Characterization of surjective partial differential operators on spaces of real analytic functions

by

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Dedicated to Professor D. Vogt on the occasion of his 60th birthday

Abstract. Let $A(\Omega)$ denote the real analytic functions defined on an open set $\Omega \subset \mathbb{R}^n$. We show that a partial differential operator $P(D)$ with constant coefficients is surjective on $A(\Omega)$ if and only if for any relatively compact open $\omega \subset \Omega$, $P(D)$ admits (shifted) hyperfunction elementary solutions on Ω which are real analytic on ω (and if the equation $P(D)f = g$, $g \in A(\Omega)$, may be solved on ω). The latter condition is redundant if the elementary solutions are defined on $\mathrm{conv}(\Omega)$. This extends and improves previous results of Andersson, Kawai, Kaneko and Zampieri. For convex Ω, a different characterization of surjective operators $P(D)$ on $A(\Omega)$ was given by Hörmander using a Phragmén–Lindelöf type condition, which cannot be extended to the case of noncovex Ω. The paper is based on a surjectivity criterion for exact sequences of projective (DFS)-spectra which improves earlier results of Braun and Vogt, and Frerick and Wengenroth.

This paper is concerned with the basic question when

(0.1) $$P(D) : A(\Omega) \to A(\Omega) \text{ is surjective.}$$

Here $P(D)$ is a partial differential operator with constant coefficients, $\Omega \subset \mathbb{R}^n$ is an open set and $A(\Omega)$ is the space of real analytic functions on Ω.

Several methods have been developed to solve this problem: The first counterexample to (0.1) is due to Piccinini [34] who showed that the heat equation is not surjective on $A(\mathbb{R}^3)$ (see also the conjecture of De Giorgi and Cattabriga [10]). Then Hörmander [13] characterized (0.1) for convex sets Ω by means of a Phragmén–Lindelöf condition valid on the complex variety of P. Since then Hörmander's method has been adapted by several authors for further studies on this problem (Miwa [31], Andreotti and Nacinovich [3], Zampieri [42], Braun [6] and the recent series of papers of Braun, Meise and Taylor [7, 8]).

Hörmander's criterion is restricted to convex sets Ω by the use of Fourier theory. On the other hand, using hyperfunctions and so-called "good elementary solutions" for $P(D)$, Kawai [18] was able to prove (0.1) for locally hyperbolic operators on special, not necessarily convex bounded open sets Ω. The assumption of boundedness was removed by Kaneko [15] (see also Kaneko [16, 17]). Andersson [2] used a similar idea to show that locally hyperbolic operators are surjective on $A(\mathbb{R}^n)$.

While Kawai mainly ignored the (complicated) topology of $A(\Omega)$, Hörmander observed that the problem is intimately related to a "decomposition with bounds" in the kernel of $P(D)$. In the present paper we will use the theory of Proj^k-functors of Palamodov [32, 33] (see also Vogt [38]), which is an abstract version of Hörmander's observation, to prove the following characterization of (0.1) for general open sets Ω in the spirit of Kawai's work: Let $\mathfrak{B}(\Omega)$ denote the hyperfunctions on Ω and for $\delta > 0$ let

$$A_\delta(\omega) := \{f \in A(\omega) \mid \sup_{x \in \omega} |f^{(a)}(x)| \delta^{|a|}/a! < \infty\}.$$

THEOREM. *Let $\Omega \subset \mathbb{R}^n$ be open. The following statements are equivalent*:

(a) $P(D) : A(\Omega) \to A(\Omega)$ *is surjective.*

(b) (i) *For any $g \in A(\Omega)$ and any $\omega \subset\subset \Omega$ there is $f \in A(\omega)$ such that*

$$(0.2) \qquad\qquad P(D)f = g|_\omega.$$

   (ii) *$P(D)$ satisfies the following condition $(\underline{B_\Omega})$: for any $\omega \subset\subset \Omega$ there is $\widetilde{\omega} \subset\subset \Omega$ with $\widetilde{\omega} \supset\supset \omega$ such that for any $\xi \in \partial\widetilde{\omega}$ and any $\widehat{\omega} \subset\subset \Omega$ there is $F \in \mathfrak{B}(\widehat{\omega})$ such that*

$$P(D)F = \delta_\xi \quad on \ \widehat{\omega} \quad and \quad F|_\omega \in A(\omega),$$

   where δ_ξ is the point evaluation at ξ.

(c) *$P(D)$ satisfies (b)(i) and the following condition $(\overline{B_\Omega})$: for any $\omega \subset\subset \Omega$ there are $\widetilde{\omega} \subset\subset \Omega$ and $\delta > 0$ such that for any $\xi \in \Omega\setminus\widetilde{\omega}$ there is $F \in \mathfrak{B}(\Omega)$ such that*

$$P(D)F = \delta_\xi \quad on \ \Omega \quad and \quad F|_\omega \in A_\delta(\omega).$$

This result is part of our Main Theorem 2.1 where further equivalent characterizations of (0.1) are given (including the important technical condition $(\underline{A_\Omega})$).

$(\overline{B_\Omega})$ is similar to the criterion for surjectivity of $P(D)$ on nonquasi-analytic classes of ultradifferentiable functions of Roumieu type which we proved in Langenbruch [22] (see also Langenbruch [23, 25]).

$(\overline{B_\Omega})$ improves the necessary condition for (0.1) from Langenbruch [24, Theorem 1.3]. We used the latter condition to show that local hyperbolicity

of the principal part P_m of P and hyperbolicity of the localizations of P_m at ∞ are necessary for (0.1) in many situations (see Langenbruch [24, 26]).

The semiglobal solvability condition (0.2) is always satisfied if Ω is convex. Therefore, (0.1) is equivalent to $(\underline{B_\Omega})$ and to $(\overline{B_\Omega})$ if Ω is convex.

(0.2) is also redundant if the shifted elementary solutions F are defined on convex sets. This gives the following two sufficient criteria:

$P(D)$ is surjective on $A(\Omega)$ if one of the following assumptions holds:

(i) For any $\omega \subset\subset \Omega$ there is $\widetilde{\omega} \subset\subset \Omega$ with $\omega \subset\subset \widetilde{\omega}$ such that for any $\widehat{\omega} \subset\subset \Omega$ and any $\xi \in \partial\widetilde{\omega}$ there is $F \in \mathfrak{B}(\mathrm{conv}(\widehat{\omega}))$ such that

$$P(D)F = \delta_\xi \quad \text{on } \mathrm{conv}(\widehat{\omega}) \quad \text{and} \quad F|_\omega \in A(\omega).$$

(ii) $P(D)$ satisfies $(\underline{B_{\mathbb{R}^n}})$ and for any $\omega \subset\subset \Omega$ and any $\zeta \in \partial\Omega$ there are a convex neighbourhood U of $\overline{\Omega}$ and $F \in \mathfrak{B}(U)$ such that

$$P(D)F = \delta_\xi \quad \text{on } U \quad \text{and} \quad F|_\omega \in A(\omega).$$

This contains and improves the corresponding results of Andersson [2], Kawai [18], Kaneko [15, 16] and Zampieri [42]. In particular, we get the following result for operators with locally hyperbolic principal part P_m (see Section 2 for the definitions):

$P(D)$ is surjective on $A(\Omega)$ if for any $\zeta \in \partial\Omega$ and any $\Theta \in S^{n-1}$ there is $N_{\Theta,\zeta} \in S^{n-1}$ such that P_m is locally hyperbolic with respect to $N_{\Theta,\zeta}$ and

$$(\zeta + K((P_m)_\Theta, N_{\Theta,\zeta})) \cap \Omega = \emptyset,$$

where $K((P_m)_\Theta, N_{\Theta,\zeta})$ is the local propagation cone of P_m at Θ with respect to $N_{\Theta,\xi}$.

The paper is organized as follows: In the first section, a short introduction to the theory of the Proj^k-functors of Palamodov (see Palamodov [32, 33] and Vogt [38]) is given. We here prove a strong surjectivity criterion for exact sequences of projective (DFS)-spectra (see Theorem 1.5) which improves the corresponding results of Braun and Vogt [9], Frerick and Wengenroth [11] and Wengenroth [39]: We will show that a continuous linear mapping is surjective if the kernel spectrum satisfies property (P_3) (instead of the apparently stronger condition (P_2) used in *loc. cit.*) and a suitable notion of reducedness. Moreover, we obtain a priori bounds for the solutions. In our concrete situation this implies that the following is also equivalent to the surjectivity of $P(D)$ on $A(\Omega)$ (see Theorem 2.1): for any neighbourhood $V \subset \mathbb{C}^n$ of Ω there is a neighbourhood $W \subset \mathbb{C}^n$ of Ω such that $P(D)f = g$ may be solved with $f \in H(W)$ if $g \in H(V)$. Hörmander [13] proved such a priori Cauchy estimates for convex Ω using his Phragmén–Lindelöf principle.

In Section 2, the main result is stated in Theorem 2.1. We then prove the above consequences and discuss their relation to the literature and some examples. In Sections 3 and 4 we use the Grothendieck–Tillmann duality to

show that the technical conditions $(\overline{A_\Omega})$ (and $(\underline{A_\Omega})$) are necessary (and sufficient, respectively) for (0.1). These are needed to prove the hyperfunction criteria $(\underline{B_\Omega})$ and $(\overline{B_\Omega})$ for (0.1) in Section 5.

1. Projective (DFS)**-spectra.** We start with some useful notations and conventions: In this paper, $n \in \mathbb{N}$ is always at least 2. The sets Ω and ω are always open in $\mathbb{R}^n$. We also assume that Ω is connected.

The real analytic functions on Ω are denoted by $A(\Omega)$. By $P(D)$ we always mean a partial differential operator in n variables with constant coefficients. We are concerned with the question when

(1.1) $P(D) : A(\Omega) \to A(\Omega)$ is surjective.

In this paper, we will extend real analytic functions on Ω to harmonic functions in $n+1$ variables defined near Ω in $\mathbb{R}^{n+1}$ (rather than to holomorphic functions defined near Ω in $\mathbb{C}^n$). To be precise, a point in $\mathbb{R}^{n+1}$ is usually written as $(x, y) \in \mathbb{R}^n \times \mathbb{R}$. As usual, $\Delta = \sum_{k \leq n} (\partial/\partial x_k)^2 + (\partial/\partial y)^2$ denotes the Laplace operator on $\mathbb{R}^{n+1}$. The harmonic germs near a set $S \subset \mathbb{R}^{n+1}$ are denoted by $C_\Delta(S)$. It is clear that (1.1) holds if and only if

(1.2) $P(D_x) : C_\Delta(\Omega \times \{0\}) \to C_\Delta(\Omega \times \{0\})$ is surjective.

Indeed, the equivalence of (1.1) and (1.2) is obtained by means of the Cauchy problem for Δ with data on $\Omega \times \{0\}$. $C_\Delta(\Omega \times \{0\})$ is the projective limit of the projective spectrum

$$C_\Delta^\Omega := (C_\Delta(K_k \times \{0\}), R_j^k),$$

where $\{K_k \mid k \in \mathbb{N}\}$ is an increasing compact exhaustion of Ω, that is,

$$K_k \subset\subset \operatorname{int}(K_{k+1}) \quad \text{for each } k \quad \text{and} \quad \bigcup_{k=1}^{\infty} K_k = \Omega.$$

The linking maps

$$R_j^k : C_\Delta(K_j \times \{0\}) \to C_\Delta(K_k \times \{0\}) \quad \text{for } j \geq k$$

are defined by restriction.

The theory of projective spectra of linear spaces and the corresponding Proj^k-functors were developed by Palamodov [32, 33] (see also Vogt [38]). We will briefly introduce the corresponding notions and facts which we need. The reader is referred to these papers for further information.

For $S \subset \mathbb{R}^{n+1}$ let

$$N_P(S) := \{f \in C_\Delta(S) \mid P(D_x)f = 0\}$$

and let
$$N_P^\Omega := (N_P(K_k \times \{0\}), R_j^k)$$
be the projective spectrum of kernels of $P(D_x)$. We thus have the short sequence of projective spectra

(1.3) $$0 \to N_P^\Omega \to C_\Delta^\Omega \xrightarrow{P(D)} C_\Delta^\Omega \to 0.$$

It is called *exact* if for any $k \in \mathbb{N}$ there is $j \geq k$ such that

(1.4) $$P(D)C_\Delta(K_k \times \{0\}) \supset R_j^k(C_\Delta(K_j \times \{0\})).$$

We now have the following key result, which is essentially Theorem 5.1 of Vogt [38] in our concrete situation.

PROPOSITION 1.1. *Let the sequence* (1.3) *of projective spectra be exact. Then*

(1.1) $$P(D) : A(\Omega) \to A(\Omega) \text{ is surjective}$$

if and only if $\mathrm{Proj}^1(N_P^\Omega) = 0$.

Proof. Necessity. Since (1.3) is exact, we have the exact sequence of linear spaces

(1.5) $$0 \to \mathrm{Proj}^0(N_P^\Omega) \to \mathrm{Proj}^0(C_\Delta^\Omega) \xrightarrow{P(D)} \mathrm{Proj}^0(C_\Delta^\Omega)$$
$$\to \mathrm{Proj}^1(N_P^\Omega) \to \mathrm{Proj}^1(C_\Delta^\Omega) \to \mathrm{Proj}^1(C_\Delta^\Omega) \to 0$$

by Palamodov [32, p. 542]. We can identify $\mathrm{Proj}^0(N_P^\Omega)$ with $N_P(\Omega \times \{0\})$ (and $\mathrm{Proj}^0(C_\Delta^\Omega)$ with $C_\Delta(\Omega \times \{0\})$). Since $A(\Omega)$ (and hence $C_\Delta(\Omega \times \{0\})$) is ultrabornological by Martineau [29], Theorem 3.5 of Wengenroth [39] implies that

(1.6) $$\mathrm{Proj}^1(C_\Delta^\Omega) = 0.$$

By (1.5) and (1.6) we have the exact sequence of linear spaces

$$0 \to N_p(\Omega \times \{0\}) \to C_\Delta(\Omega \times \{0\}) \xrightarrow{P(D)} C_\Delta(\Omega \times \{0\}) \to \mathrm{Proj}^1(N_P^\Omega) \to 0.$$

Since $P(D)(C_\Delta(\Omega \times \{0\})) = C_\Delta(\Omega \times \{0\})$ by the assumption and (1.2), this implies that $\mathrm{Proj}^1(N_P^\Omega) = 0$.

Sufficiency. By (1.5) we get the exact sequence of linear spaces

$$0 \to \mathrm{Proj}^0(N_P^\Omega) \to \mathrm{Proj}^0(C_\Delta^\Omega) \xrightarrow{P(D)} \mathrm{Proj}^0(C_\Delta^\Omega) \to 0$$

since $\mathrm{Proj}^1(N_P^\Omega) = 0$ by assumption. The claim now follows by the above identification. ∎

The exactness of the sequence (1.3) of spectra means that the equation $P(D)f = g$ can be solved semiglobally in $C_\Delta(\Omega)$. If Ω is convex, this semiglobal solvability easily follows from the solvability theory of (overdetermined) systems of partial differential equations. For general open sets Ω,

however, it must be proved before Proposition 1.1 can be used (see Proposition 3.3).

The reader is referred to Palamodov [32, 33] and Vogt [38] for the definition of the Proj^1-functor. We do not need the definition here since we will only use explicit criteria for the vanishing of the Proj^1-functor of projective (DFS)-spectra (see Theorems 1.2 and 1.5 below). We briefly introduce the corresponding notions and results in the form needed in this paper: Let $\mathfrak{X} = (X_k, R_j^k)$ be a projective (DFS)-spectrum, that is, a projective spectrum of (DFS)-spaces $X_k = \mathrm{lim\,ind}_{j \to \infty} X_{k,j}$ with Banach spaces $X_{k,j}$ and compact inclusion mappings from $X_{k,j}$ into $X_{k,j+1}$. Let $B_{k,j}$ be the unit ball in $X_{k,j}$. For $X := \mathrm{lim\,proj}_{k \to \infty} X_k$ let $R^k : X \to X_k$ be the canonical mapping.

A characterization of $\mathrm{Proj}^1(\mathfrak{X}) = 0$ has been obtained by Retakh [35]. We only need the necessity of a part of his criterion here:

THEOREM 1.2 (Retakh [35, Theorem 3]). *Let $\mathfrak{X}$ be a projective (DFS)-spectrum. If $\mathrm{Proj}^1(\mathfrak{X}) = 0$, then for any $k \in \mathbb{N}$ there is $j \in \mathbb{N}$ with $j \geq k$ such that*

$$R_j^k(X_j) \subset R^k(X) + B_{k,j}.$$

In our concrete situation we have the following:

$$\mathfrak{X} := N_P^{\Omega} = (N_P(K_k \times \{0\}), R_j^k)$$

and

$$N_P(K_k \times \{0\}) := \lim_{j \to \infty} \mathrm{ind}\, NB_P((K_k \times \{0\}) + V_{1/j}),$$

where $V_\varepsilon := \{z \in \mathbb{R}^{n+1} \mid |z| < \varepsilon\}$ and for open $V \subset \mathbb{R}^{n+1}$, $NB_P(V)$ is the Banach space

$$(1.7) \qquad NB_P(V) := \{f \in N_P(V) \mid f \text{ is bounded on } V\}.$$

Our sufficient criterion for $\mathrm{Proj}^1(\mathfrak{X}) = 0$ is a variant of the results of Braun and Vogt [9], Frerick and Wengenroth [11] and Wengenroth [39] (see Theorem 1.5 below). There are two major differences as compared with these papers: Firstly, we will use condition (P_3) defined for the spectrum $\mathfrak{X} = (X_k, R_j^k)$ as follows:

$(P_3) \qquad \forall \mu \; \exists k \; \forall K \; \exists n \; \forall m \; \exists N, S : R_k^\mu(B_{k,m}) \subset S(R_K^\mu(B_{K,N}) + B_{\mu,n}).$

Notice that the order of quantifiers for K and n has been changed compared with condition (P_2) which has been used in *loc. cit.* This weaker order of quantifiers has apparently been overlooked, though the proofs of the corresponding results from [9] needed here can be applied almost without change. The use of (P_3) is essential to obtain the sufficiency of the criterion (e)(ii) of the Main Theorem 2.1.

Secondly, we will use the following notion of *sw*-reducedness: The spectrum $\mathfrak{X} = (X_k, R_j^k)$ is called *stepwise reduced* (*sw*-reduced) if

$$(1.8) \qquad \forall j \; \exists L \geq j \; \forall J \geq j, \; \forall l \; \exists k : R_L^j(X_{L,l}) \subset R_J^j(X_{J,k}) + B_{j,k}.$$

The author is indebted to P. Domański (Poznań) for suggesting this notion.

The statement in (1.8) implies that $\mathfrak{X}$ is reduced in the sense of Braun and Vogt ([9, p. 150]), that is,

$$(1.9) \quad \forall N \; \exists L \; \forall K \geq L : \text{the closure of } R_K^N(X_K) \text{ in } X_N \text{ contains } R_L^N(X_L).$$

The use of *sw*-reducedness will give Theorem 1.5 below and thus the implication (a)$\Rightarrow$(b) in our Main Theorem 2.1. The statement in 2.1(b) is important for the transition to the hyperfunction criterion in part (e)(ii) of the Main Theorem 2.1.

We start with the following result translating the crucial points from Braun and Vogt [9] to our situation:

LEMMA 1.3. *Let $\mathfrak{X} = (X_k, R_j^k)$ be reduced and satisfy* (P_3). *Assume also that*:

(a) $\mathfrak{X}$ *satisfies the following condition* $(\overline{\text{P}_3})$:

$$\forall \mu \; \exists k \; \forall K \; \exists n \; \forall m \; \forall \varepsilon > 0 \; \exists N, S : R_k^\mu(B_{k,m}) \subset SR_K^\mu(B_{K,N}) + \varepsilon B_{\mu,n}.$$

(b) *Passing to a subsequence in $\mathfrak{X}$, for each fixed $d \in \mathbb{N}$ the following condition* $(\overline{\text{P}_3})''$ *holds*:

$$\forall \mu \; \exists n \; \forall m \; \forall \varepsilon > 0 \; \exists N, S : R_\mu^{\mu-1}(B_{\mu,m}) \subset SR_{\mu+d}^{\mu-1}(B_{\mu+d,N}) + \varepsilon B_{\mu-1,n}.$$

Then there is an increasing sequence $\widetilde{n}(k) \in \mathbb{N}$ such that for any k and m there are $\widetilde{N}(k, m) \in \mathbb{N}$ and $\widetilde{S}(k, m) > 0$ such that

$$(1.10) \qquad R_k^{k-1}(B_{k,m}) \subset \widetilde{S}(k,m)R_{k+d}^{k-1}(B_{k+d,\widetilde{N}(k,m)}) + B_{k-1},$$

where

$$B_j := \bigcap_{l=1}^{j}(R_j^l)^{-1}(B_{l,\widetilde{n}(l)}).$$

Proof. (a) This follows from the proofs of Braun and Vogt [9, Lemmata 9 and 5].

(b) Passing to a subsequence, we may assume that $(\overline{\text{P}_3})$ holds with $k = \mu + 1$. This is then applied for $K := \mu + d$ to obtain $(\overline{\text{P}_3})''$.

Since (1.10) is crucial for the present paper, we include its proof here for the convenience of the reader, though it is mainly a notational variant of the corresponding proof of [9, Lemma 7]: We define $\widetilde{S}$, $\widetilde{N}$ and $\widetilde{n}$ inductively for $k \geq 2$, using $n(k)$, $S(k, m, \varepsilon)$ and $N(k, m, \varepsilon)$ from $(\overline{\text{P}_3})''$ for k instead of μ:

$$\widetilde{S}(k,m) := S(k,m,\varepsilon_k),$$

where $\varepsilon_2 := 1$ and $\varepsilon_k := 1/(2(1 + \widetilde{S}(k-1,n(k))))$ for $k \geq 3$,

$$\widetilde{N}(2,m) := N(2,m,1),$$

$$\widetilde{N}(k,m) := \begin{cases} N(k,\widetilde{N}(k-1,n(k)),\varepsilon_k) & \text{if } m < \widetilde{N}(k-1,n(k)) \\ N(k,m,\varepsilon_k) & \text{if } m \geq \widetilde{N}(k-1,n(k)) \end{cases} \quad \text{for } k \geq 3,$$

$$\widetilde{n}(1) := n(2),$$

$$\widetilde{n}(k-1) := \max(n(k), \widetilde{N}(k-1,n(k))) \quad \text{for } k \geq 3.$$

The claim for $k = 2$ then follows from $(\overline{P_3})''$. Let $k \geq 3$ and assume that the claim holds for $k - 1$. Let $u \in B_{k,m}$ be given. By $(\overline{P_3})''$ we can decompose $R_k^{k-1}(u)$:

$$(1.11) \qquad\qquad R_k^{k-1}(u) = R_{k+d}^{k-1}(v_{k-1}) + u_{k-1}$$

for some

$$(1.12) \qquad v_{k-1} \in SB_{k+d,N(k,m,\varepsilon_k)} \quad \text{and} \quad u_{k-1} \in \varepsilon_k B_{k-1,n(k)}.$$

By the induction hypothesis for $m = n(k)$ we can decompose $R_{k-1}^{k-2}(u)$:

$$(1.13) \qquad\qquad R_{k-1}^{k-2}(u_{k-1}) = R_{k-1+d}^{k-2}(v_{k-2}) + \widetilde{u}_{k-2}$$

for some

$$(1.14) \qquad v_{k-2} \in \varepsilon_k \widetilde{S}(k-1,n(k))B_{k-1+d,\widetilde{N}(k-1,n(k))} \subset \tfrac{1}{2}B_{k-1+d,\widetilde{N}(k-1,n(k))}$$

(since $\varepsilon_k \widetilde{S}(k-1,n(k)) \leq 1/2$) and some

$$(1.15) \qquad\qquad \widetilde{u}_{k-2} \in \varepsilon_k B_{k-2}.$$

Let

$$(1.16) \qquad\qquad u_{k-2} := u_{k-1} - R_{k-1+d}^{k-1}(v_{k-2}).$$

Then

$$u_{k-2} \in \varepsilon_k B_{k-1,n(k)} + \varepsilon_k \widetilde{S}(k-1,n(k))B_{k-1,\widetilde{N}(k-1,n(k))} \subset \tfrac{1}{2}B_{k-1,\widetilde{n}(k-1)}$$

by (1.12), (1.14) and the choice of $\widetilde{n}(k-1)$. Moreover,

$$R_{k-1}^{k-2}(u_{k-2}) = R_{k-1}^{k-2}(u_{k-1}) - R_{k-1+d}^{k-2}(v_{k-2}) = \widetilde{u}_{k-2} \subset \tfrac{1}{2}B_{k-2}$$

by (1.13) and (1.15), since $\varepsilon_k \leq 1/2$. Thus,

$$(1.17) \qquad\qquad u_{k-2} \in \tfrac{1}{2}B_{k-1}.$$

By (1.11), (1.16), (1.12), (1.14) and (1.15) we get

$$R_k^{k-1}(u) = R_{k-1+d}^{k-1}(v_{k-1}) + u_{k-1} = R_{k+d}^{k-1}(v_{k-1}) + R_{k-1+d}^{k-1}(v_{k-2}) + u_{k-2}$$

$$\in SR_{k+d}^{k-1}(B_{k+d,N(k,m,\varepsilon_k)}) + \tfrac{1}{2}R_k^{k-1}(B_{k-1+d,\widetilde{N}(k-1,n(k))}) + \tfrac{1}{2}B_{k-1}$$

$$\in SR_{k+d}^{k-1}(B_{k+d,N(k,m,\varepsilon_k)}) + \tfrac{1}{2}R_k^{k-1}(B_{k,m}) + \tfrac{1}{2}B_{k-1}$$

if $m \geq \widetilde{N}(k-1, n(k))$. This defines a decomposition

$$R_k^{k-1}(u) = R_{k+d}^{k-1}(x_1) + R_k^{k-1}(y_1) + w_1$$

and by induction we can find

(1.18) $\quad x_{j+1} \in 2^{-j} S B_{k+d, N(k,m,\varepsilon_k)}, y_{j+1} \in 2^{-j-1} B_{k,m}, \quad w_{j+1} \in 2^{-j-1} B_{k-1}$

such that

$$R_k^{k-1}(y_j) = R_{k+d}^{k-1}(x_{j+1}) + R_k^{k-1}(y_{j+1}) + w_{j+1} \quad \text{for } j \geq 1.$$

Since $X_{k+d, N(k,m,\varepsilon_k)}$ and $X_{k-1, \tilde{n}(k-1)}$ are Banach spaces, the series

(1.19) $\quad v := \sum_{j \geq 1} x_j \in 2 S B_{k+d, N(k,m,\varepsilon_k)} \quad \text{and} \quad w := \sum_{j \geq 1} w_j \in B_{k-1, \tilde{n}(k-1)}$

converge and we see as in [9], by (1.18), that $w \in B_{k-1}$. Then (1.19) implies that

$$R_k^{k-1}(u) = R_{k+d}^{k-1}(v) + w \in 2 S B_{k+d, N(k,m,\varepsilon_k)} + B_{k-1}.$$

This ends the induction step since we have $B_{k,m} \subset B_{k, \widetilde{N}(k-1, n(k))}$ if $m < \widetilde{N}(k-1, n(k))$. ∎

THEOREM 1.4. *If $\mathfrak{X}$ is a reduced projective* (DFS)*-spectrum with* (P_3), *then* $\mathrm{Proj}^1(\mathfrak{X}) = 0$.

Proof. Since $\mathfrak{X}$ is reduced and satisfies (P_3) we have

$$\forall j \; \exists k \; \forall K \; \exists n : R_k^j(X_k) \subset R_K^j(X_K) + B_{j,n}$$

(use the proof of [9, Lemma 5]). Passing to a subsequence, we may thus assume that $\mathfrak{X}$ satisfies

(1.20) $\qquad \forall k \; \exists n(k) : R_k^{k-1}(X_k) \subset R_{k+2}^{k-1}(X_{k+2}) + B_{k-1, n(k)}$

and also (by Lemma 1.3 for $d = 3$)

(1.21) $\qquad \forall k, m : R_k^{k-1}(B_{k,m}) \subset R_{k+3}^{k-1}(X_{k+3}) + B_{k-1}$,

where

$$B_j := \bigcap_{l=1}^j (R_j^l)^{-1}(B_{l, \tilde{n}(l)}).$$

We now proceed as in the proof of [9, Theorem 8], using (1.20) and (1.21):

$$R_k^{k-2}(X_k) \subset R_{k+2}^{k-2}(X_{k+2}) + R_{k-1}^{k-2}(B_{k-1, n(k)}) \subset R_{k+2}^{k-2}(X_{k+2}) + B_{k-2}.$$

Passing to the subsequence with even indices, for any k we thus get

$$R_k^{k-1}(X_k) \subset R_{k+1}^{k-1}(X_{k+1}) + B_{k-1}.$$

This implies that $\mathrm{Proj}^1(\mathfrak{X}) = 0$ (see the proof of Vogt [37, Theorem 4.4]). ∎

We now consider the following situation: Let

(1.22) $\qquad\qquad\qquad 0 \to \mathfrak{X} \to \mathfrak{Y} \xrightarrow{(T_k)} \mathfrak{Z} \to 0$

be an exact sequence of projective (DFS)-spectra $\mathfrak{X} = (X_k, \; R_j^k)$, $\mathfrak{Y} = (Y_k, R_j^k)$ and $\mathfrak{Z} = (Z_k, \; R_j^k)$. Let X, Y and Z be the projective limits of $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$, respectively, and let $T : Y \to Z$ be the mapping defined by (T_k).

THEOREM 1.5. *In the situation above, let $\mathfrak{X}$ be sw-reduced and satisfy* (P$_3$). *Then for any sequence $\mu(j) \in \mathbb{N}$ there is a sequence $\widetilde{\mu}(j) \in \mathbb{N}$ such that the following holds: If $g \in Z$ satisfies $g_j := R^j(g) \in Z_{j,\mu(j)}$ for any j, then there is $f \in Y$ such that*

$$T(f) = g \quad and \quad R^j(f) \in Y_{j,\widetilde{\mu}(j)} \quad for \; any \; j.$$

Proof. (a) When proving Theorem 1.5 we may pass to equivalent projective (DFS)-spectra $\widetilde{\mathfrak{X}} = (\widetilde{X}_k, \widetilde{R}_j^k)$, $\widetilde{\mathfrak{Y}} = (\widetilde{Y}_k, \widetilde{R}_j^k)$ and $\widetilde{\mathfrak{Z}} = (\widetilde{Z}_k, \widetilde{R}_j^k)$ defining the projective limits $\widetilde{X}$, $\widetilde{Y}$ and $\widetilde{Z}$ isomorphic to X, Y and Z, respectively.

(I) Suppose the statement is proved for the spectra $\widetilde{\mathfrak{X}}$, $\widetilde{\mathfrak{Y}}$ and $\widetilde{\mathfrak{Z}}$ and the mapping $\widetilde{T} : \widetilde{Y} \to \widetilde{Z}$ corresponding to T. Let

$$\varphi := (\varphi_{\lambda(j)}^j)_j : \widetilde{\mathfrak{Y}} \to \mathfrak{Y} \quad and \quad \Phi := (\Phi_{\lambda(j)}^j)_j : \mathfrak{Y} \to \widetilde{\mathfrak{Y}}$$

and

$$\psi := (\psi_{\kappa(j)}^j)_j : \widetilde{\mathfrak{Z}} \to \mathfrak{Z} \quad and \quad \Psi := (\Psi_{\kappa(j)}^j)_j : \mathfrak{Z} \to \widetilde{\mathfrak{Z}}$$

define the respective equivalence maps. Let $\mu(j)$ and $g \in Z$ be given such that $g_j := R^j(g) \in Z_{j,\mu(j)}$ for any j. By Grothendieck's lemma (Meise and Vogt [30, 24.33]) there is $\nu(j)$ such that $\Psi_{\kappa(j)}^j(R^{\kappa(j)}(g)) =: \widetilde{g}_j \in \widetilde{Z}_{j,\nu(j)}$ for any j. Now, $(\widetilde{g}_j)_j$ defines $\widetilde{g} \in \widetilde{Z}$ and $\widetilde{R}^j(\widetilde{g}) = \widetilde{g}_j \in \widetilde{Z}_{j,\nu(j)}$. Choose $\widetilde{\mu}(j)$ for $\nu(j)$ by the statement for $\widetilde{\mathfrak{X}}$, $\widetilde{\mathfrak{Y}}$ and $\widetilde{\mathfrak{Z}}$ and then $\widetilde{f} \in \widetilde{Y}$ such that $\widetilde{T}(\widetilde{f}) = \widetilde{g}$ and $\widetilde{R}^j(\widetilde{f}) \in \widetilde{Y}_{j,\widetilde{\mu}(j)}$ for any j. Again by Grothendieck's lemma there is $\widetilde{\nu}(j)$ such that

$$f_j := \varphi_{\lambda(j)}^j(\widetilde{R}^{\lambda(j)}(\widetilde{f})) \in Y_{j,\widetilde{\nu}(j)} \quad for \; any \; j.$$

Then $(f_j)_j$ defines $f \in Y$, $T(f) = g$ and $R^j(f) = f_j \in Y_{j,\widetilde{\nu}(j)}$. The claim thus holds for the spectra $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ with $\widetilde{\nu}(j)$ instead of $\widetilde{\mu}(j)$.

(II) Using the continuity of the equivalence maps and Grothendieck's lemma as in (I), one proves that (P$_3$) and *sw*-reducedness are inherited from $\mathfrak{X}$ by the equivalent (DFS)-spectrum $\widetilde{\mathfrak{X}}$.

(b) Passing to equivalent spectra, we may assume that the following statements (i)–(iv) hold:

(i) $T_j : Y_j \to Z_j$ is continuous.

Since (1.22) is an exact sequence of spectra, we may assume that

$$T_j(Y_j) \supset R_{j+1}^j(Z_{j+1}).$$

The mapping

$$T_j^{-1} \circ R_{j+1}^j : Z_{j+1} \to Y_j / \ker T_j$$

is continuous by the closed graph theorem for (DFS)-spaces. Since $Y_j/\ker T_j$ is isomorphic to $\operatorname{lim\,ind}_k(Y_{j,k}/\ker(T_j|_{Y_{j,k}}))$, Grothendieck's lemma implies that for any k there is $\widehat{k}$ such that

$$T_j^{-1} \circ R_{j+1}^j : Z_{j+1,k} \to Y_{j,\widehat{k}}/\ker(T_j|_{Y_{j,\widehat{k}}}) \text{ is continuous.}$$

Thus, we have

(ii) $\forall k \ \exists \widehat{k} : T_j(Y_{j,\widehat{k}}) \supset R_{j+1}^j(Z_{j+1,k})$.

Since (1.22) is exact at $\mathfrak{X}$, $\mathfrak{X}$ is equivalent to a spectrum of linear subspaces of (Y_j). Since (1.22) is exact at $\mathfrak{Y}$, $\mathfrak{X}$ is equivalent to the spectrum $(\ker T_k, R_j^k)$ (compare [38, remark before 1.3]). We may therefore assume that

(iii) $X_j = \ker(T_j)$ with the step spaces induced by Y_j.

Instead of the step spaces $X_{j,1}, X_{j,2}, \ldots$ for X_j we can also take the step spaces $0, \ldots, 0, X_{j,1}, X_{j,2}, \ldots$ for X_j (with j times the null space in front, similarly for Y_j). Then passing to a subsequence (in the second index), we may also assume that for any $j \geq k$ and any n,

(1.23) $$R_j^k(X_{j,n}) \subset X_{k,n} \quad \text{and} \quad R_j^k(Y_{j,n}) \subset Y_{k,n}$$

(see [9, (1)]). Passing to a subsequence (in the first index), we assume that $\mathfrak{X}$ satisfies $\overline{(P_3)}''$ for $d = 2$ and also (since $\mathfrak{X}$ is *sw*-reduced)

(1.24) $$\forall j, l \ \exists k : R_j^{j-1}(X_{j,l}) \subset R_{j+1}^{j-1}(X_{j+1,k}) + B_{j-1,k}.$$

Now Lemma 1.3 gives

(1.25) $$R_k^{k-1}(B_{k,m}) \subset \widetilde{S}(k,m) R_{k+2}^{k-1}(B_{k+2,\widetilde{N}(k,m)}) + B_{k-1}$$
$$\subset R_{k+2}^{k-1}(X_{k+2,\widetilde{N}(k,m)}) + B_{k-1},$$

where

(1.26) $$B_j := \bigcap_{l=1}^{j} (R_j^l)^{-1}(B_{l,\widetilde{n}(l)}) \subset B_{j,\widetilde{n}(j)}.$$

(iv) We have

(1.27) $$\forall j, l \ \exists \widetilde{k} : R_j^{j-2}(X_{j,l}) \subset R_{j+1}^{j-2}(X_{j+1,\widetilde{k}}) + 2^{-j} B_{j-2}.$$

Indeed, fix j and l and choose k by (1.24). Then (1.24) and (1.26) imply

$$R_j^{j-2}(X_{j,l}) = R_{j-1}^{j-2}(R_j^{j-1}(X_{j,l})) \subset R_{j-1}^{j-2}(R_{j+1}^{j-1}(X_{j+1,k}) + B_{j-1,k})$$
$$\subset R_{j+1}^{j-2}(X_{j+1,k}) + R_{j+1}^{j-2}(X_{j+1,\widetilde{N}(j-1,k)}) + B_{j-2}$$
$$\subset R_{j+1}^{j-2}(X_{j+1,\widetilde{k}}) + B_{j-2}$$

for $\widetilde{k} := \max(k, \widetilde{N}(j-1,k))$. Since $X_{j,l}$ and $X_{j+1,\widetilde{k}}$ are linear spaces, this shows (1.27).

(c) Using the reductions and results from (b), we now start to prove the theorem: Let $\mu(j)$ and $g \in Z$ be as above. By (b)(ii) there are

$$(1.28) \qquad f_j \in Y_{j,\mu_1(j)} \quad \text{such that} \quad T_j(f_j) = R^j_{j+1}(g_{j+1}) = g_j,$$

where $\mu_1(j) \geq \mu(j+1)^\wedge$ can be chosen increasing and $g_j := R_j(g)$. We now prove by induction that for $j \geq 3$ there are $\mu_2(j) \in \mathbb{N}$ with $\mu_2(j) \geq \mu_1(j+1)$ and $h_j \in X_{j,\mu_2(j)}$ such that

$$(1.29) \qquad R^{j-2}_{j+1}(f_{j+1} + h_{j+1}) - R^{j-2}_j(f_j + h_j) \in 2^{-j}B_{j-2}.$$

Let $h_3 := 0$ and $\mu_2(3) := \mu_1(4)$. If h_j is chosen, we set $\widetilde{F} := R^j_{j+1}(f_{j+1}) - f_j - h_j$ and get $\widetilde{F} \in X_j$ by (b)(iii) and (1.28) since $h_j \in X_j$. Since $f_{j+1} \in Y_{j+1,\mu_1(j+1)}$ and $h_j \in X_{j,\mu_2(j)}$ we have $\widetilde{F} \in X_{j,\mu_2(j)}$ by (1.23) since $\mu_1(j) \leq \mu_1(j+1) \leq \mu_2(j)$. By (1.27) we can thus find $\widetilde{k} =: \mu_2(j+1) \geq \mu_1(j+2)$ and $h_{j+1} \in X_{j+1,\mu_2(j+1)}$ such that

$$R^{j-2}_j(\widetilde{F}) + R^{j-2}_{j+1}(h_{j+1}) \in 2^{-j}B_{j-2}.$$

This shows (1.29).

(d) For $j \geq k \geq 2$ we have

$$R^k_j(f_j + h_j)$$
$$= R^k_{k+2}(f_{k+2} + h_{k+2}) + \sum_{l=k+2}^{j-1} R^k_{l-2}(R^{l-2}_{l+1}(f_{l+1} + h_{l+1}) - R^{l-2}_l(f_l + h_l))$$

$$\in Y_{k,\mu_2(k+2)} + \sum_{l=k+2}^{j-1} 2^{-l} R^k_{l-2}(B_{l-2}) \subset Y_{k,\mu_2(k+2)} + \Big(\sum_{l=k+2}^{j-1} 2^{-l} \Big) B_k$$

$$\subset Y_{k,\mu_2(k+2)} + \Big(\sum_{l=k+2}^{j-1} 2^{-l} \Big) B_{k,\nu(k)}$$

by (1.23) and (1.26). Hence,

$$F_k := \lim_{j\to\infty} R^k_j(f_j + h_j) \text{ exists in } Y_{k,\widetilde{\mu}(k)} \text{ for any } k,$$

where $\widetilde{\mu}(k) := \max(\mu_2(k+2), \nu(k))$. For $l \leq k$ we have

$$R^l_k(F_k) = \lim_{j\to\infty} R^l_k(R^k_j(f_j + h_j)) = \lim_{j\to\infty} R^l_j(f_j + h_j) = F_l.$$

Thus, $(F_k)_k$ defines $F \in \operatorname{limproj}_k Y_k = Y$ with $R^k(F) = F_k \in Y_{k,\widetilde{\mu}(k)}$ for any k. We have $T(F) = g$ since $h_j \in X_j = \operatorname{Ker}(T_j)$ and therefore

$$
\begin{aligned}
T_k(F_k) &= \lim_{j\to\infty} T_k(R_j^k(f_j + h_j)) \\
&= \lim_{j\to\infty} R_j^k(T_j(f_j)) = \lim_{j\to\infty} R_j^k(R_{j+1}^j(g_{j+1})) = g_k
\end{aligned}
$$

by (1.28). ∎

Notice that condition (P_3) is weaker than Vogt's condition (P_2) from [37]. The latter condition follows from the surjectivity of $T : Y \to Z$ if $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ are projective (DFS)-spectra and if $\operatorname{Proj}^1(\mathfrak{Y}) = 0$ (use Vogt [37, Theorems 5.2 and 2.7]). Assuming that $\mathfrak{X}$ is sw-reduced, Theorem 1.5 thus provides an automatic improvement on the solutions in this situation if the surjectivity of T is known.

We will use Theorem 1.5 in Section 4 to prove that the weak technical condition $(\underline{A_\Omega})$ is sufficient for the surjectivity of $P(D)$ on $A(\Omega)$. For this we have to check first that the sequence (1.3) of projective spectra is exact. Several equivalent criteria for this are given in Proposition 3.3, including the condition (C_Ω) which obviously follows from $(\underline{A_\Omega})$. Next we have to show that the kernel spectrum N_P^Ω is sw-reduced (see Proposition 4.3, which is based on Lemma 4.2). The main point however is to prove condition (P_3) for N_P^Ω, which means that we have to prove a decomposition with bounds in N_P^Ω. This decomposition is obtained in Theorem 4.5, essentially by using a cutting off procedure, which of course leads out of N_P^Ω, and then repairing this defect by means of the existence result for $P(D)$ from Theorem 4.4, which is the analytic core of this part of our results.

2. The characterization. In this section we will state the new characterization of surjective partial differential operators on $A(\Omega)$ and prove some direct applications. The proof of the Main Theorem 2.1 is postponed to the subsequent sections. To state the theorem, we first introduce some notation: Let

$$
G(x,y) := -|(x,y)|^{1-n}/((n-1)c_{n+1})
$$

be the canonical even elementary solution of Δ (see Hörmander [14, Theorem 3.3.2], and recall that $n + 1 \geq 3$). For $T \in \mathbb{R}$ let

$$
\mathfrak{U}(\Omega \times \{T\}) := \{V \subset \mathbb{R}^{n+1} \mid V \text{ is open and } V \cap (\mathbb{R}^n \times \{T\}) = \Omega \times \{T\}\},
$$
$$
\mathfrak{U}(\Omega) := \mathfrak{U}(\Omega \times \{0\}).
$$

For $\delta > 0$ let

$$
A_\delta(\omega) := \{f \in A(\omega) \mid \sup_{x\in\omega} |f^{(a)}(x)|\delta^{|a|}/a! < \infty\}.
$$

The hyperfunctions on Ω can be defined by

(2.1) $\mathfrak{B}(\Omega) := \widetilde{C}_\Delta(\Omega \times (\mathbb{R} \setminus \{0\}))/\widetilde{C}_\Delta(\Omega \times \mathbb{R})$

(see Bengel [4] and Hörmander [14, Chapter IX]). Here $\widetilde{C}_\Delta(V)$ is the space of harmonic functions on V which are even with respect to y.

Our characterization of surjective partial differential operators on $A(\Omega)$ is contained in the following Main Theorem of this paper:

THEOREM 2.1. *Let $\Omega \subset \mathbb{R}^n$ be open. The following statements are equivalent*:

(a) $P(D) : A(\Omega) \to A(\Omega)$ *is surjective.*
(b) *For any $V \in \mathfrak{U}(\Omega)$ there is $W \in \mathfrak{U}(\Omega)$ with $W \subset V$ such that*

$$P(D)C_\Delta(W) \supset C_\Delta(V)|_W.$$

(c) $P(D)$ *satisfies the following condition* ($\underline{A_\Omega}$): *for any $\omega \subset\subset \Omega$ there is $\widetilde{\omega} \subset\subset \Omega$ with $\widetilde{\omega} \supset\supset \omega$ such that for any $\widehat{\omega} \subset\subset \Omega$ and any $\xi \in \partial\widetilde{\omega}$ there is $\delta > 0$ such that for any $0 < T \le \delta$ there are $V \in \mathfrak{U}(\widehat{\omega} \times \{T\})$ and $E \in C_\Delta(X)$ such that*

(2.2) $P(D)E = G(\cdot - \xi, \cdot)$ *on* $X := V \cup (\omega \times]T - \delta, T + \delta[).$

(d) $P(D)$ *satisfies the following condition* ($\overline{A_\Omega}$): *for any $\omega \subset\subset \Omega$ there are $\widetilde{\omega} \subset\subset \Omega$ and $\delta > 0$ such that for any $\xi \in \Omega \setminus \widetilde{\omega}$ and any $0 < T \le 1$ there are $V \in \mathfrak{U}(\Omega \times \{T\})$ and $E \in C_\Delta(X)$ such that* (2.2) *holds.*

(e) (i) *For any $g \in A(\Omega)$ and any $\omega \subset\subset \Omega$ there is $f \in A(\omega)$ such that*

(2.3) $P(D)f = g|_\omega.$

(ii) $P(D)$ *satisfies the following condition* ($\underline{B_\Omega}$): *for any $\omega \subset\subset \Omega$ there is $\widetilde{\omega} \subset\subset \Omega$ with $\widetilde{\omega} \supset\supset \omega$ such that for any $\xi \in \partial\widetilde{\omega}$ and any $\widehat{\omega} \subset\subset \Omega$ there is $F \in \mathfrak{B}(\widehat{\omega})$ such that*

$$P(D)F = \delta_\xi \quad on \ \widehat{\omega} \quad and \quad F|_\omega \in A(\omega),$$

where δ_ξ is the point evaluation at ξ.

(f) $P(D)$ *satisfies* (e)(i) *and the following condition* ($\overline{B_\Omega}$): *for any $\omega \subset\subset \Omega$ there are $\widetilde{\omega} \subset\subset \Omega$ and $\delta > 0$ such that for any $\xi \in \Omega \setminus \widetilde{\omega}$ there is $F \in \mathfrak{B}(\Omega)$ such that*

$$P(D)F = \delta_\xi \quad on \ \Omega \quad and \quad F|_\omega \in A_\delta(\omega).$$

(g) $P(D)$ *satisfies* (e)(i) *and the following condition: for any $\omega \subset\subset \Omega$ there are $\widetilde{\omega} \subset\subset \Omega$ with $\widetilde{\omega} \supset \omega$ and $\delta > 0$ such that for any $0 < \widehat{\delta} \le \delta$ there is $\gamma > 0$ such that for any $H \in \mathfrak{B}(\Omega)$ with $H|_{\widetilde{\omega}} \in A_{\widehat{\delta}}(\widetilde{\omega})$ there is $F \in \mathfrak{B}(\Omega)$ such that*

$$P(D)F = H \quad on \ \Omega \quad and \quad F|_\omega \in A_\gamma(\omega).$$

Condition $(\overline{\mathrm{B}_\varOmega})$ is similar to a condition which is equivalent to the surjectivity of $P(D)$ on nonquasianalytic classes of ultradifferentiable functions of Roumieu type (Langenbruch [22, 25, 23]).

The canonical elementary solution G of $\varDelta$ is used in (2.2) since G provides the basic isomorphism of $C_\varDelta(K)'$ and $C_{\varDelta,0}(\mathbb{R}^{n+1} \setminus K)$ in the Grothendieck–Tillmann duality (see (3.3)).

Condition $(\overline{\mathrm{B}_\varOmega})$ is an improvement of the necessary condition from Langenbruch [24, Theorem 1.3], where a weaker notion of (shifted) elementary solution was used.

The condition in Theorem 2.1(g) roughly means that the equation $P(D)F = H$ may be solved in $\mathfrak{B}(\varOmega)$ respecting lacunas in the analytic singular support. This corresponds to the characterization of surjectivity of $P(D)$ in nonquasianalytic Gevrey classes (see Langenbruch [22, Theorem 2.1(c)]).

We already used the necessary condition from Langenbruch [24, Theorem 1.3] to show that local hyperbolicity of the principal part P_m of P and hyperbolicity of the localizations of P_m at ∞ are necessary for the surjectivity of $P(D)$ on $A(\varOmega)$ in many situations (see Langenbruch [24, 26]).

In the following discussion of the Main Theorem we will therefore concentrate on obtaining weak forms of sufficient (or equivalent) conditions for surjectivity and relate these to the corresponding results in the literature.

As a first consequence, we notice that condition (e)(i) is redundant if the shifted elementary solutions F are defined on $\mathrm{conv}(\widehat{\omega})$:

COROLLARY 2.2. *$P(D)$ is surjective on $A(\varOmega)$ if for any $\omega \subset\subset \varOmega$ there is $\widetilde{\omega} \subset\subset \varOmega$ with $\widetilde{\omega} \supset\supset \omega$ such that for any $\widehat{\omega} \subset\subset \varOmega$ and any $\xi \in \partial\widetilde{\omega}$ there is $F \in \mathfrak{B}(\mathrm{conv}(\widehat{\omega}))$ such that*

$$P(D)F = \delta_\xi \quad on \ \mathrm{conv}(\widehat{\omega}) \quad and \quad F|_\omega \in A(\omega).$$

Proof. We show $(\mathrm{A}_\varOmega)$: Let $\omega \subset\subset \omega_1 \subset\subset \varOmega$. Choose $\widetilde{\omega}$ for ω_1 by the assumption. For $\xi \in \partial\widetilde{\omega}$, choose $F \in \mathfrak{B}(\widehat{\omega}_1)$ for $\widehat{\omega}_1 := \mathrm{conv}(\widehat{\omega})$ by the assumption. By the definition (2.1) of hyperfunctions, F is defined by some $F_1 \in \widetilde{C}_\varDelta(\widehat{\omega}_1 \times (\mathbb{R} \setminus \{0\}))$ such that for some $H \in D'(\widehat{\omega}_1 \times \mathbb{R})$ we have

$$(2.4) \qquad P(D)F_1 = H|_{\widehat{\omega}_1 \times (\mathbb{R} \setminus \{0\})} \quad and \quad \varDelta H = \delta_{(\xi,0)} \quad on \ \widehat{\omega}_1 \times \mathbb{R}.$$

Moreover, $F_2 := F_1|_{\widehat{\omega}_1 \times]0,\infty[}$ can be extended as a harmonic function to $V \cup (\widehat{\omega}_1 \times]0,\infty[)$ for some neighbourhood $V \in \mathfrak{U}(\omega_1)$. Hence there is $\delta > 0$ such that F_2 can be uniquely extended to a harmonic function F_3 defined on

$$W := (\omega \times]{-}\delta,\infty[) \cup (\widehat{\omega}_1 \times]0,\infty[).$$

By (2.4) there is $h \in C_\varDelta(\widehat{\omega}_1 \times \mathbb{R})$ such that $H = G(\cdot - \xi, \cdot) + h$. Since $\varDelta h = 0$

and $\widehat{\omega}_1 \times \mathbb{R}$ is convex, we may solve the overdetermined system

$$(2.5) \qquad \begin{cases} P(D)g = h \\ \Delta g = 0 \end{cases} \quad \text{on } \widehat{\omega}_1 \times \mathbb{R}$$

by the fundamental principle of Ehrenpreis/Palamodov. $E := F_3 - g|_W$ then satisfies

$$P(D)E = G(\cdot - \xi, \cdot) \quad \text{on } W$$

by harmonic continuation. Since

$$\omega \times \,]T - \delta/2, T + \delta/2[\,\subset W \in \mathfrak{U}(\widehat{\omega} \times \{T\})$$

for $0 < T \leq \delta/2$, this shows ($\underline{A_\Omega}$). ∎

COROLLARY 2.3. *Let Ω be convex. Then $P(D)$ is surjective on $A(\Omega)$ iff $P(D)$ satisfies ($\underline{B_\Omega}$) iff $P(D)$ satisfies ($\overline{B_\Omega}$).*

Proof. The necessity follows from Theorem 2.1, the sufficiency follows from Corollary 2.3 since $\widehat{\omega}$ can be chosen convex. ∎

The two main examples of convex open sets are $\mathbb{R}^n$ and a halfspace. We state the respective conditions for these cases explicitly. Let

$$U_\varepsilon(\xi) := \{x \in \mathbb{R}^n \mid |x - \xi| < \varepsilon\} \quad \text{and} \quad U_\varepsilon := U_\varepsilon(0).$$

EXAMPLE 2.4. *$P(D)$ is surjective on $A(\mathbb{R}^n)$ if and only if for any $j \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that for any $\xi \in \mathbb{R}^n$ with $|\xi| \geq k$ there is an elementary solution $F \in \mathfrak{B}(\mathbb{R}^n)$ of $P(D)$ such that F is real analytic on $U_j(\xi)$.*

Proof. The stated condition directly follows from ($\overline{B_{\mathbb{R}^n}}$) and it implies ($\underline{B_{\mathbb{R}^n}}$). So the claim is a consequence of Corollary 2.3. ∎

In the discussion of surjectivity of $P(D)$ on $A(\Omega)$, operators with locally hyperbolic principal part P_m have been studied extensively. These are defined as follows: Let

$$S^{n-1} := \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

DEFINITION 2.5. *P_m is locally hyperbolic if for any $\Theta \in S^{n-1}$ there are $N \in S^{n-1}$ and $\delta > 0$ such that for $(x, z) \in \mathbb{R}^n \times \mathbb{C}$,*

$$P_m(\Theta + x + zN) \neq 0 \quad \text{if } |(x, z)| < \delta \text{ and } \operatorname{Im} z \neq 0.$$

If P_m is locally hyperbolic at Θ with respect to N, then the localization $(P_m)_\Theta$ of P_m at Θ is hyperbolic with respect to N. Recall that $(P_m)_\Theta$ is defined by

$$(P_m)_\Theta := \lim_{s \to 0}(P_m(\theta + sx)s^{-q_\theta}),$$

where s^{q_θ} is the term of lowest order in the expansion of $(P_m)_\Theta$.

Let $\Gamma((P_m)_\Theta, N)$ be the component of N in $\{x \in \mathbb{R}^n \mid (P_m)_\Theta(x) \neq 0\}$. Then the dual cone of $\Gamma((P_m)_\Theta, N)$ is called the *local propagation cone* for P_m at Θ and is denoted by $K((P_m)_\Theta, N)$. Obviously, P_m is also locally

hyperbolic with respect to N for any $\widetilde{\Theta} \in S^{n-1}$ near Θ. Moreover, the mapping $\Theta \mapsto K((P_m)_\Theta, N)$ is locally upper semicontinuous, that is, for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$(2.6) \quad S^{n-1} \cap K((P_m)_{\widetilde{\Theta}}, N) \subset S^{n-1} \cap (K((P_m)_\Theta, N) + U_\varepsilon) \quad \text{if } |\Theta - \widetilde{\Theta}| < \delta.$$

This means that $K((P_m)_{\widetilde{\Theta}}, N)$ is contained in a conic ε-neighbourhood of $K((P_m)_\Theta, N)$ if $|\Theta - \widetilde{\Theta}| < \delta$. The following basic result is due to Andersson [1, Corollary 4.1] (use also the remark of Andersson [2, p. 2] and (2.6)).

THEOREM 2.6. *Let P_m be locally hyperbolic at any $\Theta \in S^{n-1}$ with respect to $N_\Theta \in S^{n-1}$. Then for any $\varepsilon > 0$ there are $\Theta_1, \ldots, \Theta_k \in S^{n-1}$ and a fundamental solution $F \in \mathfrak{B}(\mathbb{R}^n)$ for $P(D)$ which is analytic outside a conic ε-neighbourhood of*

$$\bigcup_{j \leq k} K((P_m)_{\Theta_j}, N_{\Theta_j}).$$

Thus, if P_m is locally hyperbolic, for any $x \in S^{n-1}$ there is an elementary solution $F \in \mathfrak{B}(\mathbb{R}^n)$ for $P(D)$ such that

$$(2.7) \qquad F \text{ is real analytic in a conic } \varepsilon\text{-neighbourhood } \Gamma \text{ of } x.$$

Therefore, the criterion of Example 2.4 is satisfied if P_m is locally hyperbolic, and $P(D)$ is thus surjective on $A(\mathbb{R}^n)$. This is the result of Andersson [2] who used (2.7) and some additional growth condition for E to get the result by a decomposition of $A(\mathbb{R}^n)$ into a finite sum of real analytic functions decaying sufficiently fast. In some sense, Example 2.4 thus turns the part (2.7) of the sufficient criterion of Andersson into a characterization. Notice that for $n \geq 4$ there are operators which are surjective on $A(\mathbb{R}^n)$ though P_m is not locally hyperbolic (Hörmander [13]).

We now discuss the case of halfspaces: For $0 \neq \eta \in \mathbb{R}^n$, $k \in \,]0, \infty]$ and $\varepsilon \in \mathbb{R}$ let

$$H(\eta)_{\varepsilon,k} := \{x \in U_k \mid \langle x, \eta \rangle > \varepsilon\},$$

where $H(\eta) := H(\eta)_{0,\infty}$ is the halfspace defined by η.

EXAMPLE 2.7. $P(D)$ is surjective on $A(H(\eta))$ if and only if $P(D)$ satisfies $(\overline{\mathrm{B}_{\mathbb{R}^n}})$ (or $(\underline{\mathrm{B}_{\mathbb{R}^n}})$) and for any $\varepsilon > 0$ there is $\delta < 0$ such that for any $k \in \mathbb{N}$ there is an elementary solution $F \in \mathfrak{B}(H(\eta)_{\delta,k})$ of $P(D)$ such that F is real analytic on $H(\eta)_{\varepsilon,1/\varepsilon}$.

Proof. "$\Rightarrow$" $P(D)$ is surjective on $A(\mathbb{R}^n)$ by Hörmander [13], hence $(\overline{\mathrm{B}_{\mathbb{R}^n}})$ holds by Theorem 2.1, and Theorem 2.1 also implies $(\underline{\mathrm{B}_{H(\eta)}})$. The second condition now follows from $(\underline{\mathrm{B}_{H(\eta)}})$.

"$\Leftarrow$" By Corollary 2.3 we have to show $(\underline{\mathrm{B}_{H(\eta)}})$. Let $\omega \subset\subset H(\eta)$. By $(\underline{\mathrm{B}_{\mathbb{R}^n}})$ there is $C > 0$ such that for any $\xi \in \partial U_C$ and any $k \in \mathbb{N}$ there is

$F \in \mathfrak{B}(U_k)$ such that

$$P(D)F = \delta_\xi \quad \text{on } U_k \quad \text{and} \quad F|_\omega \in A(\omega).$$

Choose $0 < \varepsilon \leq 1$ such that

(2.8) $\qquad H_{2\varepsilon,1/\varepsilon} \supset\supset \omega + \{x \in \overline{U}_C \mid \langle x, \eta \rangle = 0\} =: \omega + \widetilde{U}_C$

and then choose $0 < \delta < \varepsilon$ and $\widetilde{F} \in \mathfrak{B}(H(\eta)_{-\delta,k})$ for $k \in \mathbb{N}$ by the assumption. For $x \in \widetilde{U}_C$ we set $\xi := \delta\eta + x$ and $F_\xi := \widetilde{F}(\cdot - \xi)$. Then F_ξ is defined on $H(\eta)_{0,k-C}$,

$$P(D)F_\xi = \delta_\xi \quad \text{on } H_{0,k-C} \quad \text{and} \quad F|_\omega \in A(\omega)$$

by (2.8). This proves $(\underline{\mathrm{B}}_{H(\eta)})$ for $\widetilde{\omega} := H(\eta)_{\delta,C}$. ∎

Condition $(\underline{\mathrm{B}}_{\mathbb{R}^n})$ cannot be omitted in Example 2.7, since it does not follow from the second condition. Indeed, the canonical elementary solution of the heat operator $P(D) := \sum_{j \leq n-1} \partial_j^2 - \partial_n$ vanishes on $H(e_n)$, so the second condition in Example 2.7 is clearly satisfied, but $P(D)$ is not surjective on $A(\mathbb{R}^n)$ for $n \geq 3$.

We now come to a sufficient criterion in the spirit of the results of Kawai [18], Kaneko [15–17] and Zampieri [42], involving only a condition for the boundary points of Ω.

THEOREM 2.8. *$P(D)$ is surjective on $A(\Omega)$ if $P(D)$ satisfies $(\underline{\mathrm{B}}_{\mathbb{R}^n})$ and if for any $\zeta \in \partial\Omega$ and any $\omega \subset\subset \Omega$ there are a convex neighbourhood U of $\overline{\Omega}$ and $F_\zeta \in \mathfrak{B}(U)$ such that*

$$P(D)F_\zeta = \delta_\zeta \quad on \ U \quad and \quad F_\zeta|_\omega \in A(\omega).$$

If Ω is bounded, condition $(\underline{\mathrm{B}}_{\mathbb{R}^n})$ can be omitted.

Proof. We check the condition of Corollary 2.2. Fix $\omega \subset\subset \Omega$. By Theorem 2.1, $P(D)$ satisfies $(\overline{\mathrm{B}}_{\mathbb{R}^n})$, hence there is $C > 0$ such that for any $\xi \in \partial U_C$ there is $F \in \mathfrak{B}(\mathbb{R}^n)$ such that $P(D)F = \delta_\xi$ and $F|_\omega \in A(\omega)$. Therefore, the condition of Corollary 2.2 must be checked only for those points $\xi \in \Omega \cap \overline{U}_C$ which are near the boundary. But for these, suitable hyperfunctions are provided by small shifts of F_ζ for $\zeta \in \partial\Omega$ from the assumption. ∎

COROLLARY 2.9. *$P(D)$ is surjective on $A(\Omega)$ if for any $\zeta \in \partial\Omega$ and any $\Theta \in S^{n-1}$ there is $N_{\Theta,\zeta} \in S^{n-1}$ such that P_m is locally hyperbolic at Θ with respect to $N_{\Theta,\zeta}$ and*

(2.9) $\qquad (\zeta + K((P_m)_\Theta, N_{\Theta,\zeta})) \cap \Omega = \emptyset.$

Proof. Since P_m is locally hyperbolic, $P(D)$ satisfies $(\underline{\mathrm{B}}_{\mathbb{R}^n})$ by (2.7). Fix $\omega \subset\subset \Omega$ and $\zeta \in \partial\omega$. By (2.9) we may choose a conic neighbourhood Γ of $\bigcup_{\Theta \in S^{n-1}} K((P_m)_\Theta, N_{\Theta,\zeta})$ such that

(2.10) $\qquad (\zeta + \Gamma) \cap \omega = \emptyset.$

By Theorem 2.6 there is an elementary solution $\widetilde{F} \in \mathfrak{B}(\mathbb{R}^n)$ for $P(D)$ such that $\widetilde{F}$ is real analytic on $\mathbb{R}^n \setminus \Gamma$, hence $F := \widetilde{F}(\cdot - \zeta)$ is real analytic on ω by (2.10). Thus $P(D)A(\Omega) = A(\Omega)$ by Theorem 2.8. ∎

The first results in the direction of Corollary 2.9 are due to Kawai [18]. For bounded Ω, the result is due to Kaneko [16] (see also Kaneko [17]) using the approach of Kawai.

For convex Ω, Corollary 2.9 was proved by Zampieri [42] using Hörmander's approach. For nonconvex Ω, a different proof was given by Zampieri [43, Theorem 2.1] in the special case where P_m decomposes into an elliptic and a hyperbolic factor.

The case of unbounded Ω was considered by Kaneko [15] using Fourier hyperfunctions. Additionally to the pointwise condition (2.9) he needed certain extra assumptions including boundary points at ∞, i.e. points in the closure of $\partial\Omega$ in the directional compactification D^n of $\mathbb{R}^n$ (see Kaneko [16, Section 3]). These assumptions are redundant, since we only need shifted elementary solutions which are real analytic on $\omega \subset\subset \Omega$ (and not on Ω), so (2.6) and Theorem 2.6 are sufficient in our approach.

As an easy example we now consider operators of real principal type. Recall that $P(D)$ is an *operator of principal type* if P_m is real and if $\operatorname{grad} P_m(\theta) \neq 0$ whenever $0 \neq \theta \in \mathbb{R}^n$ and $P_m(\theta) = 0$. The set of *bicharacteristic directions* is then defined by

$$\mathrm{BD}(P) := \{\xi \in \mathbb{R}^n \mid \xi = \operatorname{grad} P_m(\theta) \text{ for some } 0 \neq \theta \in \mathbb{R}^n \text{ with } P_m(\theta) = 0\}.$$

Notice that operators of real principal type are locally hyperbolic with local propagation cones

$$K((P_m)_\theta, \operatorname{grad} P_m(\theta)) = \pm [0, \infty[\, \operatorname{grad} P_m(\theta).$$

$b_{(x,\xi)} := x + \xi\mathbb{R}$ is called a *bicharacteristic line* if $x \in \mathbb{R}^n$ and $\xi \in \mathrm{BD}(P)$.

EXAMPLE 2.10. Let $P(D)$ be an operator of real principal type. Then $P(D)$ is surjective on $A(\Omega)$ if any bicharacteristic line $b_{(x,\xi)}$ issuing from $x \in \partial\Omega$ intersects Ω in an interval.

Proof. The statement is obvious from Corollary 2.9 and the remarks above. ∎

The example improves on the corresponding results of Kawai [18, Theorem 3.1] and Kaneko [15, 16]. The example includes many classical operators, e.g. the wave equation $P(D) := \partial_n^2 - \sum_{j=1}^{n-1} \partial_j^2$, the ultrahyperbolic operator $P(D) := \partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2$ (for $n = 4$) and Zeylon's operator $P(D) = \sum_{j=1}^n \partial_j^3$.

In fact, our results improve on those of Kaneko [15] already in the case of two variables, where we obtain the following characterization of Zampieri [41, 43] (for bounded open sets see Kawai [18, Theorem 5.7]):

COROLLARY 2.11. *The following are equivalent for an open set $\Omega \subset \mathbb{R}^2$:*

(a) $P(D)$ *is surjective on* $A(\Omega)$.

(b) *Every characteristic line intersects Ω in an interval.*

Proof. (a)$\Rightarrow$(b). Since $P(D)$ is surjective on $A(\Omega)$, Ω is P-convex for supports by Zampieri [40]. Hence, (b) follows from Hörmander [14, Theorem 10.8.3].

(b)$\Rightarrow$(a). Since $n = 2$, any P_m is locally hyperbolic, and the notions of a characteristic line and of the union of the positive and negative propagation cones coincide. Hence, the assumptions of Corollary 2.9 are satisfied by (b), and $P(D)$ is surjective on $A(\Omega)$ by that corollary. ∎

In a remark on p. 345 of [15], Kaneko noticed that the results of that paper are not sufficient to show that the operator $P(D) = D_1$ is surjective on $A(\Omega)$ for $\Omega := \{x \in \mathbb{R}^2 \mid x_1 < -x_2^2\}$, while this result clearly follows from Corollary 2.11 (or integration).

At the end of his paper [16], Kaneko conjectured that condition (2.9) was also necessary for surjectivity of $P(D)$ on $A(\Omega)$ if Ω is bounded. The following example shows that this is not true:

EXAMPLE 2.12. For $t \in \mathbb{R}$ let $\gamma(t) := (\cos(t), \sin(t))$. Let $\widetilde{\Omega}$ be the spiral in $\mathbb{R}^3$ defined by the disjoint (!) union

$$\widetilde{\Omega} := \bigcup_{t \in \mathbb{R}} (\gamma(t)]1/2, 2[\times]t-1, t+1[)$$

and let

$$\Omega := \{x \in \widetilde{\Omega} \mid 0 < x_3 < 10\}.$$

Then $P(D) := D_3$ is surjective on $A(\Omega)$ though (2.9) is not satisfied.

Proof. The latter statement is obvious (consider $\zeta = (1, 0, 1/2)$). To prove surjectivity, we set

$$g(x) := i \int_{t_x}^{x_3} f(x_1, x_2, \xi) \, d\xi \quad \text{for } x \in \Omega \text{ and } f \in A(\Omega),$$

where t_x is the unique real number such that $x \in \gamma(t_x)]1/2, 2[\times]t_x-1, t_x+1[$. Then g is defined and real analytic on Ω and $D_3 g = f$. ∎

Example 2.12 also shows that the existence of elementary solutions defined on $\mathrm{conv}(\widehat{\omega})$ as in Corollary 2.2 rather than on $\widehat{\omega}$ as in (B_Ω) is also not necessary for surjectivity. In fact, let $(1, 0, 1/2), (1, 0, 2\pi) \in \omega \subset\subset \Omega$ and let $\xi := (1, 0, 1 - \varepsilon) \in \widehat{\omega} \subset\subset \Omega$. Then $\mathrm{conv}(\widehat{\omega})$ contains a column $C := B_\varepsilon(1, 0) \times [1/2, 2\pi]$ for large $\widehat{\omega}$. Let $F \in \mathfrak{B}(\mathrm{conv}(\widehat{\omega}))$ be as in Corollary 2.2. Then F is a hyperfunction zerosolution of D_3 on $C \setminus \{\xi\}$, hence locally constant with respect to x_3. Since F is real analytic near $(1, 0, 1/2)$ and

$(1, 0, 2\pi)$, F is real analytic on $U \setminus \{\xi\}$ for a neighbourhood U of ξ. Subtracting the usual shifted elementary solution $E := \delta_{(0,1)} \otimes \chi_{[1-\varepsilon, \infty[}$ we see in the same way that E is real analytic near ξ (except at ξ), a contradiction.

The methods and results from this paper are used in Langenbruch [27] to show several inheritance properties for surjectivity of partial differential operators on spaces of real analytic functions. We show that surjectivity is inherited similarly to the case of operators on C^∞-functions. More precisely, if $P(D)$ is surjective on $A(\Omega_j)$ for any $j \in J$ then $P(D)$ is surjective on $A(\Omega)$ for

$$\Omega := \Big(\bigcap_{j \in J} \Omega_j \Big)^{\circ}$$

and

$$\Omega := \liminf_{j} {}^{\circ}\Omega_j$$

$$:= \{\xi \in \mathbb{R}^n \mid \exists \varepsilon > 0 : B_\varepsilon(\xi) \subset \Omega_j \text{ for all but finitely many } j\}.$$

Also, if $P(D)$ is surjective on $A(\Omega)$, then $P(D)$ is surjective on $A(\Omega_\varepsilon)$ for any $\varepsilon > 0$, where

$$\Omega_\varepsilon := \{\xi \in \Omega \mid \operatorname{dist}(\xi, \partial\Omega) > \varepsilon\}.$$

As the main result of [27], we obtain

THEOREM 2.13. *$P(D)$ is surjective on $A(\mathbb{R}^n)$ if $P(D)$ is surjective on $A(\Omega)$ for some $\Omega \neq \emptyset$.*

For convex Ω, this is one of the main results of Hörmander [13], while the question had been open for general Ω.

As already mentioned in the introduction, the heat equation was the first example of a nonsurjective partial differential operator on $A(\mathbb{R}^n)$ (for $n \geq 3$). By Theorem 2.13 we obtain the following improvement of this classical result:

EXAMPLE 2.14. Let $n \geq 3$ and let $P_m(x) := \sum_{j=1}^{k} \partial_j^2$ for some $1 < k < n$. Then there is no open set $\Omega \subset \mathbb{R}^n$ such that $P(D)$ is surjective on $A(\Omega)$.

Of course, this example also applies to the Schrödinger equation.

3. Necessity. In this section, we consider condition $(\overline{A_\Omega})$ defined as follows: For any $\omega \subset\subset \Omega$ there are $\widetilde{\omega} \subset\subset \Omega$ and $\delta > 0$ such that for any $\xi \in \Omega \setminus \widetilde{\omega}$ and any $0 < T \leq 1$ there are $V \in \mathfrak{U}(\Omega \times \{T\})$ and $E_\xi \in C_\Delta(W)$, $W := V \cup (\omega \times]T - \delta, T + \delta[)$, such that

$$P(D)E_\xi = G(\cdot - \xi, \cdot)|_W.$$

Clearly, $(\overline{A_\Omega})$ is stronger than $(\underline{A_\Omega})$ from the Main Theorem 2.1 and we will show in this section that $(\overline{A_\Omega})$ holds if $P(D)$ is surjective on $A(\Omega)$ (see Theorem 3.4).

To this end, we will first discuss the operator $P(D)$ on $C_\Delta(Q)$ for a compact set $Q \subset \mathbb{R}^{n+1}$. We introduce the Grothendieck duality and condition (C_Ω) which is equivalent to the exactness of the sequence (1.3) of spectra. Since (C_Ω) is clearly weaker than (A_Ω), it can also be used in the sufficiency proof in the next section. The section ends with the proof of the necessity of $\overline{(A_\Omega)}$.

It is clear from the remarks in Section 1 that we will have to consider the operator

$$P(D) : C_\Delta(Q) \to C_\Delta(Q)$$

for compact nonconvex sets $Q \subset \mathbb{R}^{n+1}$. We start with some corresponding auxiliary results. Since the Laplacian is elliptic, $C_\Delta(\mathbb{R}^{n+1})$ is dense in $C_\Delta(Q)$ for any compact set $Q \subset \mathbb{R}^{n+1}$ such that

(3.1) $\mathbb{R}^{n+1} \setminus Q$ does not have a bounded component

(see e.g. Hörmander [14, Corollary 10.5.3]). If $L \subset Q$ is compact and also satisfies (3.1), we can thus identify $C_\Delta(L)'$ with a subset of $C_\Delta(Q)'$. This identification will be frequently used.

To solve the equation $P(D)f = g$ for $g \in C_\Delta(Q)$ by means of the Hahn–Banach theorem (see Remark 3.1 below) we need an appropriate representation for $C_\Delta(Q)_b'$. This is provided by the Grothendieck–Tillmann duality (which is also the reason why we have to use the canonical elementary solution G of Δ in (A_Ω) since G defines the canonical isomorphism in the duality): For $Q \subset \mathbb{R}^{n+1}$ compact let

$$C_{\Delta,0}(\mathbb{R}^{n+1} \setminus Q) := \{f \in C_\Delta(\mathbb{R}^{n+1} \setminus Q) \mid \lim_{\xi \to \infty} f(\xi) = 0\}$$

endowed with the topology of $C(\mathbb{R}^{n+1} \setminus Q)$. Then $C_{\Delta,0}(\mathbb{R}^{n+1} \setminus Q)$ is a Fréchet space. Furthermore,

(3.2) $C(V \setminus Q)$ also induces the topology of $C_{\Delta,0}(\mathbb{R}^{n+1} \setminus Q)$

if V is an open neighbourhood of Q. Let

$$\varkappa(\mu)(x,y) := u_\mu(x,y) := \langle \mu_{(s,t)}, G(s-x, t-y)\rangle \quad \text{for } \mu \in C_\Delta(Q)_b'.$$

Then we have the topological isomorphisms

(3.3) $\varkappa : C_\Delta(Q)_b' \to C_{\Delta,0}(\mathbb{R}^{n+1} \setminus Q) \cong C_\Delta(\mathbb{R}^{n+1} \setminus Q)/C_\Delta(\mathbb{R}^{n+1})$

by the Grothendieck–Tillmann duality (Grothendieck [12, Theorem 4], Mantovani and Spagnolo [28], Tillmann [36, Satz 6]).

A first application of (3.3) is contained in the following remark, which will be frequently used. For compact sets $Q \subset L \subset \mathbb{R}^{n+1}$ let

$$R_L^K : C_\Delta(L) \to C_\Delta(Q)$$

be the canonical mapping defined by restriction.

REMARK 3.1. *Let $Q \subset L \subset \mathbb{R}^{n+1}$ be compact sets satisfying (3.1). Then*

$$P(D)C_\Delta(Q) \supset R_L^Q(C_\Delta(L))$$

if there is a neighbourhood V of L such that for any compact set $J \subset V \setminus L$ there is a bounded set $B \subset C_\Delta(Q)$ such that for any $\mu \in C_\Delta(Q)'$,

$$(3.4) \qquad \sup_{(x,y)\in J} |u_\mu(x,y)| \leq C \sup_{f\in B} |\langle P(-D)\mu, f\rangle|.$$

Proof. The mapping $P(-D) : C_\Delta(Q)' \to C_\Delta(Q)'$ is injective by (3.4) and (3.3). Moreover,

$$T := {}^t R_L^Q \circ P(-D)^{-1} : H := P(-D)C_\Delta(Q)' \to C_\Delta(L)'$$

is continuous by (3.4) and (3.2) if H is endowed with the topology of $C_\Delta(Q)'$.
Indeed,

$$u_{T(P(-D)\mu)} = u_\mu|_{\mathbb{R}^{n+1}\setminus L}.$$

For $g \in C_\Delta(L)$, the mapping

$$h : H \to \mathbb{C}, \qquad P(-D)\mu \mapsto \langle \mu, g\rangle,$$

is therefore continuous and can be extended to $\widetilde{h} \in C_\Delta(Q)'' = C_\Delta(Q)$ by the Hahn–Banach theorem since $C_\Delta(Q)$ is a (DFS)-space and hence reflexive. Then $P(D)\widetilde{h} = R_L^Q(g)$ since

$$\langle \mu, P(D)\widetilde{h}\rangle = \langle P(-D)\mu, \widetilde{h}\rangle = h(P(-D)\mu) = \langle \mu, g\rangle \quad \text{ for } \mu \in C_\Delta(Q)'. \ \blacksquare$$

A standard estimate for u_μ is provided by the following basic

REMARK 3.2. *Let $Q \subset \mathbb{R}^{n+1}$ be compact with (3.1). For any compact set $L \subset \mathbb{R}^{n+1} \setminus \mathrm{conv}(Q)$ there is a bounded set $B \subset C_\Delta(Q)$ such that for any $\mu \in C_\Delta(\mathbb{R}^{n+1})'$ with $P(-D)\mu \in C_\Delta(Q)'$ we have $\mu \in C_\Delta(\mathrm{conv}(Q))'$ and*

$$\sup_{(x,y)\in L} |u_\mu(x,y)| \leq C \sup_{f\in B} |\langle P(-D)\mu, f\rangle|.$$

Proof. (a) By assumption, $\mu \in C_\Delta(J)'$ for some compact convex set $J \subset \mathbb{R}^{n+1}$. For $w \in \mathbb{R}^{n+1}$ let

$$V_w := \{z \in \mathbb{R}^{n+1} \mid \langle z, w\rangle > 0\}.$$

Since V_w is convex, there is $E_w \in C_\Delta(V_w)$ such that $P(D)E_w = G|_{V_w}$ (as in (2.5)). For $z \in \mathbb{R}^{n+1}$ with $\langle z, w\rangle$ sufficiently small we get

$$(3.5) \qquad u_\mu(z) = \langle \mu_\eta, G(\eta - z)\rangle = \langle \mu_\eta, P(D)E_w(\eta - z)\rangle$$
$$= \langle P(-D)\mu, E_w(\eta - z)\rangle =: v_\mu(z).$$

Then v_μ defines a harmonic function on

$$W_{w,c(w)} := \{z \in \mathbb{R}^{n+1} \mid \langle z, w \rangle < c(w)\},$$

where $c(w) := \inf\{\langle w, \eta \rangle \mid \eta \in Q\}$. By the identity theorem, u_μ can thus be extended as a harmonic function on $\mathbb{R}^{n+1} \setminus \text{conv}(Q)$, i.e. $\mu \in C_\Delta(\text{conv}(Q))'_b$ by (3.3).

(b) For each compact convex set $L \in \mathbb{R}^{n+1}$,

$$P(D) : C_\Delta(L) \to C_\Delta(L) \text{ is surjective.}$$

Since $C_\Delta(L)$ is a (DFS)-space, this implies that $P(-D) : C_\Delta(L)'_b \to C_\Delta(L)'_b$ is a topological isomorphism onto its range. The claim now follows from (3.3) since $C_\Delta(Q)'_b$ is continuously embedded in $C_\Delta(\text{conv}(Q))'_b$. ∎

The exactness of the sequence (1.3) of spectra can now be characterized as follows: For a compact set $K \subset \Omega$ let $R^K : C_\Delta(\Omega \times \mathbb{R}) \to C_\Delta(K)$ be the canonical mapping defined by restriction. Let $I(\delta) :=]-\delta, \delta[$ for $\delta > 0$.

PROPOSITION 3.3. *The following are equivalent*:

(a) *For any compact set $K \subset \Omega$,*

$$P(D)C_\Delta(K \times \{0\}) \supset R^K(C_\Delta(\Omega \times \mathbb{R})).$$

(b) *The following condition $(\overline{C_\Omega})$ holds: for any $\omega \subset\subset \Omega$ there are $\widetilde{\omega} \subset\subset \Omega$ and $\delta > 0$ such that for any $\xi \in \Omega \setminus \widetilde{\omega}$ there is $E_\xi \in C_\Delta(W)$, $W := \omega \times I(\delta)$, such that*

$$(3.6) \qquad P(D)E_\xi = G(\cdot - \xi, \cdot)|_W.$$

(c) *The following condition (C_Ω) holds: for any $\omega \subset\subset \Omega$ there is $\widetilde{\omega} \subset\subset \Omega$ with $\omega \subset\subset \widetilde{\omega}$ such that for any $\xi \in \partial\widetilde{\omega}$ there are $\delta_\xi > 0$ and $E_\xi \in C_\Delta(W)$, $W := \omega \times I(\delta_\xi)$, satisfying (3.6).*

(d) *For any compact set $K \subset \Omega$ there is a compact set $J \subset \Omega$ with $K \subset J$ such that*

$$P(D)C_\Delta(K \times \{0\}) \supset R^K_J(C_\Delta(J \times \{0\})).$$

(e) *The sequence (1.3) of projective spectra is exact.*

Proof. The statements in (d) and (e) are equivalent by the definition (1.4). Similarly, (d) clearly implies (a).

(a)⇒(b). For $K := \overline{\omega}$ the mapping

$$T := P(D)^{-1} \circ R^K : C_\Delta(\Omega \times \mathbb{R}) \to C_\Delta(K \times \{0\})/\ker P(D)$$

is defined by (a) and continuous by the closed graph theorem. The latter space is a (DFS)-space with step spaces $Y_k := BC_\Delta((K + U_{1/k}) \times I(1/k))/\ker P(D)$ of bounded harmonic functions. Thus, there is $k \in \mathbb{N}$ by Grothendieck's lemma (Meise and Vogt [30, 24.33]) such that

$$T : C_\Delta(\Omega \times \mathbb{R}) \to Y_k \text{ is continuous.}$$

Also, there are $C, c > 0$ and $\omega_1 \subset\subset \Omega$ that

$$\|T(f)\|_{Y_k} \leq C \sup_{x \in \omega_1, \, |y| < c} |f(x, y)| \quad \text{for } f \in C_\Delta(\Omega \times \mathbb{R}).$$

Since $C_\Delta(\Omega \times \mathbb{R})$ is dense in $C_\Delta(\overline{\omega}_1 \times \overline{I(c)})$ by (3.1), T can be uniquely extended to a continuous mapping $\widetilde{T} : C_\Delta(\overline{\omega_1} \times \overline{I(c)}) \to Y_k$ which also satisfies

$$P(D)g = f|_{\omega \times I(1/k)}$$

if $\widetilde{T}(f) = [g] \in Y_k$. This shows (b) for $\widetilde{\omega} \supset\supset \overline{\omega}_1$ since then $G(\cdot - \xi, \cdot) \in C_\Delta(\overline{\omega}_1 \times \overline{I(c)})$ if $\xi \notin \widetilde{\omega}$.

    (c)$\Rightarrow$(d). By Remark 3.1, we have to estimate u_μ suitably for $\mu \in C_\Delta(K \times \{0\})'$. By Remark 3.2 for any compact set $L \subset \mathbb{R}^{n+1} \setminus (\text{conv}(K) \times \{0\})$ there is a bounded set $B_1 \subset C_\Delta(K \times \{0\})$ such that

$$(3.7) \qquad \sup_{(x,y) \in L} |u_\mu(\cdot, y)| \leq C \sup_{f \in B_1} |\langle P(-D)\mu, f \rangle|.$$

Choose $\varepsilon > 0$ such that $\omega_1 := K + U_\varepsilon \subset\subset \Omega$. Choose $\widetilde{\omega} \supset\supset \omega_1$ for ω_1 by (c) and set $J := \overline{\widetilde{\omega}}$. We may assume that $X := \widetilde{\omega} + U_\varepsilon \subset\subset \Omega$. For $\xi \in \partial\widetilde{\omega}$ choose $\delta_\xi < \varepsilon$ and $E_\xi \in C_\Delta(\omega_1 \times I(\delta_\xi))$ by (C_Ω). For $(x, y) \in U_{\delta_\xi} \times I(\delta_\xi)$ we then get

$$\begin{aligned} u_\mu(x + \xi, y) &= \langle \mu_{(s,t)}, G(s - x - \xi, t - y) \rangle \\ &= \langle \mu_{(s,t)}, P(D)E_\xi(s - x, t - y) \rangle \\ &= \langle P(-D)\mu_{(s,t)}, E_\xi(s - x, t - y) \rangle. \end{aligned}$$

Choose $\xi_1, \ldots, \xi_\lambda \in \partial\widetilde{\omega}$ such that $W := \bigcup_{j \leq \lambda} U_{\delta_{\xi_j}}(\xi_j) \supset \partial\widetilde{\omega}$ and let $\delta := \min(\delta_{\xi_1}, \ldots, \delta_{\xi_\lambda})/2$. Then

$$(3.8) \qquad \sup_{(x,y) \in W \times I(\delta)} |u_\mu(x, y)| \leq \sup_{f \in B} |\langle P(-D)\mu, f \rangle|,$$

where

$$B := \{ E_{\xi_j}(\cdot - x - \xi_j, \cdot - y) \mid j \leq \lambda, \, |y| \leq \delta, \, |x - \xi_j| \leq \delta_{\xi_j} \}$$

is bounded in $C_\Delta(K \times \{0\})$ since $K + U_{\delta_{\xi_j}} \subset\subset \omega_1$. The claim in (d) follows from (3.7) and (3.8) by Remark 3.1. ∎

    We now come to the main result of this section.

THEOREM 3.4. *Let* $P(D)\colon A(\Omega) \to A(\Omega)$ *be surjective. Then* $P(D)$ *satisfies* $(\overline{\text{A}_\Omega})$.

    Proof. (i) Since $P(D)$ is surjective on $A(\Omega)$, $P(D)$ satisfies (C_Ω) and the sequence (1.3) of projective spectra is exact by Proposition 3.3. By Proposition 1.1 we therefore know that $\text{Proj}^1 N_P^\Omega = 0$.

(ii) Let $\{K_k \mid k \in \mathbb{N}\}$ be an increasing compact exhaustion of Ω. Then $N_P^\Omega = (X_k, R_j^k) = (N_P(K_k), R_j^k)$ is a projective (DFS)-spectrum with

$$X_k = \liminf_d X_{k,d} := \liminf_d NB_P(K_k + V_{1/d})$$

(see (1.7)). For $\omega \subset\subset \Omega$ choose $K_k \supset\supset \omega$. Theorem 1.2 of Retakh implies that there is $j \geq k$ such that

$$(3.9) \quad R_j^k(N_P(K_j \times \{0\})) \subset R^k(N_P(\Omega \times \{0\})) + NB_P(K_k + V_{1/j}).$$

Let $K_j \subset\subset \omega_1 \subset\subset \Omega$ and choose $\widetilde{\omega} \subset\subset \Omega$ and $\delta > 0$ for ω_1 by $(\overline{C_\Omega})$. For $\xi \in \Omega \setminus \widetilde{\omega}$ and $T > 0$ the equation

$$(3.10) \qquad\qquad P(D)F = G(\cdot - \xi, \cdot)$$

thus has solutions $f \in C_\Delta(\omega_1 \times I(\delta))$ by $(\overline{C_\Omega})$ and $f_T \in C_\Delta(\Omega \times \{T\})$ since $P(D)$ is surjective on $A(\Omega)$. Set

$$g := f - f_T \in N_P(K_j \times \{T\}).$$

By (3.9) there are $V \in \mathfrak{U}(\Omega \times \{T\})$, $g_1 \in N_P(V)$, $g_2 \in NB_P(\omega \times]T - 1/j, T + 1/j[)$ and $0 < \gamma < \delta_1 := \min(\delta, 1/j)/2$ such that $g = g_1 - g_2$ on $\omega \times]T - \gamma, T + \gamma[$ (since $\omega \subset K_k$). We may assume that $f_T \in C_\Delta(V)$ and $\omega \times]T - \gamma, T + \gamma[= V \cap (\omega \times \mathbb{R})$. For $T < \delta_1$, $f_T + g_1 \in C_\Delta(V)$ and $f + g_2 \in C_\Delta(\omega \times]T - \delta_1, T + \delta_1[)$ then coincide on $V \cap (\omega \times]T - \delta_1, T + \delta_1[) = \omega \times]T - \gamma, T + \gamma[$. They thus define a solution E of (3.10) on $V \cup (\omega \times]T - \delta_1, T + \delta_1[)$. This proves $(\overline{A_\Omega})$. ∎

We will show at the end of Section 4 that $(\overline{A_\Omega})$ is equivalent to the surjectivity of $P(D)$ on $A(\Omega)$.

Notice that the choice of δ in $(\overline{A_\Omega})$ is uniform as in condition $(\overline{B_\Omega})$ from Theorem 2.1. However, since $T > 0$ in $(\overline{A_\Omega})$, the functions E_ξ coming from $(\overline{A_\Omega})$ are not hyperfunctions (in contrast to the elementary solutions F_ξ from $(\overline{B_\Omega})$).

4. Sufficiency. In this section we will consider condition (A_Ω) from Theorem 2.1(c), i.e.: for any $\omega \subset\subset \Omega$ there is $\widetilde{\omega} \subset\subset \Omega$ with $\omega \subset\subset \widetilde{\omega}$ such that for any $\widehat{\omega} \subset\subset \Omega$ and any $\xi \in \partial\widetilde{\omega}$ there is $\delta > 0$ such that for any $0 < T \leq \delta$ there are $V \in \mathfrak{U}(\widehat{\omega} \times \{T\})$ and $E \in C_\Delta(W)$, $W := V \cup (\omega \times]T - \delta, T + \delta[)$, such that

$$P(D)E = G(\cdot - \xi, \cdot)|_W.$$

At the end of this section we will prove a first characterization of the surjectivity of $P(D)$ on $A(\Omega)$ including condition $(\underline{A_\Omega})$ and part of the Main Theorem 2.1 (see Theorem 4.6). The main tool in the proof of Theorem 4.6 will be Theorem 1.5, so this section is largely devoted to verifying the assumptions of that theorem in our concrete situation.

Our first aim is to prove that the spectrum N_P^Ω is *sw*-reduced (see Proposition 4.3 below). For this, we need two auxiliary results:

LEMMA 4.1. *Let $P(D)$ satisfy $(\underline{A}_\Omega)$.*

(a) *The sequence* (1.3) *of projective spectra is exact.*
(b) *Ω is P-convex for supports.*

Proof. (a) Condition $(\underline{A}_\Omega)$ clearly implies (C_Ω) from Proposition 3.3. Hence, the claim follows from Proposition 3.3.

(b) This follows from (a) by Zampieri [40, Theorem 1]. ■

LEMMA 4.2. *Let Ω be P-convex for supports and let $P(D)$ be nonelliptic. For any $\omega \subset\subset \Omega$ there is $\widetilde{\omega} \subset\subset \Omega$ such that for any $\mu \in C_\Delta(\mathbb{R}^{n+1})'$ the following holds: If $\nu := P(-D)\mu \in C_\Delta(\omega \times \mathbb{R})'$ and if there are $\delta > 0$ and a neighbourhood Z of $\partial\omega$ such that u_μ can be extended as a harmonic function from above to $Z \times \,]-\delta, \infty[$ (and from below to $Z \times \,]-\infty, \delta[$, respectively), then $\mu \in C_\Delta(\widetilde{\omega} \times \mathbb{R})'$.*

Proof. The proof relies on the theory of distributional and formal boundary values of harmonic functions (see e.g. Langenbruch [20, 21]).

(i) By hypothesis we may assume that u_μ is defined on $V := Z \times (\mathbb{R} \setminus \{0\})$ and that $u_\mu|_V$ can be extended as a distribution (again denoted by u_μ) to $Z \times \mathbb{R}$ such that

$$(4.1) \qquad \Delta u_\mu = g_0 \otimes \delta_y + g_1 \otimes \partial_y \delta_y \quad \text{on } Z \times \mathbb{R},$$

where

$$(4.2) \qquad g_j \in A(Z) \quad \text{and} \quad P(-D)g_j = 0 \quad \text{on } Z \text{ for } j = 0, 1,$$

since we may assume that

$$(4.3) \qquad P(-D)u_\mu = u_\nu \in C_\Delta(((\mathbb{R}^n \setminus \omega) \cup Z) \times \mathbb{R}).$$

(ii) We first cut off u_μ appropriately: Choose $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi = 1$ on a neighbourhood U of $\mathbb{R}^n \setminus \omega$ and $\varphi = 0$ near $\omega \setminus Z$. Since Δ is elliptic, we can choose $w \in D'((\omega \cup Z) \times \mathbb{R})$ such that

$$(4.4) \qquad \Delta w = (\varphi g_0) \otimes \delta_y + (\varphi g_1) \otimes \partial_y \delta_y \quad \text{on } (\omega \cup Z) \times \mathbb{R}.$$

Then $u_\mu - w \in C_\Delta(((\omega \cup Z) \cap U) \times \mathbb{R})$ and therefore there are a compact set $Q \subset \mathbb{R}^{n+1}$ and $v \in C_\Delta(\mathbb{R}^{n+1} \setminus Q)$ such that

$$(4.5) \qquad v - u_\mu \in C_\Delta(U \times \mathbb{R}) \quad \text{and} \quad v - w \in C_\Delta((\omega \cup Z) \times \mathbb{R}).$$

By (3.3) we may assume that

$$(4.6) \qquad v = u_\lambda \quad \text{for some } \lambda \in C_\Delta(\mathbb{R}^{n+1})'.$$

(iii) Let $F : H(\mathbb{C}^n)^2 \to C_\Delta(\mathbb{R}^{n+1})$ be the canonical isomorphism defined via the solution of the Cauchy problem for Δ with data on $\mathbb{R}^n \times \{0\}$. Then

$$\langle \varrho, (f_0, f_1) \rangle := \langle \lambda, F(f_0, f_1) \rangle$$

defines $\varrho = (\varrho_0, \varrho_1) \in (H(\mathbb{C}^n)')^2$ and

(4.7) $P(-D)\varrho_j = P(-D)(\varphi g_j) =: \widetilde{g}_j \in C_0^\infty(\omega \setminus U)$ for $j = 0, 1$

by (4.2)–(4.6). By Fourier transformation and Malgrange's lemma (Hörmander [14, Lemma 7.3.3]) we thus have $\varrho_j \in C_0^\infty(\mathbb{R}^n)$.

With $\widetilde{\lambda} := (\varrho_0 \otimes \delta_y + \varrho_1 \otimes \partial_y \delta_y) \in C_\Delta(\mathbb{R}^{n+1})'$ we therefore get

$$\langle \lambda, F(f_0, f_1) \rangle = \langle \varrho, (f_0, f_1) \rangle = \langle \widetilde{\lambda}, F(f_0, f_1) \rangle \quad \text{for any } (f_0, f_1) \in H(\mathbb{C}^n)^2,$$

that is, $\lambda = \widetilde{\lambda}$, and by (4.6), v can be extended to $\mathbb{R}^{n+1}$ as a distribution (denoted by v again) such that

(4.8) $$\Delta v = \varrho_0 \otimes \delta_y + \varrho_1 \otimes \partial_y \delta_y.$$

This implies by (4.1) and (4.5) that

(4.9) $$\varrho_j = g_j \text{ is real analytic on } Z \cap U.$$

(iv) ϱ_j is real analytic on U. Indeed, choose $\Phi \in C^\infty(\mathbb{R}^n)$ such that $\Phi = 1$ on a neighbourhood X of $U \setminus Z$ and $\Phi = 0$ near $\omega \setminus U$. Let H be a distributional elementary solution for $P(-D)$. For $x \in X$ and $a \in \mathbb{N}_0^n$ we then get

(4.10) $\varrho_j^{(a)}(x) = \Phi(x)\varrho_j^{(a)}(x) = (H * P(-D)(\Phi \varrho_j^{(a)}))(x)$

$$= \sum_{0 < |b| \leq m} (H * (P^{(b)}(-D)\varrho_j^{(a)} D^b \Phi))(x)(-1)^{|b|}/b!,$$

since

$$(P(-D)\varrho_j^{(a)})\Phi = \widetilde{g}_j^{(a)}\Phi = 0$$

by (4.7) and the choice of Φ. The claim follows from (4.9) and (4.10), since $\operatorname{supp} \operatorname{grad} \Phi \subset Z \cap U$.

(v) Since Ω is P-convex and P is nonelliptic, $\mathbb{R}^{n+1} \setminus \Omega$ does not have a bounded component. Since ϱ_j has compact support, we thus conclude from (iv) that $\operatorname{supp} \varrho_j \subset \Omega$. Since Ω is P-convex and $\operatorname{supp} P(-D)\varrho_j = \operatorname{supp} \widetilde{g}_j \subset \omega$, there is $\widetilde{\omega} \subset\subset \Omega$ such that $\operatorname{supp} \varrho_j \subset \widetilde{\omega}$. This proves the lemma by (4.5)–(4.8). ∎

PROPOSITION 4.3. *Let $P(D)$ satisfy $(\underline{A_\Omega})$.*

(a) *For any $\omega \subset\subset \Omega$ there is $\widetilde{\omega} \subset\subset \Omega$ with $\omega \subset\subset \widetilde{\omega}$ such that for any $\widehat{\omega} \subset\subset \Omega$ with $\widetilde{\omega} \subset\subset \widehat{\omega}$ there is $\beta > 0$ such that for any $0 < \tau < \beta$, $N_P(\widetilde{\omega} \times I(\tau))|_{\omega \times I(\tau)}$ is contained in the closure of $N_P(\widehat{\omega} \times I(\tau))|_{\omega \times I(\tau)}$ in $N_P(\omega \times I(\tau))$.*
(b) *N_P^Ω is sw-reduced.*

Proof. (a) (I) Let $P(D)$ be elliptic and let

$$C_P^\infty(U) := \{f \in C^\infty(U) \mid P(D)f = 0\}.$$

Then there is $C > 0$ such that for any $\varepsilon > 0$,

$$(4.11) \qquad \sup_a |f^{(a)}(0)|\varepsilon^{|a|}/(a!C^{|a|}) \leq C_1 \sup_{|x| \leq \varepsilon} |f(x)|$$

$$\text{if } f \in C_P^\infty(V_\varepsilon) \text{ is bounded.}$$

Let

$$\omega^B := \{(x, y) \in \omega \times \mathbb{R} \mid |y| < B \operatorname{dist}(x, \partial\omega)\}.$$

Inequality (4.11) implies that there is $B > 0$ such that for any $\omega \subset\subset \mathbb{R}^n$,

$$(4.12) \qquad C_P^\infty(\omega)^2 = N_p(\omega^B),$$

where the isomorphism is provided by the solution of the Cauchy problem for Δ with data on $\omega \times \{0\}$. Notice that we have the restrictions

$$\varrho : N_p((\omega_1)^B) \to N_P(\omega \times I(\delta)) \quad \text{if } \omega \subset\subset \omega_1 \text{ and } 0 < \tau < \gamma(\omega_1),$$
$$\widetilde{\varrho} : N_p(\omega \times I(\delta)) \to N_P(\omega^B)$$

(for $\widetilde{\varrho}$ use also (4.12)). The claim thus follows in this case from the well known fact that any open set Ω is P-convex if $P(D)$ is elliptic, hence

$$C_P^\infty(\Omega(1/n)) \text{ is dense in } C_P^\infty(\Omega(1/m)) \quad \text{if } n > m,$$

where $\Omega(1/k) := \{x \in \Omega \mid x \in U_k \text{ and } \operatorname{dist}(x, \partial\Omega) > 1/k\}$.

(II) Let $P(D)$ be nonelliptic. Let $\nu \in C_\Delta(\omega \times I(\tau))'$ and let $\nu|_{N_p(\widehat{\omega} \times I(\tau))} = 0$. We have to show that $\nu|_{N_p(\widetilde{\omega} \times I(\tau))} = 0$.

(i) Let $F : H(\mathbb{C}^n)^2 \to C_\Delta(\mathbb{R}^{n+1})$ be the canonical isomorphism defined via the solution of the Cauchy problem for Δ with data on $\mathbb{R}^n \times \{0\}$. Then

$$\langle \widetilde{\nu}, (f_0, f_1) \rangle := \langle \nu, F(f_0, f_1) \rangle$$

defines $\widetilde{\nu} = (\widetilde{\nu}_0, \widetilde{\nu}_1) \in (H(\mathbb{C}^n)')^2$ and $\widetilde{\nu}_0$ and $\widetilde{\nu}_1$ both vanish on the exponential solutions of $P(D)$. By Fourier transformation and Malgrange's lemma there is $\widetilde{\mu} = (\widetilde{\mu}_0, \widetilde{\mu}_1) \in (H((\mathbb{C}^n))')^2$ such that $P(-D)\widetilde{\mu}_d = \widetilde{\nu}_d$ for $d = 0, 1$. Then

$$\langle \mu, f \rangle := \langle \widetilde{\mu}, (f(\,,0), \partial_y f(\,,0)) \rangle \quad \text{for } f \in C_\Delta(\mathbb{R}^{n+1})$$

defines

$$(4.13) \qquad \mu \in C_\Delta(\mathbb{R}^{n+1})' \quad \text{and} \quad P(-D)\mu = \nu.$$

By Remark 3.2,

$$(4.14) \qquad \mu \in C_\Delta(\operatorname{conv}(\omega) \times I(\tau))'.$$

(ii) We now show that u_μ satisfies the assumption of Lemma 4.2.

Since $Y := \mathbb{R}^n \times\,]0, \infty[$ is convex, there is $F \in C_\Delta(Y)$ such that $P(D)F = G|_Y$ (see (2.5)). Let $\omega \subset\subset \omega_1 \subset\subset \Omega$ and choose $\widetilde{\omega}$ for ω_1 by $(\underline{A_\Omega})$. Let $\widehat{\omega} \subset\subset \Omega$ be fixed and let $\widehat{\omega} \subset\subset \widehat{\omega}_1 \subset\subset \Omega$. For $\xi \in \partial\widetilde{\omega}$ and $\widehat{\omega}_1$ choose $0 < \delta_\xi$ by $(\underline{A_\Omega})$. For $T_\xi := \delta_\xi/4$ choose $V_\xi \in \mathfrak{U}(\widehat{\omega}_1 \times \{T_\xi\})$ and $E_\xi \in C_\Delta(W_\xi)$,

$W_\xi := V_\xi \cup (\omega_1 \times I(\delta_\xi))$, by $(\underline{A_\Omega})$. Since $\widehat{\omega} \subset\subset \widehat{\omega}_1$ and $V_\xi \in \mathfrak{U}(\widehat{\omega}_1 \times \{T_\xi\})$, we may choose $0 < \alpha_\xi \leq \delta_\xi/16$ such that

$$(4.15) \qquad (\omega + U_{\alpha_\xi}) \subset \omega_1 \quad \text{and} \quad (\widehat{\omega} + U_{\alpha_\xi}) \times (T_\xi + I(2\alpha_\xi)) \subset V_\xi.$$

For $(x,y) \in U_{\alpha_\xi} \times I(\alpha_\xi)$ we then have

$$P(D)F(\cdot - x - \xi, \cdot + T_\xi - y) = G(\cdot - x - \xi, \cdot + T_\xi - y)$$
$$= P(D)E_\xi(\cdot - x, \cdot + T_\xi - y) \quad \text{on } \widehat{\omega} \times I(\tau), \text{ if } 0 < \tau \leq \alpha_\xi.$$

Hence

$$F(\cdot - x - \xi, \cdot + T_\xi - y) = E_\xi(\cdot - x, \cdot + T_\xi - y) + h_{\xi,x,y}$$

for some $h_{\xi,x,y} \in N_P(\widehat{\omega} \times I(\tau))$. Since $\nu(h_{\xi,x,y}) = 0$ by assumption, we thus get by (4.14) and (4.15), for $(x,y) \in U_{\alpha_\xi} \times I(\alpha_\xi)$,

$$
\begin{aligned}
u_\mu(\xi + x, -T_\xi + y) &= \langle \mu_{(s,t)}, G(s - \xi - x, t + T_\xi - y) \rangle \\
&= \langle \mu_{(s,t)}, P(D_s)F(s - \xi - x, T_\xi - y + t) \rangle \\
&= \langle P(-D)\mu_{(s,t)}, F(s - \xi - x, T_\xi - y + t) \rangle \\
(4.16) \qquad &= \langle \nu_{(s,t)}, F(s - \xi - x, T_\xi - y + t) \rangle \\
&= \langle \nu_{(s,t)}, E_\xi(s - x, T_\xi - y + t) \rangle =: v_\xi(x,y).
\end{aligned}
$$

Equality (4.16) follows from (4.13), since $\mathbb{R}^n \times I(\tau)$ satisfies (3.1). v_ξ is harmonic on $U := U_{\alpha_\xi} \times I(\delta_\xi/2)$ by (4.15), since $E_\xi \in C_\Delta(W_\xi)$ and $\tau \leq \alpha_\xi \leq \delta_\xi/4$. Thus u_μ can be extended as a harmonic function to $U_{\alpha_\xi}(\xi) \times]-\infty, \delta_\xi/4[$. Choose $\xi_1, \ldots, \xi_d \in \partial\widetilde{\omega}$ such that $Z := \bigcup_{j=1}^d U_{\alpha_{\xi_j}}(\xi_j) \supset \partial\widetilde{\omega}$ and let

$$\tau < \beta := \min\{\alpha_{\xi_j} \mid j \leq d\}.$$

Then

$$(4.17) \qquad u_\mu \text{ extends to } Z \times]-\infty, \delta[\text{ (and to } Z \times]-\delta, \infty[)$$

from below (and from above, respectively) for $\delta := \min\{\delta_{\xi_j}/4 \mid j \leq d\}$.

(iii) Since Ω is P-convex for supports by Lemma 4.1(b), there is $\widetilde{\omega}_1 \subset\subset \Omega$ by Lemma 4.2 and (4.17) such that $\mu \in C_\Delta(\widetilde{\omega}_1 \times \mathbb{R})'$. Hence $\mu \in C_\Delta(\widetilde{\omega}_1 \times I(\tau))'$ by (4.14) and $P(-D)\mu = \nu$ in $C_\Delta(\widetilde{\omega}_1 \times I(\tau))'$ by (4.13) since $\widetilde{\omega}_1 \times I(\tau)$ satisfies (3.1). Thus,

$$\langle \nu, f \rangle = \langle P(-D)\mu, f \rangle = \langle \mu, P(D)f \rangle = 0 \quad \text{for } f \in N_P(\widetilde{\omega}_1 \times I(\tau)) \text{ if } \tau < \beta.$$

(b) This follows from (a) and the definition in (1.8) since $NB_P(\omega \times I(\tau))$ is continuously embedded in $N_P(\omega \times I(\tau))$ and the restriction

$$R_1 : N_P(\omega \times I(\tau)) \to NB_P(\omega_1 \times I(\tau_1))$$

is continuous if $\omega_1 \subset\subset \omega$ and $0 < \tau_1 < \tau$. ∎

The next theorem is the crucial step in the proof of the sufficiency of $(\underline{A_\Omega})$, since it will give the property (P_3) from Theorem 1.5 for N_P^Ω (see the

proof of Theorem 4.5). Let

$$J(\delta) := [-\delta, \delta] \quad \text{for } \delta > 0.$$

THEOREM 4.4. *Let* $P(D)$ *satisfy* $(\underline{A_\Omega})$.

(a) *For any compact* $K \subset \Omega$ *there is a compact* $\widetilde{K} \subset \Omega$ *with* $K \subset \widetilde{K}$ *such that for any compact* $Z \subset \Omega$ *with* $\widetilde{K} \subset Z$ *there is* $\delta > 0$ *such that for any* $0 < \gamma < \delta$ *there are a compact* $\widetilde{Z} \subset \Omega$ *with* $Z \subset \widetilde{Z}$ *and* $0 < \beta < \gamma$ *such that*

$$P(D)C_\Delta((K \times J(\delta)) \cup (Z \times J(\beta)))$$
$$\supset C_\Delta((\widetilde{K} \times J(2\delta)) \cup (\widetilde{Z} \times J(\gamma)))|_{(K \times J(\delta)) \cup (Z \times J(\beta))}.$$

(b) *For any* $\omega \subset\subset \Omega$ *there is* $\widetilde{\omega} \subset\subset \Omega$ *with* $\omega \subset\subset \widetilde{\omega}$ *such that for any* $\widehat{\omega} \subset\subset \Omega$ *with* $\widetilde{\omega} \subset\subset \widehat{\omega}$ *there is* $\delta > 0$ *such that for any* $0 < \gamma < \delta$ *there are* $0 < \beta < \gamma$ *and a continuous linear operator*

$$S : C_\Delta((\widetilde{\omega} \times \mathbb{R}) \cup (\mathbb{R}^n \times I(\gamma))) \to C_\Delta((\omega \times I(\delta)) \cup (\widehat{\omega} \times I(\beta)))$$

such that

$$P(D)S(f) = f \quad \text{on } (\omega \times I(\delta)) \cup (\widehat{\omega} \times I(\beta)).$$

Proof. (a) (I) For compact $K \subset \Omega$ choose $\omega \subset\subset \Omega$ such that $K \subset \omega$. Then choose $\widetilde{\omega}$ for ω by $(\underline{A_\Omega})$. Choose a compact $\widetilde{K} \subset \Omega$ with $\overline{\widetilde{\omega}} \subset \text{int}(\widetilde{K})$. For $Z \supset \widetilde{K}$ fixed choose $\widehat{\omega} \subset\subset \Omega$ with $\widehat{\omega} \supset Z$. We will apply Remark 3.1 for $Q := (K \times J(\delta)) \cup (Z \times J(\beta))$ and $L := (\widetilde{K} \times J(2\delta)) \cup (\widetilde{Z} \times J(\gamma))$, where $\delta > \gamma > \beta > 0$ and $\widetilde{Z}$ will be specified in the proof.

Let $\mu \in C_\Delta(Q)'$. By Remark 3.2, for any compact $X \subset \mathbb{R}^{n+1} \setminus (\text{conv}(Z) \times J(\delta))$ and for any $a \in \mathbb{N}_0^n$ there is a bounded set $B \subset C_\Delta(Q)$ such that

$$(4.18) \qquad \sup_{(x,y)\in X} |D^a u_\mu(x,y)| \leq C \sup_{f\in B} |\langle P(-D)\mu, f\rangle|.$$

(II) Using $(\underline{A_\Omega})$, we now obtain additional uniform bounds for u_μ in several steps.

(i) For $\xi \in \partial\widetilde{\omega}$ choose $\delta_\xi > 0$ by $(\underline{A_\Omega})$. If $\delta \leq \delta_\xi/2$, we may choose $V_{\xi,T} \in \mathfrak{U}(\widehat{\omega} \times \{T\})$ and $E_\xi \in C_\Delta(V_{\xi,T} \cup (\overline{\omega} \times]T - \delta_\xi, T + \delta_\xi[))$ for $0 < T \leq 2\delta$ by $(\underline{A_\Omega})$. Since $V_{\xi,T} \in \mathfrak{U}(\widehat{\omega} \times \{T\})$, $K \subset \omega$ and $Z \subset \widehat{\omega}$, there are $\varrho > 0$ and $0 < \nu_{\xi,T} \leq \delta_\xi/4$ such that

$$(4.19) \qquad K + \overline{U}_\varrho \subset \omega \quad \text{and} \quad (Z + \overline{U}_\varrho) \times [T - 2\nu_{\xi,T}, T + 2\nu_{\xi,T}] \subset V_{\xi,T}.$$

For $(x,y) \in \overline{U}_\varrho \times J(\nu_{\xi,T})$ we then get, for $a \in \mathbb{N}_0^{n+1}$,

$$(4.20) \qquad D^a u_\mu(x + \xi, y - T) = \langle \mu_{(s,t)}, D^a G(s - \xi - x, t + T - y)\rangle$$
$$= \langle \mu_{(s,t)}, P(D_s)D^a E_\xi(s - x, t + T - y)\rangle$$
$$= \langle P(-D)\mu_{(s,t)}, D^a E_\xi(s - x, t + T - y)\rangle$$

if $\delta \leq \delta_\xi/4$ and $\beta \leq \nu_{\xi,T}$. We may choose $\{\xi_1, \ldots, \xi_k\} \subset \partial\widetilde{\omega}$ such that

$$\bigcup_{j \leq k} U_\varrho(\xi_j) \supset \partial\widetilde{\omega}.$$

Set

$$\delta := \min\{\delta_{\xi_j}/4 \mid j \leq k\}.$$

For $j \leq k$ we may choose $T_{j,d} \in [\gamma/2, 2\delta]$ for $d \leq \lambda(j)$ such that

$$\bigcup_{d \leq \lambda(j)}]-T_{j,d} - \nu_{\xi_j,T_{j,d}}, -T_{j,d} + \nu_{\xi_j,T_{j,d}}[\supset [-2\delta, -\gamma/2]$$

and therefore

$$V := \bigcup_{j \leq k, d \leq \lambda(j)} U_\varrho(\xi_j) \times]-T_{j,d} - \nu_{\xi_j,T_{j,d}}, -T_{j,d} + \nu_{\xi_j,T_{j,d}}[\supset \partial\widetilde{\omega} \times [-2\delta, -\gamma/2].$$

Let $\beta \leq \nu_1 := \min\{\nu_{\xi_j,T_{j,d}} \mid d \leq \lambda(j), j \leq k\}$. Then (4.21) shows that for any $a \in \mathbb{N}_0^{n+1}$,

$$\sup_{(x,y)\in V} |D^a u_\mu(x,y)| \leq \sup_{f\in\widetilde{B}} |\langle P(-D)\mu, f\rangle|,$$

where $\widetilde{B} := \{D^a E_{\xi_j}(\cdot - x, \cdot + T_{j,d} - y) \mid j \leq k, d \leq \lambda(j), |x| \leq \nu_{\xi_j,T_{j,d}}, y \in J(\nu_{\xi_j,T_{j,d}})\}$ is bounded in $C_\Delta(Q)$ by (4.19). Applying (4.21) for $u_\mu(\cdot, -\cdot)$ instead of u_μ, we get a bounded set $\widetilde{B} \subset C_\Delta(Q)$ such that for $V_1 := \{(x,y) \in \mathbb{R}^{n+1} \mid (x, |y|) \in V\}$,

$$(4.21) \qquad \sup_{(x,y)\in V_1} |D^a u_\mu(x,y)| \leq \sup_{f\in\widetilde{B}} |\langle P(-D)\mu, f\rangle|.$$

(ii) We now decompose u_μ into $v_\mu + w_\mu$, keeping track of uniform estimates.

Choose a neighbourhood U_1 of $\partial\widetilde{\omega}$ such that

$$U_1 \subset\subset \text{int}(\widetilde{K}) \quad \text{and} \quad U_1 \times]\gamma/2, 2\delta[\subset V_1$$

and set $U := U_1 \cup \omega$. Choose $\varphi \in C_0^\infty(U \times I(2\delta))$ such that $\varphi = 1$ near $\overline{\widetilde{\omega}} \times J(\delta)$. Let $\widetilde{f}_\mu := \Delta(\varphi u_\mu)|_W$ for $W := \mathbb{R}^n \times (\mathbb{R} \setminus J(3\gamma/4))$. By (4.18) and (4.21), $\widetilde{f}_\mu$ can be extended trivially (i.e. by taking its value equal to 0 outside W) to a bounded function f_μ on $\mathbb{R}^{n+1}$ such that

$$(4.22) \qquad \text{supp}\, f_\mu \subset (\text{supp}\,\text{grad}\,\varphi) \cap \overline{W} =: \widetilde{W} \subset V_1 \cup (\omega \times (I(2\delta) \setminus J(\delta))).$$

Moreover, since $P(-D)f_\mu$ is the trivial extension of $P(-D)\widetilde{f}_\mu$ also $P(-D)f_\mu$ is a bounded function and

$$(4.23) \qquad \sup\{|P(-D)f_\mu(x,y)| + |f_\mu(x,y)| \mid (x,y) \in \mathbb{R}^{n+1}\}$$
$$\leq C_1 \sup_{\substack{(x,y)\in\widetilde{W} \\ |a|\leq m+2}} |D^a u_\mu(x,y)| \leq C_2 \sup_{f\in B\cup\widetilde{B}} |\langle P(-D)\mu, f\rangle|$$

by (4.22), (4.21) and (4.18). Let $g_\mu := G * f_\mu$. Then

$$(4.24) \quad \sup\{|P(-D)g_\mu(x,y)| + |g_\mu(x,y)| \mid (x,y) \in \mathbb{R}^{n+1}\}$$
$$\leq C_3 \sup_{f \in B \cup \widetilde{B}} |\langle P(-D)\mu, f\rangle|$$

by (4.23). Set $v_\mu := -g_\mu|_{\mathbb{R}^{n+1} \setminus (\widetilde{K} \times J(2\delta))}$ and $w_\mu := (1-\varphi)u_\mu + g_\mu|_{\mathbb{R}^{n+1} \setminus (Z \times J(\gamma))}$, where w_μ is defined by trivial extension of $(1-\varphi)u_\mu$. Then we have $v_\mu \in C_{\Delta,0}(\mathbb{R}^n \setminus (\widetilde{K} \times J(2\delta)))$ and $w_\mu \in C_{\Delta,0}(\mathbb{R}^{n+1} \setminus (Z \times J(\gamma)))$. Moreover,

$$(4.25) \qquad\qquad u_\mu = v_\mu + w_\mu \quad \text{on } \mathbb{R}^{n+1} \setminus L_1$$

for $L_1 := (\widetilde{K} \times J(2\delta)) \cup (Z \times J(\gamma))$ by the choice of φ.

(iii) Let $Y \subset \mathbb{R}^{n+1} \setminus L$ be compact. Clearly,

$$(4.26) \qquad\qquad \sup_{(x,y) \in Y} |v_\mu(x,y)| \leq C_3 \sup_{f \in B \cup \widetilde{B}} |\langle P(-D)\mu, f\rangle|$$

by (4.24). Since $w_\mu \in C_{\Delta,0}(\mathbb{R}^{n+1} \setminus (Z \times J(\gamma)))$, we can find a compact $\widetilde{Z} \subset \Omega$ with $\widetilde{Z} \supset Z$ (using $(\underline{A_\Omega})$ for $\omega_1 \supset Z$ as above) such that for suitable compacts $Y_1 \subset \widetilde{Y} \subset (\mathbb{R}^n \times I(2\gamma)) \setminus (\widetilde{Z} \times J(\gamma))$ (use (3.3)), sufficiently small $\gamma > 0$ and $V_2 := (V_1 \cup \omega) \times \mathbb{R}$,

$$(4.27) \qquad \sup_{(x,y) \in Y} |w_\mu(x,y)|$$
$$\leq C_4 \sup_{(x,y) \in Y_1} |w_\mu(x,y)| \leq C_5 \sup_{(x,y) \in \widetilde{Y}} |P(-D)w_\mu(x,y)|$$
$$\leq C_6 \Big(\sup_{f \in B \cup \widetilde{B}} |\langle P(-D)\mu, f\rangle| + \sup_{(x,y) \in \widetilde{Y} \setminus V_2} |P(-D)u_\mu(x,y)|$$
$$+ \sup_{(x,y) \in \widetilde{Y} \cap V_1} |P(-D)(\varphi u_\mu)(x,y)| \Big)$$
$$\leq C_7 \sup_{f \in \widehat{B}} |\langle P(-D)\mu, f\rangle|$$

for some bounded set $\widehat{B} \subset C_\Delta(Q)$, where the last estimates follow from (4.24), the choice of φ, (4.21) and (4.22). The claim in (a) now follows from (4.25)–(4.27) by Remark 3.1.

(b) Let $\omega \subset\subset \omega_1 \subset\subset \Omega$ and set $K := \overline{\omega}_1$. Choose $\widetilde{K}$ for K by (a) and $\widetilde{\omega} \subset\subset \Omega$ such that $\widetilde{\omega} \supset\supset \widetilde{K}$. Fix $\widehat{\omega} \subset\subset \Omega$ with $\widehat{\omega} \supset \widetilde{\omega}$ and set $Z := \overline{\widehat{\omega}}$. Choose $\delta > 0$ by (a), fix $0 < \gamma < \delta$ and choose $0 < \gamma_1 < \gamma$. Finally, choose $0 < \beta < \gamma_1$ for γ_1 by (a). Then the mapping

$$P(D)^{-1} : C_\Delta((\widetilde{\omega} \times \mathbb{R}) \cup (\mathbb{R}^n \times I(\gamma))) \to C_\Delta(W)/N_P(W),$$

$W := (\omega_1 \times I(\delta)) \cup (\widehat{\omega}_1 \times I(\beta))$, is defined and linear by (a). It is continuous

by the closed graph theorem. Let

$$L_{2,\Delta} := L_2(\widetilde{W}) \cap C_\Delta(\widetilde{W}), \quad \widetilde{W} := (\omega \times I(\delta/2)) \cap (\widehat{\omega} \times I(\beta/2)).$$

Then $L_{2,\Delta}$ is a Hilbert space and $N_{P,L_2} := L_{2,\Delta} \cap \ker P(D)$ is a closed subspace. Let

$$L : L_{2,\Delta}/N_{P,L_2} \to (N_{P,L_2})^\perp$$

be the canonical topological isomorphism. Let

$$J : C_\Delta(W)/N_P(W) \to L_{2,\Delta}/N_{P,L_2}$$

be defined by restriction. Since J is continuous,

$$S := L \circ J \circ P(D)^{-1} : C_\Delta((\widetilde{\omega} \times \mathbb{R}) \cup (\mathbb{R}^n \times I(\gamma))) \to C_\Delta(\widetilde{W})$$

is continuous and $P(D)(S(f)) = f$ on $\widetilde{W} = (\omega \times I(\delta/2)) \cup (\widehat{\omega} \times I(\beta/2))$. ∎

THEOREM 4.5 (semiglobal decomposition with bounds). *Let $P(D)$ satisfy $(\underline{A_\Omega})$.*

(a) *For any compact $K \subset \Omega$ there is a compact $\widetilde{K} \subset \Omega$ with $K \subset \widetilde{K}$ such that for any compact $Q \subset \Omega$ with $\widetilde{K} \subset Q$ there is $n \in \mathbb{N}$ such that for any $m \geq n$ there are $N \geq m$ and a continuous linear operator*

$$L = (L_1, L_2) : NB_P((\widetilde{K} \times \{0\}) + V_{1/m})$$
$$\to NB_P((Q \times \{0\}) + V_{1/N}) \times NB_P((K \times \{0\}) + V_{1/n})$$

such that $L_1(f) + L_2(f) = f$ on $(K \times \{0\}) + V_{1/N}$.

(b) *N_P^Ω satisfies (P_3).*

Proof. (b) This follows directly from (a) by the definitions of N_P^Ω and (P_3).

(a) Notice that we have the continuous restrictions

$$\varrho : N_P(\omega \times I(\gamma)) \to NB_P((K \times \{0\}) + V_\delta) \quad \text{if } \delta < \gamma \text{ and } K + \overline{V}_\delta \subset \omega,$$
$$\widetilde{\varrho} : NB_P((K \times \{0\}) + V_\delta) \to N_P(\omega \times I(\gamma)) \quad \text{if } \gamma \leq \delta \text{ and } \omega \subset K.$$

The claim in (a) thus follows if we show the following

CLAIM. *For any $\omega \subset\subset \Omega$ there are $\widetilde{\omega} \subset\subset \Omega$ such that for any $\widehat{\omega} \subset\subset \Omega$ there is $\delta > 0$ such that for any $0 < \gamma < \delta$ there are $0 < \beta < \gamma$ and a continuous linear operator*

$$L = (L_1, L_2) : N_P(\widetilde{\omega} \times I(\gamma)) \to N_P(\widehat{\omega} \times I(\beta)) \times N_P(\omega \times I(\delta))$$

such that $L_1(f) + L_2(f) = f$ on $\omega \times I(\beta)$.

To prove the Claim, choose $\widetilde{\omega}$ for ω by Theorem 4.4(b).

(i) Let $\widetilde{\omega} \subset\subset \widetilde{\omega}_1 \subset\subset \Omega$ and $\gamma > 0$. There is a continuous linear operator

$$R = (R_1, R_2) : C_\Delta(\widetilde{\omega}_1 \times I(\gamma)) \to C_\Delta(\widetilde{\omega} \times \mathbb{R}) \times C_\Delta(\mathbb{R}^n \times I(\gamma/2))$$

such that $R_1(f) + R_2(f) = f$ on $\widetilde{\omega} \times I(\gamma/2)$. Indeed, let $\varphi \in D(\widetilde{\omega}_1)$ be such that $\varphi = 1$ near $\overline{\widetilde{\omega}}$. For $f \in C_\Delta(\widetilde{\omega}_1 \times I(\gamma))$, the function $\widetilde{f} := \Delta(\varphi f)|_{\mathbb{R}^n \times I(\gamma/2)}$ defines a function f_1 on $\mathbb{R}^{n+1}$ (by trivial extension) which is bounded and has compact support. Then

$$R_1(f) := G * f_1|_{\widetilde{\omega} \times \mathbb{R}} \quad \text{and} \quad R_2(f) := (\varphi f - G * f_1)|_{\mathbb{R}^n \times I(\gamma/2)}$$

have the required properties.

(ii) For $f \in N_P(\widetilde{\omega}_1 \times I(\gamma))$ we have $P(D)R_1(f) = -P(D)R_2(f)$ on $\widetilde{\omega} \times I(\gamma/2)$. Thus

$$\widetilde{R}(f) := \begin{cases} P(D)R_1(f) & \text{on } \widetilde{\omega} \times \mathbb{R}, \\ -P(D)R_2(f) & \text{on } \mathbb{R}^n \times I(\gamma/2), \end{cases}$$

defines a linear and continuous operator

$$\widetilde{R} : N_P(\widetilde{\omega}_1 \times I(\gamma)) \to C_\Delta(W), \quad W := (\widetilde{\omega} \times \mathbb{R}) \cup (\mathbb{R}^n \times I(\gamma/2)).$$

Fix $\widehat{\omega} \subset\subset \Omega$ with $\widetilde{\omega}_1 \subset \widehat{\omega}$ and choose $\delta > 0$ by Theorem 4.4(b). Let $0 < \gamma < \delta$ and choose $0 < \beta$ for $\gamma/2$ by Theorem 4.4(b). For $f \in N_P(\widetilde{\omega}_1 \times I(\gamma))$ let

$$L_1(f) := (R_2(f) + S \circ \widetilde{R}(f))|_{\widehat{\omega} \times I(\beta)}, \quad L_2(f) := (R_1(f) - S \circ \widetilde{R}(f))|_{\omega \times I(\delta)},$$

with the operator S from Theorem 4.4(b). Then L has the properties stated in the Claim for $\widetilde{\omega}_1$ in place of $\widetilde{\omega}$ (use also (i)). ∎

Summarizing the results proved so far, we get the following characterization which proves the first part of the Main Theorem 2.1.

THEOREM 4.6. *The following statements are equivalent*:

(i) $P(D)$ *is surjective on* $A(\Omega)$.
(ii) *For any* $V \in \mathfrak{U}(\Omega)$ *there is* $W \in \mathfrak{U}(\Omega)$ *with* $W \subset V$ *such that*

$$P(D)C_\Delta(W) \supset C_\Delta(V)|_W.$$

(iii) $P(D)$ *satisfies* $(\underline{\mathrm{A}_\Omega})$.
(iv) $P(D)$ *satisfies* $(\overline{\mathrm{A}_\Omega})$.

Proof. (i)⇒(iv). This was proved in Theorem 3.4.

(iv)⇒(iii) and (ii)⇒(i). These are trivial.

(iii)⇒(ii). The sequence (1.3) of spectra is exact by Lemma 4.1(a). The projective (DFS)-spectrum N_P^Ω is *sw*-reduced by Proposition 4.3(b). Furthermore, N_P^Ω satisfies (P_3) by Theorem 4.5(b). The claim thus follows from Theorem 1.5. ∎

5. Regular hyperfunction elementary solutions. In this section the proof of Theorem 2.1 will be completed. The main point is to prove that the hyperfunction criterion $(\overline{\mathrm{B}_\Omega})$ from Theorem 2.1(e) is necessary for the surjectivity of $P(D)$ on $A(\Omega)$. For this we have to solve the equation $P(D)f = g$ for $g \in C_\Delta(W)$, where W is a nonconvex open set in $\mathbb{R}^{n+1}$

(see Theorem 5.4 below). We thus need a variant of Remark 3.1 for open sets, which is provided by the following lemma. Notice that the assumptions are stronger as compared with Remark 3.1, due to the more complicated topology of $C_\Delta(W)'_\mathrm{b}$.

Similarly to Section 3 we will need the fact that $C_\Delta(\mathbb{R}^{n+1})$ is dense in $C_\Delta(W)$ for the open sets $W \subset \mathbb{R}^{n+1}$ used in this section. This again follows from Hörmander [14, Corollary 10.5.3] if W satisfies (3.1), i.e.

$$\mathbb{R}^{n+1} \setminus W \text{ does not have a bounded component.}$$

This condition holds for the open sets W needed here since always

$$W \cap (\{x\} \times \mathbb{R}) \text{ is an interval for any } x \in \mathbb{R}^n.$$

In particular, the neighbourhoods $V \in \mathfrak{U}(\Omega \times \{T\})$ can be chosen in this way (e.g. as the union of sets $\omega_j \times \ ]T - \varepsilon_j, T + \varepsilon_j[$).

LEMMA 5.1. *Let* $W \subset V \subset \mathbb{R}^{n+1}$ *be open sets satisfying* (3.1). *Then*

$$P(D)C_\Delta(W) \supset C_\Delta(V)|_W$$

if for any bounded set $B \supset C_\Delta(W)'_\mathrm{b}$ *the set*

(5.1) $\widetilde{B} := \{\mu \in C_\Delta(V)'_\mathrm{b} \mid P(-D)\mu \in B\}$ *is bounded in* $C_\Delta(V)'_\mathrm{b}$.

Proof. Let $E := C_\Delta(W)'_\mathrm{b}$ and $F := C_\Delta(V)'_\mathrm{b}$. Then (3.1) and (5.1) imply that $H := P(-D)F \cap E$ is sequentially closed in E since F is a (DFS)-space. Thus, H is closed in E (by Komatsu [19, Theorem 6′]) since E is a (DFS)-space. Therefore, H is a (DFS)-space. Since $P(-D)$ is injective on $\{\mu \in F \mid P(-D)\mu \in E\}$ by (5.1),

$${}^t R_V^W \circ P(-D)^{-1} : H \to F$$

is defined, linear and continuous by the closed graph theorem. The proof is now completed as in Remark 3.1. ∎

The assumptions of Lemma 5.1 will be checked in two steps: roughly speaking, we will first give bounds for the "supports" of $\mu \in C_\Delta(W)'$ in terms of $\mathrm{supp}(P(-D)\mu)$ (see Lemma 5.3, this step plays the role of $P(D)$-convexity for supports). In the second step, we then estimate u_μ in terms of $u_{P(-D)\mu}$. For the first step, the following remark is useful: Let

$$[\omega]_T := \omega \times \ ]T, \infty[\quad \text{for } T \in \mathbb{R}.$$

REMARK 5.2. *Let* $P(D)$ *satisfy* $(\overline{\mathrm{A}_\Omega})$. *For any* $\omega \subset\subset \Omega$ *there are* $\widetilde{\omega} \subset\subset \Omega$ *with* $\widetilde{\omega} \supset\supset \omega$ *and* $\delta > 0$ *such that for any* $\xi \in \Omega \setminus \widetilde{\omega}$ *and any* $0 < T \le 1$ *there are* $F_\xi \in C_\Delta(Y)$, $Y := [\Omega]_T \cup [\omega]_{T-\delta}$, *and* $h_\xi \in C_\Delta(\Omega \times \mathbb{R})$ *such that*

$$P(D)F_\xi = (G(\cdot - \xi, \cdot) + h_\xi)|_Y.$$

Proof. Choose $\widetilde{\omega} \subset\subset \Omega$ and $\delta > 0$ by $(\overline{\mathrm{A}_\Omega})$. For $\xi \in \Omega \setminus \widetilde{\omega}$ choose $V \in \mathfrak{U}(\Omega \times \{T\})$ and $E_\xi \in C_\Delta(W)$, $W := V \cup (\omega \times \ ]T - \delta, T + \delta[)$, by $(\overline{\mathrm{A}_\Omega})$. Take $\varphi \in C^\infty(\Omega \times \mathbb{R})$ such that $\varphi = 1$ near $\Omega \times \ ]-\infty, T]$ and $\mathrm{supp}\,\varphi \subset$

$V \cup (\Omega \times \,]-\infty, T])$. Then $\Delta(E_\xi \varphi)$ is defined on W and can be extended trivially to $g \in C^\infty(\Omega \times \mathbb{R})$. Since Δ is elliptic, there is $g_1 \in C^\infty(\Omega \times \mathbb{R})$ such that $\Delta g_1 = g$. Moreover, $E_\xi \varphi$ is defined on W and can be extended trivially to $E_1 \in C^\infty(W \cup [\Omega]_T)$. Set $Y := [\Omega]_T \cup [\omega]_{T-\delta}$ and $F_\xi := (E_1 - g_1)|_Y$. Then $F_\xi \in C_\Delta(Y)$ and $P(D)F_\xi = H_\xi|_Y$, where

$$H_\xi := G(\,\cdot - \xi, \cdot\,)\varphi - P(D)g_1 + \sum_{a \neq 0} P^{(a)}(D)E_\xi D^a \varphi/a! \in D'(\omega \times \mathbb{R})$$

(the last terms are again extended trivially). Since

$$\Delta(P(D)F_\xi) = \Delta H_\xi = \delta_{(\xi,0)} = \Delta G(\,\cdot - \xi, 0) \quad \text{on } \omega \times \mathbb{R},$$

the remark is proved. ∎

LEMMA 5.3. *Let $P(D)$ satisfy $(\overline{A_\Omega})$. Then there are a relatively compact exhaustion $\{\omega_j \mid j \in \mathbb{N}\}$ of Ω and a decreasing sequence $\delta_j > 0$ such that for any decreasing sequence $0 < \tau_j < \delta_j$ and any $\mu \in C_\Delta(\Omega \times \mathbb{R})'$,*

$$\mu \in C_\Delta\Big(\bigcup_{j \in \mathbb{N}}(\omega_{j+1} \times I(\tau_j))\Big)' \quad \text{if} \quad P(-D)\mu \in C_\Delta\Big(\bigcup_{j \in \mathbb{N}}(\omega_j \times I(\tau_j))\Big)'.$$

Proof. With $\omega \subset\subset \Omega$ fixed set $\omega_1 := \omega$. For $j \in \mathbb{N}$ choose $\widetilde{\omega} =: \omega_{j+1}$ and $\delta := \delta_j$ for ω_j by $(\overline{A_\Omega})$. We may assume that $\{\omega_j \mid j \in \mathbb{N}\}$ is a relatively compact exhaustion of Ω. Let $0 < \tau_j < \delta_j$ and $\mu \in C_\Delta(\Omega \times \mathbb{R})'$ with $P(-D)\mu \in C_\Delta(\bigcup_{j \in \mathbb{N}}(\omega_j \times I(\tau_j)))'$. Then

$$\mu \in C_\Delta(K \times J(C))' \quad \text{and} \quad P(-D)\Big(\bigcup_{j \leq J}(K_j \times J(\tau_j - \varepsilon_j))\Big)'$$

for suitable $C, J \in \mathbb{N}$, $0 < \varepsilon_j < \tau_j$ and compacts $K \subset \Omega$ and $K_j \subset \omega_j$. We may assume that

$$K_j + U_{\varepsilon_j} \subset \omega_j \quad \text{and} \quad K + U_{\varepsilon_1} \subset \Omega.$$

For $\xi \in \Omega \setminus \omega_2$ and $0 < T \leq 1$ we choose $F_\xi \in C_\Delta([\Omega]_T \cup [\omega_1]_{T-\delta_1})$ and $h_\xi \in C_\Delta(\Omega \times \mathbb{R})$ by Remark 5.2. Then we have, for $y < -C$ and $x \in U_{\varepsilon_1}$,

$$u_\mu(x + \xi, y - T)$$
$$= \langle \mu_{(s,t)}, G(s - \xi - x, t + T - y)\rangle$$
$$= \langle \mu_{(s,t)}, P(D_s)E_\xi(s - x, t + T - y) - h_\xi(s - x, t + T - y)\rangle$$
$$= \langle P(-D)\mu_{(s,t)}, E_\xi(s - x, T + t - y)\rangle - \langle \mu_{(s,t)}, h(s - x, t + T - y)\rangle$$
$$:= v_\mu(x, y).$$

Now, v_μ is a harmonic function on $U_{\varepsilon_1}(\xi) \times \,]-\infty, -\tau_2 + \varepsilon_2[$. With the same argument applied to $u_\mu(\cdot, -\cdot)$ we see that $\mu \in C_\Delta((\omega_2 \times I(\tau_1)) \cup ((\Omega \setminus \omega_2) \times I(\tau_2)))'$. As in part (a)(II)(ii) of the proof of Theorem 4.4 we now decompose

μ as $\mu = \mu_1 + \mu_2$ with $\mu_1 \in C_\Delta(\omega_2 \times I(\tau_1))'$ and $\mu_2 \in C_\Delta(\Omega \times \mathbb{R})'$, with

$$P(-D)\mu_2 \in C_\Delta\Big(\bigcup_{j \geq 2} \omega_j \times I(\tau_j)\Big)'.$$

The lemma is thus proved by applying the same argument to $\mu_2, \mu_3, \dots$ inductively. ∎

The hyperfunction criterion $(\overline{B_\Omega})$ in Theorem 2.1 will be a consequence of the result below. For $Z \subset \mathbb{R}^{n+1}$ let

$$Z_+ := \{(x,y) \in Z \mid y > 0\}.$$

THEOREM 5.4. *Let $P(D)$ be surjective on $A(\Omega)$. Then there is $Z \in \mathfrak{U}(\Omega)$ such that for any $\omega \subset\subset \Omega$ there are $\widetilde{\omega} \subset\subset \Omega$ with $\omega \subset\subset \widetilde{\omega}$ and $\delta > 0$ such that for any $0 < \widetilde{\delta} \leq \delta$,*

$$P(D)C_\Delta(Z_+ \cup (\omega \times I(\widetilde{\delta}))) \supset C_\Delta([\Omega]_0 \cup [\widetilde{\omega}]_{-6\widetilde{\delta}})|_{Z_+ \cup (\omega \times I(\widetilde{\delta}))}.$$

Proof. (a) We will apply Lemma 5.1 for $W := Z_+ \cup (\omega \times I(\widetilde{\delta}))$ and $V := [\Omega]_0 \cup [\widetilde{\omega}]_{-6\widetilde{\delta}}$. Let $B \subset C_\Delta(W)'_b$ be bounded and

$$\widetilde{B} := \{\mu \in C_\Delta(V)' \mid \nu := P(-D)\mu \in B\}.$$

Choose an increasing relatively compact exhaustion $\{\omega_j \mid j \in \mathbb{N}\}$ of Ω and a decreasing sequence $\delta_j > 0$ by Lemma 5.3. By Theorem 4.6(ii) for $\Omega \times \mathbb{R}$ there is $X \in \mathfrak{U}(\Omega)$ such that

(5.2) $$P(D)C_\Delta(X) \supset C_\Delta(\Omega \times \mathbb{R})|_X.$$

We may assume that

$$X = \bigcup_{j \geq 1} \omega_{j+1} \times I(\tau_j)$$

for a decreasing sequence $0 < \tau_j < \delta_j$. Let

$$Z := \bigcup_{j=1}^{\infty} \omega_j \times I(\tau_j).$$

We may assume that $\omega \times I(\widetilde{\delta}) \subset Z$, and therefore

$$W := Z_+ \cup (\omega \times I(\widetilde{\delta})) = \bigcup_{j=1}^{\infty} (\omega_j \times \left]0, \tau_j\right[) \cup (\omega \times I(\widetilde{\delta})).$$

Since

$$B \subset C_\Delta(W)' \subset C_\Delta(Z)',$$

we have $\widetilde{B} \subset C_\Delta(X)'$ by Lemma 5.3, and therefore

(5.3) $$\widetilde{B} = \{\mu \in C_\Delta(X)' \mid \nu := P(-D)\mu \in B\}.$$

(b) There are $J \in \mathbb{N}$ and $\varrho \geq \tau_1$ such that

(5.4) $\qquad\qquad \widetilde{B}$ is bounded in $C_\Delta(\omega_J \times I(\varrho))'$.

To prove this, we will use Bonet and Galbis [5, Lemma 2]. The corresponding assumptions are easily checked: The restriction

$$j : F := C_\Delta(\Omega \times \mathbb{R}) \to E := C_\Delta(X)$$

is injective with dense range since $C_\Delta(\mathbb{R}^{n+1})$ is dense in $C_\Delta(X)$ by (3.1). Also $P(D)C_\Delta(\Omega \times \mathbb{R}) \subset C_\Delta(\Omega \times \mathbb{R})$ and $P(D)C_\Delta(X) \subset C_\Delta(X)$ are both dense since $P(D)C_\Delta(\mathbb{R}^{n+1}) = C_\Delta(\mathbb{R}^{n+1})$. Therefore (5.2) and (5.3) imply by [5, Lemma 2] that $\widetilde{B}$ is bounded in $C_\Delta(\Omega \times \mathbb{R})'_{\mathrm{b}}$, since B is also bounded in $C_\Delta(X)'_{\mathrm{b}}$. This proves (5.4).

(c) We may assume that $\omega \subset\subset \omega_J$ and that B is bounded in $C_\Delta(K)'_{\mathrm{b}}$, where

$$K := \bigcup_{j=1}^{J} (\overline{\omega}_j \times [\gamma, \tau_j]) \cup \overline{(\omega \times I(\widetilde{\delta}))}$$

for some $0 < \gamma < \tau_{J+1}$.

Let $\mu \in \widetilde{B}$ and $\omega \subset\subset \omega_1 \subset\subset \Omega$. With $0 < \beta < \min(\gamma, \tau_{J+1} - \gamma)$ to be fixed later, we now decompose u_μ just as in the proof of Theorem 4.4: Take $\varphi \in C_0^\infty((\omega_{J+1} \times \,]\gamma - \beta, \varrho + 1[) \cup (\omega_1 \times I(2\widetilde{\delta})))$ such that $\varphi = 1$ near $(\overline{\omega}_J \times [\gamma, \varrho]) \cup (\omega \times I(\widetilde{\delta}))$. Let $f_{\mu,+}$ be the trivial extension of $\widetilde{f}_{\mu,+} := \Delta(\varphi u_\mu)|_{L_+}$, where $L_+ := \mathbb{R}^n \times \,]\gamma + \beta, \infty[$. Set

$$u_{\mu,+} := (G * f_{\mu,+})|_{\mathbb{R}^{n+1} \setminus K_+}, \qquad K_+ := \overline{\omega}_{J+1} \times [\gamma + \beta, \varrho + 1].$$

Similarly, let $f_{\mu,-}$ be the trivial extension of $\widetilde{f}_{\mu,-} := \Delta(\varphi u_\mu)|_{L_-}$, where $L_- := (\mathbb{R}^n \times \,]-\infty, \gamma + \beta[)$, and

$$u_{\mu,-} := (G * f_{\mu,-})|_{\mathbb{R}^{n+1} \setminus K_-},$$
$$K_- := (\overline{\omega_{J+1}} \times [\gamma - \beta, \gamma + \beta]) \cup (\overline{\omega}_1 \times [-2\widetilde{\delta}, \gamma + \beta]).$$

Since $f_{\mu,+}$ and $f_{\mu,-}$ are bounded,

(5.5) $\qquad u_{\mu,+} \in C_{\Delta,0}(\mathbb{R}^{n+1} \setminus K_+) \quad \text{and} \quad u_{\mu,-} \in C_{\Delta,0}(\mathbb{R}^{n+1} \setminus K_-).$

Moreover,

(5.6) $\qquad\qquad u_\mu = u_{\mu,+} + u_{\mu,-} \quad \text{on } \mathbb{R}^{n+1} \setminus (K_+ \cup K_-),$

since

$$\varphi u_\mu - G * f_{\mu,-} - G * f_{\mu,+} \in C_{\Delta,0}(\mathbb{R}^{n+1}) = \{0\}$$

by (5.5) and the relevant definitions. By (5.4) and (3.3),

$$\{u_\mu \mid \mu \in \widetilde{B}\} \text{ is bounded in } C_\Delta(\mathbb{R}^{n+1} \setminus \overline{(\omega_J \times I(\varrho))}).$$

Therefore, by the choice of φ,

$$\{f_{\mu,+} \mid \mu \in \widetilde{B}\} \text{ is uniformly bounded,}$$

hence

(5.7) $\{u_{\mu,+} \mid \mu \in \widetilde{B}\}$ is uniformly bounded in $C_\Delta(\mathbb{R}^{n+1} \setminus K_+)$.

Since $P(-D)u_{\mu,-} = P(-D)u_\mu - P(-D)u_{\mu,+}$ by (5.6),

$$\{P(-D)u_{\mu,-} \mid \mu \in \widetilde{B}\} \text{ is bounded in } C_{\Delta,0}(\mathbb{R}^{n+1} \setminus K_-)$$

by (5.7) and by the fact that

$$\{P(-D)\mu \mid \mu \in \widetilde{B}\} = B \text{ is bounded in } C_\Delta(W)'_{\mathrm{b}} \text{ and in } C_\Delta(\overline{\omega_J \times I(\widetilde{\delta})})'_{\mathrm{b}}$$

(the claim in the preceding line holds by assumption and (5.4)). The proof of Theorem 4.4(a) then shows that

(5.8) $\{u_{\mu,-} \mid \mu \in \widetilde{B}\}$ is bounded in

$$C_\Delta(\mathbb{R}^{n+1} \setminus ((Q \times [\gamma/2, 3\gamma/2]) \cup (\widetilde{K} \times J(5\widetilde{\delta}))))$$

for suitable compacts $Q \subset \Omega$ and $\widetilde{K} \subset \Omega$, if $\beta > 0$ is sufficiently small. Notice that $\delta > 0$ in $(\overline{A_\Omega})$ only depends on ω, and that $(\overline{A_\Omega})$ clearly also holds for $\widetilde{\delta} > 0$ instead of δ if $\widetilde{\delta} \leq \delta$.

The proof of the theorem is finished by choosing $\widetilde{\omega} \subset\subset \Omega$ with $\widetilde{\omega} \supset \widetilde{K}$ and applying Lemma 5.1 and (5.6)–(5.8). ∎

The proof of the Main Theorem 2.1 is now completed by the following result:

THEOREM 5.5. *The following statements are equivalent*:

(i) *$P(D)$ is surjective on $A(\Omega)$.*
(ii) *For any $g \in A(\Omega)$ and any $\omega \subset\subset \Omega$ there is $f \in A(\omega)$ such that*

(5.9) $$P(D)f = g|_\omega,$$

and $P(D)$ satisfies the following condition $(\underline{\mathrm{B}_\Omega})$: for any $\omega \subset\subset \Omega$ there is $\widetilde{\omega} \subset\subset \Omega$ with $\widetilde{\omega} \supset \omega$ such that for any $\widehat{\omega} \subset\subset \Omega$ and any $\xi \in \partial\widetilde{\omega}$ there is $F \in \mathfrak{B}(\widehat{\omega})$ such that

$$P(D)F = \delta_\xi \quad \text{on } \widehat{\omega} \quad \text{and} \quad F|_\omega \in A(\omega).$$

(iii) *$P(D)$ satisfies (5.9) and the following condition $(\overline{\mathrm{B}_\Omega})$: for any $\omega \subset\subset \Omega$ there are $\widetilde{\omega} \subset\subset \Omega$ with $\widetilde{\omega} \supset \omega$ and $\delta > 0$ such that for any $\xi \in \Omega \setminus \widetilde{\omega}$ there is $F \in \mathfrak{B}(\Omega)$ such that*

$$P(D)F = \delta_\xi \quad \text{on } \Omega \quad \text{and} \quad F|_\omega \in A_\delta(\omega).$$

(iv) *$P(D)$ satisfies (5.9) and the following condition: for any $\omega \subset\subset \Omega$ there are $\widetilde{\omega} \subset\subset \Omega$ with $\widetilde{\omega} \supset \omega$ and $\delta > 0$ such that for any $0 < \widehat{\delta} \leq \delta$ there is $\gamma > 0$ such that for any $H \in \mathfrak{B}(\Omega)$ with $H|_{\widetilde{\omega}} \in A_{\widehat{\delta}}(\widetilde{\omega})$ there is $F \in \mathfrak{B}(\Omega)$ such that*

$$P(D)F = H \quad \text{on } \Omega \quad \text{and} \quad F|_\omega \in A_\gamma(\omega).$$

Proof. (i)⇒(iv). (5.9) is clearly satisfied. $P(D)$ satisfies $(\overline{A_\Omega})$ by Theorem 4.6. To prove the claim, we take $\omega_1 \subset\subset \Omega$ with $\omega_1 \supset\supset \omega$ and then choose $\widetilde{\omega}_1$ with $\widetilde{\omega}_1 \supset\supset \omega_1$ and $\delta > 0$ for ω_1 by Theorem 5.4. Let $\widetilde{\omega}_1 \subset\subset \widetilde{\omega} \subset\subset \Omega$ and choose $H \in \mathfrak{B}(\Omega)$ as above. Then H is defined by $\widetilde{H} \in \widetilde{C}_\Delta(\Omega\times(\mathbb{R}\setminus\{0\}))$, and $\widetilde{H}|_{[\Omega]_0}$ can be extended to $H_1 \in C_\Delta([\Omega]_0 \cup [\widetilde{\omega}_1]_{-6\widetilde{\gamma}})$ for some $0 < \widetilde{\gamma} \leq \widehat{\delta}$ with $\widetilde{\gamma} \leq \delta/6$. By Theorem 5.4 there are $Z \in \mathfrak{U}(\Omega)$ and $E_1 \in C_\Delta(Z_+ \cup (\omega_1 \times I(\widetilde{\gamma})))$ such that

(5.10) $$P(D)E_1 = G(\cdot - \xi, \cdot)|_{Z_+ \cup (\omega_1 \times I(\widetilde{\gamma}))}.$$

We may assume that $Z = \{(x,y) \mid (x,|y|) \in Z_+\} \cup (\Omega \times \{0\})$. Set $E_2(x,y) := E_1(x,|y|)$ for $(x,y) \in Z \setminus (\Omega \times \{0\})$. Then E_2 satisfies (5.10) on $Z \setminus (\Omega \times \{0\})$ since $\widetilde{H}$ is even with respect to y. Thus, E_2 defines $F \in \mathfrak{B}(\Omega)$ with $P(D)F = H$, and $F|_\omega \in A_\gamma(\omega)$ for some $\gamma > 0$, since $E_1 \in C_\Delta(\omega_1 \times I(\widetilde{\gamma}))$ and Δ is elliptic (use (4.11)).

(iv)⇒(iii)⇒(ii). This is trivial.

(ii)⇒(i). By Theorem 4.6 it suffices to prove $(\underline{A_\Omega})$. Choose $\omega_1 \subset\subset \Omega$ such that $\omega_1 \supset\supset \omega$. Choose $\widetilde{\omega} \subset\subset \Omega$ for ω_1, fix $\widehat{\omega} \subset\subset \Omega$ and choose $\widehat{\omega}_1 \subset\subset \Omega$ with $\widehat{\omega}_1 \supset\supset \widehat{\omega}$. For $\xi \in \partial\widetilde{\omega}$ choose $F \in \mathfrak{B}(\widehat{\omega}_1)$ by $(\underline{B_\Omega})$. Then F is defined by $E_1 \in \widetilde{C}_\Delta(\widehat{\omega}_1 \times (\mathbb{R} \setminus \{0\}))$ with

(5.11) $$P(D)E_1 = H \quad \text{on } \widehat{\omega}_1 \times (\mathbb{R} \setminus \{0\}),$$

where $H \in D'(\widehat{\omega}_1 \times \mathbb{R})$ satisfies $\Delta H = \delta_{(\xi,0)}$ on $\widehat{\omega}_1 \times \mathbb{R}$. Therefore,

(5.12) $$H = G(\cdot - \xi, \cdot) + h \quad \text{for some } h \in C_\Delta(\widehat{\omega}_1 \times \mathbb{R}).$$

For sufficiently large $\widehat{\omega}_1$, by (5.9), Proposition 3.3 (and extension via the Cauchy problem) and Grothendieck's lemma there is $\delta_1 > 0$ such that

$$P(-D)C_\Delta(\widehat{\omega} \times I(\delta)) \supset C_\Delta(\widehat{\omega}_1 \times \mathbb{R})|_{\widehat{\omega} \times I(\delta_1)}.$$

We thus get $g \in C_\Delta(\widehat{\omega} \times I(\delta_1))$ such that

(5.13) $$P(D)g = h|_{\widehat{\omega} \times I(\delta_1)}.$$

Since $F|_{\omega_1} \in A(\omega_1)$, $E_1|_{[\Omega]_0}$ can be extended as a harmonic function (via the Cauchy problem) to $[\omega]_{-\delta_2}$ for some $0 < \delta_2 \leq \delta_1$. Equation (5.11) also holds on $[\omega]_{-\delta_2}$ by analytic continuation. Then

$$E := (E_1 - g)|_W, \quad W := (\widehat{\omega} \times \,]0, \delta_1[) \cup (\omega \times I(\delta_2)),$$

satisfies

$$P(D)E = G(\cdot - \xi, \cdot) \quad \text{on } W$$

by (5.11)–(5.13). Obviously, $W \in \mathfrak{U}(\widehat{\omega} \times \{T\})$ and $W \supset \omega \times \,]T - \delta, T + \delta[$ for $\delta := \delta_2/2$ and $0 < T \leq \delta$. This shows $(\underline{A_\Omega})$. ∎

In the statement of Theorem 5.5, (5.9) can be substituted by any of the equivalent conditions from Proposition 3.3.

By the results of this paper, it would be interesting to give precise bounds on the analytic singular support of hyperfunction elementary solutions of $P(D)$, especially for nonhomogeneous operators or operators with nonlocally hyperbolic principal part.

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