of the group of all invertible elements of A. Unital C^*-algebras are maximal unitary [31, Proposition 3].

COROLLARY 2.7 ([31, Theorem 6]). *Let A be a finite-dimensional maximal unitary complex Banach algebra. Then A is isometrically isomorphic to a C^*-algebra.*

Proof. By Corollary 2.6, there exists a norm $|||\cdot|||$ on A and an involution $*$ on A such that $(A, |||\cdot|||, *)$ becomes a C^*-algebra, and the inclusion $U_A \subseteq U_{(A,|||\cdot|||)}$ holds. Since $|||\cdot|||$ and $\|\cdot\|$ are equivalent, $U_{(A,|||\cdot|||)}$ is a bounded subgroup of the group of all invertible elements of A. Since A is maximal unitary, we deduce $U_A = U_{(A,|||\cdot|||)}$. Since both A and $(A, |||\cdot|||)$ are unitary, the above equality implies $|||\cdot||| = \|\cdot\|$. ∎

Let A and B be algebras. The vector space $A \times B$ becomes an algebra over $\mathbb{K}$ under the product defined coordinate-wise. If A and B are actually normed algebras, then the algebra $A \times B$ is a normed algebra under the norm $\|(a, b)\| := \max\{\|a\|, \|b\|\}$. Such a normed algebra will be denoted by $A \oplus_\infty B$.

PROPOSITION 2.8. *Let A and B be unitary normed algebras. Then $A \oplus_\infty B$ is unitary.*

Proof. Let f be a norm-one continuous linear functional on $A \oplus_\infty B$. Then there are continuous linear functionals g and h on A and B, respectively, satisfying $\|g\| + \|h\| = 1$ and $f(a, b) = g(a) + h(b)$ for every (a, b) in $A \oplus_\infty B$. Since (u, v) belongs to $U_{A\oplus_\infty B}$ whenever u and v are in U_A and U_B, respectively, and A and B are unitary, we have

$$\begin{aligned}1 &\geq \sup\{\Re e(f(w)) : w \in U_{A\oplus_\infty B}\} \geq \sup\{\Re e(f(u, v)) : (u, v) \in U_A \times U_B\}\\ &= \sup\{\Re e(g(u)) : u \in U_A\} + \sup\{\Re e(h(v)) : v \in U_B\} = \|g\| + \|h\| = 1.\end{aligned}$$

Therefore we obtain $\sup\{\Re e(f(w)) : w \in U_{A\oplus_\infty B}\} = 1$ for every norm-one continuous linear functional f on $A \oplus_\infty B$, and the result follows by applying the Hahn–Banach separation theorem. ∎

REMARK 2.9. (a) If we define *real C^*-algebras* as real closed $*$-invariant subalgebras of (complex) C^*-algebras, it follows from the proof of Corollary 2.6 that *every finite-dimensional unitary real Banach algebra is isomorphic to a real C^*-algebra.* Moreover, *finite-dimensional maximal unitary real Banach algebras are isometrically isomorphic to real C^*-algebras.* This follows from the above by realizing that *finite-dimensional real C^*-algebras are unitary* (as we do immediately below), and then arguing as in the proof of Corollary 2.7.

Let A be a finite-dimensional real C^*-algebra. By Wedderburn theory, A has a unit $\mathbf{1}$, and we have $A = \bigoplus_{i=1}^n A_i$ for some $n \in \mathbb{N}$ and some $*$-invariant simple ideals $A_1, \ldots, A_n$ of A. Let u be an extreme point of B_A. By [32, Lemma 3.2], we have $(\mathbf{1} - uu^*)A(\mathbf{1} - u^*u) = 0$. Writing $u = \sum_{i=1}^n u_i$ with $u_i \in A_i$ for every i, we deduce $(\mathbf{1}_i - u_iu_i^*)A_i(\mathbf{1}_i - u_i^*u_i) = 0$ for every i, where $\mathbf{1}_i$ denotes the unit of A_i. Since, for each i, A_i is finite-dimensional and prime, we obtain $u_iu_i^* = u_i^*u_i = \mathbf{1}_i$, and hence u belongs to U_A. Since u is an arbitrary extreme point of B_A, and B_A is convex and compact, it follows from the above and the Krein–Milman theorem that $B_A = \overline{\mathrm{co}}\, U_A$.

(b) In general, a unitary Banach algebra need not have a continuous involution sending each unitary element to its inverse. To realize this, keep in mind that, as a consequence of [45, Theorem 7.7.1], there exists a closed ideal (say M) of $\ell_1^{\mathbb{C}}(\mathbb{Z})$ that is not a $*$-ideal. Then, putting $A := \ell_1^{\mathbb{C}}(\mathbb{Z})/M$, A becomes a unitary commutative complex Banach algebra (by Proposition 2.1). Assume that there is a continuous involution $*$ on A satisfying $u^* = u^{-1}$ for every $u \in U_A$. Choose $\psi \in M$ with $\psi^* \notin M$, and denote by π the natural quotient homomorphism $\ell_1^{\mathbb{C}}(\mathbb{Z}) \to \ell_1^{\mathbb{C}}(\mathbb{Z})/M$. Since $\psi = \sum_{n\in\mathbb{Z}} \psi(n)\delta_n$, and $\pi(\delta_n)$ belongs to U_A for every $n \in \mathbb{Z}$, we have

$$\begin{aligned}\pi(\psi^*) &= \sum_{n\in\mathbb{Z}} \overline{\psi(n)}\pi(\delta_n^*) = \sum_{n\in\mathbb{Z}} \overline{\psi(n)}\pi(\delta_n^{-1}) = \sum_{n\in\mathbb{Z}} \overline{\psi(n)}(\pi(\delta_n))^{-1} \\ &= \sum_{n\in\mathbb{Z}} \overline{\psi(n)}(\pi(\delta_n))^* = \Big[\sum_{n\in\mathbb{Z}} \psi(n)\pi(\delta_n)\Big]^* = [\pi(\psi)]^* = 0.\end{aligned}$$

Therefore ψ^* belongs to M, a contradiction.

(c) If a unitary normed algebra A has a continuous involution $*$ satisfying $u^* = u^{-1}$ for every $u \in U_A$, then such an involution is isometric. Indeed, the set $\{a \in A : \|a^*\| \leq 1\}$ is closed and convex, and contains U_A.

(d) In general, maximal unitary complex Banach algebras need not be C^*-algebras. A large list of examples confirming the above assertion can be derived from [18]. In that paper *uniquely maximal Banach algebras* are defined as those norm-unital Banach algebras A satisfying:

(1) U_A is a maximal bounded subgroup of the group of all invertible elements of A, and

(2) $|||\cdot||| = \|\cdot\|$ whenever $|||\cdot|||$ is any equivalent norm on A such that $(A, |||\cdot|||)$ is a norm-unital normed algebra with $U_{(A,|||\cdot|||)} = U_A$.

Uniquely maximal Banach algebras are maximal unitary [18, Theorem 1]. Moreover, given a group G (with unit denoted by e), the Banach algebra $\ell_1(G)$ is uniquely maximal if and only if, for every $s \in G \setminus \{e\}$, the set $\{tst^{-1} : t \in G\}$ is infinite [18, Theorem 5].

(e) Examples of unitary complex Banach algebras which are neither C^*-algebras nor group algebras can be found in [31] and [41, Section 10.3]. Other examples of the same kind can be derived from Propositions 2.4 and 2.8, and part (b) of the present remark.

3. Characterizations. Let X be a normed space. We denote by X^* the (topological) dual space of X. Up to the canonical injection, X will be seen as a subspace of its bidual X^{**}. $\mathcal{B}(X)$ will denote the normed algebra of all bounded linear operators on X, and I_X will stand for the identity operator on X. Each continuous bilinear mapping from $X \times X$ to X will be called a *product* on X. Each product f on X has a natural norm $\|f\|$ given by

$$\|f\| := \sup\{\|f(x,y)\| : x, y \in B_X\}.$$

We denote by $\mathcal{P}(X)$ the normed space of all products on X.

Now, let e be a "distinguished" norm-one element in the normed space X. The set $D(X,e)$ of *states* of X relative to e is defined as the non-empty, convex, and weak*-compact subset of X^* given by

$$D(X,e) := \{\phi \in B_{X^*} : \phi(e) = 1\}.$$

For x in X, the *numerical range* $V(X,e,x)$ of x relative to e is given by

$$V(X,e,x) := \{\phi(x) : \phi \in D(X,e)\}.$$

Numerical ranges are preserved (respectively, contracted) under isometric (respectively, contractive) linear mappings preserving distinguished elements. We say that e is a *vertex* of B_X if the conditions $x \in X$ and $\phi(x) = 0$ for all ϕ in $D(X,e)$ imply $x = 0$. It is well known and easy to see that the vertex property for e implies that e is an extreme point of B_X. For x in X, we define the *numerical radius* $v(X,e,x)$ of x relative to e by

$$v(X,e,x) := \max\{|\varrho| : \varrho \in V(X,e,x)\}.$$

The *numerical index* $n(X,e)$ of X relative to e is the number given by

$$n(X,e) := \max\{r \geq 0 : r\|x\| \leq v(X,e,x) \text{ for all } x \text{ in } X\}.$$

We note that $0 \leq n(X,e) \leq 1$ and that the condition $n(X,e) > 0$ implies that e is a vertex of B_X. Note also that, if Y is a subspace of X containing e, then $n(Y,e) \geq n(X,e)$.

Given a set Γ and a normed space X, we denote by $\ell_\infty(\Gamma, X)$ the normed space of all functions $\psi : \Gamma \to X$ such that $\|\psi\| := \sup\{\|\psi(\gamma)\| : \gamma \in \Gamma\} < \infty$. When X is in fact a normed algebra, $\ell_\infty(\Gamma, X)$ becomes a normed algebra under the product defined point-wise. Given a normed algebra A, we denote by p_A the natural product of A. Note that $\|p_A\| = 1$ whenever A is norm-unital.

PROPOSITION 3.1. *Let A be a unitary Banach algebra. Then, for every p in $\mathcal{P}(A)$, we have*

$$V(\mathcal{P}(A), p_A, p) = \overline{\mathrm{co}}\Big[\bigcup_{(u,v)\in U_A\times U_A} V(A, \mathbf{1}, u^{-1}p(u,v)v^{-1})\Big].$$

Proof. For p in $\mathcal{P}(A)$, let $\widehat{p}$ stand for the element of $\ell_\infty(U_A \times U_A, A)$ defined by $\widehat{p}(u,v) := u^{-1}p(u,v)v^{-1}$ for every $(u,v) \in U_A \times U_A$. Then the mapping $p \mapsto \widehat{p}$ from $\mathcal{P}(A)$ to $\ell_\infty(U_A \times U_A, A)$ is linear, sends p_A to the unit $\widetilde{\mathbf{1}}$ of the Banach algebra $\ell_\infty(U_A \times U_A, A)$ (namely, the constant function equal to $\mathbf{1}$ on $U_A \times U_A$), and is an isometry (since $\overline{\mathrm{co}}\, U_A = B_A$). Therefore

$$V(\mathcal{P}(A), p_A, p) = V(\ell_\infty(U_A \times U_A, A), \widetilde{\mathbf{1}}, \widehat{p})$$

for every $p \in \mathcal{P}(A)$. Since, for p in $\mathcal{P}(A)$, we have

$$V(\ell_\infty(U_A \times U_A, A), \widetilde{\mathbf{1}}, \widehat{p}) = \overline{\mathrm{co}}\Big[\bigcup_{(u,v)\in U_A\times U_A} V(A, \mathbf{1}, \widehat{p}(u,v))\Big]$$

(see [43, Proposition 3]), the result follows. ■

In the main result of this section (see Theorem 3.8 below) we will show that the property of unitary Banach algebras proved in Proposition 3.1 actually characterizes them among norm-unital Banach algebras. This will be achieved through a general discussion in the setting of Banach spaces, involving "big points" [5] and "strong subdifferentiability of the norm" [28]. Such a discussion is motivated by the facts that, if A is a norm-unital Banach algebra, then the norm of A is strongly subdifferentiable at its unit, and that, if A is a unitary Banach algebra, then the unit of A is a big point of A. In fact, as we show in Remark 3.2 which follows, strong subdifferentiability of the norm has already been implicitly applied in the proof of Proposition 3.1.

Let X be a normed space, and e a norm-one element in X. For x in X, the number $\lim_{\alpha\to 0^+} (\|e+\alpha x\| - 1)/\alpha$ (which always exists because the mapping $\alpha \mapsto \|e+\alpha x\|$ from $\mathbb{R}$ to $\mathbb{R}$ is convex) is usually denoted by $\tau(e,x)$. We say that the norm of X is *strongly subdifferentiable* at e if

$$\lim_{\alpha\to 0^+} \frac{\|e+\alpha x\| - 1}{\alpha} = \tau(e,x) \quad \text{uniformly} \quad \text{for } x \in B_X.$$

REMARK 3.2. In the proof of Proposition 3.1 we have applied the result of [43, Proposition 3] that $V(\ell_\infty(\Gamma, A), \widetilde{\mathbf{1}}, \phi) = \overline{\mathrm{co}}[\bigcup_{\gamma\in\Gamma} V(A, \mathbf{1}, \phi(\gamma))]$ whenever Γ is any set, A is any norm-unital Banach algebra, and ϕ is any element of $\ell_\infty(\Gamma, A)$. It is worth mentioning that the result just quoted need not remain true if the couple $(A, \mathbf{1})$ above is replaced with (X, e) for an arbitrary Banach space X and an element e in S_X. Actually, if we denote by $\widetilde{e}$ the constant function equal to e on Γ, the couples (X,e) as above which satisfy $V(\ell_\infty(\Gamma, X), \widetilde{e}, \phi) = \overline{\mathrm{co}}[\bigcup_{\gamma\in\Gamma} V(X, e, \phi(\gamma))]$ for every set Γ and every ϕ in $\ell_\infty(\Gamma, X)$ are characterized as those such that the norm of X is strongly subdifferentiable at e (see [1, Theorem 2.7] or [44, Corollary 2]). According to [28, Corollary 4.4], the strong subdifferentiability of a Banach space X at an element e in S_X is also equivalent to the upper semicontinuity $(n-n)$ of the duality mapping of X at e, previously introduced in [26]. *Semicontinuity* $(n-n)$ *of the duality mapping of* X *at* e means that for every $\varepsilon > 0$ there is $\delta > 0$ such that $D(X,x) \subseteq D(X,e) + \varepsilon B_{X^*}$ whenever x lies in S_X and $\|x-e\| < \delta$. For more information about the concepts just introduced the reader is referred to [25].

As a first consequence of Remark 3.2, we obtain the following corollary.

Corollary 3.3. *Let A be a unitary Banach algebra. Then the norm of $\mathcal{P}(A)$ is strongly subdifferentiable at p_A.*

Proof. In the proof of Proposition 3.1 we have seen that $\mathcal{P}(A)$ can be identified with a subspace of $\ell_\infty(U_A \times U_A, A)$ in such a way that p_A converts into $\widetilde{\mathbf{1}}$. On the other hand, since $\ell_\infty(U_A \times U_A, A)$ is a norm-unital Banach algebra whose unit is $\widetilde{\mathbf{1}}$, the norm of $\ell_\infty(U_A \times U_A, A)$ is strongly subdifferentiable at $\widetilde{\mathbf{1}}$ (see Remark 3.2). Now the result follows from the obvious hereditary character of the strong subdifferentiability of the norm. ■

Let X be a normed space. We denote by $\mathcal{G}_X$ the group of all surjective isometries from X to X. Given e in X and a subgroup $\mathcal{G}$ of $\mathcal{G}_X$, we say that e is a *$\mathcal{G}$-big point* of X if $\overline{\mathrm{co}}(\mathcal{G}(e)) = B_X$. We call $\mathcal{G}_X$-big points of X simply *big points* of X. We note that, if e is a $\mathcal{G}$-big point of X for some $\mathcal{G}$ as above, then e is a big point of X, and e lies in S_X (unless $X = 0$). A subset M of a vector space over $\mathbb{K}$ is said to be *circled* if $S_{\mathbb{K}} M \subseteq M$.

Lemma 3.4. *Let X be a normed space over* $\mathbb{K}$, *let e be in S_X, and let $\mathcal{G}$ be a circled subgroup of $\mathcal{G}_X$. Then the following assertions are equivalent*:

(1) *e is a $\mathcal{G}$-big point of X.*

(2) *For every lower semicontinuous norm $|\!|\!|\cdot|\!|\!|$ on X satisfying* (i) $|\!|\!|e|\!|\!| \leq 1$, (ii) $\|\cdot\| \leq |\!|\!|\cdot|\!|\!|$, *and* (iii) $\mathcal{G} \subseteq \mathcal{G}_{(X,|\!|\!|\cdot|\!|\!|)}$, *we have* $|\!|\!|\cdot|\!|\!| = \|\cdot\|$.

(3) *For every continuous norm $|\!|\!|\cdot|\!|\!|$ on X satisfying* (i)–(iii) *above*, *we have* $|\!|\!|\cdot|\!|\!| = \|\cdot\|$.

Proof. (1)⇒(2). Let $|\!|\!|\cdot|\!|\!|$ be a lower semicontinuous norm on X satisfying (i)–(iii) in the statement. By (i) and (iii), we have $\mathcal{G}(e) \subseteq B_{(X,|\!|\!|\cdot|\!|\!|)}$, and from the lower semicontinuity of $|\!|\!|\cdot|\!|\!|$ we deduce that $B_{(X,|\!|\!|\cdot|\!|\!|)}$ is $\|\cdot\|$-closed in X. Therefore, by the assumption (1), we have $B_X \subseteq B_{(X,|\!|\!|\cdot|\!|\!|)}$, that is, $\|\cdot\| \geq |\!|\!|\cdot|\!|\!|$. It follows from (ii) that $|\!|\!|\cdot|\!|\!| = \|\cdot\|$.

(2)⇒(3). This is clear.

(3)⇒(1). Let $0 < \varepsilon \leq 1$. Then $\mathrm{co}[(\varepsilon B_X) \cup \mathcal{G}(e)]$ is a $\mathcal{G}$-invariant absolutely convex subset of X contained in B_X and containing εB_X. Therefore the Minkowski functional of $\mathrm{co}[(\varepsilon B_X) \cup \mathcal{G}(e)]$ (say $|\!|\!|\cdot|\!|\!|_\varepsilon$) is a norm on X

satisfying $\varepsilon\|\!|\cdot|\!\|_\varepsilon \le \|\cdot\| \le \|\!|\cdot|\!\|_\varepsilon$,

$$\{x \in X : \|\!|x|\!\|_\varepsilon < 1\} \subseteq \mathrm{co}[(\varepsilon B_X) \cup \mathcal{G}(e)] \subseteq \{x \in X : \|\!|x|\!\|_\varepsilon \le 1\}, \tag{3.1}$$

and $\mathcal{G} \subseteq \mathcal{G}_{(X,\|\!|\cdot|\!\|)}$. Now $\|\!|\cdot|\!\|_\varepsilon$ satisfies all requirements made for $\|\!|\cdot|\!\|$ in (3). Therefore, by the assumption (3), we have $\|\!|\cdot|\!\|_\varepsilon = \|\cdot\|$. Let x be in X with $\|x\| < 1$. It follows from the left inclusion in (3.1) that x belongs to $\mathrm{co}[(\varepsilon B_X)\cup\mathcal{G}(e)]$. Since $\mathrm{co}[(\varepsilon B_X)\cup\mathcal{G}(e)]$ is contained in $\varepsilon B_X+\mathrm{co}(\mathcal{G}(e))$, there exists y in $\mathrm{co}(\mathcal{G}(e))$ such that $\|x-y\| \le \varepsilon$. The arbitrariness of $\varepsilon \in \,]0,1]$ and $x \in \mathrm{int}(B_X)$ yields $\mathrm{int}(B_X) \subseteq \overline{\mathrm{co}}(\mathcal{G}(e))$. Therefore we have $\overline{\mathrm{co}}(\mathcal{G}(e)) = B_X$, that is, e is a $\mathcal{G}$-big point of X. ∎

Proposition 3.5. *For a Banach space X over $\mathbb{K}$, an element e in S_X, and a circled subgroup $\mathcal{G}$ of $\mathcal{G}_X$, consider the following assertions*:

(1) *e is a $\mathcal{G}$-big point of X, and the norm of X is strongly subdifferentiable at e.*
(2) *The set $\{T^*(f) : (T,f) \in \mathcal{G} \times D(X,e)\}$ is norm-dense in S_{X^*}.*
(3) *For every G in $\mathcal{B}(X^*)$,*

$$V(\mathcal{B}(X^*), I_{X^*}, G) = \overline{\mathrm{co}}\Big[\bigcup_{T\in\mathcal{G}} V(X^{**}, e, (T^{**}\circ G^*\circ T^{-1})(e))\Big].$$

(4) *For every F in $\mathcal{B}(X)$,*

$$V(\mathcal{B}(X), I_X, F) = \overline{\mathrm{co}}\Big[\bigcup_{T\in\mathcal{G}} V(X, e, (T\circ F\circ T^{-1})(e))\Big].$$

(5) *For every equivalent norm $\|\!|\cdot|\!\|$ on X satisfying*
   (i) $\|\!|e|\!\| = 1$,
   (ii) $V(X,e,x) \subseteq V((X,\|\!|\cdot|\!\|),e,x)$ *for all $x \in X$, and*
   (iii) $\mathcal{G} \subseteq \mathcal{G}_{(X,\|\!|\cdot|\!\|)}$,

   we have $\|\!|\cdot|\!\| = \|\cdot\|$.
(6) *e is a $\mathcal{G}$-big point of X.*

Then (1)⇒(2)⇒(3)⇒(4)⇒(5)⇒(6).

Proof. (1)⇒(2). This is proved in [8, Lemma 3] in the particular case of $\mathcal{G} = \mathcal{G}_X$, but the argument in that proof works without changes in our more general situation.

(2)⇒(3). For every normed space Y, put

$$\Pi(Y) := \{(y,g) : y \in S_Y,\ g \in D(Y,y)\}.$$

Now let Γ be the subset of $\Pi(X^*)$ defined by

$$\Gamma := \{(T^*(f), T^{-1}(e)) : (T,f) \in \mathcal{G} \times D(X,e)\}.$$

By the assumption (2), the first coordinate projection $\Gamma \to S_{X^*}$ is dense in S_{X^*}. It follows from [13, Theorem 9.3] that, for each $G \in \mathcal{B}(X^*)$, we have

$$V(\mathcal{B}(X^*), I_{X^*}, G) = \overline{\mathrm{co}}\{(T^{-1}(e))(G(T^*(f))) : T \in \mathcal{G},\ f \in D(X,e)\}.$$

Since

$$\begin{aligned}\{(T^{-1}(e))(G(T^*(f))) &: T \in \mathcal{G},\ f \in D(X,e)\} \\ &= \{((T^{**} \circ G^* \circ T^{-1})(e))(f) : T \in \mathcal{G}, f \in D(X,e)\} \\ &\subseteq \{f((T^{**} \circ G^* \circ T^{-1})(e)) : T \in \mathcal{G}, f \in D(X^{**},e)\} \\ &= \bigcup_{T\in\mathcal{G}} V(X^{**}, e, (T^{**} \circ G^* \circ T^{-1})(e)),\end{aligned}$$

we deduce the inclusion

$$V(\mathcal{B}(X^*), I_{X^*}, G) \subseteq \overline{\mathrm{co}}\Big[\bigcup_{T\in\mathcal{G}} V(X^{**}, e, (T^{**} \circ G^* \circ T^{-1})(e))\Big].$$

But the converse inclusion is trivial. Indeed, for T in $\mathcal{G}$, the mapping $G \mapsto (T^{**} \circ G^* \circ T^{-1})(e)$ from $\mathcal{B}(X^*)$ to X^{**} is a linear contraction sending I_{X^*} to e.

(3)⇒(4). The mapping $F \mapsto F^*$ from $\mathcal{B}(X)$ to $\mathcal{B}(X^*)$ is a linear isometry, and consequently we have

$$V(\mathcal{B}(X), I_X, F) = V(\mathcal{B}(X^*), I_{X^*}, F^*)$$

for every F in $\mathcal{B}(X)$. By applying the assumption (3), we obtain

$$V(\mathcal{B}(X), I_X, F) = \overline{\mathrm{co}}\Big[\bigcup_{T\in\mathcal{G}} V(X^{**}, e, (T^{**} \circ F^{**} \circ T^{-1})(e))\Big]$$

for every F in $\mathcal{B}(X)$. Finally, note that, for x in X and G in $\mathcal{B}(X)$, we have $V(X^{**}, e, x) = V(X, e, x)$ (because the inclusion of X into X^{**} is a linear isometry) and $G^{**}(x) = G(x)$.

(4)⇒(5). Let $|\!|\!|\cdot|\!|\!|$ be an equivalent norm on X satisfying (i)–(iii) of (5). By the assumption (4) and the requirements on $|\!|\!|\cdot|\!|\!|$, for F in $\mathcal{B}(X)$ we have

$$\begin{aligned}V(\mathcal{B}(X), I_X, F) &= \overline{\mathrm{co}}\Big[\bigcup_{T\in\mathcal{G}} V(X, e, (T \circ F \circ T^{-1})(e))\Big] \\ &\subseteq \overline{\mathrm{co}}\Big[\bigcup_{T\in\mathcal{G}_{(X,|\!|\!|\cdot|\!|\!|)}} V((X, |\!|\!|\cdot|\!|\!|), e, (T \circ F \circ T^{-1})(e))\Big].\end{aligned}$$

But, for T in $\mathcal{G}_{(X,|\!|\!|\cdot|\!|\!|)}$, the mapping $F \mapsto (T \circ F \circ T^{-1})(e)$ from $\mathcal{B}(X, |\!|\!|\cdot|\!|\!|)$ to $(X, |\!|\!|\cdot|\!|\!|)$ is a linear contraction sending I_X to e, and hence

$$V((X, |\!|\!|\cdot|\!|\!|), e, (T \circ F \circ T^{-1})(e)) \subseteq V(\mathcal{B}(X, |\!|\!|\cdot|\!|\!|), I_X, F).$$

It follows that, for F in $\mathcal{B}(X)$, we have

$$V(\mathcal{B}(X), I_X, F) \subseteq V(\mathcal{B}(X, |\!|\!|\cdot|\!|\!|), I_X, F),$$

and consequently

$$v(\mathcal{B}(X), I_X, F) \leq v(\mathcal{B}(X, \|\!|\cdot|\!\|), I_X, F).$$

By [40, Theorem 3], there exists a positive number k such that $|\!\|\cdot\|\!| = k\|\cdot\|$ on X. But, in fact, we have $k = 1$ because, by (i), $|\!\|e\|\!| = \|e\| = 1$. (Actually, in the proof of [40, Theorem 3] the equality $v(\mathcal{B}(X), I_X, \cdot) = v(\mathcal{B}(X, |\!\|\cdot\|\!|), I_X, \cdot)$ is assumed, but really only the inequality $v(\mathcal{B}(X), I_X, \cdot) \leq v(\mathcal{B}(X, |\!\|\cdot\|\!|), I_X, \cdot)$ is used.)

(5)⇒(6). In view of Lemma 3.4, it is enough to show that, for every continuous norm $|\!\|\cdot\|\!|$ on X satisfying (a) $|\!\|e\|\!| \leq 1$, (b) $\|\cdot\| \leq |\!\|\cdot\|\!|$, and (c) $\mathcal{G} \subseteq \mathcal{G}_{(X,|\!\|\cdot\|\!|)}$, we have $|\!\|\cdot\|\!| = \|\cdot\|$. Let $|\!\|\cdot\|\!|$ be such a norm. The continuity of $|\!\|\cdot\|\!|$ and (b) imply that $|\!\|\cdot\|\!|$ and $\|\cdot\|$ are equivalent. Moreover, from (a) and (b) we deduce that $|\!\|e\|\!| = 1$. On the other hand, by (b), the mapping $I_X : (X, |\!\|\cdot\|\!|) \to X$ is a linear contraction, and hence we have $V(X, e, \cdot) \subseteq V((X, |\!\|\cdot\|\!|), e, \cdot)$. It follows from the assumption (5) that $|\!\|\cdot\|\!| = \|\cdot\|$. ∎

COROLLARY 3.6. *Let X be a Banach space over $\mathbb{K}$, let e in S_X be such that the norm of X is strongly subdifferentiable at e, and let $\mathcal{G}$ be a circled subgroup of $\mathcal{G}_X$. Then assertions* (2)–(6) *in Proposition* 3.5 *are equivalent.* ∎

REMARK 3.7. (a) It is easy to see that assertions (1)–(6) in Proposition 3.5, for an arbitrary circled subgroup $\mathcal{G}$ of $\mathcal{G}_X$, are respectively stronger than the same assertions in the case of $\mathcal{G} = \mathcal{G}_X$

(b) The implication (1)⇒(2) in Proposition 3.5 is not reversible. To see this recall that a normed space X is said to be *transitive* if for every (equivalently, some) e in S_X we have $\mathcal{G}_X(e) = S_X$. Now let X be a transitive Banach space, let e be an arbitrary element of S_X, and take $\mathcal{G} = \mathcal{G}_X$. Then assertion (2) is true for the triple $(X, e, \mathcal{G})$ because, in this case, the set $\{T^*(f) : T \in \mathcal{G},\ f \in D(X, e)\}$ is nothing but the set of all norm-one norm attaining functionals on X, and the Bishop–Phelps theorem applies. On the other hand, if assertion (1) were true for $(X, e, \mathcal{G})$, then, by the arbitrariness of $e \in S_X$, and [25, Theorem 5.1], X would be an Asplund space. But, certainly, there exist non-Asplund transitive Banach spaces. Indeed, every Banach space can be isometrically embedded into a transitive Banach space [9, Corollary 2.21].

(c) The implication (2)⇒(4) in Proposition 3.5 is not reversible either. To see this recall that a normed space X is said to be *almost transitive* if, for every (equivalently, some) e in S_X, $\mathcal{G}_X(e)$ is dense in S_X. Now let X be a Banach space, let e be in S_X, and take $\mathcal{G} = \mathcal{G}_X$. If X is almost transitive, then an easy application of [13, Theorem 9.3] shows that assertion (4) is true for $(X, e, \mathcal{G})$. On the other hand, if X is smooth at e, and if assertion (2) is true for $(X, e, \mathcal{G})$, then, for f equal to the unique element in $D(X, e)$, we deduce that $\mathcal{G}_{X^*}(f)$ is dense in S_{X^*}, and hence X^* is almost transitive. Finally note that $L_1([0,1])$ is an almost transitive Banach space [9, Example 2.13] which is smooth at "many" points of its unit sphere (by separability) and whose dual is not almost transitive [9, Example 2.32].

Let A be a normed algebra. If X is a normed A-bimodule, then X^* is canonically a normed A-bimodule under the module operations defined by $(af)(x) := f(xa)$ and $(fa)(x) := f(ax)$ for all $f \in X^*$, $a \in A$, and $x \in X$. In this way A^* and A^{**} become normed A-bimodules.

THEOREM 3.8. *For a norm-unital Banach algebra A, the following assertions are equivalent*:

(i) *For every p in $\mathcal{P}(A)$,*

$$V(\mathcal{P}(A), p_A, p) = \overline{\mathrm{co}}\Big[\bigcup_{(u,v)\in U_A\times U_A} V(A, \mathbf{1}, u^{-1}p(u,v)v^{-1})\Big].$$

(ii) *The set $U_A D(A, \mathbf{1})$ is norm-dense in S_{A^*}.*

(iii) *For every G in $\mathcal{B}(A^*)$,*

$$V(\mathcal{B}(A^*), I_{A^*}, G) = \overline{\mathrm{co}}\Big[\bigcup_{u\in U_A} V(A^{**}, \mathbf{1}, G^*(u^{-1})u)\Big].$$

(iv) *For every F in $\mathcal{B}(A)$,*

$$V(\mathcal{B}(A), I_A, F) = \overline{\mathrm{co}}\Big[\bigcup_{u\in U_A} V(A, \mathbf{1}, F(u^{-1})u)\Big].$$

(v) *For every equivalent norm $|||\cdot|||$ on A satisfying*

(a) *$(A, |||\cdot|||)$ is a norm-unital normed algebra*,

(b) *$V(A, \mathbf{1}, a) \subseteq V((A, |||\cdot|||), \mathbf{1}, a)$ for all $a \in A$, and*

(c) *$U_A \subseteq U_{(A,|||\cdot|||)}$*,

we have $|||\cdot||| = \|\cdot\|$.

(vi) *A is unitary.*

(vii) *For every continuous norm $|||\cdot|||$ on A satisfying* (a) *and* (c) *of* (v), *and*

(b′) *$\|\cdot\| \le |||\cdot|||$*,

we have $|||\cdot||| = \|\cdot\|$.

Moreover, *if $\mathbb{K} = \mathbb{C}$, then the above assertions are also equivalent to*

(viii) *The same as* (v) *with "continuous" instead of "equivalent".*

Proof. Let X, e, and $\mathcal{G}$ stand for the Banach space of A, the unit $\mathbf{1}$ of A, and the circled subgroup of $\mathcal{G}_A$ given by

$$\mathcal{G} := \{a \mapsto au : u \in U_A\},$$

respectively. Then assertions (ii)–(iv) and (vi) in the theorem are nothing but reformulations of assertions (2)–(4) and (6), respectively, in Proposition 3.5, and assertion (5) in Proposition 3.5 implies assertion (v) in the theorem. Since the norm of A is strongly subdifferentiable at $\mathbf{1}$ (by Remark 3.2), it follows from Corollary 3.6 that (ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(vi)$\Rightarrow$(v).

(v)$\Rightarrow$(vii). Let $\|\!|\cdot|\!\|$ be a continuous norm on A satisfying (a), (b′), and (c). The continuity of $\|\!|\cdot|\!\|$ and (b′) imply that $\|\!|\cdot|\!\|$ and $\|\cdot\|$ are equivalent. On the other hand, by (b′), the mapping $I_A : (A, \|\!|\cdot|\!\|) \to A$ is a linear contraction, and hence (b) follows. Therefore, by the assumption (v), we have $\|\!|\cdot|\!\| = \|\cdot\|$.

(vii)$\Rightarrow$(vi). Let $0 < \varepsilon \leq 1$. Since $\mathrm{co}[(\varepsilon B_A) \cup U_A]$ is an absolutely convex subset of A contained in B_A and containing εB_A, the Minkowski functional of $\mathrm{co}[(\varepsilon B_A) \cup U_A]$ (say $\|\!|\cdot|\!\|_\varepsilon$) is a norm on A satisfying $\varepsilon\|\!|\cdot|\!\|_\varepsilon \leq \|\cdot\| \leq \|\!|\cdot|\!\|_\varepsilon$, and we have

$$\{a \in A : \|\!|a|\!\|_\varepsilon < 1\} \subseteq \mathrm{co}[(\varepsilon B_A) \cup U_A] \subseteq \{a \in A : \|\!|a|\!\|_\varepsilon \leq 1\}. \tag{3.2}$$

On the other hand, since $(\varepsilon B_A) \cup U_A$ is a subsemigroup of A, and the convex hull of a subsemigroup is a subsemigroup, we deduce that $\|\!|\cdot|\!\|_\varepsilon$ actually becomes an algebra norm on A [15, Proposition 1.9], and hence $1 \leq \|\!|\mathbf{1}|\!\|_\varepsilon$. Now, if u is in U_A, then, by the right inclusion in (3.2), we have

$$1 \leq \|\!|\mathbf{1}|\!\|_\varepsilon = \|\!|uu^{-1}|\!\|_\varepsilon \leq \|\!|u|\!\|_\varepsilon \|\!|u^{-1}|\!\|_\varepsilon \leq 1 \cdot 1 = 1,$$

and hence $\|\!|\mathbf{1}|\!\|_\varepsilon = \|\!|u|\!\|_\varepsilon = \|\!|u^{-1}|\!\|_\varepsilon = 1$. Therefore the normed algebra $(A, \|\!|\cdot|\!\|_\varepsilon)$ is norm-unital, and the inclusion $U_A \subseteq U_{(A,\|\!|\cdot|\!\|)}$ holds. Now $\|\!|\cdot|\!\|_\varepsilon$ satisfies all requirements for $\|\!|\cdot|\!\|$ in assertion (vii). Therefore, by the assumption (vii), we have $\|\!|\cdot|\!\|_\varepsilon = \|\cdot\|$. Let a be in A with $\|a\| < 1$. It follows from the left inclusion in (3.2) that a belongs to $\mathrm{co}[(\varepsilon B_A) \cup U_A]$. Since $\mathrm{co}[(\varepsilon B_A) \cup U_A]$ is contained in $\varepsilon B_A + \mathrm{co}(U_A)$, there exists b in $\mathrm{co}(U_A)$ such that $\|a - b\| \leq \varepsilon$. The arbitrariness of $\varepsilon \in]0, 1]$ and $a \in \mathrm{int}(B_A)$ yields $\mathrm{int}(B_A) \subseteq \overline{\mathrm{co}}(U_A)$. Therefore we have $\overline{\mathrm{co}}(U_A) = B_A$, that is, A is unitary.

(vi)$\Rightarrow$(i). By Proposition 3.1.

(i)$\Rightarrow$(iv). For $F \in \mathcal{B}(A)$, let $p_F \in \mathcal{P}(A)$ be defined by $p_F(x, y) := F(x)y$. Then $F \mapsto p_F$ is a linear isometry sending I_A to p_A. Therefore we have

$$V(\mathcal{P}(A), p_A, p_F) = V(\mathcal{B}(A), I_A, F) \quad \forall F \in \mathcal{B}(A).$$

By applying the assumption (i), assertion (iv) follows.

(viii)$\Rightarrow$(v). This is obvious (in both real and complex cases).

(v)$\Rightarrow$(viii) (in the complex case). Let $\|\!|\cdot|\!\|$ be a continuous norm on A satisfying (a), (b), and (c). By (b) and the Bohnenblust–Karlin theorem [13, Theorem 4.1], we have

$$\|\cdot\| \leq ev(A, \mathbf{1}, \cdot) \leq ev((A, \|\!|\cdot|\!\|), \mathbf{1}, \cdot) \leq e\|\!|\cdot|\!\|.$$

It follows from the continuity of $|||\cdot|||$ that $|||\cdot|||$ and $\|\cdot\|$ are equivalent. Now apply the assumption (v). ∎

REMARK 3.9. (a) A direct proof of the implication (vi)⇒(ii) in Theorem 3.8 is the one which follows. This proof is inspired by that of Lemma 4.2 of [7], where some misprints arise. Let h be in S_{A^*}, and let $0 < \varepsilon < 2$. By the assumption (vi), there exists v in U_A such that $|1 - h(v)| < \varepsilon^2/16$. By the Bishop–Phelps–Bollobás theorem [14, Theorem 16.1], there are $a \in S_A$ and $g \in D(A, a)$ satisfying $\|a - v\| < \varepsilon/2$ and $\|g - h\| < \varepsilon/2$. Put $u := v^{-1}$ and $f := ag$. Then u and f belong to U_A and $D(A, \mathbf{1})$, respectively, and we have

$$\begin{aligned}\|h - uf\| = \|h - uag\| &\le \|h - uah\| + \|uah - uag\|\\ &\le \|\mathbf{1} - ua\|\,\|h\| + \|ua\|\,\|h - g\|\\ &= \|v - a\| + \|ua\|\,\|h - g\|\\ &\le \|v - a\| + \|h - g\| < \varepsilon.\end{aligned}$$

(b) Let A be a norm-unital normed algebra. If u is in U_A, then we have $D(A, u) = u^{-1}D(A, \mathbf{1})$. Therefore assertion (ii) in Theorem 3.8 can be reformulated as

(ii′) $\bigcup_{u\in U_A} D(A, u)$ is norm-dense in S_{A^*}.

(c) Assertions (i), (ii), and (iv) in Theorem 3.8 are known to be true in the case when A is a unital C^*-algebra (see [34, Corollary 1.2], [7, Lemma 4.2], and [43, Theorem 5], respectively).

Let X be a normed space. The *normed space numerical index*, $N(X)$, of X is defined by $N(X) := n(\mathcal{B}(X), I_X)$. We always have $N(X) \ge N(X^*)$, because the mapping $F \mapsto F^*$ from $\mathcal{B}(X)$ to $\mathcal{B}(X^*)$ is a linear isometry. The converse inequality has been claimed in [23]. However, the proof of this claim has never appeared and whether $N(X) = N(X^*)$ for arbitrary X remains an open problem.

COROLLARY 3.10. *Let X be a Banach space, and let e be a big point of X such that the norm of X is strongly subdifferentiable at e. Then we have $N(X) \ge N(X^*) \ge n(X, e)$.*

Proof. In fact the desired conclusion $N(X^*) \ge n(X, e)$ is true if the requirements made on e are replaced with the strictly weaker ones that e lies in S_X and that, for every G in $\mathcal{B}(X^*)$, the equality

$$V(\mathcal{B}(X^*), I_{X^*}, G) = \overline{\mathrm{co}}\Big[\bigcup_{T\in\mathcal{G}_X} V(X^{**}, e, (T^{**}\circ G^*\circ T^{-1})(e))\Big] \tag{3.3}$$

holds (see Proposition 3.5 and Remark 3.7(b)). Indeed, by Proposition 3.5, these new requirements imply that e is a big point of X, and hence, by

Goldstein's theorem, $\mathrm{co}(\mathcal{G}_X(e))$ is w^*-dense in $B_{X^{**}}$. Since, for G in $\mathcal{B}(X^*)$, the set

$$\{x \in X^{**} : \|G^*(x)\| \leq \sup\{\|G^*(T(e))\| : T \in \mathcal{G}_X\}\}$$

is w^*-closed and convex, and contains $\mathcal{G}_X(e)$, we deduce

$$\|G\| = \sup\{\|G^*(T(e))\| : e \in \mathcal{G}_X\} \tag{3.4}$$

for every G in $\mathcal{B}(X^*)$. On the other hand, (3.3) implies

$$\begin{aligned} v(\mathcal{B}(X^*), I_{X^*}, G) &= \sup\{v(X^{**}, e, (T^{**} \circ G^* \circ T^{-1})(e)) : T \in \mathcal{G}_X\} \\ &\geq n(X^{**}, e) \sup\{\|(T^{**} \circ G^* \circ T^{-1})(e)\| : T \in \mathcal{G}_X\} \\ &= n(X^{**}, e) \sup\{\|G^*(T^{-1}(e))\| : T \in \mathcal{G}_X\} \end{aligned}$$

for every $G \in \mathcal{B}(X^*)$. Since, by [39, Lemma 4.8], the equality $n(X^{**}, e) = n(X, e)$ holds, it follows from (3.4) that

$$v(\mathcal{B}(X^*), I_{X^*}, G) \geq n(X, e)\|G\|$$

for every $G \in \mathcal{B}(X^*)$. Therefore we have $n(X^*) = n(\mathcal{B}(X^*), I_{X^*}) \geq n(X, e)$, as desired. ∎

COROLLARY 3.11. *Let X be a Banach space. Assume that there exists a big point e of X such that $n(X, e) = 1$. Then $N(X) = N(X^*) = 1$.*

Proof. The condition $n(X, e) = 1$, for e in S_X, implies that the norm of X is strongly subdifferentiable at e [1, Corollary 5.9]. Now apply Corollary 3.10. ∎

We note that the couples (X, e), where X is a real normed space and e is a norm-one element in X satisfying $n(X, e) = 1$, are nothing but the so-called "unit order spaces" (see for example [30, Section 1.2]). We also note that, if (X, e) is a complete unit order space, and if there exists a big point b of X such that $\|e - b\| < 2$, then e is big point of X [9, Proposition 5.23].

LEMMA 3.12. *Let Γ be a set, let X be a Banach space, and let e be in S_X such that the norm of X is strongly subdifferentiable at e. Then $n(X, e) = n(\ell_\infty(\Gamma, X), \widetilde{e})$.*

Proof. The mapping $x \mapsto \widetilde{x}$ from X to $\ell_\infty(\Gamma, X)$ (where, for x in X, $\widetilde{x}$ means the constant function equal to x on Γ) is a linear isometry, and hence we have $n(X, e) \geq n(\ell_\infty(\Gamma, X), \widetilde{e})$. On the other hand, by Remark 3.2, the equality $V(\ell_\infty(\Gamma, X), \widetilde{e}, \phi) = \overline{\mathrm{co}}[\bigcup_{\gamma \in \Gamma} V(X, e, \phi(\gamma))]$ holds for every ϕ in $\ell_\infty(\Gamma, X)$. Therefore, for ϕ in $\ell_\infty(\Gamma, X)$, we have

$$\begin{aligned} v(\ell_\infty(\Gamma, X), \widetilde{e}, \phi) &= \sup\{v(X, e, \phi(\gamma)) : \gamma \in \Gamma\} \\ &\geq n(X, e) \sup\{\|\phi(\gamma)\| : \gamma \in \Gamma\} = n(X, e)\|\phi\|. \end{aligned}$$

Since ϕ is arbitrary in $\ell_\infty(\Gamma, X)$, we deduce $n(X, e) \leq n(\ell_\infty(\Gamma, X), \widetilde{e})$. ∎

The following corollary is known in the case of unital C^*-algebras [34].

Corollary 3.13. *Let A be a real or complex norm-unital Banach algebra over $\mathbb{K}$. Then*

$$n(A,\mathbf{1}) \geq N(A) \geq n(\mathcal{P}(A), p_A).$$

If in addition A is unitary, then we actually have

$$n(A,\mathbf{1}) = N(A) = N(A^*) = n(\mathcal{P}(A), p_A).$$

As a consequence, if $\mathbb{K}=\mathbb{C}$ and if A is unitary, then $n(\mathcal{P}(A), p_A) \geq 1/e$.

Proof. The mapping $a \mapsto L_a$ from A to $\mathcal{B}(A)$ (where, for a in A, L_a means the operator of left multiplication by a on A) is a linear isometry sending $\mathbf{1}$ to I_A, and hence we have $n(A,\mathbf{1}) \geq n(\mathcal{B}(A), I_A) = N(A)$. Moreover, we know from the proof of (i)⇒(iv) in Theorem 3.8 that there is a linear isometry from $\mathcal{B}(A)$ to $\mathcal{P}(A)$ sending I_A to p_A, so that the inequality $N(A) \geq n(\mathcal{P}(A), p_A)$ follows.

Suppose that A is unitary. Then there exists a linear isometry from $\mathcal{P}(A)$ to $\ell_\infty(U_A \times U_A, A)$ sending p_A to $\widetilde{\mathbf{1}}$ (see the proof of Proposition 3.1). Therefore $n(\mathcal{P}(A), p_A) \geq n(\ell_\infty(U_A \times U_A, A), \widetilde{\mathbf{1}})$. Keeping in mind Remark 3.2 and Lemma 3.12, the above inequality reads $n(\mathcal{P}(A), p_A) \geq n(A,\mathbf{1})$. Since the norm of A is strongly subdifferentiable at $\mathbf{1}$, and $\mathbf{1}$ is a big point of A (see the beginning of the proof of Theorem 3.8), the equalities $n(A,\mathbf{1}) = N(A) = N(A^*)$ follow from Corollary 3.10 and the first assertion in the present corollary. If in addition $\mathbb{K}=\mathbb{C}$, then, by the Bohnenblust–Karlin theorem [13, Theorem 4.1], we have $n(\mathcal{P}(A), p_A) \geq n(A,\mathbf{1}) \geq 1/e$. ∎

If A is a unital C^*-algebra, then $n(A,\mathbf{1})$ is equal to 1 or $1/2$ depending on whether or not A is commutative [19]. If $A = \ell_1(G)$ for some group G, then it is easily checked that $n(A,\mathbf{1}) = 1$. If A is either the real algebra underlying $\mathbb{C}$ or the algebra of Hamilton's quaternions, then we have $n(A,\mathbf{1}) = 0$ (since in this case A is a real Hilbert space different from $\mathbb{R}$). We do not know other values of $n(A,\mathbf{1})$ when A is a unitary Banach algebra.

4. Dentability of balls

Lemma 4.1. *Let X be a normed space, and let $\mathcal{G}$ be a subgroup of $\mathcal{G}_X$. Then the set of all $\mathcal{G}$-big points of X is closed in X.*

Proof. Let y be in the closure of the set of all $\mathcal{G}$-big points of X, let x be in S_X, and let $\varepsilon > 0$. Then there exists a $\mathcal{G}$-big point e of X with $\|e-y\| < \varepsilon/2$, and $\alpha_1,\ldots,\alpha_n \in \mathbb{R}_0^+$ and $T_1,\ldots,T_n \in \mathcal{G}$ satisfying $\sum_{i=1}^n \alpha_i = 1$ and $\|x - \sum_{i=1}^n \alpha_i T_i(e)\| < \varepsilon/2$. Putting $z := \sum_{i=1}^n \alpha_i T_i(y)$, we have $z \in \mathrm{co}(\mathcal{G}(y))$ and $\|x-z\| < \varepsilon$. Thus $x \in \overline{\mathrm{co}}(\mathcal{G}(y))$, so y is a $\mathcal{G}$-big point. ∎

Let X be a normed space, and e an element in X. We say that e is a *denting* (respectively, *quasi-denting*) *point* of B_X if e belongs to B_X and, for every $\varepsilon > 0$, e does not belong to $\overline{\mathrm{co}}(B_X \setminus (e + \varepsilon B_X))$ (respectively, $\overline{\mathrm{co}}(B_X \setminus (e + \varepsilon B_X))$ is not equal to B_X). Denting points of B_X do exist in abundance if X is a Banach space with the Radon–Nikodym property (in particular, if X is reflexive).

In Remark 3.9(a) we noted that, if A is a unitary Banach algebra, and if h is in S_{A^*}, then 1 belongs to the closure of $h(U_A)$ in $\mathbb{K}$. This is nothing but a very particular case of the following lemma.

LEMMA 4.2. *Let X and Y be normed spaces, let C be a subset of X whose closed convex hull is B_X, let e be a quasi-denting point of B_Y, and let $F : X \to Y$ be a metric surjection. Then e belongs to the closure of $F(C)$ in Y.*

Proof. Assume that e does not belong to $\overline{F(C)}$. Then there exists $\varepsilon > 0$ satisfying $F(C) \subseteq Y \setminus (e + \varepsilon B_Y)$. Since F is contractive and $B_X = \overline{\mathrm{co}}(C)$, we have in fact $F(B_X) \subseteq \overline{\mathrm{co}}(B_Y \setminus (e + \varepsilon B_Y))$. But, since F is actually a metric surjection, the equality $\mathrm{int}(B_Y) = F(\mathrm{int}(B_X))$ holds. It follows $B_Y = \overline{\mathrm{co}}(B_Y \setminus (e + \varepsilon B_Y))$, which is not possible because e is a quasi-denting point of B_Y. ■

Let X be a normed space. A *slice* in X is a set of the form

$$\{x \in B_X : \Re e(f(x)) > \alpha\}$$

for some f in S_{X^*} and $\alpha < 1$. We say that B_X is *dentable* if for every $\varepsilon > 0$ there exists a slice in X with diameter less than ε. We note that, if there exists some quasi-denting point e of B_X, then B_X is dentable. Indeed, given $\varepsilon > 0$, there exists $z \in B_X \setminus [\overline{\mathrm{co}}(B_X \setminus (e + (\varepsilon/3)B_X))]$, so that, by the Hahn–Banach theorem, there exists $f \in S_{X^*}$ such that

$$\alpha := \sup\{\Re e(f(x)) : x \in B_X \setminus (e + (\varepsilon/3)B_X)\} < \Re e(f(z)) \leq 1,$$

and therefore the slice $\{x \in B_X : \Re e(f(x)) > \alpha\}$ is contained in $e + (\varepsilon/3)B_X$, and hence has diameter less than ε. We denote by d_X, q_X, and b_X the set of denting points of B_X, quasi-denting points of B_X, and big points of X, respectively. Moreover, given a subgroup $\mathcal{G}$ of $\mathcal{G}_X$, we denote by $\mathcal{G}$-b_X the set of all $\mathcal{G}$-big points of X, and we remark that $\mathcal{G}$-b_X is $\mathcal{G}$-invariant.

Let $\mathcal{G}$ be a group acting on a set E. We say that $\mathcal{G}$ acts transitively on E if, for every $e \in E$, we have $\mathcal{G}(e) = E$. When E is in fact a topological space, we say that $\mathcal{G}$ acts almost transitively on E if, for every $e \in E$, $\mathcal{G}(e)$ is dense in E.

PROPOSITION 4.3. *Let X be a normed space such that b_X is non-empty. Then the following assertions are equivalent*:

(1) *d_X is non-empty.*
(2) *q_X is non-empty.*
(3) *B_X is dentable.*

Moreover, if B_X is dentable and $\mathcal{G}$ is a subgroup of $\mathcal{G}_X$ such that $\mathcal{G}$-b_X is non-empty, then

$$d_X = q_X = \mathcal{G}\text{-}b_X = b_X,$$

and $\mathcal{G}$ acts almost transitively on b_X.

Proof. Let $\mathcal{G}$ be a subgroup of $\mathcal{G}_X$ such that $\mathcal{G}$-b_X is non-empty. Let us fix a $\mathcal{G}$-big point e of X. Then we have

$$d_X \subseteq q_X \subseteq \overline{\mathcal{G}(e)} \subseteq \mathcal{G}\text{-}b_X \subseteq b_X.$$

Indeed, the first and last inclusions are clear, the second follows from Lemma 4.2, and the third holds because $\mathcal{G}$-b_X is closed in X (by Lemma 4.1) and $\mathcal{G}$-invariant. Moreover, we know that B_X is dentable whenever q_X is non-empty. Thus, to conclude the proof it is enough to show that, if B_X is dentable, then the inclusion $b_X \subseteq d_X$ holds.

Suppose that B_X is dentable. Let b be a big point of X, and let $\varepsilon > 0$. Then there is a slice S in X with diameter less than ε. Since $B_X \setminus S$ is a convex, closed, and proper subset of B_X, and $\overline{\mathrm{co}}(\mathcal{G}_X(b)) = B_X$, there must exist T in $\mathcal{G}_X$ such that $T(b)$ belongs to S. Then $S' := T^{-1}(S)$ is a slice in X containing b and whose diameter is less than ε. Now we have $b \in S' \subseteq b + \varepsilon B_X$, and hence $b \notin B_X \setminus S' \supseteq \overline{\mathrm{co}}[B_X \setminus (b + \varepsilon B_X)]$ because $B_X \setminus S'$ is closed and convex. From the arbitrariness of ε we deduce that b is in d_X. ∎

The following corollary becomes a partial converse to Corollary 3.11.

Corollary 4.4. *Let X be a Banach space satisfying $N(X) = 1$ and such that B_X is dentable, and let e be a big point of X. Then $n(X, e) = 1$.*

Proof. By Proposition 4.3, e is a denting point of B_X. Since $N(X) = 1$, it follows from [38, Lemma 1] that $|f(e)| = 1$ for every extreme point of B_{X^*}. Now let x be an arbitrary element of S_X. Since $D(X, x)$ is a w^*-closed face of B_{X^*}, there exists an extreme point g of B_{X^*} which lies in $D(X, x)$. By the above, there is α in $S_{\mathbb{K}}$ such that αg belongs to $D(X, e)$. It follows that

$$1 = g(x) = |\alpha g(x)| \leq v(X, e, x). \ ∎$$

Let A be a norm-unital normed algebra, and let $\mathcal{G}$ denote the subgroup of $\mathcal{G}_A$ given by $\mathcal{G} := \{a \mapsto au : u \in U_A\}$. Then U_A is $\mathcal{G}$-invariant, and $\mathcal{G}$ acts transitively on U_A (indeed, for v, w in U_A we have $w = vu$ with $u := v^{-1}w \in U_A$). It follows that, if U_A is $\mathcal{G}_A$-invariant, then $\mathcal{G}_A$ acts transitively on U_A. Now note that U_A is closed whenever A is complete, and that, since $U_A = \mathcal{G}(\mathbf{1})$, A is unitary if and only if $\mathbf{1}$ is a $\mathcal{G}$-big point of A. If we keep in mind these comments, the next theorem follows straightforwardly from Proposition 4.3.

Theorem 4.5. *Let A be a unitary Banach algebra. Then the following assertions are equivalent*:

(1) $\mathbf{1}$ *is a denting point of* B_A.
(2) d_A *is non-empty.*
(3) q_A *is non-empty.*
(4) B_A *is dentable.*

Moreover, *if* B_A *is dentable*, *then*

$$d_A = q_A = U_A = b_A,$$

and $\mathcal{G}_A$ *acts transitively on* U_A. ■

Let X be a Banach space. We say that X^* has *Mazur's w^*-intersection property* if every bounded w^*-closed convex subset of X^* can be expressed as an intersection of closed balls in X^*. It is well known that X^* has Mazur's w^*-intersection property whenever the norm of X^* is Fréchet differentiable at every point of S_{X^*}.

The converse is no longer true. Indeed, X is reflexive when the norm of X^* is Fréchet differentiable at every point of S_{X^*} [22, Corollary 1, p. 34], whereas, for arbitrary X, there exists a Banach space Y containing X isometrically and such that Y^* has Mazur's w^*-intersection property [33, Proposition 2.9].

We denote by $\mathbb{H}$ the real algebra of Hamilton's quaternions, with norm equal to its usual module function.

Corollary 4.6. *Let A be a unitary Banach algebra over $\mathbb{K}$. Then the following assertions are equivalent*:

(1) *The norm of A^* is Fréchet differentiable at every point of S_{X^*}.*
(2) *A^* has Mazur's w^*-intersection property.*
(3) $A = \mathbb{C}$ (*respectively*, $A = \mathbb{R}$, $\mathbb{C}$, *or* $\mathbb{H}$) *if* $\mathbb{K} = \mathbb{C}$ (*respectively*, *if* $\mathbb{K} = \mathbb{R}$).

Proof. In view of the above comment and the triviality of (3)⇒(1), it is enough to prove that (2) implies (3). Assume that A satisfies (2). By [27, Theorem 3.1], the set of all denting points of B_A is dense in S_A. Then, by Theorem 4.5, we have $U_A = S_A$. This implies that A is a division algebra and that $\|a\| = r(a)$ for every a in A, where $r(a) := \lim_{n\to\infty}\{\|a^n\|^{1/n}\}$ is the spectral radius. In view of these properties and [15, Theorems 14.2 and 14.7], there exists an isometric algebra isomorphism from A onto $\mathbb{A}$, where $\mathbb{A} = \mathbb{C}$ if $\mathbb{K} = \mathbb{C}$, $\mathbb{A} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ if $\mathbb{K} = \mathbb{R}$, and in any case the norm of $\mathbb{A}$ is its usual modulus function. ■

Remark 4.7. (a) In general, for a Banach space X, quasi-denting points of B_X need not be denting points of B_X. In fact, for an arbitrary Banach space Y, all elements of S_Y can be converted into quasi-denting points of B_X

for a suitable isometric enlargement X of Y. This follows from references [33, Proposition 2.9] and [27, Theorem 3.1], and the easy fact that the set of all quasi-denting points of the closed unit ball of every normed space is closed in the space. Now choose the Banach space Y in such a way that there exists an element e in S_Y which is not a denting point of B_Y, and let X be an isometric enlargement of Y with the property quoted above. Then e is a quasi-denting point of B_X, but is not a denting point of B_X (since the denting point property is inherited by subspaces). Choosing Y two-dimensional, and looking at the arguments in [33], we realize that the Banach space X above can be chosen 3-dimensional.

(b) Let A be a unitary Banach algebra. According to Theorem 4.5, B_A is dentable if and only if the unit of A is a denting point of B_A. In the case $A = \ell_1(G)$ for some group G it is easy to see that B_A is dentable, but in general B_A need not be dentable. For example, this is the case when $A = \mathcal{B}(H)$ for an arbitrary infinite-dimensional complex Hilbert space [29]. More generally, the closed unit ball of every infinite-dimensional C^*-algebra is not dentable [4]. When B_A is dentable, Theorem 4.5 provides us with three geometric characterizations of unitary elements of A.

(c) Let A be a norm-unital complex Banach algebra. Then every unitary element of A is a vertex of B_A. Indeed, $\mathbf{1}$ is vertex of B_A (by the Bohnenblust–Karlin theorem), and U_A is contained in the orbit of $\mathbf{1}$ under $\mathcal{G}_A$ (by the comment preceding Theorem 4.5). It is a tempting conjecture that, when A is actually unitary, vertices of B_A and unitary elements of A coincide. The conjecture is right in the case where A is either a unital C^*-algebra [12, Example 4.1] or of the form $\ell_1^{\mathbb{C}}(G)$ for some group G.

(d) In general, for a unitary Banach algebra A, one cannot expect $\mathcal{G}_A$ to act transitively (nor even almost transitively) on b_A. To see this, first note that every norm-unital Banach algebra different from $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ fails to be almost transitive as a Banach space (see the comments after [6, Corollary 2.6]). Now recall that a normed space X is called *convex-transitive* if $S_X = b_X$. It follows that $\mathcal{G}_A$ does not act almost transitively on b_A whenever A is a convex-transitive (in the sense of Banach spaces) infinite-dimensional norm-unital Banach algebra. Possible choices of unitary Banach algebras A in such a situation are $A = C^{\mathbb{C}}(K)$ for K equal to either $S_{\mathbb{C}}$ or the Cantor set, $A = L_\infty^{\mathbb{C}}([0,1])$ [9, Example 2.32], and the Calkin algebra $A = \mathcal{B}(H)/\mathcal{K}(H)$ for H equal to the infinite-dimensional separable complex Hilbert space [7, Corollary 4.6]. Now note that, for every unital C^*-algebra A, U_A is $\mathcal{G}_A$-invariant (because U_A coincides with the set of vertices of B_A), so $\mathcal{G}_A$ acts transitively on U_A (by the comment preceding Theorem 4.5), and so we have $\mathcal{G}_A(\mathbf{1}) = U_A = \overline{\mathcal{G}_A(\mathbf{1})}$. But, if A is equal to $\mathcal{B}(H)$ for H as above, then there exist big points of A which are not unitary (a consequence of

[7, Theorem 4.5]), and hence, since $\mathbf{1}$ is a big point of A, $\mathcal{G}_A$ does not act almost transitively on b_A.

(e) An alternative proof of Corollary 4.6 is the following. Let A be a unitary Banach algebra satisfying condition (2) in Corollary 4.6. If we keep in mind that A has big points, Theorem 6.8 of [9] applies, giving that the Banach space of A is almost transitive and superreflexive. Then, by [21, Corollary IV.5.7], the Banach space of A is uniformly smooth. In particular, the Banach space of A is smooth at the unit of A, and condition (3) in Corollary 4.6 follows by applying [46].

(f) It would be interesting to find an example of a (necessarily non-unitary) norm-unital Banach algebra A failing condition (3) in Corollary 4.6 but satisfying condition (1) (or at least condition (2)) in that corollary.

5. Holomorphy. Let X be a complex Banach space. A *holomorphic vector field* on $\mathrm{int}(B_X)$ is nothing but a holomorphic mapping from $\mathrm{int}(B_X)$ to X. A holomorphic vector field Λ on $\mathrm{int}(B_X)$ is said to be *complete* if, for each x in $\mathrm{int}(B_X)$, there exists a differentiable function $\varphi : \mathbb{R} \to \mathrm{int}(B_X)$ satisfying

$$\varphi(0) = x \quad \text{and} \quad \frac{d}{dt}\varphi(t) = \Lambda(\varphi(t))$$

for every t in $\mathbb{R}$. Let X_{s} denote the set of values at zero of all complete holomorphic vector fields on $\mathrm{int}(B_X)$. According to [2, Theorem 3.6], X_{s} is a closed subspace of X. Such a subspace is usually called the *symmetric part* of X. By [2, Main Lemma 4.2], the orbit of zero under the group of all biholomorphic automorphisms of the open unit ball of X coincides with the open unit ball of X_{s}. The possibility $X = X_{\mathrm{s}}$ has been deeply studied by many authors since the fundamental work of W. Kaup (see [36] and [37]), who proves that such an equality is equivalent to the fact that X is (linearly isometric to) a JB*-triple. We recall that a complex Banach space X is said to be a JB*-*triple* if it is endowed with a continuous triple product $\{\ldots\} : X \times X \times X \to X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

(1) For all x in X, the mapping $y \mapsto \{xxy\}$ from X to X is a hermitian operator on X and has non-negative spectrum.

(2) The main identity

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

holds for all a, b, x, y, z in X.

(3) $\|\{xxx\}\| = \|x\|^3$ for every x in X.

Concerning condition (1) above, we also recall that a bounded linear operator F on X is said to be *hermitian* if $V(\mathcal{B}(X), I_X, F) \subseteq \mathbb{R}$ (equivalently, if $\|\exp(irF)\| = 1$ for every r in $\mathbb{R}$ [15, Corollary 10.13]).

Now let A be a norm-unital complex Banach algebra. We say that A *satisfies the von Neumann inequality* if, for every a in B_A and every complex polynomial P, we have

$$\|P(a)\| \leq \max\{|P(z)| : z \in B_{\mathbb{C}}\}.$$

The following proposition is a summary of essentially known facts.

PROPOSITION 5.1. *For a norm-unital complex Banach algebra A, consider the following assertions*:

(1) *A is a C^*-algebra* (*for some involution* $*$).
(2) *A satisfies the von Neumann inequality.*
(3) *The holomorphic vector field $a \mapsto \mathbf{1} - a^2$ on* $\operatorname{int}(B_X)$ *is complete.*
(4) $\|(a+\lambda\mathbf{1})(\mathbf{1}+\overline{\lambda}a)^{-1}\| < 1$ *for every* $\lambda \in \operatorname{int}(B_{\mathbb{C}})$ *and every* $a \in \operatorname{int}(B_A)$.
(5) *There exists* $\mu \in \operatorname{int}(B_{\mathbb{C}}) \setminus \{0\}$ *such that* $\|(a \pm \mu\mathbf{1})(\mathbf{1} \pm \overline{\mu}a)^{-1}\| < 1$ *for every* $a \in \operatorname{int}(B_A)$.
(6) $\mathbf{1}$ *belongs to the symmetric part* A_{s} *of* A.
(7) $\|\mathbf{1} + A_{\mathrm{s}}\| < 1$.

Then (1)$\Rightarrow$(2)$\Leftrightarrow$(3)$\Leftrightarrow$(4)$\Rightarrow$(5)$\Rightarrow$(6)$\Rightarrow$(7).

Proof. The implications (4)$\Rightarrow$(5) and (6)$\Rightarrow$(7) are clear.

(1)$\Rightarrow$(2). This is a celebrated theorem of von Neumann (see for example [42, Section 153]).

(2)$\Leftrightarrow$(3)$\Leftrightarrow$(4). This is the main result in [3].

(5)$\Rightarrow$(6). For λ in $\operatorname{int}(B_{\mathbb{C}})$, consider the holomorphic mapping $g_\lambda : \operatorname{int}(B_A) \mapsto A$ defined by $g_\lambda(a) := (a+\lambda\mathbf{1})(\mathbf{1}+\overline{\lambda}a)^{-1}$ for every a in $\operatorname{int}(B_A)$. If for some $a \in \operatorname{int}(B_A)$ and some $\lambda \in \operatorname{int}(B_{\mathbb{C}})$ we have $\|g_\lambda(a)\| < 1$, then one easily checks that the equality $g_{-\lambda}(g_\lambda(a)) = a$ holds. It follows that, if assertion (5) is true for A, then g_μ becomes a biholomorphic automorphism of $\operatorname{int}(B_A)$ satisfying $g_\mu(0) = \mu\mathbf{1}$. Since $\mu \neq 0$, we deduce that $\mathbf{1}$ lies in A_{s}. ∎

The implication (1)$\Rightarrow$(2) in Proposition 5.1 is not reversible. Indeed, the disc algebra satisfies the von Neumann identity and cannot be a C^* algebra for any involution. On the other hand, it is worth mentioning that the reversibility of the implication (3)$\Rightarrow$(6) in Proposition 5.1 is raised as an open problem in [3].

THEOREM 5.2. *Let A be a norm-unital complex Banach algebra. Then A is a C^*-algebra if and only if A is unitary and one of the assertions* (2)–(7) *in Proposition* 5.1 *is true for A.*

Proof. In view of Proposition 5.1, it is enough to show that, if A is unitary, and if $\|\mathbf{1} + A_{\mathrm{s}}\| < 1$, then A is a C^*-algebra. Put $\varrho := \|\mathbf{1} + A_{\mathrm{s}}\|$. Then, since A_{s} is $\mathcal{G}_A$-invariant [3, Proposition 1.2], and right multiplications by unitary elements of A are elements of $\mathcal{G}_A$, we have $\|u + A_{\mathrm{s}}\| = \varrho$ for every u in U_A.

Suppose that A is unitary. Then, since $\overline{\mathrm{co}}(U_A) = B_A$, and the set $\{x \in A : \|x + A_{\mathrm{s}}\| \le \varrho\}$ is closed and convex, it follows from the above that $\|a + A_{\mathrm{s}}\| \le \varrho$ for every a in B_A. Assume additionally that $\varrho < 1$. Then, by Riesz's lemma, we have $A_{\mathrm{s}} = A$. Therefore A_{s} is linearly isometric to a JB*-triple, and hence, by [35, Corollary 3.4], A is a C^*-algebra. ∎

Remark 5.3. (a) According to Theorem 5.2, a norm-unital complex Banach algebra is a C^*-algebra if (and only if) it is unitary and satisfies the von Neumann inequality. It is worth mentioning that this part of Theorem 5.2 is implicitly contained in [3]. Indeed, if A is a norm-unital complex Banach algebra satisfying the von Neumann inequality, then, by [3, Theorem 2.6], A_{s} is a C^*-algebra for the restrictions of the norm and the product of A, and contains the unit of A. But, if in addition A is unitary, then one easily sees that $A_{\mathrm{s}} = A$ (since A_{s} is $\mathcal{G}_A$-invariant [3, Proposition 1.2], and right multiplications by unitary elements of A are elements of $\mathcal{G}_A$).

(b) The condition $A_{\mathrm{s}} \neq 0$ on a unitary complex Banach algebra A does not imply that A is a C^*-algebra. Indeed, if we take $A := \ell_1^{\mathbb{C}}(\mathbb{Z}_2) \oplus_\infty \mathbb{C}$, then A is a unitary complex Banach algebra (by Proposition 2.8), is not a C^*-algebra, and $A_{\mathrm{s}} \supseteq \{0\} \times \mathbb{C}$ (since $\mathrm{int}(B_{\ell_1^{\mathbb{C}}(\mathbb{Z}_2)\oplus_\infty\mathbb{C}}) = \mathrm{int}(B_{\ell_1^{\mathbb{C}}(\mathbb{Z}_2)}) \times \mathrm{int}(B_{\mathbb{C}})$).

(c) For a norm-unital complex Banach algebra A, consider the assertions:

(i) A is unitary.
(ii) A satisfies the von Neumann inequality.

We know that (i) + (ii) is equivalent to the fact that A is a C^*-algebra, and that neither (i) nor (ii) implies that A is a C^*-algebra. In the case of $A = \mathcal{B}(X)$ for some complex Banach space X, assertion (ii) alone is equivalent to the fact that A is a C^*-algebra [24], whereas the same conclusion, with (i) instead of (ii), remains an open problem. Some partial answers to this problem can be found in [10].

Acknowledgements. The authors are grateful to H. G. Dales, A. Kaidi, A. Moreno Galindo, J. C. Navarro, and M. V. Velasco for their interesting remarks concerning the matter of the paper.

References

[1] C. Aparicio, F. Ocaña, R. Payá and Á. Rodríguez Palacios, *A non-smooth extension of Fréchet differentiability of the norm with applications to numerical ranges*, Glasgow Math. J. 28 (1986), 121–137.

[2] J. Arazy, *An application of infinite dimensional holomorphy to the geometry of Banach spaces*, in: Geometrical Aspects of Functional Analysis, J. Lindenstrauss and V. D. Milman (eds.), Lecture Notes in Math. 1267, Springer, Berlin, 1980, 122–150.

[3] —, *Isometries of Banach algebras satisfying the von Neumann identity*, Math. Scand. 74 (1994), 137–151.

[4] J. Becerra Guerrero, G. López, and Á. Rodríguez Palacios, *Relatively weakly open sets in closed balls of C^*-algebras*, J. London Math. Soc., to appear.

[5] J. Becerra Guerrero and Á. Rodríguez Palacios, *The geometry of convex transitive Banach spaces*, Bull. London Math. Soc. 31 (1999), 323–331.

[6] —, —, *Isometric reflections on Banach spaces after a paper of A. Skorik and M. Zaidenberg*, Rocky Mountain J. Math. 30 (2000), 63–83.

[7] —, —, *Transistivity of the norm on Banach spaces having a Jordan structure*, Manuscripta Math. 102 (2000), 111–127.

[8] —, —, *Characterizations of almost transitive superreflexive Banach spaces*, Comment. Math. Univ. Carolin. 42 (2001), 629–636.

[9] —, —, *Transitivity of the norm on Banach spaces*, Extracta Math. 17 (2002), 1–58.

[10] J. Becerra Guerrero, Á. Rodríguez Palacios, and G. Wood, *Banach spaces whose algebras of operators are unitary*: *a holomorphic approach*, Bull. London Math. Soc. 35 (2003), 218–224.

[11] S. K. Berberian, *Lectures in Functional Analysis and Operator Theory*, Grad. Texts in Math. 15, Springer, New York, 1974.

[12] H. F. Bohnenblust and S. Karlin, *Geometrical properties of the unit sphere of a Banach algebra*, Ann. of Math. 62 (1955), 217–229.

[13] F. F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, Cambridge Univ. Press, Cambridge, 1971.

[14] —, —, *Numerical Ranges II*, Cambridge Univ. Press, Cambridge, 1973.

[15] —, —, *Complete Normed Algebras*, Springer, Berlin, 1973.

[16] G. Choquet, *Cours d'analyse*, *Tome II*: *Topologie*, Masson, Paris, 1964.

[17] E. R. Cowie, *Isometries in Banach algebras*, Ph.D. thesis, Swansea, 1981.

[18] —, *An analytic characterization of groups with no finite conjugacy classes*, Proc. Amer. Math. Soc. 87 (1983), 7–10.

[19] M. J. Crabb, J. Duncan, and C. M. McGregor, *Characterizations of commutativity for C^*-algebras*, Glasgow Math. J. 15 (1974), 172–175.

[20] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Math. Soc. Monogr. (N.S.) 24, Oxford Science Publications, Clarendon Press, Oxford, 2000.

[21] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman Monogr. Surveys Pure Appl. Math. 64, Wiley, New York, 1993.

[22] J. Diestel, *Geometry of Banach Spaces. Selected Topics*, Lecture Notes in Math. 48, Springer, Berlin, 1975.

[23] J. Duncan, Review of [39], Math. Rev. 87e:46067.

[24] C. Foiaş, *Sur certains théorèmes de J. von Neumann concernant les ensembles spectraux*, Acta Sci. Math. 18 (1957), 15–20.

[25] C. Franchetti and R. Payá, *Banach spaces with strongly subdifferentiable norm*, Boll. Un. Mat. Ital. B (7) 7 (1993), 45–70.

[26] J. R. Giles, D. A. Gregory, and B. Sims, *Geometrical implications of upper semi-continuity of the duality mapping of a Banach space*, Pacific J. Math. 79 (1978), 99–109.

[27] —, —, —, *Characterisation of normed linear spaces with Mazur's intersection property*, Bull. Austral. Math. Soc. 18 (1978), 105–123.

[28] D. A. Gregory, *Upper semi-continuity of subdifferential mappings*, Canad. Math. Bull. 23 (1980), 11–19.

[29] R. Grząślewicz and P. Scherwentke, *On strongly extreme and denting points in $\mathcal{L}(H)$*, Math. Japon. 41 (1995), 283–284.

[30] H. Hanche-Olsen and E. Størmer, *Jordan Operator Algebras*, Monographs Stud. Math. 21, Pitman, Boston, 1984.

[31] M. L. Hansen and R. V. Kadison, *Banach algebras with unitary norms*, Pacific J. Math. 175 (1996), 535–552.

[32] J. M. Isidro, W. Kaup, and Á. Rodríguez Palacios, *On real forms of* JB**-triples*, Manuscripta Math. 86 (1995), 311–335.

[33] M. Jiménez Sevilla and J. P. Moreno, *Renorming Banach spaces with the Mazur intersection property*, J. Funct. Anal. 144 (1997), 486–504.

[34] A. Kaidi, A. Morales Campoy, and A. Rodriguez Palacios, *Geometrical properties of the product of a* C^**-algebra*, Rocky Mountain J. Math. 31 (2001), 197–213.

[35] —, —, —, *A holomorphic characterization of* C^**- and* JB**-algebras*, Manuscripta Math. 104 (2001), 467–478.

[36] W. Kaup, *Algebraic characterization of symmetric complex Banach manifolds*, Math. Ann. 228 (1977), 39–64.

[37] —, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. 183 (1983), 503–529.

[38] G. López, M. Martín, and R. Payá, *Real Banach spaces with numerical index 1*, Bull. London Math. Soc. 31 (1999), 207-212.

[39] J. Martínez-Moreno, J. F. Mena-Jurado, R. Payá-Albert, and Á. Rodríguez-Palacios, *An approach to numerical ranges without Banach algebra theory*, Illinois J. Math. 29 (1985), 609–626.

[40] C. M. McGregor, *Operator norms determined by their numerical ranges*, Proc. Edinburgh Math. Soc. 17 (1971), 249–255.

[41] T. W. Palmer, *Banach Algebras and the General Theory of* $*$*-Algebras II*, Cambridge Univ. Press, Cambridge, 2001.

[42] F. Riesz and B. Sz.-Nagy, *Leçons d'analyse fonctionnelle*, Gauthier-Villars, Paris, 1968.

[43] Á. Rodríguez Palacios, *A Vidav–Palmer theorem for Jordan* C^**-algebras and related topics*, J. London Math. Soc. 22 (1980), 318–332.

[44] —, *A numerical range characterization of uniformly smooth Banach spaces*, Proc. Amer. Math. Soc. 129 (2001), 815–821.

[45] W. Rudin, *Fourier Analysis on Groups*, Interscience Tracts in Pure and Appl. Math. 12, Interscience, New York, 1967.

[46] I. N. Spatz, *Smooth Banach algebras*, Proc. Amer. Math. Soc. 22 (1969), 328–329.

[47] G. V. Wood, *Maximal algebra norms*, in: Contemp. Math. 321, Amer. Math. Soc., Providence, RI, 2003, 335–345.

Departamento de Matemática Aplicada
Facultad de Ciencias
Universidad de Granada
18071-Granada, Spain
E-mail: juliobg@ugr.es

Department of Mathematics
University of Wales Swansea
Swansea SA2 8PP, Wales, UK
E-mail: sr.cowell@virgin.net
G.V.Wood@swansea.ac.uk

Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071-Granada, Spain
E-mail: apalacio@ugr.es

Received September 24, 2002
Revised version September 25, 2003 (5039)