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Limiting behaviour of intrinsic seminorms in fractional order Sobolev spaces

by

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Dedicated to the memory of Charles Goulaouic (1938–1983)

Abstract. We collect and extend results on the limit of $\sigma^{1-k}(1-\sigma)^k|v|^p_{l+\sigma,p,\Omega}$ as $\sigma\to 0^+$ or $\sigma\to 1^-$, where Ω is $\mathbb{R}^n$ or a smooth bounded domain, $k\in\{0,1\}$, $l\in\mathbb{N}$, $p\in[1,\infty)$, and $|\cdot|_{l+\sigma,p,\Omega}$ is the intrinsic seminorm of order $l+\sigma$ in the Sobolev space $W^{l+\sigma,p}(\Omega)$. In general, the above limit is equal to $c[v]^p$, where c and $[\cdot]$ are, respectively, a constant and a seminorm that we explicitly provide. The particular case $p=2$ for $\Omega=\mathbb{R}^n$ is also examined and the results are then proved by using the Fourier transform.

1. Introduction. Bourgain, Brézis and Mironescu (cf. [6, 7]) proved that, for any $p\in[1,\infty)$ and any v belonging to the Sobolev space $W^{1,p}(\Omega)$,

$$\lim_{\sigma\to 1^-}(1-\sigma)|v|^p_{\sigma,p,\Omega}=p^{-1}K_{p,n}\int_\Omega|\nabla v(x)|^p\,dx, \tag{1.1}$$

where Ω is either $\mathbb{R}^n$ or a smooth bounded domain in $\mathbb{R}^n$, with $n\geq 1$, $|\cdot|_{\sigma,p,\Omega}$ is the intrinsic or Gagliardo seminorm of order σ in the Sobolev space $W^{\sigma,p}(\Omega)$ (see Section 2 for the precise definitions), and $K_{p,n}$ is a constant that only depends on p and n. Likewise, Maz'ya and Shaposhnikova [14] showed that

$$\lim_{\sigma\to 0^+}\sigma|v|^p_{\sigma,p,\mathbb{R}^n}=2p^{-1}|S_{n-1}|\,|v|^p_{0,p,\mathbb{R}^n}, \tag{1.2}$$

where S_{n-1} stands for the unit sphere in $\mathbb{R}^n$ (i.e. $S_{n-1}=\{x\in\mathbb{R}^n\mid |x|=1\}$) and $|S_{n-1}|$ is its area.

These results have been extended and completed by several authors. Let us mention, for example, Milman [15], who placed them in the framework of interpolation spaces, or Karadzhov, Milman and Xiao [11], Kolyada and Lerner [12] and Triebel [21], who generalized them to Besov spaces.

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Our interest in this subject comes from the study of sampling inequalities involving Sobolev seminorms. In [5], we have extended previous results (cf. [3, 4]) in order to allow fractional order Sobolev seminorms on the left-hand side of sampling inequalities. We have then realized that the complete comprehension of the constants involved in sampling inequalities needs an understanding of the asymptotic behaviour of the corresponding fractional order Sobolev seminorms. In fact, we need extensions of (1.1) and (1.2) having the following form:

$$\lim_{\sigma\to\ell} \sigma^{1-k}(1-\sigma)^k |v|^p_{l+\sigma,p,\Omega} = c[v]^p, \tag{1.3}$$

where $\ell = 0^+$ or 1^-, Ω is $\mathbb{R}^n$ or a smooth bounded domain, $k \in \{0,1\}$, $l \in \mathbb{N}$, $p \in [1,\infty)$, and $|\cdot|_{l+\sigma,p,\Omega}$ is the intrinsic seminorm of order $l+\sigma$ in the Sobolev space $W^{l+\sigma,p}(\Omega)$. On the right-hand side of (1.3), the notations $[\cdot]$ and c stand, respectively, for a seminorm and a constant to be specified.

This paper is organized as follows. After recalling some basic facts and definitions in Section 2, we devote Section 3 to establishing (1.3). Most of the work may be routine, but anyway we find it useful to collect and state in one place this kind of results and to provide explicit expressions for the constants and seminorms involved in the limits. In Section 4, we focus on the case of $p = 2$ and $\Omega = \mathbb{R}^n$. We show that (1.3) can be obtained by means of the Fourier transform. This line of reasoning was suggested in [6, Remark 2] starting from a result by Maz'ya and Nagel [13]. As a by-product, for $m \in \mathbb{N}$ and $s \geq 0$, we establish a relationship between the Sobolev space $W^{m+s,2}(\mathbb{R}^n)$ and the Beppo Levi space $X^{m,s}$, arising in spline theory (cf. [2, Chapter I]). Finally, in Section 5, we show an application of the limiting results of Section 3 to the study of sampling inequalities, which was the original motivation of this paper.

2. Preliminaries. For any $x \in \mathbb{R}$, we shall write $\lfloor x \rfloor$ and $\lceil x \rceil$ for the *floor* (or integer part) and *ceiling* of x, that is, the unique integers satisfying $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ and $\lceil x \rceil - 1 < x \leq \lceil x \rceil$. The letter n will always stand for an integer belonging to $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ (by convention, $0 \in \mathbb{N}$). The Euclidean norm in $\mathbb{R}^n$ will be denoted by $|\cdot|$.

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$, $x_1, \ldots, x_n$ being the generic independent variables in $\mathbb{R}^n$. In addition, given $l \in \mathbb{N}$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we write $\binom{l}{\alpha} = l!/(\alpha_1! \cdots \alpha_n!)$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We shall make frequent use of the relation

$$|x|^{2l} = \sum_{|\alpha|=l} \binom{l}{\alpha} x^{2\alpha}, \tag{2.1}$$

valid for any $l \in \mathbb{N}$ and $x \in \mathbb{R}^n$.

Let Ω be a nonempty open set in $\mathbb{R}^n$. For any $r \in \mathbb{N}$ and $p \in [1,\infty)$, we shall denote by $W^{r,p}(\Omega)$ the usual Sobolev space defined by

$$W^{r,p}(\Omega) = \{v \in L^p(\Omega) \mid \partial^\alpha v \in L^p(\Omega),\ \forall \alpha \in \mathbb{N}^n,\ |\alpha| \le r\}.$$

We recall that the derivatives $\partial^\alpha v$ are taken in the distributional sense. The space $W^{r,p}(\Omega)$ is equipped with the seminorms $|\cdot|_{j,p,\Omega}$, with $j \in \{0,\dots,r\}$, and the norm $\|\cdot\|_{r,p,\Omega}$ given by

$$|v|_{j,p,\Omega} = \Big(\sum_{|\alpha|=j} \int_\Omega |\partial^\alpha v(x)|^p\,dx\Big)^{1/p} \quad \text{and} \quad \|v\|_{r,p,\Omega} = \Big(\sum_{j=0}^{r} |v|^p_{j,p,\Omega}\Big)^{1/p}.$$

For any $r \in (0,\infty) \setminus \mathbb{N}$ and $p \in [1,\infty)$, we shall denote by $W^{r,p}(\Omega)$ the Sobolev space of noninteger order r, formed by the (equivalence classes of) functions $v \in W^{\lfloor r \rfloor,p}(\Omega)$ such that

$$|v|^p_{r,p,\Omega} = \sum_{|\alpha|=\lfloor r \rfloor} \int_{\Omega\times\Omega} \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^p}{|x-y|^{n+p(r-\lfloor r \rfloor)}}\,dx\,dy < \infty.$$

Besides the seminorms $|\cdot|_{j,p,\Omega}$, with $j \in \{0,\dots,\lfloor r \rfloor\}$, and $|\cdot|_{r,p,\Omega}$, the space $W^{r,p}(\Omega)$ is endowed with the norm

$$\|v\|_{r,p,\Omega} = (\|v\|^p_{\lfloor r \rfloor,p,\Omega} + |v|^p_{r,p,\Omega})^{1/p}.$$

Given $j \in \mathbb{N}$ and $v \in W^{j+1,p}(\Omega)$, we put

$$|\nabla v|_{0,p,\Omega} = \Big(\int_\Omega |\nabla v(x)|^p\,dx\Big)^{1/p} \quad \text{and} \quad |\nabla v|_{j,p,\Omega} = \Big(\sum_{|\alpha|=j} |\nabla(\partial^\alpha v)|^p_{0,p,\Omega}\Big)^{1/p}.$$

The mapping $v \mapsto |\nabla v|_{j,p,\Omega}$ is a seminorm in $W^{j+1,p}(\Omega)$ equivalent to $|\cdot|_{j+1,p,\Omega}$.

We shall use the following definition of the Fourier transform $\hat{v}$ of a function $v \in L^1(\mathbb{R}^n)$:

$$\hat{v}(\xi) = \int_{\mathbb{R}^n} v(x) e^{-ix\cdot\xi}\,dx, \qquad \xi \in \mathbb{R}^n,$$

where the dot symbol $\cdot$ denotes the Euclidean scalar product in $\mathbb{R}^n$. We refer to standard textbooks for the properties of the Fourier transform and their extension to the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions. We just recall the following result:

$$\forall v \in \mathcal{S}'(\mathbb{R}^n),\ \forall \alpha \in \mathbb{N}^n, \qquad i^{|\alpha|}\xi^\alpha \hat{v} = \widehat{\partial^\alpha v}. \tag{2.2}$$

3. General results for $p \in [1,\infty)$. As mentioned in the introduction, for a smooth bounded domain Ω or for $\Omega = \mathbb{R}^n$, we are interested in calculating the following limit:

$$\lim_{\sigma\to\ell} \sigma^{1-k}(1-\sigma)^k |v|^p_{l+\sigma,p,\Omega}, \tag{3.1}$$

with $\ell \in \{0^+, 1^-\}$, $k \in \{0,1\}$, $l \in \mathbb{N}$, $p \in [1,\infty)$ and v belonging to a suitable Sobolev space. For $\Omega = \mathbb{R}^n$, we shall study the cases $(\ell, k) = (0^+, 0)$ and $(1^-, 1)$, whereas for Ω bounded, we shall consider the cases $(\ell, k) = (0^+, 1)$ and $(1^-, 1)$, taking into account that $\lim_{\sigma\to 0^+}(1-\sigma) = 1$. The limit corresponding to any other combination of ℓ and k follows trivially from the above cases.

THEOREM 3.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz continuous boundary. Let $p \in [1,\infty)$ and $l \in \mathbb{N}$. Then, for any $v \in W^{l+1,p}(\Omega)$,*

$$\lim_{\sigma\to 1^-}(1-\sigma)|v|^p_{l+\sigma,p,\Omega} = p^{-1}K_{p,n}|\nabla v|^p_{l,p,\Omega}, \tag{3.2}$$

where

$$K_{p,n} = \int_{S_{n-1}} |\omega\cdot\nu|^p \, d\omega, \tag{3.3}$$

ν being any unit vector in $\mathbb{R}^n$.

Proof. The case $l = 0$ is a result by Bourgain, Brézis and Mironescu (cf. [6]). For the sake of completeness, we just clarify here some details of their proof. We use, however, the notations in [7], which are slightly simpler. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be any family of nonnegative functions, contained in $L^1_{\mathrm{loc}}(0,\infty)$, such that

$$\int_0^\infty \rho_\varepsilon(t)t^{n-1}\,dt = 1, \ \forall\varepsilon>0, \quad \text{and} \quad \lim_{\varepsilon\to 0}\int_\delta^\infty \rho_\varepsilon(t)t^{n-1}\,dt = 0, \ \forall\delta>0.$$

It follows from Theorems 2 and 3 in [6] that, for any $v \in W^{1,p}(\Omega)$,

$$\lim_{\varepsilon\to 0}\int_{\Omega\times\Omega}\frac{|v(x)-v(y)|^p}{|x-y|^p}\rho_\varepsilon(|x-y|)\,dx\,dy = K_{p,n}|\nabla v|^p_{0,p,\Omega}, \tag{3.4}$$

where $K_{p,n}$ is defined by (3.3). Let us choose the family $(\rho_\varepsilon)_{\varepsilon>0}$ given by

$$\rho_\varepsilon(t) = \begin{cases} \varepsilon d^{-\varepsilon}t^{\varepsilon-n}, & \text{if } t \le d,\\ 0, & \text{if } t > d,\end{cases}$$

d being the diameter of Ω. Then (3.4) becomes

$$\lim_{\varepsilon\to 0}\varepsilon d^{-\varepsilon}\int_{\Omega\times\Omega}\frac{|v(x)-v(y)|^p}{|x-y|^{n+p-\varepsilon}}\,dx\,dy = K_{p,n}|\nabla v|^p_{0,p,\Omega},$$

which implies (3.2), for $l = 0$, if we replace ε by $p(1-\sigma)$.

Let us now consider the case $l \ge 1$. Since the lth-order derivatives of functions in $W^{l+1,p}(\Omega)$ belong to $W^{1,p}(\Omega)$, by the case $l = 0$, for any $v \in W^{l+1,p}(\Omega)$, we have

$$\begin{aligned}\lim_{\sigma\to 1^-}(1-\sigma)|v|^p_{l+\sigma,p,\Omega} &= \lim_{\sigma\to 1^-}(1-\sigma)\sum_{|\alpha|=l}|\partial^\alpha v|^p_{\sigma,p,\Omega}\\ &= \sum_{|\alpha|=l}\lim_{\sigma\to 1^-}(1-\sigma)|\partial^\alpha v|^p_{\sigma,p,\Omega}\\ &= \sum_{|\alpha|=l}p^{-1}K_{p,n}|\nabla(\partial^\alpha v)|^p_{0,p,\Omega} = p^{-1}K_{p,n}|\nabla v|^p_{l,p,\Omega},\end{aligned}$$

which yields (3.2). ■

REMARK 3.2. Let us provide the explicit value of the constant $K_{p,n}$ given by (3.3). Since the definition of $K_{p,n}$ is independent of the unit vector ν, we can take $\nu=(1,0,\dots,0)$. On the one hand, we have

$$\begin{aligned}\int_{x_1^2+\dots+x_n^2\le 1}|x_1|^p\,dx &= \int_0^1\Big(\int_{S_{n-1}}t^{n-1}|t\omega_1|^p\,d\omega\Big)dt\\ &= \Big(\int_{S_{n-1}}|\omega\cdot\nu|^p\,d\omega\Big)\int_0^1 t^{n-1+p}\,dt = \frac{K_{p,n}}{n+p}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{x_1^2+\dots+x_n^2\le 1}|x_1|^p\,dx &= \int_{-1}^1|x_1|^p\Big(\int_{x_2^2+\dots+x_n^2\le 1-x_1^2}dx_2\cdots dx_n\Big)\,dx_1\\ &= \vartheta_{n-1}\int_{-1}^1|x_1|^p(1-x_1^2)^{(n-1)/2}\,dx_1 = 2\vartheta_{n-1}\int_0^1 x_1^p\,(1-x_1^2)^{(n-1)/2}\,dx_1\\ &= \vartheta_{n-1}\int_0^1 t^{(p-1)/2}\,(1-t)^{(n-1)/2}\,dt = \vartheta_{n-1}B\Big(\frac{p+1}{2},\frac{n+1}{2}\Big),\end{aligned}$$

where ϑ_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$ and B is the Euler Beta function. Hence,

$$K_{p,n} = (n+p)\vartheta_{n-1}B\Big(\frac{p+1}{2},\frac{n+1}{2}\Big) = \frac{2\pi^{(n-1)/2}\Gamma((p+1)/2)}{\Gamma((n+p)/2)}, \tag{3.5}$$

where Γ stands for the Euler Gamma function. Although Theorem 3.1 only requires the value of $K_{p,n}$ for $p\ge 1$, the above expression is valid, in fact, for any $p\ge 0$.

THEOREM 3.3. *Let $p\in[1,\infty)$ and $l\in\mathbb{N}$. Then, for any $v\in W^{l+1,p}(\mathbb{R}^n)$,*

$$\lim_{\sigma\to 1^-}(1-\sigma)|v|^p_{l+\sigma,p,\mathbb{R}^n} = p^{-1}K_{p,n}|\nabla v|^p_{l,p,\mathbb{R}^n}, \tag{3.6}$$

where $K_{p,n}$ is given by (3.3).

Proof. This result, for $l = 0$, is usually credited to Bourgain, Brézis and Mironescu [6], since it is implicitly contained in their paper. It can be proved using Theorem 3.1, first for smooth functions with compact support and then, by density, for any element in $W^{l+1,p}(\mathbb{R}^n)$. An explicit proof is given by Milman [15, Subsection 3.1], but without providing the precise definition of $K_{p,n}$, which can be deduced from Karadzhov, Milman and Xiao [11, p. 332]. The case $l > 0$ is identical to that in the proof of Theorem 3.1. ∎

THEOREM 3.4. *Let $p \in [1,\infty)$, $l \in \mathbb{N}$ and $\sigma_0 \in (0,1)$. Then, for any $v \in W^{l+\sigma_0,p}(\mathbb{R}^n)$,*

$$\lim_{\sigma\to 0^+} \sigma |v|^p_{l+\sigma,p,\mathbb{R}^n} = \frac{4\pi^{n/2}}{p\,\Gamma(n/2)} |v|^p_{l,p,\mathbb{R}^n}. \tag{3.7}$$

Proof. Maz'ya and Shaposhnikova proved in [14, Theorem 3] that (1.2) holds for any v belonging to $\bigcup_{0<\sigma<1} W_0^{\sigma,p}(\mathbb{R}^n)$, where $W_0^{\sigma,p}(\mathbb{R}^n)$ stands for the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to $|\cdot|_{\sigma,p,\mathbb{R}^n}$ (which is a norm in this last space). The condition on v can be relaxed to $v \in \bigcup_{0<\sigma<\sigma_0} W_0^{\sigma,p}(\mathbb{R}^n)$ for some $\sigma_0 \in (0,1)$. Likewise, since $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{\sigma,p}(\mathbb{R}^n)$ with respect to $\|\cdot\|_{\sigma,p,\mathbb{R}^n} = (|\cdot|^p_{0,p,\mathbb{R}^n} + |\cdot|^p_{\sigma,p,\mathbb{R}^n})^{1/p}$, it follows that $W^{\sigma,p}(\mathbb{R}^n) \subset W_0^{\sigma,p}(\mathbb{R}^n)$. Thus, taking into account the embedding $W^{\sigma_0,p}(\mathbb{R}^n) \hookrightarrow W^{\sigma,p}(\mathbb{R}^n)$ if $\sigma_0 \ge \sigma$, and the fact that $|S_{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$, we conclude that, for $l = 0$, (3.7) follows from Maz'ya and Shaposhnikova's result.

Now, let us assume that $l \ge 1$. Given $v \in W^{l+\sigma_0,p}(\mathbb{R}^n)$, it is clear that any lth derivative $\partial^\alpha v$ belongs to $W^{\sigma_0,p}(\mathbb{R}^n)$. The case $l = 0$ implies that

$$\begin{aligned}\lim_{\sigma\to 0^+} \sigma |v|^p_{l+\sigma,p,\mathbb{R}^n} &= \lim_{\sigma\to 0^+} \sigma \sum_{|\alpha|=l} |\partial^\alpha v|^p_{\sigma,p,\mathbb{R}^n} = \sum_{|\alpha|=l} \lim_{\sigma\to 0^+} \sigma |\partial^\alpha v|^p_{\sigma,p,\mathbb{R}^n} \\ &= \sum_{|\alpha|=l} \frac{4\pi^{n/2}}{p\Gamma(n/2)} |\partial^\alpha v|^p_{0,p,\mathbb{R}^n} = \frac{4\pi^{n/2}}{p\Gamma(n/2)} |v|^p_{l,p,\mathbb{R}^n}.\end{aligned}$$

The theorem follows. ∎

As we shall next see, there exists a qualitative difference in the behaviour of $|v|_{l+\sigma,p,\Omega}$ as $\sigma \to 0^+$ depending on whether Ω is $\mathbb{R}^n$ or a bounded set. Theorem 3.4 implies that the seminorm $|v|_{l+\sigma,p,\mathbb{R}^n}$ blows up to infinity (except for polynomials of degree $\le l$) as $\sigma \to 0^+$. However, for a bounded set Ω, a priori, the seminorm $|v|_{l+\sigma,p,\Omega}$ may remain bounded. In fact, this is always the case. Even more, as $\sigma \to 0^+$, that seminorm tends to Dini's seminorm $|v|_{l,\mathrm{Dini}(p),\Omega}$, defined, following Milman [15], by

$$|v|^p_{l,\mathrm{Dini}(p),\Omega} = \sum_{|\alpha|=l} \int_{\Omega\times\Omega} \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^p}{|x-y|^n}\, dx\, dy.$$

Let us state and establish this result. We borrow the arguments from Milman [15, Theorem 3 and Example 2].

THEOREM 3.5. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz continuous boundary. Let $p \in [1,\infty)$, $l \in \mathbb{N}$ and $\sigma_0 \in (0,1)$. Then, for any $v \in W^{l+\sigma_0,p}(\Omega)$, we have $|v|_{l,\mathrm{Dini}(p),\Omega} < \infty$ and*

$$\lim_{\sigma\to 0^+} |v|_{l+\sigma,p,\Omega} = |v|_{l,\mathrm{Dini}(p),\Omega}.$$

Proof. As in previous results, it suffices to prove the case $l = 0$. Let R be the diameter of Ω. We consider the bijective linear mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ given by $F(\hat{x}) = R\,\hat{x}$ and we write $\widehat{\Omega} = F^{-1}(\Omega)$. Since $R = \operatorname{diam}\Omega$, it is clear that $\operatorname{diam}\widehat{\Omega} = 1$. Thus,

$$\forall\sigma \in (0,\sigma_0),\ \forall\hat{x},\hat{y} \in \widehat{\Omega}, \quad 1 \ge |\hat{x}-\hat{y}|^{\sigma} \ge |\hat{x}-\hat{y}|^{\sigma_0}.$$

Consequently, given $\hat{v} \in W^{\sigma_0,p}(\widehat{\Omega})$, we have

$$\forall\sigma \in (0,\sigma_0), \quad |\hat{v}|^p_{0,\mathrm{Dini}(p),\widehat{\Omega}} \le |\hat{v}|^p_{\sigma,p,\widehat{\Omega}} \le |\hat{v}|^p_{\sigma_0,p,\widehat{\Omega}} < \infty.$$

Hence, by Lebesgue's Dominated Convergence Theorem, we get

$$\begin{aligned}\lim_{\sigma\to 0^+} |\hat{v}|^p_{\sigma,p,\widehat{\Omega}} &= \lim_{\sigma\to 0^+} \int\limits_{\widehat{\Omega}\times\widehat{\Omega}} \frac{|\hat{v}(\hat{x})-\hat{v}(\hat{y})|^p}{|\hat{x}-\hat{y}|^{n+p\sigma}}\,d\hat{x}\,d\hat{y} \\ &= \int\limits_{\widehat{\Omega}\times\widehat{\Omega}} \lim_{\sigma\to 0^+} \frac{|\hat{v}(\hat{x})-\hat{v}(\hat{y})|^p}{|\hat{x}-\hat{y}|^{n+p\sigma}}\,d\hat{x}\,d\hat{y} \\ &= \int\limits_{\widehat{\Omega}\times\widehat{\Omega}} \frac{|\hat{v}(\hat{x})-\hat{v}(\hat{y})|^p}{|\hat{x}-\hat{y}|^{n}}\,d\hat{x}\,d\hat{y} = |\hat{v}|^p_{0,\mathrm{Dini}(p),\widehat{\Omega}}.\end{aligned}$$

Now, for any $v \in W^{\sigma_0,p}(\Omega)$, the function $\hat{v} = v \circ F$ belongs to $W^{\sigma_0,p}(\widehat{\Omega})$, since

$$|v|_{\sigma_0,p,\Omega} = R^{-\sigma_0+n/p}|\hat{v}|_{\sigma_0,p,\widehat{\Omega}}.$$

Likewise,

$$|v|_{0,\mathrm{Dini}(p),\Omega} = R^{n/p}|\hat{v}|_{0,\mathrm{Dini}(p),\widehat{\Omega}}$$

and, for any $\sigma \in (0,\sigma_0)$,

$$|v|_{\sigma,p,\Omega} = R^{-\sigma+n/p}|\hat{v}|_{\sigma,p,\widehat{\Omega}}.$$

From these relations, we deduce that $|v|_{0,\mathrm{Dini}(p),\Omega}$ is finite and that

$$\begin{aligned}\lim_{\sigma\to 0^+} |v|_{\sigma,p,\Omega} &= \lim_{\sigma\to 0^+} R^{-\sigma+n/p}|\hat{v}|_{\sigma,p,\widehat{\Omega}} = R^{n/p}|\hat{v}|_{0,\mathrm{Dini}(p),\widehat{\Omega}} \\ &= |v|_{0,\mathrm{Dini}(p),\Omega}. \ \blacksquare\end{aligned}$$

REMARK 3.6. It is worth noting that, under the conditions of Theorem 3.5, the arguments in its proof lead, in general, to the following bound:

$$\forall v \in W^{l+\sigma_0,p}(\Omega), \qquad |v|_{l,\mathrm{Dini}(p),\Omega} \le R^{\sigma}|v|_{l+\sigma,p,\Omega} \le R^{\sigma_0}|v|_{l+\sigma_0,p,\Omega},$$

with $R = \operatorname{diam}\Omega$.

Remark 3.7. By a change of variables and Fubini's Theorem, it can be seen that

$$|v|_{0,\mathrm{Dini}(p),\Omega} = \left(n \int_0^{\infty} \frac{\overline{\omega}(v,t)_p^p}{t}\, dt \right)^{1/p},$$

where $\overline{\omega}(v,t)_p$ is the averaged modulus of smoothness, given by

$$\overline{\omega}(v,t)_p^p = t^{-n} \int_{|h|\le t} |\Delta_h v|_{0,p,\Omega}^p \, dh, \qquad t > 0,$$

with $\Delta_h v(x) = v(x+h) - v(x)$, if $x, x+h \in \Omega$, and $\Delta_h f(x) = 0$ otherwise. Hence, for $l = 0$, Theorem 3.5 establishes that, for any $v \in W^{\sigma_0,p}(\Omega)$, the function $\overline{\omega}(v,\,\cdot\,)_p$ satisfies a Dini-type condition. Analogous comments can be made for $l > 0$. This justifies the name given to the seminorm $|\cdot|_{l,\mathrm{Dini}(p),\Omega}$. Likewise, since $\overline{\omega}(v,t)_p$ is equivalent to the usual modulus of smoothness $\omega(v,t)_p = \sup_{|h|\le t} |\Delta_h v|_{0,p,\Omega}$, Theorem 3.5 includes as a particular case the result given by Milman (cf. [15, Example 2]).

Remark 3.8. The seminorm $|\cdot|_{r,p,\mathbb{R}^n}$ can be normalized as follows:

$$[v]_{r,p,\mathbb{R}^n} = \lambda_{\sigma,p}|v|_{r,p,\mathbb{R}^n}, \tag{3.8}$$

where $\sigma = r - \lfloor r \rfloor$ and

$$\lambda_{\sigma,p} = \begin{cases} (\sigma(1-\sigma))^{1/p} & \text{if } \sigma \in (0,1), \\ 1 & \text{if } \sigma = 0. \end{cases} \tag{3.9}$$

Then the seminorm $[\,\cdot\,]_{r,p,\mathbb{R}^n}$ is continuous in the scale of Sobolev spaces $\big(W^{r,p}(\mathbb{R}^n)\big)_{r\ge 0}$ in the following sense:

$$\forall r > 0,\ \forall v \in W^{r,p}(\mathbb{R}^n), \qquad \lim_{s\to r^-} [v]_{s,p,\mathbb{R}^n} \approx [v]_{r,p,\mathbb{R}^n},$$
$$\forall r \ge 0,\ \forall \epsilon > 0,\ \forall v \in W^{r+\epsilon,p}(\mathbb{R}^n), \qquad \lim_{s\to r^+} [v]_{s,p,\mathbb{R}^n} \approx [v]_{r,p,\mathbb{R}^n},$$

where the symbol $\approx$ means that there exist positive constants c_1 and c_2, independent of v, such that

$$c_1[v]_{r,p,\mathbb{R}^n} \le \lim_{s\to r^{\pm}} [v]_{s,p,\mathbb{R}^n} \le c_2[v]_{r,p,\mathbb{R}^n}.$$

In fact, if $r \notin \mathbb{N}$, both lateral limits are equal to $[v]_{r,p,\mathbb{R}^n}$. For $r \in \mathbb{N}$, these relations are direct consequences of Theorems 3.3 and 3.4, whereas, for $r \notin \mathbb{N}$, they come from Lebesgue's Dominated Convergence Theorem.

For a bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz continuous boundary, we could also consider the normalization $[v]_{r,p,\Omega} = \lambda_{\sigma,p}|v|_{r,p,\Omega}$. But, due to Theorem 3.5, for any $r \in \mathbb{N}$, we would get

$$\forall \epsilon > 0,\ \forall v \in W^{r+\epsilon,p}(\Omega), \qquad \lim_{s\to r^+} [v]_{s,p,\Omega} = 0,$$

which is quite unnatural. A better normalization is

$$\lceil v \rceil_{r,p,\Omega} = (1-\sigma)^{1/p} |v|_{r,p,\Omega}$$

with $\sigma = r - \lfloor r \rfloor$. We now have

$$\forall r > 0,\ r \notin \mathbb{N},\ \forall v \in W^{r,p}(\Omega), \qquad \lim_{s\to r^-} \lceil v \rceil_{s,p,\Omega} \approx \lceil v \rceil_{r,p,\Omega},$$

$$\forall r \ge 0,\ \forall \epsilon > 0,\ \forall v \in W^{r+\epsilon,p}(\Omega), \qquad \lim_{s\to r^+} \lceil v \rceil_{s,p,\Omega} \approx \begin{cases} \lceil v \rceil_{r,p,\Omega} & \text{if } r \notin \mathbb{N}, \\ |v|_{r,\mathrm{Dini}(p),\Omega} & \text{if } r \in \mathbb{N}. \end{cases}$$

Observe that, given $r \in \mathbb{N}$ and $\epsilon > 0$, the seminorms $|\cdot|_{r,\mathrm{Dini}(p),\Omega}$ and $|\cdot|_{r,p,\Omega}$ are not equivalent on $W^{r+\epsilon,p}(\Omega)$ ($|\cdot|_{r,\mathrm{Dini}(p),\Omega}$ is null for polynomials of degree $\le r$, while $|\cdot|_{r,p,\Omega}$ is null only for polynomials of degree $\le r-1$). Consequently, the seminorm $\lceil \cdot \rceil_{r,p,\Omega}$ is not right-continuous for $r \in \mathbb{N}$.

4. The particular case $p = 2$. The purpose of this section is to provide an alternative proof of Theorems 3.3 and 3.4 based on the Fourier transform. We start with two results which just reformulate a well-known characterization of the space $W^{\sigma,2}(\mathbb{R}^n)$ for $\sigma \in (0,1)$. See, for example, Goulaouic [9, Theorem VIII.4], Stein [19, Chapter V, Proposition 4] or Tartar [20, Lemma 16.3]. We also seize the opportunity to compute explicitly a certain integral appearing in these results.

LEMMA 4.1. *For any $n \in \mathbb{N}$ and $\sigma \in (0,1)$, let*

$$G_{\sigma,n} = \frac{\pi K_{2\sigma,n}}{\Gamma(1+2\sigma)\sin(\pi\sigma)}, \tag{4.1}$$

where $K_{2\sigma,n}$ is given by (3.3) (*or* (3.5)) *with $p = 2\sigma$. Then, for any $\xi \in \mathbb{R}^n$,*

$$\int_{\mathbb{R}^n} \frac{|e^{i\xi\cdot y} - 1|^2}{|y|^{n+2\sigma}}\, dy = G_{\sigma,n} |\xi|^{2\sigma}. \tag{4.2}$$

Proof. The relation (4.2) is obviously true if $\xi = 0$, so let us assume that $\xi \ne 0$. Let $\nu = \xi/|\xi|$. By the change of variables $x = |\xi| y/2$, we get

$$\int_{\mathbb{R}^n} \frac{|e^{i\xi\cdot y} - 1|^2}{|y|^{n+2\sigma}}\, dy = \frac{|\xi|^{2\sigma}}{2^{2\sigma}} \int_{\mathbb{R}^n} \frac{|e^{2i\nu\cdot x} - 1|^2}{|x|^{n+2\sigma}}\, dx = \frac{|\xi|^{2\sigma}}{2^{2\sigma}} \int_{\mathbb{R}^n} \frac{4\sin^2(\nu\cdot x)}{|x|^{n+2\sigma}}\, dx.$$

Using spherical integrals and the change $t = \rho|\nu\cdot\omega|$, we derive that

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{4\sin^2(\nu\cdot x)}{|x|^{n+2\sigma}}\, dx &= \int_{S_{n-1}} \Big(\int_0^\infty \rho^{n-1} \frac{4\sin^2(\rho\nu\cdot\omega)}{|\rho\omega|^{n+2\sigma}}\, d\rho \Big)\, d\omega \\ &= \int_{S_{n-1}} |\nu\cdot\omega|^{2\sigma} \Big(\int_0^\infty \frac{4\sin^2 t}{t^{1+2\sigma}}\, dt \Big)\, d\omega = K_{2\sigma,n} \int_0^\infty \frac{4\sin^2 t}{t^{1+2\sigma}}\, dt. \end{aligned}$$

Hence,

$$\int_{\mathbb{R}^n} \frac{|e^{i\xi\cdot y}-1|^2}{|y|^{n+2\sigma}}\,dy = |\xi|^{2\sigma}\,K_{2\sigma,n}2^{2-2\sigma}\int_0^\infty \frac{\sin^2 t}{t^{1+2\sigma}}\,dt.$$

Taking (4.1) into account, in order to prove (4.2), it suffices to see that

$$2^{2-2\sigma}\int_0^\infty \frac{\sin^2 t}{t^{1+2\sigma}}\,dt = \frac{\pi}{\Gamma(1+2\sigma)\sin(\pi\sigma)}. \tag{4.3}$$

This relation holds if $\sigma = 1/2$, since the integral on the left-hand side then equals $\pi/2$ (see, for example, Gradshteyn and Ryzhik [10, relation 3.821.9]). Now, let $\sigma \in (0,1/2)\cup(1/2,1)$. From relation 3.823 in [10], we obtain

$$2^{2-2\sigma}\int_0^\infty \frac{\sin^2 t}{t^{1+2\sigma}}\,dt = -2\Gamma(-2\sigma)\cos(\pi\sigma),$$

from which, together with the identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \qquad z\notin\mathbb{Z},$$

we deduce (4.3). The lemma follows. ∎

Proposition 4.2. *Let $\sigma\in(0,1)$. Then*

$$W^{\sigma,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)\cap \widetilde{H}^\sigma(\mathbb{R}^n), \tag{4.4}$$

with

$$\widetilde{H}^\sigma(\mathbb{R}^n) = \Big\{v\in\mathcal{S}'(\mathbb{R}^n)\ \Big|\ \hat v\in L^1_{\mathrm{loc}}(\mathbb{R}^n),\ \int_{\mathbb{R}^n}|\xi|^{2\sigma}|\hat v(\xi)|^2\,d\xi<\infty\Big\}. \tag{4.5}$$

In fact, for any $v\in W^{\sigma,2}(\mathbb{R}^n)$,

$$|v|^2_{\sigma,2,\mathbb{R}^n} = (2\pi)^{-n}G_{\sigma,n}\int_{\mathbb{R}^n}|\xi|^{2\sigma}|\hat v(\xi)|^2\,d\xi, \tag{4.6}$$

where $G_{\sigma,n}$ is the constant given by (4.1).

Proof. Let $v\in L^2(\mathbb{R}^n)$. We first remark that v is, in particular, a tempered distribution and, by Plancherel's Theorem, $\hat v\in L^2(\mathbb{R}^n)$, so $\hat v\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Thus, to prove (4.4), it suffices to see that the seminorm $|v|_{\sigma,2,\mathbb{R}^n}$ is finite if and only if the integral $\int_{\mathbb{R}^n}|\xi|^{2\sigma}|\hat v(\xi)|^2\,d\xi$ is finite. But this is a consequence of (4.6). So let us show that (4.6) holds.

To this end, we follow, for example, the reasoning of Goulaouic [9, p. 101]. For any $y\in\mathbb{R}^n$, the Fourier transform of the translated function $x\mapsto v(x+y)$ is the function $\xi\mapsto e^{iy\cdot\xi}\hat v(\xi)$. Hence, by Parseval's identity, we have

$$\int_{\mathbb{R}^n}|v(x+y)-v(x)|^2\,dx = (2\pi)^{-n}\int_{\mathbb{R}^n}|\hat v(\xi)|^2|e^{iy\cdot\xi}-1|^2\,d\xi.$$

Then, by Fubini's Theorem and Lemma 4.1, we finally deduce that

$$
\begin{aligned}
|v|^2_{\sigma,2,\mathbb{R}^n} &= \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v(x+y)-v(x)|^2}{|y|^{n+2\sigma}}\,dx\,dy \\
&= (2\pi)^{-n} \int\limits_{\mathbb{R}^n} |\hat{v}(\xi)|^2 \bigg(\int\limits_{\mathbb{R}^n} \frac{|e^{iy\cdot\xi}-1|^2}{|y|^{n+2\sigma}}\,dy \bigg)\,d\xi \\
&= (2\pi)^{-n}\, G_{\sigma,n} \int\limits_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2\,d\xi,
\end{aligned}
$$

which yields (4.6) and completes the proof. ■

The following two lemmas concern, for a suitable function v, the behaviour of the integral $\int_{\mathbb{R}^n} |\xi|^{2\sigma}|\hat{v}(\xi)|^2\,d\xi$ as $\sigma\to 0^+$ or $\sigma \to 1^-$.

Lemma 4.3. *Let $\sigma_0 \in (0,1)$. Then, for any $v \in W^{\sigma_0,2}(\mathbb{R}^n)$,*

$$
\lim_{\sigma\to 0^+} \int\limits_{\mathbb{R}^n} |\xi|^{2\sigma}|\hat{v}(\xi)|^2\,d\xi = (2\pi)^n |v|^2_{0,2,\mathbb{R}^n}.
$$

Proof. Let $v \in W^{\sigma_0,2}(\mathbb{R}^n)$. For any $\sigma \in (0,\sigma_0]$, let us consider the integral $I_\sigma = \int_{\mathbb{R}^n} g_\sigma(\xi)\,d\xi$, where $g_\sigma(\xi) = (1-|\xi|^{2\sigma})|\hat{v}(\xi)|^2$. This integral is well defined: since $v\in W^{\sigma_0,2}(\mathbb{R}^n)$, v also belongs to $L^2(\mathbb{R}^n)$ and $W^{\sigma,2}(\mathbb{R}^n)$, so $\hat{v}\in L^2(\mathbb{R}^n)$ and, by Proposition 4.2, $v \in \widetilde{H}^\sigma(\mathbb{R}^n)$.

Let $0 < r \le 1 < R$. We set

$$
I_\sigma = \int\limits_{|\xi|\le r} g_\sigma(\xi)\,d\xi + \int\limits_{r<|\xi|<R} g_\sigma(\xi)\,d\xi + \int\limits_{|\xi|\ge R} g_\sigma(\xi)\,d\xi = J_1+J_2+J_3.
$$

Let $\varepsilon > 0$ be given. Let us show that we can choose r, R and $\sigma\in(0,\sigma_0)$ such that $|I_\sigma| < \varepsilon$. We have

$$
|J_1| \le \int\limits_{|\xi|\le r} |\hat{v}(\xi)|^2\,d\xi.
$$

Clearly, $|J_1|\le \varepsilon/3$ for r small enough, since $\hat{v}\in L^2(\mathbb{R}^n)$. Moreover,

$$
|J_3| \le \int\limits_{|\xi|\ge R} |\hat{v}(\xi)|^2\,d\xi + \int\limits_{|\xi|\ge R} |\xi|^{2\sigma_0}|\hat{v}(\xi)|^2\,d\xi,
$$

and the two terms on the right are arbitrarily small when R is large enough: the first because $\hat{v}\in L^2(\mathbb{R}^n)$, and the second because, by Proposition 4.2, $v\in \widetilde{H}^{\sigma_0}(\mathbb{R}^n)$. So, $|J_3| < \varepsilon/3$ for R sufficiently large. Once r and R have been chosen, it suffices to take σ small enough to achieve $|J_2| < \varepsilon/3$.

The preceding reasoning implies that

$$
\lim_{\sigma\to 0^+} \int\limits_{\mathbb{R}^n} g_\sigma(\xi)\,d\xi = 0.
$$

Consequently, taking Plancherel's Theorem into account, we conclude that

$$\lim_{\sigma\to 0^+} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2 \, d\xi = \int_{\mathbb{R}^n} |\hat{v}(\xi)|^2 \, d\xi = (2\pi)^n |v|^2_{0,2,\mathbb{R}^n}. \blacksquare$$

Lemma 4.4. *For any* $v \in W^{1,2}(\mathbb{R}^n)$,

$$\lim_{\sigma\to 1^-} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2 \, d\xi = (2\pi)^n |\nabla v|^2_{0,2,\mathbb{R}^n}.$$

Proof. Let $v \in W^{1,2}(\mathbb{R}^n)$. For any $\sigma \in (0,1)$, we now consider the integral $I_\sigma = \int_{\mathbb{R}^n} g_\sigma(\xi)\, d\xi$, with $g_\sigma(\xi) = (|\xi|^2 - |\xi|^{2\sigma})|\hat{v}(\xi)|^2$. It is clear that $|I_\sigma| < \infty$: on the one hand, the embedding $W^{1,2}(\mathbb{R}^n) \hookrightarrow W^{\sigma,2}(\mathbb{R}^n)$ and Proposition 4.2 imply that $v \in \widetilde{H}^\sigma(\mathbb{R}^n)$; on the other hand, since $v \in W^{1,2}(\mathbb{R}^n)$,

$$\begin{aligned}
(4.7) \qquad \int_{\mathbb{R}^n} |\xi|^2 |\hat{v}(\xi)|^2 \, d\xi &= \sum_{|\beta|=1} \int_{\mathbb{R}^n} \xi^{2\beta} |\hat{v}(\xi)|^2 \, d\xi = \sum_{|\beta|=1} \int_{\mathbb{R}^n} |i\xi^\beta \hat{v}(\xi)|^2 \, d\xi \\
&= \sum_{|\beta|=1} \int_{\mathbb{R}^n} |\widehat{\partial^\beta v}(\xi)|^2 \, d\xi = \sum_{|\beta|=1} (2\pi)^n \int_{\mathbb{R}^n} |\partial^\beta v(x)|^2 \, dx \\
&= (2\pi)^n |\nabla v|^2_{0,2,\mathbb{R}^n},
\end{aligned}$$

which is finite.

As in the proof of Lemma 4.3, we set

$$I_\sigma = \int_{|\xi|\le r} g_\sigma(\xi)\, d\xi + \int_{r<|\xi|<R} g_\sigma(\xi)\, d\xi + \int_{|\xi|\ge R} g_\sigma(\xi)\, d\xi = J_1 + J_2 + J_3,$$

with $0 < r \le 1 < R$. Let $\varepsilon > 0$ be given. Clearly, we have

$$|J_1| \le 2 \int_{|\xi|\le r} |\hat{v}(\xi)|^2 \, d\xi \quad \text{and} \quad |J_3| \le 2 \int_{|\xi|\ge R} |\xi|^2 |\hat{v}(\xi)|^2 \, d\xi.$$

Then the assumption $v \in W^{1,2}(\mathbb{R}^n)$ implies that r and R can be chosen in such a way that $|J_1|$ and $|J_3|$ be $\le \varepsilon/3$. We have just to take σ sufficiently close to 1 to achieve $|J_2| < \varepsilon/3$. Consequently,

$$\lim_{\sigma\to 1^+} \int_{\mathbb{R}^n} g_\sigma(\xi)\, d\xi = 0.$$

From this relation and (4.7), we finally derive that

$$\lim_{\sigma\to 1^-} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2 \, d\xi = \int_{\mathbb{R}^n} |\xi|^2 |\hat{v}(\xi)|^2 \, d\xi = (2\pi)^n |\nabla v|^2_{0,2,\mathbb{R}^n}. \blacksquare$$

We are now ready to prove the main result in this section, which establishes Theorems 3.3 and 3.4 in the particular case $p = 2$. The reader may want to check that the constants on the right-hand side of (3.6) and (3.7) are, for $p = 2$, equal to those in (4.9) and (4.8), respectively.

THEOREM 4.5. *Let $l \in \mathbb{N}$.*

(i) *Let $\sigma_0 \in (0,1)$. Then, for any $v \in W^{l+\sigma_0,2}(\mathbb{R}^n)$,*

$$\lim_{\sigma\to 0^+} \sigma |v|^2_{l+\sigma,2,\mathbb{R}^n} = \frac{2\pi^{n/2}}{\Gamma(n/2)} |v|^2_{l,2,\mathbb{R}^n}. \tag{4.8}$$

(ii) *For any $v \in W^{l+1,2}(\mathbb{R}^n)$,*

$$\lim_{\sigma\to 1^-} (1-\sigma) |v|^2_{l+\sigma,2,\mathbb{R}^n} = \frac{\pi^{n/2}}{n\Gamma(n/2)} |\nabla v|^2_{l,2,\mathbb{R}^n}. \tag{4.9}$$

Proof. Let us first assume that $l=0$. It readily follows from (3.5), (4.1) and the properties of the Γ function that

$$\lim_{\sigma\to 0^+} \sigma G_{\sigma,n} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad \text{and} \quad \lim_{\sigma\to 1^-} (1-\sigma) G_{\sigma,n} = \frac{\pi^{n/2}}{n\Gamma(n/2)}.$$

Consequently, by Proposition 4.2 and Lemma 4.3, we have

$$\lim_{\sigma\to 0^+} \sigma |v|^2_{\sigma,2,\mathbb{R}^n} = \lim_{\sigma\to 0^+} \sigma (2\pi)^{-n} G_{\sigma,n} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2 \, d\xi = \frac{2\pi^{n/2}}{\Gamma(n/2)} |v|^2_{0,2,\mathbb{R}^n}.$$

Likewise, by Proposition 4.2 and Lemma 4.4,

$$\lim_{\sigma\to 1^-} (1-\sigma) |v|^2_{\sigma,2,\mathbb{R}^n}$$
$$= \lim_{\sigma\to 1^-} (1-\sigma)(2\pi)^{-n} G_{\sigma,n} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{v}(\xi)|^2 \, d\xi = \frac{\pi^{n/2}}{n\Gamma(n/2)} |\nabla v|^2_{0,2,\mathbb{R}^n}.$$

The reasoning for $l \ge 1$ follows the same pattern already shown in Theorems 3.3 and 3.4. ∎

In the proof of Theorem 4.5 and the preceding lemmas, Proposition 4.2 plays a fundamental role. This result can be extended to characterize the space $W^{r,2}(\mathbb{R}^n)$ for any $r \ge 0$. Although it is not required here, we include such an extension in this section for the sake of completeness.

THEOREM 4.6. *Let $r \in [0,\infty)$. Then*

$$W^{r,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n) \cap \widetilde{H}^r(\mathbb{R}^n), \tag{4.10}$$

where $\widetilde{H}^r(\mathbb{R}^n)$ is given by (4.5) with r instead of σ. Moreover, for any $m \in \mathbb{N}$ and $s \ge 0$ such that $r = m+s$,

$$W^{r,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n) \cap X^{m,s}, \tag{4.11}$$

where $X^{m,s} = \{v \in \mathcal{D}'(\mathbb{R}^n) \mid \partial^\alpha v \in \widetilde{H}^s(\mathbb{R}^n), \forall \alpha \in \mathbb{N}^n, |\alpha| = m\}$, $\mathcal{D}'(\mathbb{R}^n)$ being the space of distributions on $\mathbb{R}^n$.

Proof. We put $r = l + \sigma$, with $l = \lfloor r \rfloor$ and $\sigma \in [0,1)$. Let $m \in \mathbb{N}$ and $s \geq 0$ be such that $r = m + s$. We remark that $m \leq l$.

Since $L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$, it is clear that

$$L^2(\mathbb{R}^n) \cap \widetilde{H}^r(\mathbb{R}^n) = \Big\{ v \in L^2(\mathbb{R}^n) \;\Big|\; \int_{\mathbb{R}^n} |\xi|^{2r} |\hat{v}(\xi)|^2 \, d\xi < \infty \Big\} \tag{4.12}$$

and

$$\begin{aligned} L^2(\mathbb{R}^n) \cap X^{m,s} = \Big\{ v \in L^2(\mathbb{R}^n) \;\Big|\; &\forall \alpha \in \mathbb{N}^n,\ |\alpha| = m, \\ &\widehat{\partial^\alpha v} \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ and } \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{\partial^\alpha v}(\xi)|^2 \, d\xi < \infty \Big\}. \end{aligned} \tag{4.13}$$

We divide the proof into several steps: Steps 1 and 2 prove (4.10), whereas Steps 3 and 4 establish (4.11).

Step 1: $W^{r,2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap \widetilde{H}^r(\mathbb{R}^n)$. Let $v \in W^{r,2}(\mathbb{R}^n)$. By (4.12), we have just to show that $\int_{\mathbb{R}^n} |\xi|^{2r} |\hat{v}(\xi)|^2 \, d\xi$ is finite. Let us first consider the case $\sigma \in (0,1)$. Every lth derivative $\partial^\alpha v$ belongs to $W^{\sigma,2}(\mathbb{R}^n)$. By Proposition 4.2, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} |\xi|^{2r} |\hat{v}(\xi)|^2 \, d\xi \\ &\quad = \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\xi|^{2l} |\hat{v}(\xi)|^2 \, d\xi = \sum_{|\alpha|=l} \binom{l}{\alpha} \int_{\mathbb{R}^n} |\xi|^{2\sigma} \xi^{2\alpha} |\hat{v}(\xi)|^2 \, d\xi \\ &\quad = \sum_{|\alpha|=l} \binom{l}{\alpha} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\widehat{\partial^\alpha v}(\xi)|^2 \, d\xi = \sum_{|\alpha|=l} \binom{l}{\alpha} (2\pi)^n G_{\sigma,n}^{-1} |\partial^\alpha v|^2_{\sigma,2,\mathbb{R}^n} \\ &\quad \leq M (2\pi)^n \, G_{\sigma,n}^{-1} \sum_{|\alpha|=l} |\partial^\alpha v|^2_{\sigma,2,\mathbb{R}^n} = M (2\pi)^n G_{\sigma,n}^{-1} |v|^2_{r,2,\mathbb{R}^n} < \infty, \end{aligned}$$

with $M = \max\big\{ \binom{l}{\alpha} \;\big|\; \alpha \in \mathbb{N}^n,\ |\alpha| = l \big\}$. If $\sigma = 0$, the above reasoning is still valid, by taking $G_{\sigma,n} = 1$ and using Plancherel's Theorem instead of Proposition 4.2.

Step 2: $L^2(\mathbb{R}^n) \cap \widetilde{H}^r(\mathbb{R}^n) \subset W^{r,2}(\mathbb{R}^n)$. Let $v \in L^2(\mathbb{R}^n) \cap \widetilde{H}^r(\mathbb{R}^n)$. For any $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq l$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi|^{2r} |\hat{v}(\xi)|^2 \, d\xi &= \int_{\mathbb{R}^n} |\xi|^{2(r-|\alpha|)} |\xi|^{2|\alpha|} |\hat{v}(\xi)|^2 \, d\xi \\ &= \sum_{|\beta|=|\alpha|} \binom{|\alpha|}{\beta} \int_{\mathbb{R}^n} |\xi|^{2(r-|\alpha|)} \xi^{2\beta} |\hat{v}(\xi)|^2 \, d\xi. \end{aligned}$$

Consequently, $\widehat{\partial^\alpha v} = i^{|\alpha|}\xi^\alpha \hat{v}$ belongs to $L^2(\mathbb{R}^n)$, since

$$\begin{aligned}\int_{\mathbb{R}^n} |\xi^\alpha \hat{v}(\xi)|^2\, d\xi &= \int_{|\xi|<1} \xi^{2\alpha}|\hat{v}(\xi)|^2\, d\xi + \int_{|\xi|\geq 1} \xi^{2\alpha}|\hat{v}(\xi)|^2\, d\xi \\ &\leq \int_{|\xi|<1} |\hat{v}(\xi)|^2\, d\xi + \binom{|\alpha|}{\alpha} \int_{|\xi|\geq 1} |\xi|^{2(r-|\alpha|)}\xi^{2\alpha}|\hat{v}(\xi)|^2\, d\xi \\ &\leq \int_{\mathbb{R}^n} |\hat{v}(\xi)|^2\, d\xi + \int_{\mathbb{R}^n} |\xi|^{2r}|\hat{v}(\xi)|^2\, d\xi < \infty.\end{aligned}$$

We deduce from Plancherel's Theorem that $v \in W^{l,2}(\mathbb{R}^n)$. If $\sigma \in (0,1)$, we still have to see that $|v|_{r,2,\mathbb{R}^n}$ is finite. But a reasoning analogous to that in Step 1 shows, as desired, that

$$|v|^2_{r,2,\mathbb{R}^n} \leq (2\pi)^{-n}\, G_{\sigma,n} \int_{\mathbb{R}^n} |\xi|^{2r}|\hat{v}(\xi)|^2\, d\xi < \infty.$$

Step 3: $L^2(\mathbb{R}^n) \cap X^{m,s} \subset W^{r,2}(\mathbb{R}^n)$. Let $v \in L^2(\mathbb{R}^n) \cap X^{m,s}$. Then, taking (4.13) into account, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\xi|^{2r}|\hat{v}(\xi)|^2\, d\xi &= \int_{\mathbb{R}^n} |\xi|^{2s}|\xi|^{2m}|\hat{v}(\xi)|^2\, d\xi = \sum_{|\alpha|=m} \binom{m}{\alpha} \int_{\mathbb{R}^n} |\xi|^{2s}\xi^{2\alpha}|\hat{v}(\xi)|^2\, d\xi \\ &= \sum_{|\alpha|=m} \binom{m}{\alpha} \int_{\mathbb{R}^n} |\xi|^{2s}|\widehat{\partial^\alpha v}(\xi)|^2\, d\xi < \infty.\end{aligned}$$

Thus, it follows from (4.10) and (4.12) that $v \in W^{r,2}(\mathbb{R}^n)$.

Step 4: $W^{r,2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \cap X^{m,s}$. Let $v \in W^{r,2}(\mathbb{R}^n)$. Using (4.10), the reasoning in Step 2 shows that, for any $\alpha \in \mathbb{N}^n$ such that $|\alpha| = m$, $\widehat{\partial^\alpha v}$ belongs to $L^2(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\begin{aligned}\int_{\mathbb{R}^n} |\xi|^{2s}|\widehat{\partial^\alpha v}(\xi)|^2\, d\xi &= \int_{\mathbb{R}^n} |\xi|^{2s}\xi^{2\alpha}|\hat{v}(\xi)|^2\, d\xi \\ &\leq \sum_{|\beta|=m} \binom{m}{\beta} \int_{\mathbb{R}^n} |\xi|^{2s}\xi^{2\beta}|\hat{v}(\xi)|^2\, d\xi \\ &= \int_{\mathbb{R}^n} |\xi|^{2r}|\hat{v}(\xi)|^2\, d\xi < \infty.\end{aligned}$$

We conclude that, by (4.13), $v \in L^2(\mathbb{R}^n) \cap X^{m,s}$. ∎

Remark 4.7. For any $r \geq 0$ and $v \in L^2(\mathbb{R}^n)$, it is clear that $|\cdot|^r\hat{v} \in L^2(\mathbb{R}^n)$ if and only if $(1 + |\cdot|^2)^{r/2}\hat{v} \in L^2(\mathbb{R}^n)$, thanks to Plancherel's Theorem, and that

$$\begin{aligned}\int_{\mathbb{R}^n} |\xi|^{2r} |\hat{v}(\xi)|^2 \, d\xi &\le \int_{\mathbb{R}^n} (1+|\xi|^2)^r |\hat{v}(\xi)|^2 \, d\xi \\ &= \int_{|\xi|\le 1} (1+|\xi|^2)^r |\hat{v}(\xi)|^2 \, d\xi + \int_{|\xi|\ge 1} (1+|\xi|^2)^r |\hat{v}(\xi)|^2 \, d\xi \\ &\le 2^r \int_{\mathbb{R}^n} |\hat{v}(\xi)|^2 \, d\xi + 2^r \int_{\mathbb{R}^n} |\xi|^{2r} |\hat{v}(\xi)|^2 \, d\xi.\end{aligned}$$

Hence, the relation (4.10) follows immediately from the well-known characterization of the space $W^{r,2}(\mathbb{R}^n)$ in terms of Bessel potentials (see, for example, Adams [1, Theorem 7.63(f)] or Goulaouic [9, Corollary VIII.2]). To our knowledge, however, the relation (4.11) is new. It involves the Beppo Levi space $X^{m,s}$, which plays an essential role in spline theory (cf. [2]).

Remark 4.8. Let $r > 0$. Theorem 4.6 allows us to endow $W^{r,2}(\mathbb{R}^n)$ with seminorms defined in $\widetilde{H}^r(\mathbb{R}^n)$ or $X^{m,s}$. For example, the mapping

$$|\cdot|_{0,r} : v \mapsto \Big(\int_{\mathbb{R}^n} |\xi|^{2r} |\hat{v}(\xi)|^2 \, d\xi \Big)^{1/2} \tag{4.14}$$

is a seminorm in $\widetilde{H}^r(\mathbb{R}^n)$ (in fact, a hilbertian norm if $r < n/2$; cf. [2]), so it is in $W^{r,2}(\mathbb{R}^n)$. It follows from Steps 1 and 2 in the proof of Theorem 4.6 that $|\cdot|_{0,r}$ and $|\cdot|_{r,2,\mathbb{R}^n}$ are equivalent seminorms. The equivalence constants depend on σ, since they contain $G_{\sigma,n}$. In fact, taking into account (3.5), (4.1) and the continuity of the Gamma function, it is readily seen that, given $l \in \mathbb{N}$, there exist constants C_1 and C_2, depending on n and l, such that, for all $\sigma \in (0,1)$ and $v \in W^{l+\sigma,2}(\mathbb{R}^n)$,

$$C_1 |v|_{0,l+\sigma} \le (2\sigma(1-\sigma))^{1/2} |v|_{l+\sigma,2,\mathbb{R}^n} \le C_2 |v|_{0,l+\sigma}.$$

5. Application to sampling inequalities. Sampling inequalities in Sobolev spaces have become an essential tool for error and convergence analysis in fields like interpolation and smoothing by radial basis functions or meshless methods for the numerical solution of partial differential equations (cf., for example, [3, 16, 17, 18]). Given a domain Ω and suitable values of $p, q \in [1,\infty]$, these inequalities typically yield bounds of the $|\cdot|_{s,q,\Omega}$ Sobolev seminorm of a function $u \in W^{r,p}(\Omega)$ in terms of the $|\cdot|_{r,p,\Omega}$ seminorm of u, with $r \ge s \ge 0$, the values of u in a discrete set $A \subset \overline{\Omega}$, and the *fill distance* d between $\overline{\Omega}$ and A given by

$$d = \sup_{x\in\Omega} \inf_{a\in A} |x-a|. \tag{5.1}$$

We remark that, since A has no accumulation points and d must be finite, the set A must also be finite if Ω is bounded, and countably infinite if Ω is unbounded.

Let us consider a simplified version of the sampling inequalities proven in [5], which, however, will suffice for our purposes here. Assume that Ω is $\mathbb{R}^n$ or a bounded domain with a Lipschitz continuous boundary. Assume also that $1 \le p \le q$, $r > n/p$, and $0 \le s \le \lceil \ell_0 \rceil - 1$, with $\ell_0 = r - n(1/p - 1/q)$. Then, for any discrete set $A \subset \overline{\Omega}$ having a sufficiently small fill distance d and for any $u \in W^{r,p}(\Omega)$, the following sampling inequality holds:

$$|u|_{s,q,\Omega} \le \mathfrak{C}_s (d^{r-s-n(1/p-1/q)} |u|_{r,p,\Omega} + d^{n/q-s} \|u|_A\|_p), \tag{5.2}$$

where

$$\|u|_A\|_p = \Big(\sum_{a \in A} |u(a)|^p\Big)^{1/p}$$

and $\mathfrak{C}_s$ is a constant independent of u and A. The subscript s on $\mathfrak{C}_s$ indicates that, *a priori*, this constant depends on s.

The above inequality is first proven for nonnegative integer values of s following a strategy which dates back to Duchon [8]. Then, to derive (5.2) for noninteger values of s, we apply the K-method for real interpolation between Sobolev spaces. See [3] and [5] for details.

Likewise, we have shown in [5] that the constant $\mathfrak{C}_s$ can be written as

$$\mathfrak{C}_s = \mathfrak{C}^* \lambda_{\sigma,q}^{-1} \tag{5.3}$$

where $\sigma = s - \lfloor s \rfloor$, $\lambda_{\sigma,q}$ is given by (3.9) with q instead of p, and $\mathfrak{C}^*$ is a constant that only depends on n, r, p, q and also on Ω if this set is bounded. Since $\lambda_{\sigma,q} = (\sigma(1-\sigma))^{1/q}$ for $\sigma \in (0,1)$, it is clear that $\lambda_{\sigma,q}^{-1} \to \infty$ as $\sigma \to 0^+$ or $\sigma \to 1^-$. This fact may reduce the usefulness of the sampling inequality for values of s near integers or cause difficulties from a theoretical or a numerical standpoint. Thus, one may wonder whether the presence of $\lambda_{\sigma,q}^{-1}$, through $\mathfrak{C}_s$, on the right-hand side of (5.2) is an intrinsic feature of the sampling inequality or an undesirable by-product of the interpolation technique used to derive it. In other words, we must pay attention to the problem stated below, which we shall partially solve with the help of the limiting results in Section 3:

Problem 5.1. *Is it possible to obtain the sampling inequality* (5.2) *with a constant $\mathfrak{C}_s$ not growing to ∞ as $\sigma \to 0^+$ or $\sigma \to 1^-$, or growing, respectively, at rates lower than those of $\sigma^{-1/q}$ or $(1-\sigma)^{-1/q}$? In this sense, does the relation* (5.3) *really provide a sharp expression of $\mathfrak{C}_s$?*

Let us first consider the case $\Omega = \mathbb{R}^n$. We rewite (5.2) as

$$|u|_{l+\sigma,q,\mathbb{R}^n} \le \mathfrak{C}_{l+\sigma}\, d^{-(l+\sigma)} [\![u]\!]_A, \tag{5.4}$$

where $l = \lfloor s \rfloor$, $\sigma = s - l$ and

$$[\![u]\!]_A = d^{r-n(1/p-1/q)} |u|_{r,p,\mathbb{R}^n} + d^{n/q} \|u|_A\|_p. \tag{5.5}$$

Let us fix a set A with small fill distance d and a function $u \in W^{r,p}(\mathbb{R}^n)$ such that $[\![u]\!]_A < \infty$ (since, in this case, the set A is countably infinite, the second term of $[\![u]\!]_A$ contains a series whose convergence has to be ensured). By Theorem 3.3, the product $(1-\sigma)^{1/q}|u|_{l+\sigma,q,\mathbb{R}^n}$ has a finite limit as $\sigma \to 1^-$. Hence, unless u is a polynomial of degree $\leq l$,

$$\lim_{\sigma\to 1^-} |u|_{l+\sigma,q,\mathbb{R}^n} = \infty,$$

from which, by taking limits in (5.4) as $\sigma \to 1^-$, we derive that

$$\lim_{\sigma\to 1^-} \mathfrak{C}_{l+\sigma} = \infty.$$

In fact, since $|u|_{l+\sigma,q,\mathbb{R}^n}$ grows to ∞ as $\sigma \to 1^-$ at the same rate as $(1-\sigma)^{-1/q}$, this should also be the case of $\mathfrak{C}_{l+\sigma}$. The same kind of reasoning applies when $\sigma \to 0^+$. By Theorem 3.4, except for polynomials of degree $\leq l-1$, the seminorm $|u|_{l+\sigma,q,\mathbb{R}^n}$ blows up as $\sigma \to 0^+$, forcing $\mathfrak{C}_{l+\sigma}$ to tend to ∞ at the same rate as $\sigma^{-1/q}$. This solves Problem 5.1: the presence of both factors $(1-\sigma)^{-1/q}$ and $\sigma^{-1/q}$ on the right-hand side of (5.2) is absolutely required by the intrinsic nature of the seminorm $|u|_{s,q,\mathbb{R}^n}$; thus, the expression for the constant $\mathfrak{C}_s$ given by (5.3) is sharp.

The preceding discussion suggests that, to avoid $(1-\sigma)^{-1/q}$ and $\sigma^{-1/q}$, it is worth rewriting the sampling inequality as

$$[u]_{s,q,\mathbb{R}^n} \leq \mathfrak{C}^\star (d^{r-s-n(1/p-1/q)}|u|_{r,p,\mathbb{R}^n} + d^{n/q-s}\|u|_A\|_p), \tag{5.6}$$

where $[\,\cdot\,]_{s,q,\mathbb{R}^n} = \lambda_{\sigma,q}|\cdot|_{s,q,\mathbb{R}^n}$ is the first of the normalized seminorms considered in Remark 3.8. Of course, for $p=2$, we could use instead the seminorm $|\cdot|_{0,s}$ given by (4.14) (with r replaced by s), as justified by Remark 4.8.

Let us now assume that Ω is a bounded domain with a Lipschitz continuous boundary. Again, we express (5.2) as

$$|u|_{l+\sigma,q,\Omega} \leq \mathfrak{C}_{l+\sigma}\, d^{-(l+\sigma)}[\![u]\!]_A, \tag{5.7}$$

where $[\![\,\cdot\,]\!]_A$ is given by (5.5) with Ω instead $\mathbb{R}^n$. By Theorem 3.1, which is formally identical to Theorem 3.3, we arrive here at the same conclusions, when $\sigma \to 1^-$, as in the case $\Omega = \mathbb{R}^n$: the constant $\mathfrak{C}_{l+\sigma}$ must grow to ∞ at the rate of $(1-\sigma)^{-1/q}$ as $\sigma \to 1^-$. However, things are quite different when $\sigma \to 0^+$. Theorem 3.5 implies that

$$\lim_{\sigma\to 0^+} |u|_{l+\sigma,q,\Omega} = |u|_{l,\mathrm{Dini}(q),\Omega}.$$

Hence, by taking limits in (5.7), nothing can be deduced about the asymptotic behaviour of $\mathfrak{C}_{l+\sigma}$ as $\sigma \to 0^+$: it may well tend to ∞, but it also may remain bounded. To our knowledge, there is no objective reason for the presence of the factor $\sigma^{-1/q}$ on the right-hand side of the sampling inequality. We state this as follows:

Conjecture. *When Ω is a domain with a Lipschitz continuous boundary, the sampling inequality* (5.2) *holds with* $\mathfrak{C}_s = \mathfrak{C}^*(1-\sigma)^{-1/q}$, *where* $\mathfrak{C}^*$ *is a constant independent of* u, A *and* s, *and* $\sigma = s - \lfloor s \rfloor$.

In the present case, we could also rewrite (5.2) by using the seminorm $[\,\cdot\,]_{s,q,\Omega} = \lambda_{\sigma,q}|\cdot|_{s,q,\Omega}$ on the left-hand side. However, this seminorm is not satisfactory, since, by Theorem 3.5, as already noted in Remark 3.8,

$$\lim_{\sigma\to 0^+}[u]_{l+\sigma,q,\Omega} = 0.$$

Of course, if the above conjecture were true, a suitable normalized seminorm would be $\lceil\,\cdot\,\rceil_{s,q,\Omega} = (1-\sigma)^{1/q}|\cdot|_{s,q,\Omega}$ (cf. the second part of Remark 3.8). In such a situation, the sampling inequality would become

$$\lceil u\rceil_{s,q,\Omega} \le \mathfrak{C}^\star(d^{r-s-n(1/p-1/q)}|u|_{r,p,\Omega} + d^{n/q-s}\|u|_A\|_p).$$

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