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Dual spaces to Orlicz–Lorentz spaces

by

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Abstract. For an Orlicz function φ and a decreasing weight w, two intrinsic exact descriptions are presented for the norm in the Köthe dual of the Orlicz–Lorentz function space $\Lambda_{\varphi,w}$ or the sequence space $\lambda_{\varphi,w}$, equipped with either the Luxemburg or Amemiya norms. The first description is via the modular $\inf\{\int \varphi_*(f^*/|g|)|g| : g \prec w\}$, where f^* is the decreasing rearrangement of f, $\prec$ denotes submajorization, and φ_* is the complementary function to φ. The second description is in terms of the modular $\int_I \varphi_*((f^*)^0/w)w$, where $(f^*)^0$ is Halperin's level function of f^* with respect to w. That these two descriptions are equivalent results from the identity $\inf\{\int \psi(f^*/|g|)|g| : g \prec w\} = \int_I \psi((f^*)^0/w)w$, valid for any measurable function f and any Orlicz function ψ. An analogous identity and dual representations are also presented for sequence spaces.

1. Introduction. The main goal of the paper is to give an isometric description of the Köthe dual space of the Orlicz–Lorentz space $\Lambda_{\varphi,w}$, where φ is an Orlicz function and w is a decreasing locally integrable weight function. Orlicz–Lorentz spaces have been studied extensively for the past two decades, since when their basic properties were established in [7]. So far, however, no satisfactory isometric description of their dual spaces has been given. There are several different isomorphic representations of the Köthe dual space $(\Lambda_{\varphi,w})'$ given for example in [6] or [8]. Problem XIV in [3] asks for an isometric representation of $(\Lambda_{\varphi,w})'$.

Orlicz–Lorentz spaces can be treated as a special case of more general Calderón–Lozanovskiĭ spaces. Lozanovskiĭ [18] (see also [13]–[17], [20] and [21]) proved a duality theorem, which in particular can be applied to Orlicz–Lorentz spaces. However his original formulas are too general and not explicit enough for application in the setting of Lorentz type spaces. Here we show that Lozanovskiĭ's formulas for dual norms and Köthe dual spaces can be expressed in terms of the recently introduced modular $P_{\varphi,w}$ and the

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corresponding modular space $\mathcal{M}_{\varphi,w}$ (see [9]). In fact $\mathcal{M}_{\varphi,w} = \{f \in L^0 : P_{\varphi,w}(\lambda f) < \infty$ for some $\lambda > 0\}$, where L^0 is the space of Lebesgue measurable real functions on $I = [0, \alpha)$ and

$$P_{\varphi,w}(f) = \inf\Big\{\int_I \varphi(f^*/|g|)|g| : g \prec w\Big\}.$$

The notation $g \prec w$ means that g is *submajorized* by w, that is, $\int_0^t g^* \le \int_0^t w$ for all $t \in I$.

In the case $\varphi(u) = u^p$, $1 < p < \infty$, when $\Lambda_{\varphi,w}$ becomes the classical Lorentz space $\Lambda_{p,w}$, a different explicit isometric description of its dual was given by Halperin [4]. He introduced the notion of level intervals and level functions with respect to w, and applied them to obtain a formula for the norm of the dual space. Here we study level functions and modulars in the environment of Orlicz–Lorentz spaces, which allows us to extend Halperin's theorem to those spaces.

Consequently, we give two different isometric representations of dual spaces of Orlicz–Lorentz spaces, one by means of submajorization by the weight w, and the other by level functions with respect to w. They are valid for both function and sequence spaces.

The paper is organized as follows. In Section 1 we give basic notations and notions needed further. Among others we define Calderón–Lozanovskiĭ spaces and Orlicz–Lorentz spaces equipped with the standard Amemiya and Luxemburg norms.

In Section 2 we recall the definition of the function space $\mathcal{M}_{\varphi,w}$ and then applying the general duality theorem of Lozanovskiĭ, we prove that the Köthe dual space $(\Lambda_{\varphi,w})'$ is $\mathcal{M}_{\varphi_*,w}$ with equality of norms. In the case when the space $\Lambda_{\varphi,w}$ is separable, it is also an isometric representation of its dual space. This representation is given for both the Amemiya and Luxemburg norms.

Section 3 is devoted to a number of specific properties of the modular $P_{\varphi,w}(f)$. A sequence of technical results leads to the main theorem describing an algorithm for calculating the infimum in the formula for $P_{\varphi,w}(f)$ when f is a simple decreasing function. This is Theorem 3.9 which states that the function g^f produced by Algorithm A minimizes the modular $P_{\varphi,w}(f)$. It is interesting to observe that g^f depends only on f and w, but not on φ.

In Section 4 we give another isometric representation of the Köthe dual spaces using the so called level functions f^0 with respect to w, introduced by Halperin [4]. Applying the results of the previous section, in particular Algorithm A, we first prove that $P_{\varphi,w}(f) = \int_I \varphi(f/g^f)g^f = \int_\varphi(f^0/w)w$ for a decreasing simple function f. In the next step we extend this result to any $f \in \Lambda_{\varphi,w}$, which in fact yields the second duality theorem. Theorem 4.8 summarizes all Köthe duality formulas for the function space $\Lambda_{\varphi,w}$ equipped with either the Amemiya or Luxemburg norm. Halperin's duality result for $\Lambda_{p,w}$, $1 < p < \infty$, is then a corollary from Theorem 4.8.

In the last fifth section we present analogous results for Orlicz–Lorentz sequence spaces $\lambda_{\varphi,w}$. We show first that the sequence spaces as well as their Köthe dual spaces can be isometrically embedded into appropriate Orlicz–Lorentz function spaces. Next applying the results of the previous sections for function spaces we quickly obtain the analogous isometric representations of the dual spaces of $\lambda_{\varphi,w}$ in terms of the sequence spaces $\mathfrak{m}_{\varphi_*,w}$ introduced in [9] as well as in terms of the spaces generated by φ_*, w and level sequences.

Let us agree first on the notation and basic notions used in this paper. We denote by φ an *Orlicz function*, that is, $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(0) = 0$, φ is convex and φ is strictly increasing. Let φ_* be the *complementary* function to φ, that is, $\varphi_*(s) = \sup_{t\geq 0}\{st - \varphi(t)\}$, $s \geq 0$. We denote by φ^{-1} the inverse function to φ. We say that φ is an *N-function* whenever $\lim_{t\to 0+} \varphi(t)/t = 0$ and $\lim_{t\to\infty} \varphi(t)/t = \infty$. It is well known that φ_* is an N-function whenever φ is [10]. Recall also that the function $t \mapsto \varphi(a/t)t$ is decreasing and convex on $(0,\infty)$ for every $a > 0$. The first property results from $\varphi(t)/t$ being increasing for $t > 0$, while for the second one we see, by convexity of φ, that for any $t_1, t_2 \geq 0$,

$$\varphi\left(\frac{2a}{t_1+t_2}\right)\frac{t_1+t_2}{2} = \varphi\left(\frac{at_1}{(t_1+t_2)t_1} + \frac{at_2}{(t_1+t_2)t_2}\right)\frac{t_1+t_2}{2} \leq \frac{t_1}{2}\varphi\left(\frac{a}{t_1}\right) + \frac{t_2}{2}\varphi\left(\frac{a}{t_2}\right).$$

This also shows that $t \mapsto \varphi(a/t)t$ is strictly convex if φ is strictly convex. We say that φ satisfies the Δ_2-*condition* for all arguments, respectively for large arguments, whenever $\varphi(2u) \leq K\varphi(u)$ for all $u \geq 0$, respectively for all $u \geq u_0$ and some $u_0 \geq 0$.

Given an Orlicz function φ, define its associated *Calderón–Lozanovskiĭ function* as

$$\rho(t,s) = \rho_\varphi(t,s) = \varphi^{-1}(s/t)t, \qquad s \geq 0,\ t > 0, \tag{1.1}$$

and the *conjugate function* to ρ as

$$\hat{\rho}(t,s) = \hat{\rho}_\varphi(t,s) = \inf_{u,v>0} \frac{us+vt}{\rho(u,v)}, \qquad s,t \geq 0.$$

It is well known that the function $\rho_\varphi(t,s)$ is concave on $(0,\infty) \times [0,\infty)$. Moreover, if φ is an N-function then

$$\hat{\rho}_\varphi(t,s) = \varphi_*^{-1}(t/s)s, \qquad t \geq 0,\ s > 0, \tag{1.2}$$

(see [17, Example 3], or [21, Example 7]).

Let further $I = [0,\alpha)$ where $0 < \alpha \leq \infty$, and let $|\cdot|$ be the Lebesgue measure on I. Denote by L^0 the set of all Lebesgue measurable real-valued functions on I. Given $f \in L^0$ define its *distribution function* as

$$d_f(\lambda) = |\{t \in I : |f(t)| > \lambda\}|, \qquad \lambda \geq 0,$$

and its *decreasing rearrangement* f^* as

$$f^*(t) = \inf\{\lambda > 0 : d_f(\lambda) \leq t\}, \qquad t \in I.$$

Here decreasing or increasing means non-increasing or non-decreasing, respectively. We say that $f \in L^0$ is *submajorized* by $g \in L^0$ and we write

$$f \prec g \quad \text{whenever} \quad \int_0^t f^* \leq \int_0^t g^* \quad \text{for every } t \in I.$$

For any decreasing locally integrable function h let

$$H(t) = \int_0^t h, \qquad t \in I.$$

A Banach space $(E, \|\cdot\|_E)$ is called a *Banach function space* (or a *Köthe space*) if $E \subset L^0$ and whenever $f \in L^0$, $g \in E$ and $|f| \leq |g|$ a.e. then $f \in E$ and $\|f\|_E \leq \|g\|_E$. We will also assume that each Banach function space contains a *weak unit*, i.e. there is $f \in E$ such that $f(t) > 0$ for a.a. $t \in I$. We denote by E' the *Köthe dual* space to E, which consists of all $f \in L^0$ such that $\|f\|_{E'} = \sup\{\int_I fg : \|g\|_E \leq 1\} < \infty$. The space E' equipped with the norm $\|\cdot\|_{E'}$ is a Banach function space. It is well known that E' is non-trivial and contains a weak unit [24, Ch. 15, §71, Theorem 4(a)].

Given a Calderón–Lozanovskiĭ function ρ and a couple of Banach function spaces E, F, the *Calderón–Lozanovskiĭ space* is defined as

$$\rho^0(E,F) = \{f \in L^0 : \|f\|_\rho^0 = \inf\{\|g\|_E + \|h\|_F : |f| = \rho(|g|,|h|)\} < \infty\},$$

$$\rho(E,F) = \{f \in L^0 : \|f\|_\rho = \inf\{\max(\|g\|_E, \|h\|_F) : |f| = \rho(|g|,|h|)\} < \infty\}.$$

Recall that the spaces $\rho_\varphi^0(L^\infty, L^1)$ and $\rho_\varphi(L^\infty, L^1)$ coincide isometrically with the Orlicz space L^φ equipped with its *Amemiya* and *Luxemburg norm* respectively [18]. Moreover, in the above definitions one may take equivalently $|f| \leq \rho(|g|,|h|)$ instead of $|f| = \rho(|g|,|h|)$. In fact it is enough to apply Lemma 1 from [21], which states that if $\|g\|_E, \|h\|_F \leq 1$ are such that $0 \leq f \leq \rho(g,h)$, then there exist $0 \leq g_1 \leq g$ and $0 \leq f_1 \leq f$ satisfying $f = \rho(g_1, h_1)$. It is also known [19] that

$$\begin{aligned}\|f\|_{\rho(E,F)} &= \inf\{C > 0 : |f| \leq C\rho(|g|,|h|),\ \|g\|_E \leq 1,\ \|h\|_F \leq 1\} \\ &= \inf\{C > 0 : |f| = C\rho(|g|,|h|),\ \|g\|_E \leq 1,\ \|h\|_F \leq 1\}.\end{aligned}$$

Both spaces $\rho(E,F)$ and $\rho^0(E,F)$ coincide as sets, and the norms $\|\cdot\|_\rho$ and $\|\cdot\|_\rho^0$ are equivalent. The spaces $\hat{\rho}(E,F)$, $\hat{\rho}^0(E,F)$ are defined analogously to $\rho(E,F)$ and $\rho^0(E,F)$ with ρ replaced by $\hat{\rho}$. Moreover, the notation $\rho_\varphi(E,F)$ stands for the function $\rho = \rho_\varphi$ defined by (1.1).

Let w be a *weight function* on I, that is, $w \in L^0$, w is positive and decreasing on I, and locally integrable, i.e. $W(t) = \int_0^t w < \infty$, $t \in I$. Denote $W(\infty) = \int_0^\infty w$ when $\alpha = \infty$. The *Lorentz space* Λ_w is classically defined as

$$\Lambda_w = \Big\{ f \in L^0 : \|f\|_{\Lambda_w} = \int_I f^* w = \int_I f^*\, dW < \infty \Big\},$$

and the *Marcinkiewicz space* M_W is

$$M_W = \Big\{ f \in L^0 : \|f\|_{M_W} = \sup_{t\in I} \Big(\int_0^t f^*/W(t) \Big) < \infty \Big\}.$$

Both are Banach function spaces and each is the Köthe dual space of the other [11]. Note that $\|f\|_{M_W} \le 1$ if and only if $f \prec w$. Let φ be an Orlicz function and w a decreasing weight function on I. Then the *Orlicz–Lorentz space* [7] is the set

$$\Lambda_{\varphi,w} = \{ f \in L^0 : \exists_{\lambda>0}\ I_{\varphi,w}(\lambda f) < \infty \},$$

where $I_{\varphi,w}(f) = \int_I \varphi(f^*) w$. It is equipped with either the *Luxemburg norm*

$$\|f\|_\Lambda = \inf\{\epsilon > 0 : I_{\varphi,w}(f/\epsilon) \le 1\},$$

or the *Amemiya norm*

$$\|f\|_\Lambda^0 = \inf_{k>0} \frac{1}{k}(1 + I_{\varphi,w}(kf)).$$

We denote by $\Lambda_{\varphi,w}$ the Orlicz–Lorentz space equipped with the Luxemburg norm $\|\cdot\|_\Lambda$, and by $\Lambda^0_{\varphi,w}$ the same space equipped with the Amemiya norm $\|\cdot\|^0_\Lambda$. Orlicz–Lorentz spaces are Calderón–Lozanovskiĭ spaces relative to the couple (L^∞, Λ_w), and we have

$$\Lambda_{\varphi,w} = \rho_\varphi(L^\infty, \Lambda_w), \qquad \Lambda^0_{\varphi,w} = \rho^0_\varphi(L^\infty, \Lambda_w) \tag{1.3}$$

with equality of norms. The first equality may be found in [19] (cf. [5] and [6]). As for the second, letting $f \in \rho^0_\varphi(L^\infty, E)$, we have

$$\begin{aligned} \|f\|^0_{\rho_\varphi(L^\infty,E)} &= \inf\{\|x\|_E + \|y\|_{L^\infty} : |x| = |y|\varphi(|f|/|y|)\} \\ &= \inf_{k>0}\big\{\inf\{\|\,|y|\varphi(|f|/|y|)\|_E + \|y\|_{L^\infty} : \|y\|_{L^\infty} = k\}\big\} \\ &= \inf_{k>0}\{\|k\varphi(|f|/k)\|_E + k\} = \inf_{k>0} \frac{1}{k}(\|\varphi(k|f|)\|_E + 1), \end{aligned}$$

where the third equality is a consequence of $|y| \le \|y\|_{L^\infty}$ and the monotonicity of $s \mapsto s\varphi(a/s)$. The desired equality follows for $E = \Lambda_w$.

2. The dual space of an Orlicz–Lorentz space. In this section we will show that the Köthe dual spaces to the Orlicz–Lorentz spaces $\Lambda_{\varphi,w}$ and $\Lambda^0_{\varphi,w}$ coincide isometrically with $\mathcal{M}^0_{\varphi_*,w}$ and $\mathcal{M}_{\varphi_*,w}$, respectively. The spaces

$\mathcal{M}_{\varphi,w}$ have been recently introduced in [9]. Given an Orlicz function φ and a weight w let

$$\mathcal{M}_{\varphi,w} = \{f \in L^0 : \exists_{\lambda>0}\ P_{\varphi,w}(\lambda f) < \infty\},$$

where the modular $P_{\varphi,w}$ is defined as

$$P_{\varphi,w}(f) = \inf\left\{\int_I \varphi\left(\frac{f^*}{|g|}\right)|g| : g \prec w\right\} = \inf\left\{\left\|\varphi\left(\frac{f^*}{|g|}\right)g\right\|_1 : g \prec w\right\}.$$

Here and further, we denote by $\|\cdot\|_1$ the norm in the space L^1 of integrable functions on I. To avoid any ambiguity in the definition of $P_{\varphi,w}$ let us agree that for any measurable functions $f, g \geq 0$ on I, if $g(t) = 0$ then

$$\varphi\left(\frac{f(t)}{g(t)}\right)g(t) = \begin{cases} 0 & \text{if } f(t) = 0, \\ \infty & \text{if } f(t) \neq 0. \end{cases}$$

It is also worth observing that

$$P_{\varphi,w}(f) = \inf\left\{\left\|\varphi\left(\frac{f^*}{|g|}\right)g\right\|_1 : \|g\|_W = 1\right\}. \tag{2.1}$$

In fact by convexity of φ one has $\frac{1}{a}\varphi(at) \leq \varphi(t)$ for each $t > 0$ and $0 < a \leq 1$. Therefore, if $\|g\|_{M_W} = a < 1$ then

$$\left\|\varphi\left(\frac{af^*}{|g|}\right)\frac{g}{a}\right\|_1 \leq \left\|\varphi\left(\frac{f^*}{|g|}\right)g\right\|_1.$$

We introduce two equivalent norms on $\mathcal{M}_{\varphi,w}$. The first one is of Luxemburg type,

$$\|f\|_{\mathcal{M}} = \|f\|_{\mathcal{M}_{\varphi,w}} = \inf\{\lambda > 0 : P_{\varphi,w}(f/\lambda) \leq 1\},$$

and the second one is of Amemiya type,

$$\|f\|_{\mathcal{M}}^0 = \|f\|_{\mathcal{M}_{\varphi,w}}^0 = \inf_{k>0} \frac{1}{k}(P_{\varphi,w}(kf) + 1).$$

We denote by $\mathcal{M}_{\varphi,w}$ the space equipped with the norm $\|\cdot\|_{\mathcal{M}}$, and by $\mathcal{M}_{\varphi,w}^0$ the same space endowed with the norm $\|\cdot\|_{\mathcal{M}}^0$. Our first result expresses the spaces $\mathcal{M}_{\varphi,w}$ and $\mathcal{M}_{\varphi,w}^0$ as Calderón–Lozanovskiĭ spaces relative to the couple (M_W, L^1).

Proposition 2.1. *Let φ be an N-function. Then*

$$\rho_\varphi(M_W, L^1) = \mathcal{M}_{\varphi,w}, \qquad \rho_\varphi^0(M_W, L^1) = \mathcal{M}_{\varphi,w}^0,$$
$$\hat{\rho}_\varphi(L^1, M_W) = \mathcal{M}_{\varphi_*,w}, \qquad \hat{\rho}_\varphi^0(L^1, M_W) = \mathcal{M}_{\varphi_*,w}^0,$$

with equality of norms.

Proof. Let $f \in \rho_\varphi(M_W, L^1)$. Then

$$\begin{aligned}
\|f\|_{\rho_\varphi(M_W,L^1)} & \\
&= \|f^*\|_{\rho_\varphi(M_W,L^1)} = \inf\big\{\max\{\|g\|_{M_W}, \|h\|_1\} : f^* = \rho_\varphi(|g|,|h|)\big\} \\
&= \inf\{C>0 : f^* = C\rho_\varphi(|g|,|h|),\ \|g\|_{M_W} \le 1,\ \|h\|_1 \le 1\} \\
&= \inf\bigg\{C>0 : \varphi\bigg(\frac{f^*}{C|g|}\bigg)|g| = |h|,\ \|g\|_{M_W} \le 1,\ \|h\|_1 \le 1\bigg\} \\
&= \inf\bigg\{C>0 : \bigg\|\varphi\bigg(\frac{f^*}{C|g|}\bigg)g\bigg\|_1 \le 1,\ \|g\|_{M_W} \le 1\bigg\} \\
&= \inf\bigg\{C>0 : \inf\bigg\{\bigg\|\varphi\bigg(\frac{f^*}{C|g|}\bigg)g\bigg\|_1 : g \prec w\bigg\} \le 1\bigg\} = \|f\|_{\mathcal{M}_{\varphi,w}}.
\end{aligned}$$

Applying (2.1) we also get the second of the first two equalities:

$$\begin{aligned}
\|f\|^0_{\rho_\varphi(M_W,L^1)} &= \|f^*\|^0_{\rho_\varphi(M_W,L^1)} = \inf\{\|g\|_{M_W} + \|h\|_1 : f^* = \rho_\varphi(|g|,|h|)\} \\
&= \inf\bigg\{\|g\|_{M_W} + \bigg\|\varphi\bigg(\frac{f^*}{|g|}\bigg)g\bigg\|_1 : g \in M_W\bigg\} \\
&= \inf_{k>0}\bigg\{\inf\bigg\{k + \bigg\|\varphi\bigg(\frac{f^*}{k|g|}\bigg)kg\bigg\|_1 : \|g\|_{M_W} = 1\bigg\}\bigg\} \\
&= \inf_{k>0}\bigg\{\inf\bigg\{k + \bigg\|\varphi\bigg(\frac{f^*}{k|g|}\bigg)kg\bigg\|_1 : \|g\|_{M_W} \le 1\bigg\}\bigg\} \\
&= \inf_{k>0} \frac{1}{k}(P_{\varphi,w}(kf) + 1) = \|f\|^0_{\mathcal{M}_{\varphi,w}}.
\end{aligned}$$

The remaining equalities are proved analogously by (1.2). ∎

Now we are ready to state an isometric characterization of the (Köthe) dual spaces of Orlicz–Lorentz spaces.

THEOREM 2.2. *Let w be a decreasing weight and φ be an N-function. Then:*

(1) *The Köthe dual spaces to the Orlicz–Lorentz spaces $\Lambda_{\varphi,w}$ and $\Lambda^0_{\varphi,w}$ are*

$$(\Lambda_{\varphi,w})' = \mathcal{M}^0_{\varphi_*,w} \quad \text{and} \quad (\Lambda^0_{\varphi,w})' = \mathcal{M}_{\varphi_*,w},$$

with equality of norms.

(2) *Let φ satisfy the appropriate Δ_2-condition, that is:* (i) *for large arguments if $I = [0,\alpha)$ with $\alpha < \infty$;* (ii) *for all arguments if $I = [0,\infty)$ and $W(\infty) = \infty$. Then the dual spaces $(\Lambda_{\varphi,w})^*$ and $(\Lambda^0_{\varphi,w})^*$ are isometrically isomorphic to the respective Köthe dual spaces. In fact for any functional $\Phi \in (\Lambda_{\varphi,w})^*$ (resp., $\Phi \in (\Lambda^0_{\varphi,w})^*$) there exists*

$\phi \in \mathcal{M}^0_{\varphi_*,w}$ (resp., $\phi \in \mathcal{M}_{\varphi_*,w}$) such that

$$\Phi(f) = \int_I f\phi, \qquad f \in \Lambda_{\varphi,w},$$

and $\|\Phi\|_{(\Lambda_{\varphi,w})^*} = \|\phi\|^0_{\mathcal{M}_{\varphi_*,w}}$ (resp., $\|\Phi\|_{(\Lambda^0_{\varphi,w})^*} = \|\phi\|_{\mathcal{M}_{\varphi_*,w}}$).

Proof. By Lozanovskiĭ's representation theorem [18], for any Banach function spaces E, F we have

$$(\rho(E,F))' = \hat{\rho}^0(E',F') \quad \text{and} \quad (\rho^0(E,F))' = \hat{\rho}(E',F'),$$

with norm equalities. Notice that $(\Lambda_w)' = M_W$. This was proved in [11, Theorem 5.2, p. 112] under the assumption that $W(\infty) = \infty$ in the case of $I = [0,\infty)$, but the same proof works for any decreasing locally integrable weight function w. Thus by (1.2), (1.3) and Proposition 2.1 we get

$$(\Lambda_{\varphi,w})' = (\rho(L^\infty, \Lambda_w))' = \hat{\rho}^0(L^1, M_W) = \mathcal{M}^0_{\varphi_*,w}.$$

The second equality of (1) can be shown analogously.

The second statement follows from the well known fact that Orlicz–Lorentz spaces are order continuous [7] under the appropriate Δ_2-condition, and the general theorem stating that the Köthe dual space E' of an order continuous Banach function space E is isometrically isomorphic via integral functionals to the dual space E^* [1, Theorem 4.1, p. 20]. ∎

3. An algorithm for computing $P_{\varphi,w}(f)$ for a decreasing simple function f. In this section our goal is to find a function g which minimizes $P_{\varphi,w}(f)$ for a given decreasing simple function $f = \sum_{i=1}^n a_i \chi_{[t_{i-1},t_i)}$ with $a_1 > \cdots > a_n > 0$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$. This process consists of several steps and leads to an algorithm which reveals that such a function g exists and depends only on f and w, but not on φ.

First in Lemma 3.1 we show that the minimizing function g has to be also simple and decreasing. In the second step in Lemma 3.3 we show that such a g exists. Next, in Lemma 3.4 it is proved that $G(t_n) = W(t_n)$, and then Theorem 3.7 demonstrates that

$$g = \sum_{j=0}^{m-1} \lambda_j f \chi_{[t_{i_j}, t_{i_{j+1}})}$$

for some $(\lambda_i)_{j=0}^{m-1}$, $(t_{i_j})_{j=0}^{m-1}$ and $m \le n$, where $W(t_{i_j}) = G(t_{i_j})$. This shows that g has to be piecewise proportional to f and the ratios λ_j are determined by the points t_{i_j}. Therefore in order to find g it is sufficient to determine the points t_{i_j}. This process will be described by Algorithm A. Applying finally Theorem 3.7 and Lemma 3.6, we finish by proving that Algorithm A produces a function g that minimizes $P_{\varphi,w}(f)$.

LEMMA 3.1. *If* $f = \sum_{i=1}^n a_i \chi_{A_i}$ *where* $a_1 > \cdots > a_n > 0$ *and* $A_i = [t_{i-1}, t_i)$ *with* $0 = t_0 < t_1 < \cdots < t_n < \infty$, *then*

$$(3.1)\qquad P_{\varphi,w}(f) = \inf \left\{ \begin{matrix} \|\varphi(f/g)g\|_1 : g \prec w, \\ \textit{and } g = \sum_{i=1}^n b_i \chi_{A_i} \textit{ with } b_1 \ge \cdots \ge b_n > 0 \end{matrix} \right\}.$$

Proof. Let $f = \sum_{i=1}^n a_i \chi_{A_i}$ satisfy the assumptions. Corollary 4.5 in [9] states that

$$P_{\varphi,w}(f) = \inf\{\|\varphi(f/g)g\|_1 : g \prec w,\ 0 \le g \downarrow\},$$

where $g \downarrow$ means that g is decreasing. Fix some $g \prec w$, $g \downarrow$ and put

$$h = \varphi\left(\frac{f}{g}\right) g \quad \text{and} \quad \tilde{h} = \varphi\left(\frac{f}{Tg}\right) Tg,$$

where

$$T : g \mapsto \sum_{i=1}^n \left(\frac{1}{|A_i|} \int_{A_i} g \right) \chi_{A_i}.$$

Since g is decreasing, so is Tg. Hence it is enough to show that $\|\tilde{h}\|_1 \le \|h\|_1$ and $Tg \prec w$.

By Proposition 3.7 in [1, Chap. 2] we have $Tg \prec g$ and so $Tg \prec w$. By convexity of the function $s \mapsto \varphi(a/s)s$, $a > 0$, and Jensen's inequality for convex functions we have, for every $i = 1, \ldots, n$,

$$\varphi\left(\frac{a_i}{(1/|A_i|)\int_{A_i} g} \right) \frac{1}{|A_i|} \int_{A_i} g \le \frac{1}{|A_i|} \int_{A_i} \varphi\left(\frac{a_i}{g}\right) g,$$

which gives

$$\|\tilde{h}\|_1 = \sum_{i=1}^n |A_i| \varphi\left(\frac{a_i}{(1/|A_i|)\int_{A_i} g} \right) \frac{1}{|A_i|} \int_{A_i} g \le \sum_{i=1}^n \int_{A_i} \varphi\left(\frac{a_i}{g}\right) g = \|h\|_1,$$

and the proof is finished. ∎

LEMMA 3.2. *Suppose that* $g = g^* = \sum_{i=1}^n b_i \chi_{[t_{i-1}, t_i)}$ *where* $0 = t_0 < t_1 < \cdots < t_n < \infty$. *Then*

$$\inf_{0 < t \le t_n} \frac{W(t)}{G(t)} = \min_{i=1,\ldots,n} \frac{W(t_i)}{G(t_i)}.$$

In particular, $g \prec w$ *if and only if* $G(t_i) \le W(t_i)$ *for each* $i = 1, \ldots, n$.

Proof. The left-hand side is clearly majorized by the right-hand side. Conversely, if for some $\theta \ge 0$,

$$W(t_i) \ge \theta G(t_i), \qquad i = 1, \ldots, n,$$

which remains trivially true for $i = 0$, then by concavity of W and the fact that G is affine on each segment $[t_i, t_{i+1}]$, we have, for every $\lambda \in [0, 1]$ and

$i = 0, \ldots, n-1$,

$$W((1-\lambda)t_i + \lambda t_{i+1}) \geq (1-\lambda)W(t_i) + \lambda W(t_{i+1}) \\ \geq (1-\lambda)\theta G(t_i) + \lambda\theta G(t_{i+1}) = \theta G((1-\lambda)t_i + \lambda t_{i+1}),$$

which proves the converse inequality. ■

LEMMA 3.3. *Let φ be an N-function. Then for a simple function $f = \sum_{i=1}^n a_i\chi_{A_i}$ such that $a_1 > \cdots > a_n > 0$ and $A_i = [t_{i-1}, t_i)$ where $0 = t_0 < t_1 < \cdots < t_n < \infty$, there exists $g = \sum_{i=1}^n b_i\chi_{A_i}$ with $b_1 \geq \cdots \geq b_n > 0$, $\|g\|_{M_W} = 1$ and such that $P_{\varphi,w}(f) = \|\varphi(f/g)g\|_1$. Consequently, the infimum in the definition of $P_{\varphi,w}(f)$ is attained.*

Proof. In fact, by Lemma 3.1 the above infimum may be taken over $g = \sum_{i=1}^n b_i\chi_{A_i}$ with $b_1 \geq \cdots \geq b_n > 0$ such that $g \prec w$. By Lemma 3.2 the condition $g \prec w$ is equivalent to $G(t_k) = \sum_{i=1}^k b_i|A_i| \leq W(t_k)$ for each $k = 1, \ldots, n$. But those constraints define the set

$$C = \Big\{b = (b_i)_{i=1}^n : b_1 \geq \cdots \geq b_n > 0, \sum_{i=1}^k b_i|A_i| \leq W(t_k),\ k = 1, \ldots, n\Big\},$$

which is relatively compact in $\mathbb{R}^n$. Hence if the sequence $(b^k) = (b_i^k)_{i,k=1}^{n,\infty} \subset \mathbb{R}^n$ is minimizing for the infimum in (3.1), then there is a subsequence (b^{k_j}) such that $b^{k_j} \to b \in \bar{C}$, where $\bar{C}$ denotes the closure of C in $\mathbb{R}^n$.

Let us show that $b \in C$. Indeed, if $b \in \bar{C} \setminus C$ then $b_i = 0$ for some $i = 1, \ldots, n$, which means that $b_i^{k_j} \to 0$ if $j \to \infty$. Setting $g_{k_j} = \sum_{i=1}^n b_i^{k_j}\chi_{A_i}$, since φ is an N-function we get, for $j \to \infty$,

$$\left\|\varphi\left(\frac{f}{g_{k_j}}\right)g_{k_j}\right\|_1 \geq \varphi\left(\frac{a_i}{b_i^{k_j}}\right)b_i^{k_j}|A_i| \to \infty,$$

so (b^{k_j}) cannot be minimizing for the relevant infimum, a contradiction. ■

EXAMPLE. We present an example which shows that for decreasing simple functions f the functions g that minimize $P_{\varphi,w}(f)$ depend on f.

Let $\varphi(t) = t^2$, $w(t) = 1/(2\sqrt{t})$, $t > 0$. Define the family of functions $f_x := x\chi_{(0,1)} + 1\chi_{(1,4)}$ on $(0,\infty)$ for $x \geq 1$. Then by Lemmas 3.1–3.3,

$$P_{\varphi,w}(f_x) = \min\begin{Bmatrix} x^2/b_1 + 3/b_2 : g = b_1\chi_{[0,1)} + b_2\chi_{[1,4)} \\ \text{with } 1 \geq b_1 \geq b_2 \text{ and } b_1 + 3b_2 = 2 \end{Bmatrix}.$$

Applying Lagrange multipliers to minimize the function $\psi(b_1, b_2) = x^2/b_1 + 3/b_2$ with constraint $b_1 + 3b_2 = 2$ gives the solution $b_1 = 2x/(x+3)$, $b_2 = 2/(x+3)$. We have $b_1 \geq b_2$ for $x \geq 1$. Moreover, if $1 \leq x \leq 3$ then $b_1 \leq 1$. If $x \geq 3$ then there is no extremum in the set defined by the constraints $1 \geq b_1 \geq b_2$ and $b_1 + 3b_2 = 2$, so ψ attains its minimum at $(1, 1/3)$ or $(1/2, 1/2)$.

Finally, we get

$$P_{\varphi,w}(f_x) = \begin{cases} x(x+3)/2 + 3(x+3)/2 & \text{for } 1 \le x \le 3, \\ x^2 + 9 & \text{for } x \ge 3, \end{cases}$$

and it is clear that g cannot be chosen independently of f_x.

LEMMA 3.4. *Let φ be an N-function. Let $f = \sum_{i=1}^n a_i \chi_{A_i}$ for some a_i with $a_1 > \cdots > a_n > 0$ and $A_i = [t_{i-1}, t_i)$ where $0 = t_0 < t_1 < \cdots < t_n < \infty$. If $g = \sum_{i=1}^n b_i \chi_{A_i} = g^*$ is a minimizing function for $P_{\varphi,w}(f)$ then*

$$G(t_n) = \int_0^{t_n} g = \int_0^{t_n} w = W(t_n). \tag{3.2}$$

Proof. One may assume that $\|g\|_{M_W} = 1$. By Lemma 3.3 we also have $b_1 \ge \cdots \ge b_n > 0$, so $g > 0$ on $[0, t_n)$. Suppose g does not satisfy (3.2), that is, $G(t_n) < W(t_n)$. We will then find a new function h such that $h \prec w$ and $\|\varphi(f/h)h\|_1 < \|\varphi(f/g)g\|_1$, contradicting the minimality of g.

In fact, by Lemma 3.2, $\|g\|_{M_W} = 1$ is equivalent to

$$1 = \sup_{t_n \ge t > 0} \frac{G(t)}{W(t)} = \inf_{t_n \ge t > 0} \frac{W(t)}{G(t)} = \min_{i=1,\ldots,n} \frac{W(t_i)}{G(t_i)}.$$

It follows that $\{i > 0 : W(t_i) = G(t_i)\} \ne \emptyset$. Let

$$i_1 = \max\{i > 0 : W(t_i) = G(t_i)\} \quad \text{and} \quad \gamma_1 = \min_{i_1 < i \le n} \frac{W(t_i) - W(t_{i_1})}{G(t_i) - G(t_{i_1})}.$$

Since $G(t_n) < W(t_n)$ we have $i_1 < n$. Then $W(t_i) > G(t_i)$ for all $i > i_1$, and thus it is clear that $\gamma_1 > 1$. Set

$$g_1 = g\chi_{[0,t_{i_1})} + \gamma_1 g \chi_{[t_{i_1}, t_n)}.$$

Note that $g_1 = g_1^*$. In fact, since g is decreasing, it is sufficient to show that $b_{i_1} \ge \gamma_1 b_{i_1+1}$. We have $G(t_{i_1}) = G_1(t_{i_1}) = W(t_{i_1})$ and $G_1(t_{i_1+1}) \le W(t_{i_1+1})$ by definition of γ_1. Then

$$\begin{aligned} w(t_{i_1})(t_{i_1+1} - t_{i_1}) &\ge W(t_{i_1+1}) - W(t_{i_1}) \\ &\ge G_1(t_{i_1+1}) - G_1(t_{i_1}) = \gamma_1 b_{i_1+1}(t_{i_1+1} - t_{i_1}). \end{aligned}$$

On the other hand, $G(t_{i_1-1}) = G_1(t_{i_1-1}) \le W(t_{i_1-1})$, so

$$\begin{aligned} b_{i_1}(t_{i_1} - t_{i_1-1}) &= G_1(t_{i_1}) - G_1(t_{i_1-1}) \\ &\ge W(t_{i_1}) - W(t_{i_1-1}) \ge w(t_{i_1})(t_{i_1} - t_{i_1-1}). \end{aligned}$$

Therefore $b_{i_1} \ge w(t_{i_1}) \ge \gamma_1 b_{i_1+1}$.

We also have $g_1 \prec w$. Indeed, in view of Lemma 3.2 and the definition of i_1 it is enough to check that $G_1(t_i) \le W(t_i)$ for $i > i_1$. We have

$$\begin{aligned} G_1(t_i) &= G(t_{i_1}) + \gamma_1(G(t_i) - G(t_{i_1})) \\ &\le G(t_{i_1}) + \frac{W(t_i) - W(t_{i_1})}{G(t_i) - G(t_{i_1})}(G(t_i) - G(t_{i_1})) = W(t_i). \end{aligned}$$

However, $\|\varphi(f/g_1)g_1\|_1 < \|\varphi(f/g)g\|_1$ in view of $g_1 \geq g$ and $g_1 \neq g$ and the fact that $\varphi(a/t)t$ is a strictly decreasing function of t for each $a > 0$ (by the assumption that φ is an N-function). This contradicts the fact that g realizes the infimum in the definition of $P_{\varphi,w}(f)$. ∎

LEMMA 3.5. *Let φ be an N-function. Let $f = \sum_{i=1}^n a_i\chi_{A_i}$ for some a_i with $a_1 > \cdots > a_n > 0$, $A_i = [t_{i-1}, t_i)$ where $0 = t_0 < t_1 < \cdots < t_n < \infty$, and $g = \sum_{i=1}^n b_i\chi_{A_i} = g^*$ be a minimizing function for $P_{\varphi,w}(f)$. Assume also that for some $k = 1, \ldots, n-1$ we have $G(t_k) = W(t_k)$. Then $g\chi_{[0,t_k)}$ is a minimizing function for $P_{\varphi,w}(f\chi_{[0,t_k)})$, that is,*

$$\|\varphi(f/g)g\chi_{[0,t_k)}\|_1 = P_{\varphi,w}(f\chi_{[0,t_k)}),$$

while

$$\|\varphi(f/g)g\chi_{[t_k,t_n)}\|_1 = \inf\left\{\begin{matrix}\sum_{i=k+1}^n \varphi(a_i/c_i)c_i|A_i| : c_{k+1} > \cdots > c_n > 0, \\ \sum_{i=k+1}^j c_i|A_i| \leq W(t_j) - W(t_k),\ k < j \leq n\end{matrix}\right\}. \tag{3.3}$$

Proof. By Lemma 3.3 we have $b_1 \geq \cdots \geq b_n > 0$, and from $g \prec w$ and $G(t_k) = W(t_k)$ we deduce that

$$\sum_{i=k+1}^j b_i|A_i| \leq W(t_j) - W(t_k), \quad k < j \leq n.$$

Note that minimizing (3.3) is equivalent to minimizing $P_{\varphi,w_k}(f_k)$, where for $t \in I$ we let

$$f_k(t) = (f\chi_{[t_k,t_n)})(t + t_k), \qquad w_k(t) = (w\chi_{[t_k,t_n)})(t + t_k).$$

By Lemma 3.3 applied to the interval $[0, t_n - t_k)$ there is a decreasing simple function $h^{(1)} = \sum_{i=k+1}^n h_i\chi_{A_i - t_k} \prec w_k$ that minimizes $P_{\varphi,w_k}(f_k)$. On the other hand, by Lemmas 3.3 and 3.4 applied to $[0, t_k)$ there is a decreasing simple function $h^{(2)} = \sum_{i=1}^k h_i\chi_{A_i} \prec w\chi_{[0,t_k)}$ that minimizes $P_{\varphi,w}(f\chi_{[0,t_k)})$, and we have $H^{(2)}(t_k) = \int_0^{t_k} h^{(2)} = W(t_k)$. Thus

$$\sum_{i=1}^k \varphi(a_i/h_i)h_i|A_i| = P_{\varphi,w}(f\chi_{[0,t_k)}) \leq \sum_{i=1}^k \varphi(a_i/b_i)b_i|A_i|, \tag{3.4}$$

$$\sum_{i=k+1}^n \varphi(a_i/h_i)h_i|A_i| = P_{\varphi,w_k}(f_k) \leq \sum_{i=k+1}^n \varphi(a_i/b_i)b_i|A_i|, \tag{3.5}$$

and so

$$\sum_{i=1}^n \varphi(a_i/h_i)h_i|A_i| \leq \sum_{i=1}^n \varphi(a_i/b_i)b_i|A_i|. \tag{3.6}$$

Now let $h := \sum_{i=1}^n h_i\chi_{A_i}$ and note that h is decreasing. Indeed, we will show that $h_k \ge h_{k+1}$. Since $H(t_{k-1}) = H^{(2)}(t_{k-1}) \le W(t_{k-1})$, and $H(t_k) = H^{(2)}(t_k) = W(t_k)$ we have

$$h_k|A_k| = H(t_k) - H(t_{k-1}) \ge W(t_k) - W(t_{k-1}) = w_k|A_k|,$$

while

$$\begin{aligned} h_{k+1}|A_{k+1}| = H(t_{k+1}) - H(t_k) &= \int_0^{t_{k+1}-t_k} h^{(1)} \\ &\le W_k(t_{k+1}-t_k) = W(t_{k+1}) - W(t_k) \le w_k|A_{k+1}|, \end{aligned}$$

and thus $h_k \ge w_k \ge h_{k+1}$. Note also that $h \prec w$, since: $H(t_i) \le W(t_i)$, $i = 1, \ldots, k$, by $h^{(2)} \prec w$; $H(t_k) = H^{(2)}(t_k) = W(t_k)$; and $H(t_i) - H(t_k) \le W(t_i) - W(t_k)$, $i = k+1, \ldots, n$, in view of $h^{(1)} \prec w_k$. Then

$$\sum_{i=1}^n \varphi(a_i/h_i)h_i|A_i| \ge P_{\varphi,w}(f) = \sum_{i=1}^n \varphi(a_i/b_i)b_i|A_i|.$$

Consequently, we have equality in (3.6) and also in both (3.4) and (3.5), completing the proof. ∎

LEMMA 3.6. *Let φ be a strictly convex N-function. Let $f = \sum_{i=1}^n a_i\chi_{A_i}$ for some a_i with $a_1 > \cdots > a_n > 0$ and $A_i = [t_{i-1}, t_i)$ where $0 = t_0 < t_1 < \cdots < t_n < \infty$ and $g = \sum_{i=1}^n b_i\chi_{A_i}$ with $b_1, \ldots, b_n > 0$. If $g \ne \lambda f$ for $\lambda = G(t_n)/F(t_n)$ then*

$$\|\varphi(f/g)g\|_1 > \|\varphi(f/[\lambda f])\lambda f\|_1.$$

Proof. Set $\lambda_i = b_i/a_i$, $i = 1, \ldots, n$. If all the λ_i are equal to, say, λ' then $g = \lambda' \sum_{i=1}^n a_i\chi_{A_i} = \lambda' f$ and $G(t_n) = \lambda' F(t_n)$, so $\lambda' = \lambda$. If $g \ne \lambda f$, then not all λ_i are equal and by Jensen's inequality and strict convexity of φ we have

$$\begin{aligned} \frac{1}{G(t_n)}\left\|\varphi\left(\frac{f}{g}\right)g\right\|_1 &= \sum_{i=1}^n \varphi\left(\frac{1}{\lambda_i}\right)\frac{\lambda_i a_i|A_i|}{G(t_n)} > \varphi\left(\sum_{i=1}^n \frac{1}{\lambda_i}\,\frac{\lambda_i a_i|A_i|}{G(t_n)}\right) \\ &= \varphi\left(\frac{1}{\lambda}\right) = \frac{1}{G(t_n)}\varphi\left(\frac{1}{\lambda}\right)\|\lambda f\|_1 = \frac{1}{G(t_n)}\left\|\varphi\left(\frac{f}{\lambda f}\right)\lambda f\right\|_1. \ ∎ \end{aligned}$$

THEOREM 3.7. *Let φ be a strictly convex N-function. Let $f = \sum_{i=1}^n a_i\chi_{A_i}$ for some a_i with $a_1 > \cdots > a_n > 0$ and $A_i = [t_{i-1}, t_i)$ where $0 = t_0 < t_1 < \cdots < t_n < \infty$. If $g = \sum_{i=1}^n b_i\chi_{A_i} = g^*$ is a simple function realizing the infimum in the definition of $P_{\varphi,w}(f)$ then*

$$g = \sum_{j=0}^{m-1} \lambda_j f \chi_{[t_{i_j}, t_{i_{j+1}})} \tag{3.7}$$

for some λ_j, $j = 0, 1, \ldots, m-1$, where $0 = i_0 < i_1 < \cdots < i_m = n$ and

$$G(t_{i_j}) = W(t_{i_j})$$

for each $j = 0, 1, \ldots, m$.

Proof. Let g satisfy the assumptions of the theorem. By Lemma 3.3 we have $b_1 \geq \cdots \geq b_n > 0$, and by Lemma 3.4, $G(t_n) = W(t_n)$. Define a finite sequence $(i_0, i_1, \ldots, i_m)$ by

$$i_0 = 0 \quad \text{and} \quad i_j = \min\{i > i_{j-1} : G(t_i) = W(t_i)\}, \qquad j = 1, \ldots, m.$$

Applying Lemma 3.5 $m-1$ times, that is, decomposing f first as $f\chi_{[0,t_{i_{m-1}})} + f\chi_{[t_{i_{m-1}},t_n]}$, then $f\chi_{[0,t_{i_{m-1}})}$ as $f\chi_{[0,t_{i_{m-2}})} + f\chi_{[t_{i_{m-2}},t_{i_{m-1}})}$, etc., we find that if $g = g^* = \sum_{i=1}^n b_i \chi_{A_i}$ minimizes $P_{\varphi,w}(f)$ and $G(t_{i_j}) = W(t_{i_j})$, $G(t_{i_{j+1}}) = W(t_{i_{j+1}})$, then $(b_{i_j+1}, \ldots, b_{i_{j+1}})$ also minimizes the sum

$$\sum_{i=i_j+1}^{i_{j+1}} \varphi(a_i/b_i) b_i |A_i|$$

under the constraints $B_j(k) := \sum_{i=i_j+1}^{k} b_i |A_i| \leq W(t_k) - W(t_{i_j})$ for $k = i_j + 1, \ldots, i_{j+1}$.

Therefore we may consider each interval $[t_{i_j}, t_{i_{j+1}})$, $j = 0, 1, \ldots, m-1$, separately and we will show that $g\chi_{[t_{i_j},t_{i_{j+1}})} = \lambda_j f \chi_{[t_{i_j},t_{i_{j+1}})}$ where

$$\lambda_j = \frac{W(t_{i_{j+1}}) - W(t_{i_j})}{F(t_{i_{j+1}}) - F(t_{i_j})}.$$

If $t_{i_{j+1}} = t_{i_j+1}$ then

$$g\chi_{[t_{i_j},t_{i_{j+1}})} = \frac{F(t_{i_{j+1}}) - F(t_{i_j})}{t_{i_{j+1}} - t_{i_j}} \lambda_j \chi_{[t_{i_j},t_{i_{j+1}})} = \lambda_j f \chi_{[t_{i_j},t_{i_{j+1}})}.$$

If $t_{i_{j+1}} > t_{i_j+1}$ then for all $t_{i_j} < t_i < t_{i_{j+1}}$ one has $G(t_i) < W(t_i)$. In this case consider the function $\psi_j : \mathbb{R}_+^{i_{j+1}-i_j} \to \mathbb{R}_+$ defined for $b = (b_{i_j+1}, \ldots, b_{i_{j+1}})$ by

$$\psi_j(b) = \psi_j(b_{i_j+1}, \ldots, b_{i_{j+1}}) = \sum_{i=i_j+1}^{i_{j+1}} \varphi(a_i/b_i) b_i |A_i|,$$

and define the set

$$C_j = \left\{ \begin{array}{c} b \in \mathbb{R}_+^{i_{j+1}-i_j} : b_{i_j+1} \geq b_{i_j+2} \geq \cdots \geq b_{i_{j+1}} > 0, \\ \forall_{i_j+1 \leq k < i_{j+1}} \ B_j(k) < W(t_k) - W(t_{i_j}), \\ B_j(i_{j+1}) = W(t_{i_{j+1}}) - W(t_{i_j}) \end{array} \right\}.$$

Notice that the condition

$$B_j(k) = \sum_{i=i_j+1}^{k} b_i |A_i| < W(t_k) - W(t_{i_j}), \qquad k = i_j + 1, \ldots, i_{j+1} - 1,$$

is a consequence of the relation $g \prec w$ and the definition of i_j and i_{j+1}. In fact, by Lemma 3.2, $g \prec w$ is equivalent to $G(t_i) \leq W(t_i)$ for each

$i=1,\dots,n$, and by definition of i_j and i_{j+1} we have $G(t_k)<W(t_k)$ for each $k=i_j+1,\dots,i_{j+1}-1$. It follows that for $k=i_j+1,\dots,i_{j+1}-1$,

$$G(t_{i_j})+\sum_{i=i_j+1}^{k} b_i|A_i| = G(t_k) < W(t_k) = W(t_{i_j}) + W(t_k) - W(t_{i_j}),$$

and so $\sum_{i=i_j+1}^{k} b_i|A_i| < W(t_k)-W(t_{i_j})$.

We need to show now that ψ_j attains its minimum over C_j at the point $\lambda_j a$, $a=(a_{i_j+1},\dots,a_{i_{j+1}})$.

Consider first the simplex $S_j=\mathbb{R}_+^{i_{j+1}-i_j}\cap H_j$, where H_j is the hyperplane in $\mathbb{R}^{i_{j+1}-i_j}$ given by the equation $\sum_{i=i_j+1}^{i_{j+1}}|A_i|x_i = W(t_{i_{j+1}})-W(t_{i_j})$. Lemma 3.6 tells us that $\lambda_j a$ is the unique minimizer of ψ_j over S_j. It remains to show that $\lambda_j a\in C_j\subset S_j$. Suppose for a contradiction that $\lambda_j a\notin C_j$. By the previous reasoning, there exists $\bar b\in C_j$ that minimizes ψ_j over C_j. Define $b(\lambda)=\lambda\bar b+(1-\lambda)\lambda_j a$ for $0\le\lambda\le 1$. Then by Lemma 3.6 and since $\lambda_j a\neq\bar b$, we get $\psi_j(\bar b)>\psi_j(\lambda_j a)$. Moreover the strict convexity of $t\mapsto\varphi(d/t)t$ for each $d>0$ implies the strict convexity of ψ_j. Therefore for each $0<\lambda<1$,

$$\psi_j(b(\lambda)) < \lambda\psi_j(\bar b)+(1-\lambda)\psi_j(\lambda_j a) < \psi_j(\bar b).$$

Notice that for every $0<\lambda<1$, $b_{i_j+1}(\lambda)>\dots>b_{i_{j+1}}(\lambda)$, where $b(\lambda)=(b_{i_j+1}(\lambda),\dots,b_{i_{j+1}}(\lambda))$, and

$$\begin{aligned}\sum_{i=i_j+1}^{i_{j+1}} b_i(\lambda)|A_i| &= \lambda\sum_{i=i_j+1}^{i_{j+1}}\bar b_i|A_i| + (1-\lambda)\lambda_j(F(t_{i_{j+1}})-F(t_{i_j}))\\ &= W(t_{i_{j+1}})-W(t_{i_j}).\end{aligned}$$

Moreover for each $k=i_j+1,\dots,i_{j+1}-1$,

$$\begin{aligned}\sum_{i=i_j+1}^{k} b_i(\lambda)|A_i| &= \lambda\sum_{i=i_j+1}^{k}\bar b_i|A_i| + (1-\lambda)\lambda_j(F(t_k)-F(t_{i_j}))\\ &< \lambda(W(t_k)-W(t_{i_j}))+(1-\lambda)\lambda_j(F(t_k)-F(t_{i_j})).\end{aligned}$$

This implies that for $0<\lambda<1$ sufficiently close to 1, $b(\lambda)\in C_j$. Since $\psi_j(b(\lambda))<\psi_j(\bar b)$, the element $\bar b$ cannot minimize ψ_j over C_j, which gives the desired contradiction. We have thus shown that $g=\lambda_j f$ on $[t_{i_j},t_{i_{j+1}})$, and since j was arbitrary, the proof is finished. ∎

The following algorithm will be crucial for proving the main Theorem 3.9 which provides a procedure to obtain a minimizing function g for the modular $P_{\varphi,w}(f)$.

Algorithm A. Let $f=\sum_{i=1}^n a_i\chi_{A_i}$ for some $a_1>\dots>a_n>0$ and $A_i=[t_{i-1},t_i)$ where $0=t_0<t_1<\dots<t_n<\infty$. Define first

$$g_{-1}=f,\qquad \gamma_0=\lambda_0=\min_{1\le i\le n}\frac{W(t_i)}{F(t_i)},\qquad g_0=\gamma_0 f=\lambda_0 f,\qquad i_0=0.$$

Then for $j > 0$ let

$$(3.8)\qquad \begin{aligned} i_j &= \max\left\{ i > i_{j-1} : \gamma_{j-1} = \frac{W(t_i) - W(t_{i_{j-1}})}{G_{j-2}(t_i) - G_{j-2}(t_{i_{j-1}})} \right\}, \\ \gamma_j &= \min_{i_j < i \le n} \frac{W(t_i) - W(t_{i_j})}{G_{j-1}(t_i) - G_{j-1}(t_{i_j})}, \\ g_j &= g_{j-1}\chi_{[0,t_{i_j})} + \gamma_j g_{j-1}\chi_{[t_{i_j},t_n)}. \end{aligned}$$

Continue the recurrent steps until $i_m = n$ for some m and denote $g^f = g_{m-1}$. Clearly $\gamma_j > 1$ for $j = 1, \ldots, m-1$, and

$$g^f = \sum_{j=0}^{m-1} \lambda_j f \chi_{[t_{i_j},t_{i_{j+1}})}, \qquad \lambda_j = \prod_{i=0}^{j} \gamma_i.$$

Hence $\lambda_0 < \lambda_1 < \cdots < \lambda_{m-1}$. We also have, for $j = 0, 1, \ldots, m-1$,

$$\begin{aligned} \gamma_j &= \frac{W(t_{i_{j+1}}) - W(t_{i_j})}{G_{j-1}(t_{i_{j+1}}) - G_{j-1}(t_{i_j})} \\ &= \frac{W(t_{i_{j+1}}) - W(t_{i_j})}{\gamma_{j-1}(G_{j-2}(t_{i_{j+1}}) - G_{j-2}(t_{i_j}))} = \frac{W(t_{i_{j+1}}) - W(t_{i_j})}{\prod_{i=0}^{j-1} \gamma_i (F(t_{i_{j+1}}) - F(t_{i_j}))}. \end{aligned}$$

Hence

$$\lambda_j = \prod_{i=0}^{j} \gamma_i = \frac{W(t_{i_{j+1}}) - W(t_{i_j})}{F(t_{i_{j+1}}) - F(t_{i_j})}.$$

It follows that for each $j = 0, 1, \ldots, m-1$,

$$G^f(t_{i_j}) := \int_0^{t_{i_j}} g^f = W(t_{i_j}).$$

Now we will show that g^f is decreasing and $g^f \prec w$. Evidently $g_0 = \gamma_0 f \prec w$.

Reasoning as in Lemma 3.4 we can show that $g_j = g_j^*$ for each j. In fact, since f is decreasing, it is sufficient to show that $\lambda_{j-1} a_{i_j} \ge \lambda_j a_{i_j+1}$ for each $j = 1, \ldots, m-1$. Fix $j = 1, \ldots, m-1$. We have $G_{j-1}(t_{i_j}) = G_j(t_{i_j}) = W(t_{i_j})$ and $G_j(t_{i_j+1}) \le W(t_{i_j+1})$ by definition of γ_j. Then

$$\begin{aligned} w(t_{i_j})(t_{i_j+1} - t_{i_j}) &\ge W(t_{i_j+1}) - W(t_{i_j}) \\ &\ge G_j(t_{i_j+1}) - G_j(t_{i_j}) = \lambda_j a_{i_j+1}(t_{i_j+1} - t_{i_j}). \end{aligned}$$

On the other hand $G_{j-1}(t_{i_j-1}) = G_j(t_{i_j-1}) \le W(t_{i_j-1})$, so

$$\begin{aligned} \lambda_{j-1} a_{i_j}(t_{i_j} - t_{i_j-1}) &= G_j(t_{i_j}) - G_j(t_{i_j-1}) \\ &\ge W(t_{i_j}) - W(t_{i_j-1}) \ge w(t_{i_j})(t_{i_j} - t_{i_j-1}). \end{aligned}$$

Therefore $\lambda_{j-1}a_{i_j} \geq w(t_{i_j}) \geq \lambda_j a_{i_j+1}$. It remains to prove that $g_{j-1} \prec w$ implies $g_j \prec w$. By (3.8),

$$g_j = g_{j-1}\chi_{[0,t_{i_j})} + \gamma_j g_{j-1}\chi_{[t_{i_j},t_n)}.$$

Hence for $k \leq i_j$,

$$G_j(t_k) = G_{j-1}(t_k) \leq W(t_k).$$

If $k > i_j$, then by definition of γ_j,

$$\begin{aligned} G_j(t_k) &= G_{j-1}(t_{i_j}) + \gamma_j(G_{j-1}(t_k) - G_{j-1}(t_{i_j})) \\ &\leq W(t_{i_j}) + W(t_k) - W(t_{i_j}) = W(t_k), \end{aligned}$$

and then by Lemma 3.2 we have $g_j \prec w$, which proves that $g^f \prec w$. It is also worth noticing that since $\lambda_0 < \lambda_1 < \cdots < \lambda_{m-1}$, the function f/g^f is decreasing.

Remark 3.8. The function g^f produced by Algorithm A is of the form (3.7), but the sequence (t_{i_j}) obtained in this way need not be maximal in the sense that there may exist $t_i \notin (t_{i_j})$ such that $G^f(t_i) = \int_0^{t_i} g^f = W(t_i)$.

Now we are ready for our main result describing how to calculate the infimum in $P_{\varphi,w}(f)$ for a decreasing simple function f.

Theorem 3.9. *Let φ be an N-function and let $f = \sum_{i=1}^n a_i\chi_{A_i}$ for some a_i with $a_1 > \cdots > a_n > 0$ and $A_i = [t_{i-1}, t_i)$ where $0 = t_0 < t_1 < \cdots < t_n < \infty$. Then the function g^f produced by Algorithm A is a minimizing function for $P_{\varphi,w}(f)$, that is,*

$$P_{\varphi,w}(f) = \left\|\varphi\left(\frac{f}{g^f}\right)g^f\right\|_1.$$

The function g^f is independent of φ and depends only on f and w.

Proof. We divide the proof into two parts.

(I) Assume first that φ is, strictly convex. Let g^f be produced by Algorithm A. Suppose that a function h is minimizing as in Theorem 3.7. We will prove that $h = g^f$. This will be done by induction on the number s of steps in Algorithm A.

(a) Assume first that $s = 1$, that is,

$$\min_{1\leq i\leq n}\{W(t_i)/F(t_i)\} = W(t_n)/F(t_n).$$

Then $g^f = \lambda_0 f$ with $\lambda_0 = W(t_n)/F(t_n)$. On the other hand, by Theorem 3.7, $h = \sum_{j=0}^{p-1}\mu_j f\chi_{[t_{k_j},t_{k_{j+1}})}$, where $0 = k_0 < k_1 < \cdots < k_p = n$ and $H(t_{k_j}) = W(t_{k_j})$, $j = 1,\ldots,p$. But since $\lambda_0 f \prec w$, Lemma 3.6 shows that $\mu_0 = \cdots = \mu_{p-1} = \lambda_0$, that is, $h = \lambda_0 f = g^f$.

(b) Assume now that $s > 1$ and that Algorithm A is valid for $s-1$ steps. We claim first that

$$W(t_{i_1}) = H(t_{i_1}), \tag{3.9}$$

where $i_1 = \max\{i > 0 : \lambda_0 = W(t_i)/F(t_i)\}$, $\lambda_0 = \min_{1\le i\le n}\{W(t_i)/F(t_i)\}$. Clearly $i_1 < n$. If the claim is false then $H(t_{i_1}) < W(t_{i_1})$. Now since $H(t_i) \le W(t_i)$ for all $i = 1, \ldots, n$, two cases are possible:

(i) $H(t_i) < W(t_i)$ for each $0 < i \le i_1$, or
(ii) $H(t_k) = W(t_k)$ for some $k < i_1$ and $H(t_{i_1}) < W(t_{i_1})$.

In case (i), by (3.7), $h\chi_{[0,t_m)} = \lambda f\chi_{[0,t_m)}$ with $H(t_m) = W(t_m)$ for some $\lambda > 0$ and $t_m > t_{i_1}$. Hence $\lambda F(t_{i_1}) = H(t_{i_1}) < W(t_{i_1}) = \lambda_0 F(t_{i_1})$ and thus $\lambda < \lambda_0$. It follows that

$$H(t_m) = \lambda F(t_m) < \lambda_0 F(t_m) \le W(t_m),$$

which contradicts $H(t_m) = W(t_m)$.

In case (ii), suppose that $k < i_1$ is the largest index such that $W(t_k) = H(t_k)$. Since h is assumed to be a minimizing function, by Theorem 3.7 there exist $t_{i_1} < t_m \le t_n$ and $\lambda > 0$ such that

$$h\chi_{[t_k,t_m)} = \lambda f\chi_{[t_k,t_m)} \quad \text{and} \quad H(t_m) = W(t_m).$$

Since $\lambda_0 F \le W$ and $\lambda_0 F(t_{i_1}) = W(t_{i_1})$, we have

$$\begin{aligned}\lambda(F(t_{i_1}) - F(t_k)) &= H(t_{i_1}) - W(t_k)\\ &< W(t_{i_1}) - W(t_k) \le \lambda_0(F(t_{i_1}) - F(t_k))\end{aligned}$$

and

$$\begin{aligned}\lambda(F(t_m) - F(t_{i_1})) &= W(t_m) - H(t_{i_1})\\ &> \lambda_0 F(t_m) - W(t_{i_1}) = \lambda_0(F(t_m) - F(t_{i_1})),\end{aligned}$$

which gives a contradiction. Thus we have shown the claim (3.9).

Next we will show that

$$h\chi_{[0,t_{i_1})} = \lambda_0 f\chi_{[0,t_{i_1})} = g^f\chi_{[0,t_{i_1})}. \tag{3.10}$$

Suppose for a contradiction that

$$h\chi_{[0,t_{i_1})} = \sum_{j=0}^{r-1} \delta_j f\chi_{[t_{k(j)},t_{k(j+1)})},$$

where $0 = t_{k(0)} < t_{k(1)} < \cdots < t_{k(r)} = t_{i_1}$ with $\delta_j \ne \lambda_0$ for some $j = 0, 1, \ldots, r-1$. Then by Lemma 3.6 applied to the interval $[0, t_{i_1})$ and λ_0, we have

$$\begin{aligned}\|\varphi(f/h)h\|_1 &= \|\varphi(f/h)h\chi_{[0,t_{i_1})}\|_1 + \|\varphi(f/h)h\chi_{[t_{i_1},t_n)}\|_1\\ &> \|\varphi(f/(\lambda_0 f))\lambda_0 f\chi_{[0,t_{i_1})}\|_1 + \|\varphi(f/h)h\chi_{[t_{i_1},t_n)}\|_1.\end{aligned}$$

It follows that h is not minimizing for $P_{\varphi,w}(f)$, which contradicts our assumption and proves (3.10).

Now in view of $H(t_{i_1}) = W(t_{i_1})$ we see by the proof of Lemma 3.5 that $h_{i_1}(t) = h\chi_{[t_{i_1},t_n)}(t+t_{i_1})$ is a minimizing function for $P_{\varphi,w_{i_1}}(f_{i_1})$, where for $t \in I$,

$$f_{i_1}(t) = (f\chi_{[t_{i_1},t_n)})(t+t_{i_1}), \qquad w_{i_1}(t) = (w\chi_{[t_{i_1},t_n)})(t+t_{i_1}).$$

On the other hand, it is straightforward to see that Algorithm A for f_{i_1} and the weight w_{i_1} has $s-1$ steps and that it yields a function $g^{f_{i_1}}$ which is nothing else than $g^f\chi_{[t_{i_1},t_n)}$ shifted backward by t_{i_1}, that is,

$$g^{f_{i_1}}(t) = g^f\chi_{[t_{i_1},t_n)}(t+t_{i_1}).$$

Now by induction hypothesis we have $g^{f_{i_1}} = h_{i_1}$ and thus

$$g^f\chi_{[t_{i_1},t_n)} = h\chi_{[t_{i_1},t_n)}.$$

We also know by Lemma 3.5 that $h\chi_{[0,t_{i_1})}$ is minimizing for $P_{\varphi,w}(f\chi_{[0,t_{i_1})})$, while clearly Algorithm A for $f\chi_{[0,t_{i_1})}$ has only one step and yields $g^f\chi_{[0,t_{i_1})}$. Thus by part (a), $g^f\chi_{[0,t_{i_1})} = h\chi_{[0,t_{i_1})}$. Therefore $g^f = h$, and this finishes the proof of case (I).

(II) Assume now that φ is any N-function. Let $\varphi_m(t) = \varphi(t) + (1/m)t^2$. Then each φ_m is a strictly convex N-function and $\varphi_m \to \varphi$ uniformly on compact sets. Let g^f be produced by Algorithm A. Suppose g^f is not minimizing for $P_{\varphi,w}(f)$, i.e. there is $h = \sum_{i=1}^n b_i\chi_{A_i} \prec w$ such that for some $\delta > 0$ we have

$$\|\varphi(f/h)h\|_1 + \delta \le \|\varphi(f/g^f)g^f\|_1.$$

Since h, f and g^f are simple functions,

$$\|\varphi_m(f/h)h\|_1 \to \|\varphi(f/h)h\|_1 \quad \text{and} \quad \|\varphi_m(f/g^f)g^f\|_1 \to \|\varphi(f/g^f)g^f\|_1.$$

Hence there is N such that for $m > N$,

$$\|\varphi_m(f/h)h\|_1 \le \|\varphi(f/h)h\|_1 + \delta/3,$$
$$\|\varphi_m(f/g^f)g^f\|_1 \ge \|\varphi(f/g^f)g^f\|_1 - \delta/3.$$

Therefore for each $m > N$,

$$\|\varphi_m(f/h)h\|_1 + \delta/3 \le \|\varphi_m(f/g^f)g^f\|_1,$$

which means that g^f does not minimize $P_{\varphi_m,w}(f)$. This contradicts case (I), and the proof is complete. ∎

4. Dual norms of $\Lambda_{\varphi,w}$ in terms of level functions. In this section we develop formulas for the Köthe duals of Orlicz–Lorentz spaces equipped with the Luxemburg or Amemiya norms in terms of level functions. Let w

be a weight function on I as defined in the introduction. For $f = f^*$ locally integrable on I, define after Halperin [4] for $0 \le a < b < \infty$, $a, b \in I$,

$$W(a,b) = \int_a^b w, \qquad F(a,b) = \int_a^b f, \qquad R(a,b) = \frac{F(a,b)}{W(a,b)},$$

and for $b = \infty$,

$$R(a,b) = R(a,\infty) = \limsup_{t\to\infty} R(a,t).$$

Then $(a,b) \subset I$ is called a *level interval* (resp. *degenerate level interval*) *of f with respect to w* if $b < \infty$ (resp. $b = \infty$) and for each $t \in (a,b)$,

$$R(a,t) \le R(a,b) \quad \text{and} \quad R(a,b) > 0.$$

It is easy to see that the restriction $R(a,b) > 0$ ensures that any level interval of $f = f^*$ is in fact included in the support of f^*, and this is the only difference with the original definition from [4]. Level intervals can be equivalently assumed to be open, closed or half-closed. If the weight w is fixed then we will speak of *level intervals of f*, or just l.i. for simplicity. If a level interval is not contained in any larger level interval, then it is called a *maximal level interval of f with respect to w*, or just a maximal level interval, for short m.l.i. In [4], Halperin proved that maximal level intervals of f with respect to w are pairwise disjoint and unique and so there are at most countably many such intervals.

Remark 4.1. (1) The whole semiaxis $(0,\infty)$ may be a degenerate l.i. Take for example any weight function w and let $f = w$.

(2) Given any weight w, if a decreasing function f is constant on (a,b) then (a,b) is a l.i. of f with respect to w.

First we make a simple observation that the function $t \mapsto (\int_a^t h)/(t-a)$ is decreasing for $t > a$ whenever h is decreasing and locally integrable. Letting now $f(t) = c$ for $t \in (a,b)$, as w is decreasing, the inequality $R(a,t) \le R(a,b)$ on (a,b) is equivalent to $\frac{1}{b-a}\int_a^b w \le \frac{1}{t-a}\int_a^t w$ on (a,b).

(3) If w is a constant weight then (a,b) is a l.i. of f if and only if f is constant on (a,b). Consequently, any decreasing function with countably many different values has infinitely many m.l.i. with respect to a constant weight.

Let indeed w be constant on I, and (a,b) be a l.i. of f with respect to w. Therefore $F(a,t)/(t-a) \le F(a,b)/(b-a)$ on (a,b), and since f is decreasing we have equality, that is, $F(a,t) = F(a,b)(t-a)/(b-a)$ on (a,b). Hence $f(t) = F(a,b)/(b-a)$ for all $t \in (a,b)$, and so f is constant on (a,b).

Definition 4.2 ([4]). Let $f \in L^0$ be decreasing and locally integrable on I. Then the *level function* f^0 of f with respect to w is defined as

$$f^0(t) = \begin{cases} R(a_n, b_n)w(t) & \text{for } t \in (a_n, b_n), \\ f(t) & \text{otherwise,} \end{cases}$$

where (a_n, b_n) is an enumeration of all maximal level intervals of f.

LEMMA 4.3. *Let* $f = f^* = \sum_{i=1}^n a_i \chi_{A_i} \in \mathcal{M}_{\varphi,w}$, *where* $A_i = [t_{i-1}, t_i)$ *and* $0 = t_0 < t_1 < \cdots < t_n < \infty$. *Then*

$$P_{\varphi,w}(f) = \int_I \varphi\left(\frac{f}{g^f}\right) g^f = \int_I \varphi\left(\frac{f^0}{w}\right) w.$$

In particular, the intervals (t_{i-1}, t_i) *are level intervals of* f *with respect to* w. *Moreover, the maximal level intervals of* f *with respect to* w *are* $(t_{i_j}, t_{i_{j+1}})$, *where*

$$g^f = \sum_{j=0}^{m-1} \lambda_j f \chi_{[t_{i_j}, t_{i_{j+1}})}$$

is from Algorithm A.

Proof. Let $g^f = \sum_{j=0}^{m-1} \lambda_j f \chi_{[t_{i_j}, t_{i_{j+1}})}$ be as in Algorithm A, where $\lambda_0 < \lambda_1 < \cdots < \lambda_{m-1}$ and

$$\lambda_j = \frac{W(t_{i_j}, t_{i_{j+1}})}{F(t_{i_j}, t_{i_{j+1}})}, \qquad j = 0, 1 \ldots, m-1.$$

Hence by Theorem 3.9,

$$P_{\varphi,w}(f) = \sum_{j=0}^{m-1} \int_{t_{i_j}}^{t_{i_{j+1}}} \varphi\left(\frac{f}{\lambda_j f}\right) \lambda_j f = \sum_{j=0}^{m-1} \varphi\left(\frac{F(t_{i_j}, t_{i_{j+1}})}{W(t_{i_j}, t_{i_{j+1}})}\right) W(t_{i_j}, t_{i_{j+1}}). \tag{4.1}$$

We will now compute the level function f^0 with respect to w. Suppose first that

$$w = Tw = \sum_{i=1}^n \left(\frac{1}{|A_i|} \int_{A_i} w\right) \chi_{A_i}.$$

We shall show that every $(t_{i_j}, t_{i_{j+1}})$ is a maximal level interval of f with respect to w. By Remark 4.1(2) each (t_i, t_{i+1}) is a level interval of f. Moreover one can check that for $i < k \le n$,

$$\forall_{t \in (t_i, t_k)} \ \frac{F(t_i, t)}{W(t_i, t)} \le \frac{F(t_i, t_k)}{W(t_i, t_k)} \ \Leftrightarrow \ \forall_{i<j<k} \ \frac{F(t_i, t_j)}{W(t_i, t_j)} \le \frac{F(t_i, t_k)}{W(t_i, t_k)}.$$

Let us show that each interval $(t_{i_j}, t_{i_{j+1}})$ is a level interval for f with respect to w. In fact, we only need to show that

$$R(t_{i_j}, t_k) \le R(t_{i_j}, t_{i_{j+1}}) \quad \text{for each } i_j < k < i_{j+1}.$$

In the notation of Algorithm A (see (3.8)), we have, for $j = 0, 1, \ldots, m-1$,

$$\lambda_j = \prod_{i=0}^{j} \gamma_i = \frac{W(t_{i_{j+1}}) - W(t_{i_j})}{F(t_{i_{j+1}}) - F(t_{i_j})} = \frac{1}{R(t_{i_j}, t_{i_{j+1}})},$$

$$g_{j-1} = \lambda_0 f\chi_{[0,t_{i_1})} + \lambda_1 f\chi_{[t_{i_1},t_{i_2})} + \cdots + \lambda_{j-2} f\chi_{[t_{i_{j-2}},t_{i_{j-1}})} + \lambda_{j-1} f\chi_{[t_{i_{j-1}},t_n)}.$$

Hence $G_{j-1}(t_k) - G_{j-1}(t_{i_j}) = \lambda_{j-1}(F(t_k) - F(t_{i_j}))$ for $i_j < k < i_{j+1}$, and so

$$\frac{1}{R(t_{i_j}, t_{i_{j+1}})} = \lambda_j = \gamma_j \lambda_{j-1} = \lambda_{j-1} \min_{i_j < i \le n} \frac{W(t_i) - W(t_{i_j})}{G_{j-1}(t_i) - G_{j-1}(t_{i_j})}$$
$$\le \lambda_{j-1} \frac{W(t_k) - W(t_{i_j})}{G_{j-1}(t_k) - G_{j-1}(t_{i_j})} = \frac{W(t_k) - W(t_{i_j})}{F(t_k) - F(t_{i_j})} = \frac{1}{R(t_{i_j}, t_k)},$$

which proves that $(t_{i_j}, t_{i_{j+1}})$ is a level interval.

To see that each $(t_{i_j}, t_{i_{j+1}})$ is a maximal level interval we will need Theorem 3.1 from [4], which states that if $a_1 < a_2 < b_1 < b_2$ and $(a_1, b_1), (a_2, b_2)$ are level intervals of f with respect to w, then so is (a_1, b_2). We also need a simple observation that (a, b) is a level interval if and only if

$$R(a, b) \le R(s, b) \quad \text{for each } s \in (a, b). \tag{4.2}$$

The latter follows from the elementary fact that for $v, x, y, z > 0$,

$$\frac{y}{z} \le \frac{v+y}{x+z} \ \Leftrightarrow\ \frac{v+y}{x+z} \le \frac{v}{x},$$

and from

$$R(a, b) = \frac{F(a, s) + F(s, b)}{W(a, s) + W(s, b)}.$$

Suppose therefore that $(t_{i_j}, t_{i_{j+1}})$ is not maximal. Then there is another level interval (a, b) such that $(t_{i_j}, t_{i_{j+1}}) \subsetneq (a, b)$. It follows that $a < t_{i_j}$ or $t_{i_{j+1}} < b$. Suppose $t_{i_{j+1}} < b$ (in the other case the proof is similar). Then by Halperin's above mentioned result, $(t_{i_j}, t_{i_{j+2}})$ is a level interval. But then by definition of level intervals we get

$$\frac{1}{\lambda_j} = R(t_{i_j}, t_{i_{j+1}}) \le R(t_{i_j}, t_{i_{j+2}}) \le R(t_{i_{j+1}}, t_{i_{j+2}}) = \frac{1}{\lambda_{j+1}},$$

which means that $\lambda_j \ge \lambda_{j+1}$. However by Algorithm A, $\lambda_{j+1} = \gamma_{j+1}\lambda_j$ with $\gamma_{j+1} > 1$, which gives a contradiction.

Let now w be arbitrary. Denote $TW(t) = \int_0^t Tw$. Notice that $TW(t_i) = W(t_i)$ for each i, and $TW(t) \le W(t)$ for any $t > 0$, since for any $t \in (t_{k-1}, t_k)$,

$$TW(t) = W(t_{k-1}) + \bigg(\frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} w\bigg)(t - t_{k-1}) \le W(t_{k-1}) + \int_{t_{k-1}}^{t} w = W(t).$$

Then for each j and each $t \in (t_{i_j}, t_{i_{j+1}})$ one has $W(t_{i_j}, t) = W(t) - W(t_{i_j}) \ge TW(t) - TW(t_{i_j}) = TW(t_{i_j}, t)$. Therefore, since by the first part of the proof

$(t_{i_j}, t_{i_{j+1}})$ is a l.i. of f with respect to Tw, we have

$$R(t_{i_j}, t) = \frac{F(t_{i_j}, t)}{W(t_{i_j}, t)} \le \frac{F(t_{i_j}, t)}{TW(t_{i_j}, t)} \le \frac{F(t_{i_j}, t_{i_{j+1}})}{TW(t_{i_j}, t_{i_{j+1}})}$$
$$= \frac{F(t_{i_j}, t_{i_{j+1}})}{W(t_{i_j}, t_{i_{j+1}})} = R(t_{i_j}, t_{i_{j+1}}),$$

which shows that also $(t_{i_j}, t_{i_{j+1}})$ is a l.i. of f with respect to w. By the previous reasoning it is also a m.l.i. of f with respect to w.

Thus the level function f^0 with respect to w is given by

$$f^0(t) = \begin{cases} R(t_{i_j}, t_{i_{j+1}})w(t) & \text{for } t \in (t_{i_j}, t_{i_{j+1}}),\ j = 0, 1, \ldots, m-1, \\ 0 & \text{for } t \ge t_n. \end{cases}$$

Then, by (4.1),

$$\int_I \varphi(f^0/w)w = \sum_{j=0}^{m-1} \varphi(R(t_{i_j}, t_{i_{j+1}}))W(t_{i_j}, t_{i_{j+1}}) = P_{\varphi,w}(f),$$

and the proof is finished. ■

Remark 4.4. Algorithm A and Lemma 4.3 also suggest another point of view. Namely, rather than changing the function f, we may change the weight according to the definition of $P_{\varphi,w}(f)$. Let us define the *inverse level function* of w with respect to a decreasing function f as follows:

$$w^f(t) = \begin{cases} f(t)/R(a_n, b_n) & \text{for } t \in (a_n, b_n), \\ w(t) & \text{otherwise}, \end{cases}$$

where (a_n, b_n) is an enumeration of all maximal level intervals of f with respect to w. Then by definition of w^f,

$$\int_I \varphi\left(\frac{f^0}{w}\right) w = \int_I \varphi\left(\frac{f}{w^f}\right) w^f.$$

Notice also that $w^f \prec w$. In fact, for each m.l.i. (a, b) of f with respect to w one has $W(a, b) = W^f(a, b) = \int_a^b w^f$. Moreover, for $t \in (a, b)$,

$$W^f(a, t) = \frac{F(a, t)}{R(a, b)} \le \frac{F(a, t)}{R(a, t)} = W(a, t).$$

If t is in no m.l.i., then $W(t) = W^f(t)$. Indeed,

$$W^f(t) = \sum_{b_n \le t} W^f(a_n, b_n) + \int_{E_t} w = \sum_{b_n \le t} W(a_n, b_n) + \int_{E_t} w = W(t),$$

where $E_t = (0, t) \setminus \bigcup_{b_n \le t}(a_n, b_n)$. If t is in some m.l.i., then we have only $W^f(t) \le W(t)$.

The next result is a representation of the modular $P_{\varphi,w}(f)$ via the level function of f^* in the case when its support is a finite interval.

Proposition 4.5. *Let φ be an N-function. Then for any $f = f^* \in \mathcal{M}_{\varphi,w}$ such that* $\operatorname{supp} f = (0,s)$ *where $s < \infty$, we have*

$$P_{\varphi,w}(f) = \int_I \varphi\left(\frac{f^0}{w}\right) w = \int_I \varphi\left(\frac{f}{w^f}\right) w^f.$$

Proof. Let $f = f^* \in \mathcal{M}_{\varphi,w}$ and let (C_j) be an enumeration of all m.l.i. of f with respect to w. For every $n \in \mathbb{N}$, let

$$\mathcal{D}_n = \{(k2^{-n}s, (k+1)2^{-n}s] : 0 \le k < 2^n\}$$

be the set of dyadic subdivisions of the interval $(0,s]$, and $\mathcal{C}_n = \{C_j : j \le n\}$. The endpoints of all the intervals in $\mathcal{D}_n \cup \mathcal{C}_n$, when arranged in increasing order, define a finite partition $\mathcal{A}_n$ of $(0,s]$ into subintervals $A_k^n, k = 1, \ldots, K(n)$. In other words, $\mathcal{A}_n$ and $\mathcal{D}_n \cup \mathcal{C}_n$ generate the same algebra $\mathcal{F}_n$ of subsets of $(0,s]$. Moreover

$$|A_k^n| \le s/2^n, \qquad k = 1, \ldots, K(n). \tag{4.3}$$

Then for each $j \in \mathbb{N}$ and $n \ge j$ there is a finite set $I(j,n) \subset \mathbb{N}$ such that

$$C_j = \bigcup_{k \in I(j,n)} A_k^n.$$

Since $\mathcal{M}_{\varphi,w} \subset L^1 + L^\infty$ (see [1]), the function f is integrable on $[0,s)$, so we may define for each n the simple function

$$f_n = \sum_{k=1}^{K(n)} \left(\frac{1}{|A_k^n|} \int_{A_k^n} f\right) \chi_{A_k^n},$$

which is the conditional expectation of f with respect to the algebra $\mathcal{F}_n$.

We will show that $f_n^0 \to f^0$ a.e., where f_n^0, f^0 are the level functions for f_n, f, respectively. Fix some m.l.i. $C_j = (d,e]$ of f with respect to w. Then $(d,e] = \bigcup_{k \in I(j,n)} A_k^n$ for each $n \ge j$. Thus since f is decreasing, as in Remark 4.1(2) we have for each $t \in (d,e]$,

$$F_n(d,t) \le F(d,t), \qquad F_n(d,e) = F(d,e).$$

Therefore $(d,e]$ is a l.i. of f_n with respect to w for all $n \ge j$. Clearly for each $n \ge j$, the set C_j is contained in some m.l.i. $C^n = (d_n, e_n]$ of f_n. We claim that

$$|C^n - C_j| \to 0 \quad \text{as } n \to \infty. \tag{4.4}$$

Indeed, if (4.4) does not hold, there exist a subsequence (n_k) and numbers $d_0, e_0 \in [0,s]$ such that $d_{n_k} \to d_0$ and $e_{n_k} \to e_0$, and $d_0 < d$ or $e < e_0$. Moreover $(d_n, e_n]$ is a union of some intervals A_k^n and so $R_n(d_n, e_n) = R(d_n, e_n)$, where R_n is defined as

$$R_n(s,u) = F_n(s,u)/W(s,u).$$

For each $t \in (d_0, e_0)$ we have $t \in (d_{n_k}, e_{n_k}]$ for large k. Choose $A^n_{i(n)} = (w_n, v_n]$ and $A^n_{j(n)} = (p_n, r_n]$ in such a way that $t \in A^n_{i(n)}$ and $d_0 \in A^n_{j(n)}$ for each n. If $d_0 < d_{n_k}$, then

$$\begin{aligned}|F(d_0,t) - F_{n_k}(d_{n_k},t)| &\\ \le |F(d_0,d_{n_k})| + |F(d_{n_k},w_{n_k}) - F_{n_k}(d_{n_k},w_{n_k})| + |F(w_{n_k},t) - F_{n_k}(w_{n_k},t)| &\\ \le \int_{p_{n_k}}^{d_{n_k}} f + \int_{A^{n_k}_{i(n_k)}} f + \int_{A^{n_k}_{i(n_k)}} f_{n_k} = \int_{p_{n_k}}^{d_{n_k}} f + 2\int_{A^{n_k}_{i(n_k)}} f.\end{aligned}$$

Similarly, if $d_0 > d_{n_k}$, then

$$|F(d_0,t) - F_{n_k}(d_{n_k},t)| \le \int_{d_{n_k}}^{r_{n_k}} f + 2\int_{A^{n_k}_{i(n_k)}} f.$$

Consequently, in view of (4.3) we get

$$|F(d_0,t) - F_{n_k}(d_{n_k},t)| \to 0,$$

and so

$$R(d_0,t) \leftarrow R_{n_k}(d_{n_k},t) \le R_{n_k}(d_{n_k},e_{n_k}) = R(d_{n_k},e_{n_k}) \to R(d_0,e_0).$$

It follows that (d_0, e_0) is a l.i. of f, which contradicts our assumption on maximality of C_j and proves (4.4).

Let $t \in C_j$ for some j. Then keeping the notation as above we have

$$f^0_n(t) = R_n(d_n,e_n)w(t) = R(d_n,e_n)w(t) \to R(d,e)w(t) = f^0(t). \tag{4.5}$$

Suppose now $t \in [0,s) \setminus \bigcup_j \overline{C_j}$. Then for all $n \in \mathbb{N}$ there exists $k_0 = k_0(n)$ such that $t \in A^n_{k_0}$. Since $A^n_{k_0}$ are l.i. of f_n, there are m.l.i. M^n of f_n such that $A^n_{k_0} \subset M^n = (m_n, h_n]$. Clearly $(m_n, h_n]$ is a union of some sets A^n_k. One can also establish as in (4.4) that $|M^n| \to 0$ as $n \to \infty$, and so for a.a. t,

$$\begin{aligned} f^0_n(t) &= R_n(m_n,h_n)w(t) \\ &= R(m_n,h_n)w(t) = \frac{F(m_n,h_n)/|M^n|}{W(m_n,h_n)/|M^n|}w(t) \to f(t).\end{aligned} \tag{4.6}$$

Thus we infer from (4.5) and (4.6) that $f^0_n \to f^0$ a.e.

Notice that $P_{\varphi,w}(f_n) \le P_{\varphi,w}(f)$. In fact, suppose $P_{\varphi,w}(f) = k$. Consider the space $\mathcal{M}_{\psi,w}$, where $\psi(t) = \varphi(t)/k$, with the Luxemburg norm $\|\cdot\|$ given by the modular

$$P_{\psi,w}(f) = \frac{1}{k}P_{\varphi,w}(f).$$

This is a r.i. Banach function space with the Fatou property by Proposition 2.1. Since $f_n \prec f$ we have $\|f_n\|_{\mathcal{M}_{\psi,w}} \le \|f\|_{\mathcal{M}_{\psi,w}} = 1$ for each n. It follows from the left continuity of the function $(0,\infty) \ni \lambda \mapsto P_{\psi,w}(\lambda f)$ (see [9, Lemma 4.6]) that $P_{\psi,w}(f_n) \le 1$, and so

$$P_{\varphi,w}(f_n) \le P_{\varphi,w}(f).$$

Applying this, the convergence $f_n^0 \to f^0$ a.e. and $w^f \prec w$ by Remark 4.4, we get

$$P_{\varphi,w}(f) \ge \liminf P_{\varphi,w}(f_n) \overset{\text{Lemma 4.3}}{=} \liminf \int_I \varphi(f_n^0/w)w$$
$$\overset{\text{Fatou Lemma}}{\ge} \int_I \varphi(f^0/w)w = \int_I \varphi(f/w^f)w^f \ge P_{\varphi,w}(f),$$

which finishes the proof. ∎

Lemma 4.6. *Let φ be an N-function and $W(\infty) = \infty$. If $f = f^* \in \mathcal{M}_{\varphi,w}$ then it has no degenerate level interval.*

Proof. Suppose there is a degenerate m.l.i. (a, ∞) of f, that is,

$$R(a,t) \le \limsup_{x\to\infty} R(a,x) = R(a,\infty) \quad \text{for each } t > a, \quad \text{and} \quad R(a,\infty) > 0.$$

Without loss of generality we can also suppose that $P_{\varphi,w}(f) < \infty$, since level intervals of f are the same for all kf, where $k > 0$.

We will consider three cases.

(a) Suppose $R(a,t) < \limsup_{x\to\infty} R(a,x)$ for each $t > a$. Define

$$x_n = \max\bigl\{x \in [a, a+n] : R(a,x) = \sup\{R(a,t) : t \in [a, a+n]\}\bigr\}.$$

We have $x_n \nearrow \infty$ and $R(a,\infty) = \lim_{n\to\infty} R(a,x_n)$. In fact if $x_n \to x_0 < \infty$ then by assumption $R(a,x_0) = \lim_{n\to\infty} R(a,x_n) = \sup_{t\in(a,\infty)} R(a,t) < R(a,\infty)$, which is impossible. Therefore $x_n \nearrow \infty$ and $\lim_{n\to\infty} R(a,x_n) = \sup_{t\in(a,\infty)} R(a,t) = \limsup_{t\to\infty} R(a,t) = R(a,\infty)$.

Set $g_n = f\chi_{(0,x_n]}$. Clearly $R(a,t) \le R(a,x_n)$ for each $a < t < x_n$. Hence $(a, x_n]$ is a l.i. of f and thus a m.l.i. of g_n. Therefore $g_n^0 = f^0\chi_{(0,a)} + R(a,x_n)w\chi_{[a,x_n]} \to f^0\chi_{(0,a)} + R(a,\infty)w\chi_{[a,\infty)} = f^0$, and by Proposition 4.5 applied to g_n we have

$$P_{\varphi,w}(g_n) = \int_0^{x_n} \varphi(g_n^0/w)w = \int_0^a \varphi(f^0/w)w + \int_a^{x_n} \varphi(R(a,x_n))w$$
$$= \int_0^a \varphi(f^0/w)w + \varphi(R(a,x_n))(W(x_n) - W(a)) \to \infty,$$

since $\int_a^\infty w = \infty$ in view of $W(\infty) = \infty$. On the other hand $P_{\varphi,w}(g_n) \le P_{\varphi,w}(f)$ and so $P_{\varphi,w}(f) = \infty$, which contradicts our assumption.

Consider now the set

$$B = \{z > a : R(a,z) = R(a,\infty)\}.$$

If case (a) does not hold then $B \ne \emptyset$.

(b) Let first $\sup B = \infty$. Then there exists a sequence (x_n) such that $a < x_n \nearrow \infty$ and $R(a, x_n) = R(a, \infty)$ for each $n \in \mathbb{N}$, and we proceed as in (a).

(c) Suppose now that $\sup B = b < \infty$. Clearly $R(a, b) = R(a, \infty)$. Let $b < y_n \nearrow \infty$ be such that $R(a, y_n) \nearrow R(a, \infty)$. Then for each $\sigma > 1$ there exists N such that for $n > N$ we have

$$R(a, y_n) \le R(a, b) \le \sigma R(a, y_n).$$

We will show that for sufficiently large n,

$$R(b, y_n) \le R(a, y_n) \le \sigma R(b, y_n). \tag{4.7}$$

The left inequality follows immediately from (4.2). In order to get the right one, notice first that

$$\frac{F(a,b)}{W(a,b)} = R(a,b) \le \sigma R(a, y_n) = \sigma \frac{F(a,b) + F(b, y_n)}{W(a,b) + W(b, y_n)}.$$

Then

$$F(a,b)W(b,y_n) \le \sigma F(b,y_n)W(a,b) + (\sigma - 1)F(a,b)W(a,b),$$

and since $W(b, y_n) \to \infty$, we have

$$F(a,b)W(b,y_n) \le \sigma F(b,y_n)W(a,b) + (\sigma - 1)F(b,y_n)W(b,y_n)$$

for n large enough. Hence

$$\begin{aligned} F(a,b)W(b,y_n) &+ F(b,y_n)W(b,y_n) \\ &\le \sigma[F(b,y_n)W(a,b) + F(b,y_n)W(b,y_n)] \end{aligned}$$

and so

$$R(a, y_n) = \frac{F(a,b) + F(b,y_n)}{W(a,b) + W(b,y_n)} \le \sigma \frac{F(b,y_n)}{W(b,y_n)} = \sigma R(b, y_n),$$

and the right inequality of (4.7) is proved.

Therefore $R(b, y_n) \to R(a, b) = R(a, \infty) = R(b, \infty)$. Moreover, once again using (4.2) for each $b < t$ from $R(a, t) < R(a, b)$ we have

$$R(b,t) < R(a,t) < R(a,b) = R(a,\infty) = R(b,\infty),$$

where the second inequality follows from the definition of B. Hence (b, ∞) is a l.i. of f. Notice also that (b, ∞) is of the same type as (a, ∞) in case (a). Choosing (x_n) as in that case for b instead of a we define $g_n = f\chi_{[0,x_n]}$. Then the m.l.i. of g_n are the same as for f in $[0, a]$. Moreover, by the assumption $\sup B = b < \infty$, the interval $(a, b]$ is a m.l.i. of g_n, and by definition of (x_n), so is $(b, x_n]$. Hence (g_n) is increasing and

$$g_n^0 = f^0\chi_{(0,a]} + R(a,b)w\chi_{(a,b]} + R(b,x_n)w\chi_{(b,x_n]}.$$

Thus

$$g_n \nearrow f^0\chi_{(0,a]} + R(a,b)w\chi_{(a,b]} + R(b,\infty)w\chi_{(b,\infty)} = f^0 \quad \text{a.e.},$$

and we conclude as in (a). ∎

Now we state the main theorem of this section.

THEOREM 4.7. *Let φ be an N-function and $W(\infty)=\infty$. Then for any $f=f^*\in\mathcal{M}_{\varphi,w}$ we have*

$$P_{\varphi,w}(f)=\int_I\varphi\Big(\frac{f^0}{w}\Big)w=\int_I\varphi\Big(\frac{f}{w^f}\Big)w^f.$$

Proof. Let $f=f^*\in\mathcal{M}_{\varphi,w}$ with $P_{\varphi,w}(f)<\infty$. In view of Proposition 4.5 we may assume that $\operatorname{supp} f=(0,\infty)$. By Lemma 4.6, f has no degenerate level interval. Thus it remains to consider the following two cases.

First suppose there is a sequence $s_n\nearrow\infty$ such that each s_n is on the boundary of some m.l.i. of f. Define $g_n=f\chi_{[0,s_n]}$. Then $g_n\nearrow f$ a.e. and by Lemma 4.6 in [9],

$$P_{\varphi,w}(g_n)\to P_{\varphi,w}(f).$$

Moreover, for such (s_n) each m.l.i. of g_n is also a m.l.i. of f, and therefore $g_n^0=f^0\chi_{[0,s_n]}$. Then $g_n^0\nearrow f^0$ and by Proposition 4.5 and the Lebesgue Convergence Theorem,

$$P_{\varphi,w}(g_n)=\int_I\varphi(g_n^0/w)w\to\int_I\varphi(f^0/w)w,$$

which gives the claim.

Now assume there is s such that each l.i. of f is in $[0,s]$. Take (s_n) satisfying $s<s_n\nearrow\infty$ and put $g_n=f\chi_{[0,s_n]}$. Then once again $g_n^0=f^0\chi_{[0,s_n]}$, because there is no l.i. of g_n in (s,s_n), and we conclude as above. ∎

Summarizing the main results of Sections 2 and 4 (especially Theorems 2.2 and 4.7) we get the following theorem.

THEOREM 4.8. *Let w be a decreasing weight and φ be an N-function. Then the Köthe dual spaces to the Orlicz–Lorentz spaces $\Lambda_{\varphi,w}$ and $\Lambda^0_{\varphi,w}$ are expressed as*

$$(\Lambda_{\varphi,w})'=\mathcal{M}^0_{\varphi_*,w}\quad\textit{and}\quad(\Lambda^0_{\varphi,w})'=\mathcal{M}_{\varphi_*,w},$$

with

$$\|f\|_{(\Lambda_{\varphi,w})'}=\|f\|^0_{\mathcal{M}_{\varphi_*,w}}=\inf_{k>0}\frac{1}{k}(P_{\varphi_*,w}(kf)+1),$$
$$\|f\|_{(\Lambda^0_{\varphi,w})'}=\|f\|_{\mathcal{M}_{\varphi_*,w}}=\inf\{\lambda>0:P_{\varphi_*,w}(f/\lambda)\le1\},$$

where

$$P_{\varphi_*,w}(f)=\inf\Big\{\int_I\varphi_*(f^*/|g|)|g|:g\prec w\Big\}.$$

If in addition we assume that $W(\infty)=\infty$ for $I=[0,\infty)$ then also

$$P_{\varphi_*,w}(f)=\int_I\varphi_*((f^*)^0/w)w=\int_I\varphi_*(f^*/w^{f^*})w^{f^*},$$

where $(f^*)^0$ is the level function of f^* with respect to w, and w^{f^*} is the inverse level function of w with respect to f^*.

For $\varphi(u) = (1/p)u^p$, $1 < p < \infty$, we denote the space $\Lambda_{\varphi,w}$ by $\Lambda_{p,w}$. The next corollary provides an isometric description of $(\Lambda_{p,w})'$. The second formula recovers Halperin's Theorem 6.1 and Corollary on page 288 in [4].

Corollary 4.9. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then for any $f \in (\Lambda_{p,w})'$ we have*

$$\|f\|_{(\Lambda_{p,w})'} = \inf\Big\{\Big(\int_I (f^*/|g|)^q|g|\Big)^{1/q} : g \prec w\Big\}.$$

If in addition $W(\infty) = \infty$ in the case of $I = [0,\infty)$, then

$$\|f\|_{(\Lambda_{p,w})'} = \Big(\int_I ((f^*)^0/w)^q w\Big)^{1/q}.$$

Proof. The first equality follows from Theorem 2.2, while the second one from Theorem 4.7. ∎

Remark 4.10. In Lorentz's paper [12] the theorem (Theorem 3.6.5) on duality of the space $\Lambda_{p,w}$ for $1 < p < \infty$ was also stated in terms of "level functions", but his definition of a level function is different from the one introduced earlier by Halperin. A similar notion of level function was later used by Sinnamon (see [23, Chapter 2.9]). Both Lorentz's and Halperin's representations suggest that $f^0/w = (f/w)^L$ for every non-negative decreasing function f, where the right side means the level function of f/w in the Lorentz sense. It is straightforward to check this equality for a decreasing characteristic function.

5. Sequence case. We complete the discussion on duals of Orlicz–Lorentz spaces by considering the discrete case. All results given above for function spaces are valid in Orlicz–Lorentz sequence spaces as well. Recall that for a given sequence $x = (x_i)$, its decreasing rearrangement $x^* = (x_i^*)$ is defined as $x_i^* = \inf\{\lambda > 0 : d_x(\lambda) < i\}$, $i \in \mathbb{N}$, where $d_x(\lambda) = |\{i \in \mathbb{N} : |x_i| > \lambda\}|$ for $\lambda > 0$, and $|\cdot|$ is the counting measure on $\mathbb{N}$. Then given an Orlicz function φ and a decreasing positive weight sequence $w = (w_i)$, the Orlicz–Lorentz sequence space $\lambda_{\varphi,w}$ is defined as

$$\lambda_{\varphi,w} = \Big\{x = (x_i) \in l^0 : \exists_{\delta>0} \sum_{i=1}^{\infty} \varphi(\delta x_i^*)w_i < \infty\Big\},$$

where l^0 is the space of all real-valued sequences. We consider the space $\lambda_{\varphi,w}$ with the Luxemburg norm $\|\cdot\|_{\lambda_{\varphi,w}}$, denoted further by $\lambda_{\varphi,w}$, or with the Amemiya norm $\|\cdot\|^0_{\lambda_{\varphi,w}}$, denoted by $\lambda^0_{\varphi,w}$. Those norms are defined analogously to those for function spaces. Orlicz–Lorentz sequence spaces are

Köthe spaces as subspaces of l^0, and their Köthe dual spaces are defined analogously to the function case. To each $x \in \lambda_{\varphi,w}$ we assign an element $\bar{x} \in \Lambda_{\varphi,\bar{w}}$ on $[0,\infty)$, where

$$\bar{x} = \sum_{i=1}^{\infty} x_i \chi_{[i-1,i)} \quad \text{and} \quad \bar{w} = \sum_{i=1}^{\infty} w_i \chi_{[i-1,i)}.$$

The above correspondence between x and $\bar{x}$ is a linear isometry between $\lambda_{\varphi,w}$ and a closed subspace of $\Lambda_{\varphi,\bar{w}}$. Evidently

$$\|x\|_{\lambda_{\varphi,w}} = \|\bar{x}\|_{\Lambda_{\varphi,\bar{w}}} \quad \text{and} \quad \|x\|^0_{\lambda_{\varphi,w}} = \|\bar{x}\|^0_{\Lambda_{\varphi,\bar{w}}}.$$

The lemma below ensures that the correspondence remains true in the dual space.

LEMMA 5.1. *Let* $y = (y_i) \in (\lambda_{\varphi,w})'$. *Then*

$$\|y\|_{(\lambda_{\varphi,w})'} = \|\bar{y}\|_{(\Lambda_{\varphi,\bar{w}})'} \quad \textit{and} \quad \|y\|_{(\lambda^0_{\varphi,w})'} = \|\bar{y}\|_{(\Lambda^0_{\varphi,\bar{w}})'}.$$

Proof. Define an averaging operator T on $\Lambda_{\varphi,\bar{w}}$ by

$$T : h \mapsto \sum_{i=1}^{\infty} \Big(\int_{[i-1,i)} h \Big) \chi_{[i-1,i)}.$$

Then by [1, Theorem 4.8], $\|Th\|_{\Lambda_{\varphi,\bar{w}}} \le \|h\|_{\Lambda_{\varphi,\bar{w}}}$ for each $h \in \Lambda_{\varphi,\bar{w}}$. Moreover, for any $y \in (\lambda_{\varphi,w})'$,

$$\int_0^{\infty} \bar{y}h = \int_0^{\infty} \bar{y}(Th).$$

Therefore

$$\begin{aligned} \|\bar{y}\|_{(\Lambda_{\varphi,\bar{w}})'} &= \sup\Big\{ \int_0^{\infty} \bar{y}h : \|h\|_{\Lambda_{\varphi,\bar{w}}} \le 1 \Big\} = \sup\Big\{ \int_0^{\infty} \bar{y}(Th) : \|h\|_{\Lambda_{\varphi,\bar{w}}} \le 1 \Big\} \\ &= \sup\Big\{ \int_0^{\infty} \bar{y}(Th) : \|Th\|_{\Lambda_{\varphi,\bar{w}}} \le 1 \Big\} = \sup\Big\{ \int_0^{\infty} \bar{y}\bar{z} : \|z\|_{\lambda_{\varphi,w}} \le 1 \Big\} \\ &= \sup\Big\{ \sum_{i=1}^{\infty} y_i z_i : \|z\|_{\lambda_{\varphi,w}} \le 1 \Big\} = \|y\|_{(\lambda_{\varphi,w})'}. \end{aligned}$$

Similarly we prove the second equality. ∎

By analogy to the function case the following space has been defined in [9]:

$$\mathfrak{m}_{\varphi,w} = \{x \in l^0 : \exists_{\lambda>0}\ p_{\varphi,w}(\lambda x) < \infty\},$$

with the modular

$$p_{\varphi,w}(x) = \inf\Big\{ \sum_{i=1}^{\infty} \varphi(x_i^*/|y_i|)|y_i| : y \prec w \Big\},$$

where the submajorization of sequences $y \prec w$ means that $\sum_{i=1}^n y_i^* \le \sum_{i=1}^n w_i$ for all $n \in \mathbb{N}$. We denote by $\mathfrak{m}_{\varphi,w}$ the above space equipped with the Luxemburg norm $\|\cdot\|_{\mathfrak{m}}$, and by $\mathfrak{m}^0_{\varphi,w}$ the space endowed with the Amemiya norm $\|\cdot\|^0_{\mathfrak{m}}$. Moreover, we adapt the definitions of the previous section to the sequence case by setting, for every non-negative decreasing sequence $x = (x_i)$ and $a, b \in \mathbb{N} \cup \{0\}$, $a < b$,

$$w(a,b) = \sum_{i=a+1}^{b} w_i, \qquad x(a,b) = \sum_{i=a+1}^{b} x_i, \qquad r(a,b) = \frac{x(a,b)}{w(a,b)}.$$

Then $(a,b] = \{a+1, \ldots, b\} \subset \mathbb{N}$ is called a *level interval of x with respect to w* if for each $j = a+1, \ldots, b$,

$$r(a,j) \le r(a,b) \quad \text{and} \quad 0 < r(a,b),$$

and the *level sequence x^0 of x with respect to w* is defined as

$$x_i^0 = \begin{cases} r(a_n,b_n)w_i & \text{for } i \in (a_n,b_n], \\ x_i & \text{otherwise,} \end{cases}$$

where $(a_n, b_n]$ is an enumeration of all maximal level intervals of x. Notice that the results of the previous section ensure that the correspondence between x and $\bar{x}$ preserves the level intervals. In fact (see the proofs of Lemmas 3.2 and 4.3) we have for any $a \in \mathbb{N} \cup \{0\}$, $r(a,j) \le r(a,b)$ for all $j = a+1, \ldots, b$ if and only if $\bar{x}(a,t)/\bar{w}(a,t) \le \bar{x}(a,b)/\bar{w}(a,b)$ for all $t \in (a,b)$. Hence $(a,b] \subset \mathbb{N}$ is a m.l.i. of x with respect to w if and only if (a,b) is a m.l.i. of $\bar{x}$ with respect to $\bar{w}$. Therefore

$$\int_0^\infty \varphi((\bar{x}^*)^0/\bar{w})\bar{w} = \sum_{i=1}^\infty \varphi((x_i^*)^0/w_i)w_i. \tag{5.1}$$

Moreover, in view of Lemma 3.2, $y \prec w$ if and only if $\bar{y} \prec \bar{w}$ and thus

$$p_{\varphi,w}(x) = \inf\Big\{\int_0^\infty \varphi(\bar{x}^*/|\bar{y}|)|\bar{y}| : \bar{y} \prec \bar{w}\Big\}.$$

Hence by Lemma 3.1 applied to the step function $\bar{x}^*$ we obtain

$$P_{\varphi,\bar{w}}(\bar{x}) = p_{\varphi,w}(x). \tag{5.2}$$

Finally, employing equalities (5.1), (5.2), Lemma 5.1 and Theorem 4.8, we can state a duality result for the Orlicz–Lorentz sequence space $\lambda_{\varphi,w}$.

THEOREM 5.2. *Let w be a decreasing weight sequence and φ be an N-function. Then*

$$(\lambda_{\varphi,w})' = \mathfrak{m}^0_{\varphi_*,w} \quad \textit{and} \quad (\lambda^0_{\varphi,w})' = \mathfrak{m}_{\varphi_*,w}.$$

If in addition $\sum_{i=1}^{\infty} w_i = \infty$, *then*

$$p_{\varphi_*,w}(x) = \sum_{i=1}^{\infty} \varphi_*\left(\frac{(x_i^*)^0}{w_i}\right) w_i.$$

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