Spectral and homological properties
of Hilbert modules over the disc algebra

by

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Abstract. We study general Hilbert modules over the disc algebra and exhibit necessary spectral conditions for the vanishing of certain associated extension groups. In particular, this sheds some light on the problem of identifying the projective Hilbert modules. Part of our work also addresses the classical derivation problem.

1. Introduction. This paper is concerned with polynomially bounded operators and some of their spectral properties. Recall that a bounded linear operator T acting on some Hilbert space $\mathcal{H}$ is said to be *polynomially bounded* if there exists a constant $C > 0$ such that for every polynomial p, we have

$$\|p(T)\| \leq C\|p\|_\infty \quad \text{where} \quad \|p\|_\infty = \sup_{|z|<1} |p(z)|.$$

This inequality allows one to extend continuously the polynomial functional calculus $p \mapsto p(T)$ to all functions f in the *disc algebra* $A(\mathbb{D})$, which consists of the holomorphic functions on $\mathbb{D}$ that are continuous on $\overline{\mathbb{D}}$ (throughout the paper $\mathbb{D}$ denotes the open unit disc and $\mathbb{T}$ denotes the unit circle). The point of view we adopt is that of Douglas and Paulsen (see [13]) where these operators are studied as modules over the disc algebra: the map

$$A(\mathbb{D}) \times \mathcal{H} \to \mathcal{H}, \quad (f, h) \mapsto f(T)h,$$

gives rise to a module structure on $\mathcal{H}$, and we say that $(\mathcal{H}, T)$ is a *Hilbert module*. We only deal with modules over $A(\mathbb{D})$ in this paper, so no confusion may arise regarding the underlying function algebra, and we usually do not mention it explicitly. Moreover, when the underlying Hilbert space is understood, we slightly abuse terminology and say that T is a Hilbert module. Using these notions, the authors of [13] reformulated several interesting operator-theoretic questions in the language of module theory, and in doing so suggested the use of cohomological methods. Accordingly, we

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phrase most of our results using extension groups of Hilbert modules, and thus we start by briefly reviewing the definition of these groups.

Given two Hilbert modules $(\mathcal{H}_1, T_1)$ and $(\mathcal{H}_2, T_2)$, the *extension group*

$$\mathrm{Ext}^1_{A(\mathbb{D})}(T_2, T_1)$$

consists of equivalence classes of exact sequences

$$0 \to \mathcal{H}_1 \to \mathcal{K} \to \mathcal{H}_2 \to 0$$

where $\mathcal{K}$ is another Hilbert module and each map is a module morphism. Rather than formally defining the equivalence relation and the group operation, we simply use the following characterization from [5].

THEOREM 1.1. *Let $(\mathcal{H}_1, T_1)$ and $(\mathcal{H}_2, T_2)$ be Hilbert modules. Then the group $\mathrm{Ext}^1_{A(\mathbb{D})}(T_2, T_1)$ is isomorphic to $\mathcal{A}/\mathcal{J}$, where $\mathcal{A}$ is the space of operators $X : \mathcal{H}_2 \to \mathcal{H}_1$ for which the operator*

$$\begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$$

is polynomially bounded, and $\mathcal{J}$ is the space of operators of the form $T_1 L - L T_2$ for some bounded operator $L : \mathcal{H}_2 \to \mathcal{H}_1$.

If the operator $X : \mathcal{H}_2 \to \mathcal{H}_1$ belongs to the space $\mathcal{A}$, we denote by $[X]$ its equivalence class in

$$\mathcal{A}/\mathcal{J} = \mathrm{Ext}^1_{A(\mathbb{D})}(T_2, T_1).$$

It is well-known that given $[X] \in \mathrm{Ext}^1_{A(\mathbb{D})}(T_2, T_1)$, we have $[X] = 0$ if and only if the operator

$$\begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$$

is similar to $T_1 \oplus T_2$. Moreover, extension groups are invariant under similarity, so if $(\mathcal{H}_1', T_1')$ and $(\mathcal{H}_2', T_2')$ are Hilbert modules which are similar to $(\mathcal{H}_1, T_1)$ and $(\mathcal{H}_2, T_2)$ respectively, then the groups $\mathrm{Ext}^1_{A(\mathbb{D})}(T_2, T_1)$ and $\mathrm{Ext}^1_{A(\mathbb{D})}(T_2', T_1')$ are isomorphic. A Hilbert module $(\mathcal{H}_2, T_2)$ is said to be *projective* if

$$\mathrm{Ext}^1_{A(\mathbb{D})}(T_2, T_1) = 0$$

for every Hilbert module $(\mathcal{H}_1, T_1)$. It is easy to verify using Theorem 1.1 that the map $[X] \mapsto [X^*]$ establishes an isomorphism between the groups $\mathrm{Ext}^1_{A(\mathbb{D})}(T_2, T_1)$ and $\mathrm{Ext}^1_{A(\mathbb{D})}(T_1^*, T_2^*)$, so T_2 is projective if and only if

$$\mathrm{Ext}^1_{A(\mathbb{D})}(T_1, T_2^*) = 0$$

for every Hilbert module $(\mathcal{H}_1, T_1)$.

Because of their connection with commutant lifting properties, those Hilbert modules which are projective are of special interest from the point of view of operator theory. In fact, an important question is whether or not the projectivity of a module can be detected from its basic operator-theoretic properties. This problem attracted a lot of interest (see [5]–[10], [12], [14]–[16] for partial results), but to this day the complete picture is still unclear and the full answer unknown.

Most of the main results about projective modules over the disc algebra focus on the case where the said modules are assumed to be similar to a contraction. The only known instance of a projective Hilbert module is when the underlying operator is (similar to) a unitary (see [7]). On the other hand, Ferguson showed in [15] that any module which is projective and similar to a contraction must in fact be similar to an isometry. Of course, this does not tell the whole story, as it is known that there exist polynomially bounded operators that are not similar to a contraction (see [18]) and thus Hilbert modules that are not similar to a contractive module.

Our aim is to exhibit necessary conditions for a general Hilbert module $(\mathcal{H}, T)$ to be projective. Our main results in this direction say that for such a module, the left spectrum $\sigma_l(T)$ must be contained in the unit circle. This fact can be recovered from Ferguson's result for contractive Hilbert modules, but again the point here is that we do not assume that the module T is similar to a contraction. Furthermore, we obtain those restrictions on the spectrum of the operator T under a variety of assumptions which are formally weaker than projectivity. More precisely, we prove the following in Section 2.

THEOREM 1.2. *Let $\lambda \in \mathbb{D}$, let $(\mathcal{H}, T)$ be a Hilbert module and let P be the orthogonal projection of $\mathcal{H}$ onto $\ker(T - \lambda)$. If $\ker(T - \lambda) \neq 0$, then*

$$\mathrm{Ext}^1_{A(\mathbb{D})}(T, (I - P)T^*(I - P)) \neq 0.$$

THEOREM 1.3. *Let $\lambda \in \mathbb{D}$ and let $(\mathcal{H}, T)$ be a Hilbert module. Then*

$$\mathrm{Ext}^1_{A(\mathbb{D})}(T, \lambda) = 0 \quad \textit{if and only if} \quad \lambda \notin \sigma_l(T).$$

A notion related to the study of extension groups is that of a derivation of the disc algebra. Recall that given a Hilbert module $(\mathcal{H}, T)$, a bounded linear map

$$\delta : A(\mathbb{D}) \to B(\mathcal{H})$$

is called a *derivation* if it satisfies

$$\delta(fg) = f(T)\delta(g) + \delta(f)g(T)$$

for all $f, g \in A(\mathbb{D})$. A derivation is *inner* if there exists $\Delta \in B(\mathcal{H})$ such that

$$\delta(f) = f(T)\Delta - \Delta f(T)$$

for every $f \in A(\mathbb{D})$. The connection between derivations and extension groups is realized as follows. Let $X \in B(\mathcal{H})$ and set

$$R = \begin{pmatrix} T & X \\ 0 & T \end{pmatrix}.$$

For every polynomial p, we have

$$p(R) = \begin{pmatrix} p(T) & \delta_X(p) \\ 0 & p(T) \end{pmatrix}$$

for some operator $\delta_X(p)$. Then, the operator R is polynomially bounded if and only if the map $p \mapsto \delta_X(p)$ extends to a derivation on $A(\mathbb{D})$. Moreover, $[X] = 0$ in $\mathrm{Ext}^1_{A(\mathbb{D})}(T, T)$ if and only if δ_X is an inner derivation.

It is an interesting and non-trivial issue to determine the modules T for which every derivation is inner, or equivalently for which the group $\mathrm{Ext}^1_{A(\mathbb{D})}(T, T)$ is trivial. In relation to this problem, in Section 3 we investigate the condition $\mathrm{Ext}^1_{A(\mathbb{D})}(T, T) = 0$ and its consequences on the operator T, and establish the following theorem.

THEOREM 1.4. *Let $(\mathcal{H}, T)$ be a Hilbert module such that $\mathcal{H}$ is infinite-dimensional and*

$$\mathrm{Ext}^1_{A(\mathbb{D})}(T, T) = 0.$$

Then the subspaces $\ker T$ *and* $\ker T^*$ *are orthogonal, and the subspaces*

$$\{h \in \mathcal{H} \ominus \ker T : Th \in \mathcal{H} \ominus \ker T\}$$

and

$$\{h \in \mathcal{H} \ominus \ker T^* : T^*h \in \mathcal{H} \ominus \ker T^*\}$$

are infinite-dimensional.

A natural strengthening of this result would read as follows: if

$$\mathrm{Ext}^1_{A(\mathbb{D})}(T, T) = 0$$

then T has no eigenvalues inside the unit disc. We verify this in the special cases of normal operators in Section 3 (Theorem 3.3), and of matrices and C_0 contractions in Section 4 (Lemma 4.3 and Theorem 4.6 respectively).

2. Spectral properties and the vanishing of extension groups. Let $(\mathcal{H}, T)$ be a Hilbert module. Since the powers of T are uniformly bounded, it is a trivial consequence of the spectral radius formula that $\sigma(T) \subset \overline{\mathbb{D}}$. The aim of this section is to investigate the relation between the spectrum of T and the vanishing of the group $\mathrm{Ext}^1_{A(\mathbb{D})}(T, X)$ where X is some fixed module.

Recall now that the *left* (respectively *right*) *spectrum* of an element a in a unital Banach algebra is the set of complex numbers λ with the property that $a - \lambda$ is not left (respectively right) invertible. These sets are denoted by

$\sigma_l(a)$ and $\sigma_r(a)$ respectively. If we are dealing with a bounded operator T on some Banach space, then it is well-known that $\sigma_l(T)$ coincides with the set of complex numbers λ such that $T - \lambda I$ is not bounded below, while $\sigma_r(T)$ coincides with the set of complex numbers λ with $T - \lambda I$ not surjective.

We first reformulate a result of [11] which yields a sufficient spectral condition for the vanishing of an extension group.

THEOREM 2.1. *Let $(\mathcal{H}_1, T_1)$ and $(\mathcal{H}_2, T_2)$ be Hilbert modules. If the sets $\sigma_l(T_2)$ and $\sigma_r(T_1)$ are disjoint, then*
$$\mathrm{Ext}^1_{A(\mathbb{D})}(T_2, T_1) = 0.$$

Proof. It follows at once from Theorem 5 of [11] that the map
$$B(\mathcal{H}_2, \mathcal{H}_1) \to B(\mathcal{H}_2, \mathcal{H}_1), \quad L \mapsto T_1 L - L T_2,$$
is surjective under our assumption. The conclusion is then an immediate consequence of Theorem 1.1. ∎

Before giving an easy consequence of Theorem 2.1, we need some notation. Let $\mathcal{E}$ be a separable Hilbert space and let $H^2(\mathcal{E})$ be the Hardy space of (weakly) holomorphic $\mathcal{E}$-valued functions on the unit disc with square summable Taylor coefficients at the origin. Let $S = S_{\mathcal{E}}$ be the unilateral shift on $H^2(\mathcal{E})$ which acts by multiplication by the variable.

COROLLARY 2.2. *Let $(\mathcal{H}, T)$ be a Hilbert module such that $\sigma(T) \subset \mathbb{D}$. Then $\mathrm{Ext}^1_{A(\mathbb{D})}(S_{\mathcal{E}}, T) = 0$.*

Proof. This follows immediately from Theorem 2.1 and from the classical fact that the left spectrum of the unitaleral shift is the unit circle $\mathbb{T}$. ∎

This result contrasts nicely with a result of Carlson and Clark [7, Corollary 3.4] which says that if $\sigma(T) \subset \mathbb{D}$, then the group $\mathrm{Ext}^1_{A(\mathbb{D})}(T, S)$ is isomorphic to $\mathcal{H}$, where S denotes the shift of multiplicity one.

The remainder of the section is devoted to finding conditions on the spectrum of a module that are necessary for the vanishing of certain extension groups. We first need an auxiliary result which will simplify some proofs. For $\lambda \in \mathbb{D}$, we set
$$\varphi_\lambda(z) = \frac{z - \lambda}{1 - \overline{\lambda} z}.$$
If $(\mathcal{H}, T)$ is a Hilbert module, then the operator $\varphi_\lambda(T)$ is bounded since $\sigma(T) \subset \overline{\mathbb{D}}$, as was observed at the beginning of the section.

LEMMA 2.3. *Let $(\mathcal{H}_1, T_1)$ and $(\mathcal{H}_2, T_2)$ be Hilbert modules such that*
$$\mathrm{Ext}^1_{A(\mathbb{D})}(T_2, T_1) = 0.$$
Then
$$\mathrm{Ext}^1_{A(\mathbb{D})}(\varphi_\lambda(T_2), \varphi_\lambda(T_1)) = 0 \quad \text{for every } \lambda \in \mathbb{D}.$$

Proof. Assume that the operator

$$R = \begin{pmatrix} \varphi_\lambda(T_1) & X \\ 0 & \varphi_\lambda(T_2) \end{pmatrix}$$

is polynomially bounded, so that there exists a constant $C > 0$ such that

$$\|f(R)\| \leq C\|f\|_\infty$$

for every $f \in A(\mathbb{D})$. Using the fact that $\varphi_{-\lambda} \circ \varphi_\lambda(z) = z$ for every $z \in \mathbb{D}$, we find

$$\varphi_{-\lambda}(R) = \begin{pmatrix} T_1 & Y \\ 0 & T_2 \end{pmatrix}$$

for some operator Y. But since $\varphi_{-\lambda}$ is an automorphism of the unit disc, we see that $\varphi_{-\lambda}(R)$ is also polynomially bounded. Indeed, if $f \in A(\mathbb{D})$ then

$$\|f(\varphi_{-\lambda}(R))\| = \|(f \circ \varphi_{-\lambda})(R)\| \leq C\|f \circ \varphi_{-\lambda}\|_\infty = C\|f\|_\infty.$$

Now, $\mathrm{Ext}^1_{A(\mathbb{D})}(T_2, T_1)$ is assumed to be trivial, so there exists an invertible operator W with the property that

$$W\varphi_{-\lambda}(R)W^{-1} = T_1 \oplus T_2,$$

whence

$$WRW^{-1} = \varphi_\lambda(T_1) \oplus \varphi_\lambda(T_2)$$

and the element $[X]$ is trivial in $\mathrm{Ext}^1_{A(\mathbb{D})}(\varphi_\lambda(T_2), \varphi_\lambda(T_1))$. ∎

Another preliminary lemma is required. Its proof can be found in [10].

LEMMA 2.4. *Let $(\mathcal{H}_1, T_1)$ and $(\mathcal{H}_2, T_2)$ be Hilbert modules. Let $X : \mathcal{H}_2 \to \mathcal{H}_1$ be a bounded operator such that $T_1^N X T_2^N = 0$ for some integer $N \geq 0$. Then the operator $R : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ defined as*

$$R = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$$

is polynomially bounded.

We now come to the first main result of this section.

THEOREM 2.5. *Let $\lambda \in \mathbb{D}$, let $(\mathcal{H}, T)$ be a Hilbert module and let P be the orthogonal projection of $\mathcal{H}$ onto $\ker(T - \lambda)$. If $\ker(T - \lambda) \neq 0$, then*

$$\mathrm{Ext}^1_{A(\mathbb{D})}(T, (I - P)T^*(I - P)) \neq 0.$$

Proof. Let

$$T_\lambda = (I - P)\varphi_{\bar{\lambda}}(T)(I - P).$$

Since $\ker \varphi_\lambda(T)$ is clearly invariant for T we have

(1) $$T_\lambda = \varphi_{\bar{\lambda}}((I - P)T(I - P)).$$

Moreover,
$$f(T_\lambda) = (I - P)(f \circ \varphi_{\overline{\lambda}})(T)(I - P)$$
for every $f \in A(\mathbb{D})$ and thus T_λ is polynomially bounded. The operator
$$R = \begin{pmatrix} \varphi_{\overline{\lambda}}(T^*) & P \\ 0 & T_\lambda \end{pmatrix}$$
acting on $\mathcal{H} \oplus \mathcal{H}$ is also seen to be polynomially bounded in view of Lemma 2.4 and of the fact that $P T_\lambda = 0$.

We now proceed to show that $[P]$ gives rise to a non-trivial element of $\operatorname{Ext}^1_{A(\mathbb{D})}(T_\lambda, \varphi_{\overline{\lambda}}(T^*))$. Assume on the contrary that there exists $L \in B(\mathcal{H})$ such that
$$P = \varphi_{\overline{\lambda}}(T^*)L - L T_\lambda.$$
Note that $T_\lambda P = 0$ and
$$P \varphi_{\overline{\lambda}}(T^*) = ((\varphi_{\overline{\lambda}}(T^*))^* P)^* = (\varphi_\lambda(T) P)^* = 0,$$
hence
$$P = P^3 = P(\varphi_{\overline{\lambda}}(T^*)L - L T_\lambda)P = 0,$$
which is equivalent to $\ker(T - \lambda)$ being trivial, contrary to assumption. Therefore,
$$\operatorname{Ext}^1_{A(\mathbb{D})}(T_\lambda, \varphi_{\overline{\lambda}}(T^*)) \neq 0.$$

Note now that equation (1) implies that
$$\operatorname{Ext}^1_{A(\mathbb{D})}(\varphi_{\overline{\lambda}}((I - P)T(I - P)), \varphi_{\overline{\lambda}}(T^*)) = \operatorname{Ext}^1_{A(\mathbb{D})}(T_\lambda, \varphi_{\overline{\lambda}}(T^*)) \neq 0.$$
Lemma 2.3 therefore guarantees that
$$\operatorname{Ext}^1_{A(\mathbb{D})}((I - P)T(I - P), T^*) \neq 0,$$
which is equivalent to
$$\operatorname{Ext}^1_{A(\mathbb{D})}(T, (I - P)T^*(I - P)) \neq 0$$
and the proof is complete. ∎

Notice that this theorem offers a simple necessary condition for a Hilbert module $(\mathcal{H}, T)$ to be projective, namely that the point spectrum $\sigma_p(T)$ (the set of eigenvalues of T) be contained in the unit circle $\mathbb{T}$. The following is the second main result of this section.

THEOREM 2.6. *Let $\lambda \in \mathbb{D}$ and let $(\mathcal{H}, T)$ be a Hilbert module. Then*
$$\operatorname{Ext}^1_{A(\mathbb{D})}(T, \lambda) = 0 \quad \textit{if and only if} \quad \lambda \notin \sigma_l(T).$$

Proof. Assume that $\operatorname{Ext}^1_{A(\mathbb{D})}(T, \lambda) = 0$. The operator
$$R = \begin{pmatrix} 0 & I \\ 0 & \varphi_\lambda(T) \end{pmatrix}$$

acting on $\mathcal{H} \oplus \mathcal{H}$ is easily seen to be polynomially bounded by virtue of Lemma 2.4. Now, Lemma 2.3 implies that

$$\mathrm{Ext}^1_{A(\mathbb{D})}(\varphi_\lambda(T), 0) = 0$$

and thus we can find $L \in B(\mathcal{H})$ such that $I = -L\varphi_\lambda(T)$. Consequently,

$$T - \lambda = \varphi_\lambda(T)(1 - \overline{\lambda}T)$$

is left invertible and $\lambda \notin \sigma_l(T)$. The converse statement follows immediately from Theorem 2.1. ∎

This theorem shows in particular that in order for a Hilbert module $(\mathcal{H}, T)$ to be projective, it must satisfy $\sigma_p(T) \subset \sigma_l(T) \subset \mathbb{T}$. Now, the reader might wonder about the relevance of Theorem 2.5 in view of the corresponding statement in Theorem 2.6: the latter is much simpler to prove and has a more satisfactory conclusion than the former, while the assumption might not look stronger. However, the assumption that

$$\mathrm{Ext}^1_{A(\mathbb{D})}(T, \lambda) = 0$$

is indeed quite strong, and we proceed to illustrate why. The following proposition will be needed later as well.

PROPOSITION 2.7. *Let $T_1 \in B(\mathcal{H}_1)$ and $T_2 \in B(\mathcal{H}_2)$ be Hilbert modules and assume that $\|T_1\| < 1$. Then the operator*

$$R = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$$

is polynomially bounded for every bounded operator $X : \mathcal{H}_2 \to \mathcal{H}_1$.

Proof. Let $p(z) = \sum_{k=0}^d a_k z^k$. Then a quick calculation shows that

$$p(R) = \begin{pmatrix} p(T_1) & \delta_X(p) \\ 0 & p(T_2) \end{pmatrix}$$

where

$$\delta_X(p) = \sum_{k=1}^d a_k \sum_{j=0}^{k-1} T_1^j X T_2^{k-1-j}.$$

Since T_1 and T_2 are polynomially bounded by assumption, to establish that R is also polynomially bounded we need to show that that there exists a constant $C > 0$ independent of p such that $\|\delta_X(p)\| \leq C\|p\|_\infty$. We see that

$$\delta_X(p) = \sum_{k=1}^d a_k \sum_{j=0}^{k-1} T_1^j X T_2^{k-1-j} = \sum_{j=0}^{d-1} T_1^j X \Big(\sum_{k=j+1}^d a_k T_2^{k-1-j} \Big)$$

$$= \sum_{j=0}^{d-1} T_1^j X \Pi_j(T_2)$$

where

$$\Pi_j(z) = \sum_{k=j+1}^{d} a_k z^{k-1-j} \quad \text{for every } 0 \le j \le d-1.$$

We denote by $D : A(\mathbb{D}) \to A(\mathbb{D})$ the difference quotient operator defined as

$$Df(z) = \frac{f(z) - f(0)}{z} \quad \text{for every } f \in A(\mathbb{D}).$$

It is well-known that there exists a constant $M > 0$ such that

$$\|D^n\| \le M(1 + \log n) \quad \text{for every } n \ge 1,$$

but we sketch the argument for the convenience of the reader.

Given $f \in A(\mathbb{D})$, one verifies inductively that

$$D^n f = \frac{1}{z^n} \left(f(z) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} z^k \right) \quad \text{for every } n \ge 1,$$

whence

$$\|D^n f\|_\infty = \left\| f(z) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} z^k \right\|_\infty.$$

On the other hand, for every $\theta \in \mathbb{R}$ we see that

$$\sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} e^{ik\theta} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \left(\sum_{k=-(n-1)}^{n-1} e^{-ikt} e^{ik\theta} \right) dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \mathcal{D}_{n-1}(\theta - t) \, dt$$

where

$$\mathcal{D}_n(t) = \sum_{k=-n}^{n} e^{ikt}$$

is the Dirichlet kernel. Therefore,

$$\|D^n f\|_\infty \le (1 + \|\mathcal{D}_{n-1}\|_1) \|f\|_\infty$$

for every $n \ge 1$. It is a classical fact that $\|\mathcal{D}_n\|_1$ is comparable to $\log n$ as $n \to \infty$, so there exists a constant $M > 0$ such that

$$\|D^n\| \le M(1 + \log n) \quad \text{for every } n \ge 1.$$

Back to the problem at hand, we know that there exists a constant $C_2 > 0$ such that

$$\|f(T_2)\| \le C_2 \|f\|_\infty \quad \text{for every } f \in A(\mathbb{D}).$$

Using $\Pi_j = D^{j+1}p$ for every $0 \le j \le d-1$, we find

$$\|\delta_X(p)\| \le \sum_{j=0}^{d-1} \|T_1\|^j \|X\| \, \|\Pi_j(T_2)\| \le C_2 \sum_{j=0}^{d-1} \|T_1\|^j \|X\| \, \|D^{j+1}p\|_\infty$$

$$\le \left(\sum_{j=0}^{d-1}(1+\log(j+1))\|T_1\|^j\right) C_2 M \|X\| \, \|p\|_\infty$$

$$\le \left(\sum_{j=0}^{\infty}(1+\log(j+1))\|T_1\|^j\right) C_2 M \|X\| \, \|p\|_\infty$$

and we are done since the series $\sum_{j=0}^{\infty}(1+\log(j+1))\|T_1\|^j$ is convergent by assumption. ∎

We wish to mention that the general philosophy behind the calculations above can be extracted from the proof of Lemma 2.3 in [17].

Going back to the discussion started before the proposition, let $(\mathcal{H}, T)$ be a Hilbert module and $\lambda \in \mathbb{D}$. If we write

$$\mathrm{Ext}^1_{A(\mathbb{D})}(T, \lambda) = \mathcal{A}/\mathcal{J}$$

as in Theorem 1.1, then we see that $\mathcal{A}$ is very large. Indeed, it is as large as possible since by Proposition 2.7 it coincides with $B(\mathcal{H})$. Thus, the vanishing of the quotient $\mathcal{A}/\mathcal{J}$ is a rather strong condition. Moreover, the corresponding space $\mathcal{A}$ for

$$\mathrm{Ext}^1_{A(\mathbb{D})}(T, (I-P)T^*(I-P))$$

(see Theorem 2.5) is not as large a priori and thus the vanishing of that extension group appears to be a weaker condition. We feel this provides some intuition as to why the assumption of Theorem 2.6 may indeed be stronger than that of Theorem 2.5, and that it explains in part the difference in strength of their conclusions.

3. The derivation problem: a structure theorem. The rest of the paper is devoted to the study of Hilbert modules $(\mathcal{H}, T)$ for which $\mathrm{Ext}^1_{A(\mathbb{D})}(T, T) = 0$. As was mentioned in the introduction, this is directly related to the derivation problem, and in fact this is one of the motivations for our investigation. First, we prove a structure theorem for such Hilbert modules. We focus here on the case where $\mathcal{H}$ is infinite-dimensional. The easier finite-dimensional case is fully solved later on in Lemma 4.3.

THEOREM 3.1. *Let $(\mathcal{H}, T)$ be a Hilbert module such that $\mathcal{H}$ is infinite-dimensional and*

$$\mathrm{Ext}^1_{A(\mathbb{D})}(T, T) = 0.$$

Then the subspaces $\ker T$ *and* $\ker T^*$ *are orthogonal, and the subspaces*

$$\{h \in \mathcal{H} \ominus \ker T : Th \in \mathcal{H} \ominus \ker T\}$$

and

$$\{h \in \mathcal{H} \ominus \ker T^* : T^*h \in \mathcal{H} \ominus \ker T^*\}$$

are infinite-dimensional.

Proof. Throughout the proof, we may assume without loss of generality that both $\ker T$ and $\ker T^*$ are non-trivial. We write $\mathcal{H} = \ker T \oplus (\mathcal{H} \ominus \ker T)$ and with respect to this decomposition of the space we have

$$T = \begin{pmatrix} 0 & X \\ 0 & Y \end{pmatrix}.$$

Let

$$P = P_{\ker T} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

be the orthogonal projection of $\mathcal{H}$ onto $\ker T$ and consider the operator

$$R = \begin{pmatrix} T & P \\ 0 & T \end{pmatrix}$$

which acts on $\mathcal{H} \oplus \mathcal{H}$. Using Lemma 2.4 and the fact that $TP = 0$, we see that R is polynomially bounded. By assumption, there exists $L \in B(\mathcal{H})$ such that $P = TL - LT$. If we write

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

then we find

$$(2) \qquad \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} XL_{21} & XL_{22} - L_{11}X - L_{12}Y \\ YL_{21} & YL_{22} - L_{22}Y - L_{21}X \end{pmatrix}.$$

In particular, there must exist a bounded linear operator

$$L_{21} : \ker T \to \mathcal{H} \ominus \ker T$$

satisfying

$$(3) \qquad\qquad\qquad XL_{21} = I,$$
$$(4) \qquad\qquad\qquad YL_{21} = 0.$$

A consequence of (3) is that X is surjective, or X^* is bounded below. Taking adjoints in (3) and (4) we find that

$$(5) \qquad\qquad\qquad L_{21}^*X^* = I,$$
$$(6) \qquad\qquad\qquad L_{21}^*Y^* = 0.$$

Choose $h \in \overline{X^* \ker T} \cap \overline{Y^*(\mathcal{H} \ominus \ker T)}$. Then

$$h = \lim_{n \to \infty} X^* v_n = \lim_{n \to \infty} Y^* w_n$$

for some sequences $\{v_n\}_n \subset \ker T$ and $\{w_n\}_n \subset \mathcal{H} \ominus \ker T$. From (5) and (6) we get

$$L_{21}^* h = \lim_{n \to \infty} v_n = 0,$$

so that $h = 0$. This shows that

$$\overline{X^* \ker T} \cap \overline{Y^*(\mathcal{H} \ominus \ker T)} = \{0\}.$$

Now, a vector $h = h_1 \oplus h_2 \in \ker T \oplus (\mathcal{H} \ominus \ker T)$ lies in $\ker T^*$ if and only if

$$X^* h_1 = -Y^* h_2 \in X^* \ker T \cap Y^*(\mathcal{H} \ominus \ker T).$$

Since this intersection was already found to be zero, we see that $h_1 \in \ker X^*$ and $h_2 \in \ker Y^*$. But X^* is bounded below, whence $h_1 = 0$ and therefore

$$\ker T^* = 0 \oplus \ker Y^* \subset \mathcal{H} \ominus \ker T,$$

which establishes the first statement.

  We now turn to the proof of the second statement. Notice that in view of (3) the operator

$$L_{21} X : \mathcal{H} \ominus \ker T \to \mathcal{H} \ominus \ker T$$

is a non-zero idempotent, which we denote henceforth by E. With respect to the decomposition

$$\mathcal{H} \ominus \ker T = \operatorname{ran} E \oplus (\operatorname{ran} E)^\perp,$$

where $\operatorname{ran} E$ denotes the range of E, we can write

$$E = \begin{pmatrix} I & F \\ 0 & 0 \end{pmatrix}.$$

If we consider the invertible operator

$$W = \begin{pmatrix} I & F \\ 0 & I \end{pmatrix}$$

then we have

$$WEW^{-1} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Now, using (2) we see that

$$Y L_{22} - L_{22} Y = E.$$

Since $\mathcal{H}$ is infinite-dimensional, by a classical theorem of Wintner [23] we know that E cannot be written as the sum of a non-zero scalar multiple

of the identity and a compact operator. The same is necessarily true for WEW^{-1}, whence the orthogonal projection onto

$$\ker(WEW^{-1}) = W \ker E = W \ker X$$

cannot be compact. In other words,

$$\ker X = \{h \in \mathcal{H} \ominus \ker T : Th \in \mathcal{H} \ominus \ker T\}$$

is infinite-dimensional. We can apply the same argument to T^* to conclude that

$$\{h \in \mathcal{H} \ominus \ker T^* : T^*h \in \mathcal{H} \ominus \ker T^*\}$$

is also infinite-dimensional, which finishes the proof. ∎

We make a few comments about this result. By Lemma 2.3, we may replace T by $\varphi_\lambda(T)$ everywhere in the statement of Theorem 3.1 and thus obtain information about $\ker(T - \lambda)$ and $\ker(T^* - \overline{\lambda})$ for each $\lambda \in \mathbb{D}$. Interestingly, the theorem provides evidence that the spaces $\ker(T - \lambda)$ and $\ker(T^* - \overline{\lambda})$ cannot be too large under the condition $\mathrm{Ext}^1_{A(\mathbb{D})}(T, T) = 0$. While we do not know at the moment whether or not these spaces must be trivial in general, the following conjecture seems natural: if $\mathrm{Ext}^1_{A(\mathbb{D})}(T, T) = 0$, then T has no eigenvalues inside the unit disc.

Next, we consider a special class of operators and prove a weaker version of this conjecture for them. We restrict our attention to the so-called *D-symmetric* operators which were introduced and studied in [1], [19] and [21]. Recall that an operator $T \in B(\mathcal{H})$ is said to be *D-symmetric* if

$$\overline{\{TL - LT : L \in B(\mathcal{H})\}} = \overline{\{T^*L - LT^* : L \in B(\mathcal{H})\}}.$$

It was proved in [1] that the class of *D*-symmetric operators includes all normal operators and isometries.

THEOREM 3.2. *Let $(\mathcal{H}, T)$ be a Hilbert module satisfying $\mathrm{Ext}^1_{A(\mathbb{D})}(T, T) = 0$. If T is D-symmetric, then one of the spaces $\ker T$ and $\ker T^*$ is trivial.*

Proof. Assume that we can find unit vectors $f \in \ker T$ and $g \in \ker T^*$, and define $V \in B(\mathcal{H})$ as $Vx = \langle x, g \rangle f$ for every $x \in \mathcal{H}$. Consider the operator

$$R = \begin{pmatrix} T & V \\ 0 & T \end{pmatrix},$$

which is polynomially bounded by virtue of Lemma 2.4 since $TV = 0$. Notice now that for every $L \in B(\mathcal{H})$ we have

$$\langle (T^*L - LT^*)g, f \rangle = 0$$

by choice of f and g, while $\langle Vg, f \rangle = 1$. Thus, V lies outside the set

$$\overline{\{T^*L - LT^* : L \in B(\mathcal{H})\}}.$$

Since T is assumed to be D-symmetric, this set coincides with

$$\overline{\{TL - LT : L \in B(\mathcal{H})\}},$$

and therefore V cannot be written as $TL - LT$ for some $L \in B(\mathcal{H})$, whence $[V]$ is a non-trivial element in $\mathrm{Ext}^1_{A(\mathbb{D})}(T, T)$. ■

Note that the trick used in the proof above to construct the operator V lying outside the set

$$\overline{\{T^*L - LT^* : L \in B(\mathcal{H})\}}$$

is due to Stampfli [20]. We close this section by specializing even further and verifying the full conjecture for normal operators.

THEOREM 3.3. *Let $(\mathcal{H}, T)$ be a Hilbert module such that T is normal. Then T is unitary if and only if $\mathrm{Ext}^1_{A(\mathbb{D})}(T, T) = 0$.*

Proof. If T is unitary then the module $(\mathcal{H}, T)$ is projective by Theorem 4.1 of [7], so in particular $\mathrm{Ext}^1_{A(\mathbb{D})}(T, T) = 0$.

Conversely, assume that this extension group vanishes. If $\lambda \in \sigma(T) \cap \mathbb{D}$, then via the spectral theorem for normal operators we can find a non-zero reducing subspace $M \subset \mathcal{H}$ for T such that $\|T|M\| < 1$. With respect to the decomposition $\mathcal{H} = M \oplus M^\perp$, we have $T = T|M \oplus T|M^\perp$. Consider the operator $X = I \oplus 0$. It is easy to verify that the operator

$$R = \begin{pmatrix} T & X \\ 0 & T \end{pmatrix}$$

is unitarily equivalent to

$$\begin{pmatrix} T|M & I \\ 0 & T|M \end{pmatrix} \oplus \begin{pmatrix} T|M^\perp & 0 \\ 0 & T|M^\perp \end{pmatrix}.$$

Using Proposition 2.7, we see that R is polynomially bounded and thus

$$[X] \in \mathrm{Ext}^1_{A(\mathbb{D})}(T, T).$$

Since we assume that this extension group is zero, we can write

$$I \oplus 0 = X = TL - LT$$

for some $L \in B(\mathcal{H})$. A straightforward calculation shows that this implies

$$I = (T|M)L' - L'(T|M)$$

for some operator $L' : M \to M$, which is impossible since the identity is well-known not to be a commutator (see [23]). This contradiction shows that $\sigma(T) \subset \mathbb{T}$, and thus the normal operator T is actually unitary. ■

4. Contractions of class C_0. In this final section, we verify the conjecture made in Section 3 for another special class of operators: the C_0 contractions. We start with some background (see [2] or [22] for greater detail).

Let H^∞ be the algebra of bounded holomorphic functions on the open unit disc. A completely non-unitary contraction $T \in B(\mathcal{H})$ is said to be *of class C_0* if the associated Sz.-Nagy–Foiaş H^∞ functional calculus has non-trivial kernel. It is known in that case that

$$\{u \in H^\infty : u(T) = 0\} = \theta H^\infty$$

for some inner function θ, called the *minimal function* of T, which is uniquely determined up to a scalar factor of absolute value one. Moreover, we have

$$\sigma_p(T) = \sigma(T) \cap \mathbb{D},$$

and this set coincides with the set of zeros of θ on $\mathbb{D}$.

For any inner function $\theta \in H^\infty$, the space $H(\theta) = H^2 \ominus \theta H^2$ is closed and invariant for S^*, the adjoint of the shift operator S on H^2. The operator $S(\theta)$ defined by $S(\theta)^* = S^*|(H^2 \ominus \theta H^2)$ is called a *Jordan block*; it is of class C_0 with minimal function θ. We record a well-known elementary property of these operators.

LEMMA 4.1. *Let $\theta_1, \theta_2 \in H^\infty$ be inner functions such that $\theta_1 H^\infty + \theta_2 H^\infty = H^\infty$. Then $S(\theta_1 \theta_2)$ is similar to $S(\theta_1) \oplus S(\theta_2)$.*

A more general family of operators consists of the so-called Jordan operators. Start with a collection $\Theta = \{\theta_\alpha\}_\alpha$ of inner functions indexed by the ordinal numbers such that $\theta_\alpha = 1$ for α large enough, and θ_β divides θ_α whenever $\mathrm{card}(\beta) \geq \mathrm{card}(\alpha)$ (recall that a function $u \in H^\infty$ divides another function $v \in H^\infty$ if $v = uf$ for some $f \in H^\infty$). Let γ be the first ordinal such that $\theta_\gamma = 1$. Then the associated *Jordan operator* is $J_\Theta = \bigoplus_{\alpha < \gamma} S(\theta_\alpha)$.

Jordan operators are of fundamental importance in the study of operators of class C_0, as the following theorem from [3] illustrates. Recall here that an injective bounded linear operator with dense range is called a *quasiaffinity*. Two operators $T \in B(\mathcal{H})$ and $T' \in B(\mathcal{H}')$ are said to be *quasisimilar* if there exist quasiaffinities $X : \mathcal{H} \to \mathcal{H}'$ and $Y : \mathcal{H}' \to \mathcal{H}$ such that $XT = T'X$ and $TY = YT'$.

THEOREM 4.2. *For any operator T of class C_0 there exists a unique Jordan operator J which is quasisimilar to T.*

With these preliminaries out of the way, we return to the problem at hand. We start with the simple case where the space $\mathcal{H}$ is finite-dimensional, thus complementing Theorem 3.1.

LEMMA 4.3. *Let $(\mathbb{C}^n, T)$ be a Hilbert module. Then $\mathrm{Ext}^1_{A(\mathbb{D})}(T, T) = 0$ if and only if T is similar to a unitary.*

Proof. As before, if T is similar to a unitary then by Theorem 4.1 of [7] we know that the module $(\mathbb{C}^n, T)$ is projective and thus $\operatorname{Ext}^1_{A(\mathbb{D})}(T, T) = 0$.

Assume conversely that this extension group vanishes. This condition is invariant under similarity, so we may assume in addition that

$$T = J_{\lambda_1, m_1} \oplus \cdots \oplus J_{\lambda_d, m_d}$$

where $J_{\lambda, m}$ is the usual $m \times m$ Jordan cell with eigenvalue λ. Suppose that one of the eigenvalues lies inside $\mathbb{D}$. In other words, $T = J \oplus T'$ where $J = J_{\lambda, m}$ for some $\lambda \in \mathbb{D}$ and $1 \le m \le n$. Correspondingly, define $X = I \oplus 0$. It is easy to verify that the operator

$$R = \begin{pmatrix} T & X \\ 0 & T \end{pmatrix}$$

is unitarily equivalent to

$$\begin{pmatrix} J & I \\ 0 & J \end{pmatrix} \oplus \begin{pmatrix} T' & 0 \\ 0 & T' \end{pmatrix}.$$

Applying a polynomial p to the operator

$$\begin{pmatrix} J & I \\ 0 & J \end{pmatrix}$$

yields

$$\begin{pmatrix} p(J) & p'(J) \\ 0 & p(J) \end{pmatrix}.$$

On the other hand, an easy computation shows that

$$f(J) = \begin{pmatrix} f(\lambda) & f'(\lambda) & f''(\lambda)/2 & \ldots & f^{(m-1)}(\lambda)/(m-1)! \\ 0 & f(\lambda) & f'(\lambda) & \ldots & f^{(m-2)}(\lambda)/(m-2)! \\ 0 & 0 & f(\lambda) & \ldots & f^{(m-3)}(\lambda)/(m-3)! \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f(\lambda) \end{pmatrix}$$

for every $f \in A(\mathbb{D})$. Since $|\lambda| < 1$, the classical Cauchy estimates for derivatives of holomorphic functions imply that the operator

$$\begin{pmatrix} J & I \\ 0 & J \end{pmatrix}$$

is polynomially bounded, and thus so is R.

Now, X has non-zero trace and thus cannot be written as $TL - LT$ for some $L \in B(\mathbb{C}^n)$. Equivalently, X gives rise to a non-trivial element of $\operatorname{Ext}^1_{A(\mathbb{D})}(T, T)$, which is a contradiction. Thus, $\sigma(T) \subset \mathbb{T}$. Since a Jordan cell

$J_{\lambda,m}$ is power-bounded only when $|\lambda| < 1$ or $m = 1$, we conclude that every Jordan cell of T has size one, whence T is diagonalizable and hence similar to a unitary. ∎

We now tackle the general case where $T \in B(\mathcal{H})$ is of class C_0. We begin with an elementary fact.

LEMMA 4.4. *Let $M_1, M_2 \subset \mathcal{H}$ be two closed subspaces with trivial intersection such that M_1 has finite dimension. Then the operator $R : M_1 \oplus M_2 \to M_1 + M_2$ defined by $R(m_1 \oplus m_2) = m_1 + m_2$ is bounded and invertible.*

Proof. It is clear R is surjective, and it is injective as well since $M_1 \cap M_2 = \{0\}$. A straightforward estimate shows that R is bounded. Since M_1 is finite-dimensional and M_2 is closed, the algebraic sum $M_1 + M_2$ is closed and thus R is invertible. ∎

We need one more preliminary tool. The result is well-known but we provide a proof for the reader's convenience.

LEMMA 4.5. *Let $T \in B(\mathcal{H})$ be an operator of class C_0 such that $\lambda \in \sigma(T) \cap \mathbb{D}$. Then T is similar to $J_{\lambda,n} \oplus T'$ for some $n \geq 1$ and some operator T'.*

Proof. If we denote by θ the minimal function of T and we set as before

$$\varphi_\lambda(z) = \frac{z - \lambda}{1 - \bar{\lambda}z},$$

then we can write $\theta = \varphi_\lambda^n \psi$ where $\psi(\lambda) \neq 0$. It is clear that

$$\inf_{z \in \mathbb{D}}\{|\psi(z)| + |\varphi_\lambda^n(z)|\} > 0,$$

so by Carleson's corona theorem (see [4]) we conclude that

$$\varphi_\lambda^n H^\infty + \psi H^\infty = H^\infty.$$

By Lemma 4.1, $S(\theta)$ is similar to $S(\varphi_\lambda^n) \oplus S(\psi)$. Now, if J denotes the Jordan model of T, then this discussion shows that J is similar to $S(\varphi_\lambda^n) \oplus J'$ for some operator J', and by Theorem 4.2 we find that T is quasisimilar to $S(\varphi_\lambda^n) \oplus J'$. If we denote by $\mathcal{K} = H(\varphi_\lambda^n) \oplus \mathcal{K}'$ the space on which $S(\varphi_\lambda^n) \oplus J'$ acts, then we can find a quasiaffinity $Y : \mathcal{K} \to \mathcal{H}$ such that

$$Y(S(\varphi_\lambda^n) \oplus J') = TY.$$

Let

$$M_1 = \overline{Y(H(\varphi_\lambda^n) \oplus 0)} \quad \text{and} \quad M_2 = \overline{Y(0 \oplus \mathcal{K}')}.$$

By Lemma 4.4, the operator $R : M_1 \oplus M_2 \to M_1 + M_2$ defined by $R(m_1 \oplus m_2) = m_1 + m_2$ is bounded and invertible, and it obviously intertwines T with $T|M_1 \oplus T|M_2$. Hence, T is similar to $T|M_1 \oplus T|M_2$. But M_1 is finite-dimensional and the minimal polynomial of $T|M_1$ is clearly $(z - \lambda)^n$, so we find that T is similar to $J_{\lambda,n} \oplus T'$. ∎

Finally, we come to the main result of this section. Although weaker, it is reminiscent of both Lemma 4.3 and Theorem 3.3.

THEOREM 4.6. *Let $T \in B(\mathcal{H})$ be an operator of class C_0 such that $\mathrm{Ext}^1_{A(\mathbb{D})}(T, T) = 0$. Then the spectrum of T lies on the unit circle.*

Proof. Assume that $\lambda \in \sigma(T) \cap \mathbb{D}$. The condition $\mathrm{Ext}^1_{A(\mathbb{D})}(T, T) = 0$ is invariant under similarity, so by Lemma 4.5 we may assume that $T = J_{\lambda,n} \oplus T'$ for some $n \geq 1$ and some operator T'. By Lemma 4.3, we have

$$\mathrm{Ext}^1_{A(\mathbb{D})}(J_{\lambda,n}, J_{\lambda,n}) \neq 0,$$

so that there exists an operator X such that

$$\begin{pmatrix} J_{\lambda,n} & X \\ 0 & J_{\lambda,n} \end{pmatrix}$$

is polynomially bounded but

$$X \neq J_{\lambda,n} L - L J_{\lambda,n} \quad \text{for every } L.$$

Consider now $Y = X \oplus 0$ and

$$R = \begin{pmatrix} T & Y \\ 0 & T \end{pmatrix}.$$

The operator R is unitarily equivalent to

$$\begin{pmatrix} J_{\lambda,n} & X \\ 0 & J_{\lambda,n} \end{pmatrix} \oplus \begin{pmatrix} T' & 0 \\ 0 & T' \end{pmatrix}$$

and thus it is polynomially bounded. Suppose now that there exists an operator A such that

$$Y = TA - AT.$$

A straightforward calculation shows that this relation implies

$$X = J_{\lambda,n} A' - A' J_{\lambda,n}$$

for some operator A', which is absurd. Hence, $[Y]$ yields a non-trivial element of $\mathrm{Ext}^1_{A(\mathbb{D})}(T, T)$. ∎

In conclusion, we remark that the main results obtained in this paper extend what was already known about the spectrum of *contractive* projective modules. Indeed, we mentioned in the introduction that every such module is (similar to) an isometry, and isometries do not have point spectrum in the unit disc. This is exactly the type of behavior described in Theorems 2.5, 2.6, 3.1, 3.3 and 4.6. Moreover, we reiterate that our results were obtained for modules which are not necessarily similar to a contractive one, and under conditions that are formally weaker than projectivity.

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