STUDIA MATHEMATICA 214 (2) (2013)

Weak compactness of solutions for fourth order elliptic systems with critical growth

by

Paweł Goldstein, Paweł Strzelecki and Anna Zatorska-Goldstein (Warszawa)

Abstract. We consider a class of fourth order elliptic systems which include the Euler–Lagrange equations of biharmonic mappings in dimension 4 and we prove that a weak limit of weak solutions to such systems is again a weak solution to a limit system.

1. Introduction. In this paper we consider fourth order elliptic systems of equations of the form

$$\Delta^2 u = \Delta(D\cdot\nabla u) + \operatorname{div}(E\cdot\nabla u) + G\cdot\nabla u + \Delta\Omega\cdot\nabla u \tag{1.1}$$

for an unknown map $u\colon \mathbb{B}^4\subset\mathbb{R}^4\to\mathbb{R}^m$, i.e., in components,

$$\Delta^2 u^i = \Delta(D^{ij}_\alpha\partial_\alpha u^j) + \partial_\alpha(E^{ij}_{\alpha\beta}\partial_\beta u^j) + G^{ij}_\alpha\partial_\alpha u^j + \Delta(\Omega^{ij}_\alpha)\partial_\alpha u^j, \tag{1.2}$$

where $\alpha,\beta = 1,2,3,4$ and $i,j=1,\dots,m$. The coefficient functions D,E,G,Ω are assumed to satisfy

$$D\in W^{1,2},\quad E\in L^2,\quad G\in L^{4/3},\quad \Omega\in W^{1,2}(\mathbb{B}^4;\mathrm{so}(m)\otimes\Lambda^1\mathbb{R}^4). \tag{1.3}$$

(See Section 2 for the necessary prerequisites on the term Ω, whose structure is crucial here; for the sake of this Introduction, the reader is invited to think of each Ω_α, $1\le\alpha\le4$, as a matrix-valued function whose values (Ω^{ij}_α) are in $\mathrm{so}(m)$, i.e., are antisymmetric $m\times m$ matrices.) We study compactness of the space of solutions in the weak sequential topology of the Sobolev space $W^{2,2}$.

Let us note immediately that under the above assumptions $G\cdot\nabla u$ is just in L^1, as $\nabla u\in W^{1,2}\subset L^4$ by the Sobolev inequality in dimension 4. Thus, (1.1) is critical; by this we mean that *even* if one knew, due to some additional information, that the right hand side is in the Hardy space $\mathcal{H}^1$, and not just in L^1, then the application of the Calderón–Zygmund theory and the Sobolev imbedding theorem to the solution u of $\Delta^2 u = f\in\mathcal{H}^1$ would

2010 *Mathematics Subject Classification*: Primary 35J48; Secondary 35J60, 58E20.

Key words and phrases: fourth order nonlinear elliptic systems, weak convergence, concentration–compactness.

DOI: 10.4064/sm214-2-3

lead us back to square one, i.e. we would not learn *anything* new about ∇u and D^2u besides the starting information, $\nabla u \in L^4$ and $D^2u \in L^2$.

However, it is important that D, E, G, Ω are allowed to depend nonlinearly on u. The class of systems we consider contains, in particular, the Euler–Lagrange equations of biharmonic maps from domains in $\mathbb{R}^4$ into compact Riemannian manifolds. Our approach relies in a crucial way on the antisymmetry of the 1-form Ω and on the use of nonlinear counterparts of the Hodge decomposition, originating in gauge theory. This key idea is due to T. Rivière; it has been first used in his pioneering paper [7] on conformally invariant second order elliptic systems in the plane, with harmonic maps from planar domains into compact Riemannian manifolds serving as the crucial example. Later on, Rivière–Struwe [8], Lamm–Rivière [3], and Struwe [10] extended this approach to stationary harmonic maps in higher dimensions and to biharmonic maps.

Let us also note that the class of solutions of (1.1) in full generality is wider than the class of biharmonic maps. It can happen in dimension 4 that a solution of (1.1) is continuous, even $C^{1,\lambda}$, but still not C^2. We shall comment on that later on; let us now state the main result.

THEOREM 1.1. *Suppose (u_k) is a sequence in $W^{2,2}(\mathbb{B}^4, \mathbb{R}^m)$ of weak solutions to the system*

$$\Delta^2 u_k = \Delta(D_k \cdot \nabla u_k) + \operatorname{div}(E_k \cdot \nabla u_k) + G_k \cdot \nabla u_k + \Delta\Omega_k \cdot \nabla u_k \quad \text{in } \mathbb{B}^4. \tag{1.4}$$

Suppose $u_k \rightharpoonup u$ in $W^{2,2}(\mathbb{B}^4, \mathbb{R}^m)$. If the coefficients D_k, E_k, G_k, Ω_k are weakly convergent in their respective Sobolev spaces, i.e.

$$D_k \xrightharpoonup{W^{1,2}} D, \quad \Omega_k \xrightharpoonup{W^{1,2}} \Omega, \quad E_k \xrightharpoonup{L^2} E, \quad G_k \xrightharpoonup{L^{4/3}} G, \tag{1.5}$$

then u is a weak solution to the limit system

$$\Delta^2 u = \Delta(D \cdot \nabla u) + \operatorname{div}(E \cdot \nabla u) + G \cdot \nabla u + \Delta\Omega \cdot \nabla u \quad \text{in } \mathbb{B}^4. \tag{1.6}$$

Related compactness results for biharmonic maps, along with an energy identity accounting for the possible 'energy loss' under the limit passage, have been obtained by Hornung and Moser [2] and Laurain and Rivière [4] ([1]). (For second order elliptic systems $\Delta u_k = \Omega_k \cdot \nabla u_k$, Sharp and Topping [9] provide a compactness theorem under an assumption that rules out the concentration of the energy $\int |\nabla u_k|^2$ but allows for concentration of $\int |\nabla^2 u_k|$.)

One of our main points is that the proof in this paper is different from [2] and [4]: in contrast to these two papers, we rely neither on any continuity of solutions (or their first and second order derivatives) nor on any other improved regularity properties (like higher integrability of $\nabla^2 u$), working all

([1]) After this article was submitted for publication, the authors have learned about other results in the same spirit, obtained by Wang and Zheng [15, 14].

the time just in $W^{2,2}$ and using the Sobolev imbedding $W^{2,2} \subset W^{1,4}$ (2). Besides K. Uhlenbeck's Theorem 2.1, the main tool is the concentration–compactness method of P.-L. Lions. The combination of the two allows us, very roughly speaking, to reduce the complexity of (1.4)–(1.6) to the case

$$\Delta^2 u_k = G_k \cdot \nabla u_k, \qquad u_k \rightharpoonup u \quad \text{in } W^{2,2},$$

where G_k is a bounded sequence in $L^{4/3}$ with $G_k \rightharpoonup G$. Then, a passage to the limit (in dimension 4) can be justified by an application of the Sobolev inequality, the concentration–compactness lemma [5, Lemma 1.2], and a standard capacity type argument.

REMARK 1.2. Conditions (1.3) imposed on the coefficients allow for nonlinear dependence on u. Roughly speaking, the class (1.1) contains systems of the form $\Delta^2 u = R(u, \nabla u, \nabla^2 u)$ where the right hand side R depends polynomially on (the entries of) ∇u and $\nabla^2 u$, with coefficients that are smooth and bounded in u, so that

$$|R(u, \nabla u, \nabla^2 u)| \lesssim |\nabla^2 u|^2 + |\nabla u|^4 + 1.$$

The point is that the terms of R depending quadratically on $\nabla^2 u$ *need to have some structure*, whereas all the other terms are allowed to be arbitrary. A model case looks as follows (summation over repeated indices is understood):

$$D = D(u, \nabla u) \in W^{1,2}, \quad D^{ij}_\alpha = d^{ij}_l(u)\partial_\alpha u^l; \tag{1.7}$$

$$E = E(u, \nabla u, \nabla^2 u) \in L^2, \tag{1.8}$$

$$E^{ij}_{\alpha\beta} = e^{ij,(1)}_l(u)\partial_\alpha(\partial_\beta u^l) + e^{ij,(2)}_{ls}(u)\partial_\alpha u^l \partial_\beta u^s; \tag{1.9}$$

$$G = G(u, \nabla u, \nabla^2 u) \in L^{4/3}, \tag{1.10}$$

$$G^{ij}_\alpha = g^{ij}_{lsp,\beta\gamma}(u)\big(\partial_\beta(\partial_\gamma u^l) + \partial_\beta u^l \partial_\gamma u^s\big)\partial_\alpha u^p; \tag{1.11}$$

finally, the antisymmetric term $\Omega = \Omega(u, \nabla u) \in W^{1,2}(\mathbb{B}^4; \mathrm{so}(m) \otimes \Lambda^1\mathbb{R}^4)$ is given by

$$\Omega^{ij}_\alpha = w^i \partial_\alpha w^j - w^j \partial_\alpha w^i = -\Omega^{ji}_\alpha, \tag{1.12}$$

where $w = \nu \circ u$ for some bounded smooth map $\nu\colon \mathbb{R}^m \to \mathbb{R}^m$.

All the coefficients d, e, g (with various indices) in (1.7)–(1.11) are assumed to be of class $C^1 \cap W^{1,\infty}$ on $\mathbb{R}^m$. Struwe [10, Section 2] explains that the biharmonic equation can be written in that form, with ν being the normal to the target manifold. In that case, d, e, g depend explicitly on ν and

(2) In Section 4 we provide an example showing that (1.1) admits weak solutions that are C^1 but not C^2.

its derivatives, and the growth estimates

$$\begin{aligned} |D| + |\Omega| &\lesssim |\nabla u|, \\ |E| + |\nabla D| + |\nabla \Omega| &\lesssim |\nabla^2 u| + |\nabla u|^2, \\ |G| &\lesssim |\nabla^2 u|\,|\nabla u| + |\nabla u|^3 \end{aligned} \tag{1.13}$$

follow from (1.7)–(1.12). Under these assumptions, for every weakly convergent sequence $u_k \rightharpoonup u$ in $W^{2,2}$ we have convergence (1.5), with

$$L^2 \ni E_k = E(u_k, \nabla u_k, \nabla^2 u_k) \rightharpoonup E = E(u, \nabla u, \nabla^2 u)$$

in L^p for $p < 2$, and $L^{4/3} \ni G_k = G(u_k, \nabla u_k, \nabla^2 u_k) \rightharpoonup G = G(u, \nabla u, \nabla^2 u)$ in L^s for $s < 4/3$. Since E_k is bounded in L^2 and G_k is bounded in $L^{4/3}$, and we deal *with a bounded domain*, it is an exercise to see that in fact $E_k \rightharpoonup E$ and $G_k \rightharpoonup G$ also for the limiting exponents $p = 2$ and $s = 4/3$.

We do not need the full strength of (1.7)–(1.13).

REMARK 1.3. Our proof depends in a crucial way on the Sobolev inequality in dimension 4. It would be interesting to know what happens in higher dimensions. For example: is the convergence $u_k \to u$ in BMO, combined with the boundedness of (u_k) in $W^{2,2}$, sufficient to guarantee that (a) u solves the limiting system, or (b) $u_k \to u$ strongly in $W^{2,2}$? It is possible to check, using the sharp version of the Gagliardo–Nirenberg inequality $\|\nabla w\|^2_{L^4} \lesssim \|w\|_{\mathrm{BMO}} \|w\|_{W^{2,2}}$ (cf. Meyer–Rivière [6], or [11]), that both answers would be positive for uniformly bounded weak solutions of the simplified system $\Delta^2 u_k = G_k \nabla u_k$, with $G_k = G(u_k, \nabla u_k, \nabla^2 u_k)$, where G is of the form (1.11). It seems plausible that convergence of, say, biharmonic maps in BMO prevents bubbling and loss of energy in the limit.

The rest of the paper is organized as follows. In Section 2, we recall Uhlenbeck's result in the form that is used later on. Section 3 forms the bulk of the paper. There, we first explain the strategy of the proof in more detail, and then carry out the necessary estimates, pass to the limit and remove the singularities of the limiting system. Finally, in Section 4 we give an example showing that (1.1) may have solutions in $C^1 \setminus C^2$.

2. Uhlenbeck's result. We recall that we consider a mapping u of a ball $\mathbb{B}^4 \subset \mathbb{R}^4$ into $\mathbb{R}^m$. Below, we state the theorem of Uhlenbeck in a form adjusted to the situation. Note that if $\Omega \in W^{1,2}(\mathbb{B}^4)$ then $\Omega \in L^4(\mathbb{B}^4)$, because of the Sobolev imbedding theorem.

Everywhere below, $\mathrm{so}(m) \otimes \Lambda^k \mathbb{R}^4$ stands for the space of antisymmetric k-forms on $\mathbb{R}^4$ with matrix coefficients in $\mathrm{so}(m)$.

THEOREM 2.1. *There exist a number $\varepsilon = \varepsilon(m) > 0$ and a constant $C > 0$ such that for any $\Omega \in W^{1,2}(\mathbb{B}^4; \mathrm{so}(m) \otimes \Lambda^1 \mathbb{R}^4)$ which satisfies*

$$\|\nabla \Omega\|_{L^2} + \|\Omega\|_{L^4}^2 \le \varepsilon$$

there exist $P \in W^{2,2}(\mathbb{B}^4, \mathrm{SO}(m))$ *and* $\xi \in W^{2,2}(\mathbb{B}^4, \mathrm{so}(m) \otimes \Lambda^2\mathbb{R}^4)$ *such that*

$$(dP)P^{-1} + P\Omega P^{-1} = *d\xi \quad \text{on } \mathbb{B}^4 \tag{2.1}$$

and $d(*\xi) = 0$ *on* $\mathbb{B}^4$, $\xi|_{\partial\mathbb{B}^4} = 0$. *Moreover*

$$\|\nabla^2 P\|_{L^2} + \|\nabla P\|_{L^4} + \|\nabla^2 \xi\|_{L^2} + \|\nabla\xi\|_{L^4} \lesssim \|\nabla\Omega\|_{L^2} + \|\Omega\|_{L^4}^2. \tag{2.2}$$

Uhlenbeck's Theorem is, in fact, a local theorem in the sense that we can use it not only on the unit ball $\mathbb{B}^4$, but on any ball, and, as long as we consider balls with uniformly bounded radii, we can choose the constant ε in a uniform way (i.e. independently of the radius of the ball). This is in accordance with the original use of this theorem to prove the existence of *global* Coulomb gauges on compact manifolds. Indeed, a look at the proof of Lemma 2.5 in [13] shows that we can choose $\varepsilon = (2(C_P r + 1)C_S)^{-1}$, where C_P and C_S are the constants in the Poincaré and Sobolev inequalities for the unit ball, and r denotes the radius of the ball B. Thus $\varepsilon = (2(C_P R + 1)C_S)^{-1}$ can be chosen as a uniform estimate for all balls with radius bounded by R.

Corollary 2.2. *Theorem* 2.1 *holds for any ball* $B \subset \mathbb{B}^4$ *in place of* $\mathbb{B}^4$, *and the constant* ε *can be chosen uniformly for all such balls.*

Another corollary deals with the problem of weak continuity of P and ξ with respect to Ω. Note that Theorem 2.1 does not claim that either of them is uniquely defined.

Corollary 2.3. *Suppose* $\{\Omega_k\}$ *is a sequence in* $W^{1,2}(\mathbb{B}^4; \mathrm{so}(m) \otimes \Lambda^1\mathbb{R}^4)$. *Assume*

$$\Omega_k \rightharpoonup \Omega \quad \text{in } W^{1,2}. \tag{2.3}$$

Assume that P_k *and* ξ_k *are chosen so that* (2.1) *and* (2.2) *of Theorem* 2.1 *hold with* Ω_k *in place of* Ω.

Then both P_k *and* ξ_k *are uniformly bounded in* $W^{2,2}$, *and for any subsequence of* (Ω_k) *for which* P_k *and* ξ_k *are weakly convergent in* $W^{2,2}$ *to* P *and* ξ, *respectively, conditions* (2.1) *and* (2.2) *of Uhlenbeck's Theorem hold for* Ω, P *and* ξ.

In other words, the decomposition for the limit matrix Ω can be effectuated with (any) weak limit of the transformations P_k and forms ξ_k.

Indeed, assume $\Omega_k \rightharpoonup \Omega$ in $W^{1,2}$ and that Ω_k satisfy, uniformly with respect to k, the assumptions of Theorem 2.1. We then obtain, by the theorem, transformations P_k and ξ_k such that

$$dP_k + P_k\Omega_k = *d\xi_k \, P_k \quad \text{on } \mathbb{B}^4 \tag{2.4}$$

and

(2.5)
$$\|\nabla^2 P_k\|_{L^2} + \|\nabla P_k\|_{L^4} + \|\nabla^2 \xi_k\|_{L^2} + \|\nabla \xi_k\|_{L^4} \lesssim \|\nabla \Omega_k\|_{L^2} + \|\Omega_k\|^2_{L^4} \le M,$$
where $M > 0$ is a constant which is independent of k.

Since, by (2.5), the sequences (ξ_k) and (P_k) are bounded in $W^{2,2}$, we can choose subsequences (for simplicity still indexed by k) such that
$$\xi_k \rightharpoonup \xi \quad \text{weakly in } W^{2,2},$$
$$P_k \rightharpoonup P \quad \text{weakly in } W^{2,2} \text{ and a.e.}$$
Thus (after again choosing a subsequence) we may assume that
$$d\xi_k \to d\xi \quad \text{strongly in } L^{4-\delta},$$
$$dP_k \to dP \quad \text{strongly in } L^{4-\delta},$$
$$\Omega_k \to \Omega \quad \text{strongly in } L^{4-\delta},$$
for any small $\delta > 0$. Since P_k are also uniformly bounded in L^∞, we can take the $L^{4-\delta}$-limit on both sides of (2.4) to obtain
$$dP + P\Omega = *d\xi \qquad \text{on } \mathbb{B}^4 \tag{2.6}$$
in the sense of distributions, which proves that P, ξ and Ω indeed satisfy (2.1). The remaining estimates and boundary conditions in Theorem 2.1 are obvious.

This sort of continuity of the decomposition of Ω, i.e. the fact that the transformation of the limit Ω may be attained by taking the weak limits of the elements of the decomposition of Ω_k, allows us later to estimate the $W^{2,2}$-norm of differences between the elements of decompositions of Ω_k and Ω.

3. Proof. Let us first give a rough sketch and plan of the proof. The key idea is to prove that u solves the limiting system (1.6) outside a countable set of points and then to remove these possible singularities with the use of a properly chosen test function. A standard argument (cf. Section 3.1) shows that $\mathbb{B}^4 \setminus A_1$, where A_1 is finite, can be covered by balls B_j such that $\|\Omega_k\|_{W^{1,2}(B_j)}$ is small, so that Uhlenbeck's decomposition can be applied inside each B_j separately. Next, in Section 3.2, we fix one of the B_j and, following the crucial ideas of Lamm–Rivière [3] and Struwe [10], we use the equation $(dP_k)P_k^{-1} + P_k\Omega_k P_k^{-1} = *d\xi_k$ on B_j to rewrite (1.4) as
$$\Delta(P_k \Delta u_k) + \text{a perturbation} = H_k \cdot \nabla u_k + *d\Delta\xi_k \cdot (P_k \nabla u_k) \quad \text{on } B_j, \tag{3.1}$$
where H_k depends on $D_k, E_k, G_k, P_k, \Omega_k$ and their derivatives.

It is a purely routine matter to pass to the weak limit on the left hand side of (3.1). On the right hand side, after passing to subsequences, H_k is bounded in $L^{4/3}$ and converges strongly in all L^s for $s < 4/3$, and ∇u_k is bounded in

L^4 by the Sobolev inequality, and converges strongly in L^q for all $q < 4$. Thus, $H_k \cdot \nabla u_k$ is bounded in L^1, but a priori does not have to converge in $\mathcal{D}'$. The second term, $*d\Delta\xi_k \cdot (P_k \nabla u_k)$, presents a similar difficulty. To cope with that, we apply in Section 3.3 P.-L. Lions' concentrated compactness method [5], following earlier work of Freire, Müller and Struwe [1] on wave maps and harmonic maps, and the second and third authors' work [12] on H-systems. The key idea is to exploit the existence of *second* order derivatives of u. This yields

$$H_k \cdot \nabla u_k + *d\Delta\xi_k \cdot (P_k \nabla u_k) \to H \cdot \nabla u + *d\Delta\xi \cdot (P\nabla u) + S_j \quad \text{on } B_j,$$

where $H \cdot \nabla u + *d\Delta\xi \cdot (P\nabla u)$ is the desired term of the limit system (1.6) rewritten in the (P, ξ) gauge, and S_j is a combination of Dirac delta measures, supported on a countable subset A of B_j.

To complete the proof, in Section 3.4 we show that each S_j must be zero, using a capacity argument, based on the fact that $W^{2,2}(\mathbb{R}^4)$ contains unbounded functions. Thus, the limit u of u_k satisfies (1.6) in $\mathbb{B}^4 \setminus A_1$; another application of the same argument shows that A_1 must be empty.

3.1. Preparation to Uhlenbeck's transformation. Since the sequence Ω_k is weakly convergent, it is bounded in $W^{1,2}$; we shall denote the bound on its norm by M.

In what follows, we want to cover the ball $\mathbb{B}^4$ by balls B_j in such a way that in each B_j we may, after passing to a subsequence, assume that $\|\Omega_k\|_{W^{1,2}(B_j)} \le \varepsilon$, where ε is as in Corollary 2.2 to Uhlenbeck's Theorem. This might not be possible for the whole $\mathbb{B}^4$, but it is so outside a finite set of points. To visualize this better, replace $\mathbb{B}^4$ by a four-dimensional cube and consider its dyadic decomposition into cubes $C_{i,j}$, where the second subscript j counts subsequent cubes of a specified generation i.

A cube C is *bad* if one cannot choose a subsequence of Ω_k with $W^{1,2}$-norms bounded on C by ε, i.e. for all k sufficiently large, $\|\Omega_k\|_{W^{1,2}(C)} > \varepsilon$. Notice that, in every generation of the dyadic decomposition, the number of bad cubes $C_{i,j}$ is bounded by the same constant $N = \lceil M\varepsilon^{-1} \rceil$, and if a cube is not bad, i.e. it is *good*, so are all its descendants. The intersection of all bad cubes,

$$A_1 := \bigcap_{i=1}^{\infty} \Big(\bigcup_{1 \le j \le N} \text{bad cubes } C_{i,j} \Big) \tag{3.2}$$

is a finite set of (at most N) points, and any point not in A_1 lies in a cube that is good.

The points in A_1 are, in fact, accumulation points of the $W^{1,2}$-norm of the weakly convergent sequence (Ω_k), more precisely, those accumulation points for which the energy loss exceeds ε.

For the next two subsections, let us fix an arbitrary good ball $B = B_j$ contained in $\mathbb{B}^4 \setminus A_1$. Instead of the whole sequence u_k, we only consider the subsequence (not relabeled) of u_k (and of D_k, E_k, G_k and Ω_k) for which $\|\Omega_k\|_{W^{1,2}(B_j)} \le \varepsilon$. By Corollary 2.2, we can assume that Uhlenbeck's theorem holds for B, so there exist P_k and ξ_k such that $dP_k\, P_k^{-1} + P_k \Omega_k P_k^{-1} = *d\xi_k$.

3.2. Transformation of the equation: calculations. The calculations below follow closely and with more detail the brief calculations by Struwe in [10]. We provide them for the reader's convenience, and also because we shall need some more knowledge on the structure of certain terms in our reasoning.

We recall the indices of the multidimensional objects that appear in our system:

$$u = (u^i), \quad \nabla u = (\partial_\alpha u^i), \quad P = (P^{ij}), \quad D = (D^{ij}_\alpha),$$
$$E = (E^{ij}_{\alpha\beta}), \quad G = G^{ij}_\alpha, \quad \Omega = (\Omega^{ij}_\alpha),$$

with $i, j = 1, \dots, n$ and $\alpha, \beta = 1, 2, 3, 4$. To simplify the notation without making the calculations ambiguous we shall use the standard summation convention.

Furthermore, it is often convenient to omit at least some of the indices. In that case,

- multiplication of tensor objects that is denoted by a dot $(\cdot)$ is a standard scalar product in $\mathbb{R}^4 \times \mathbb{R}^m$, e.g.

$$E \cdot \nabla u = (E^{ij}_{\alpha\beta} \partial_\beta u^j)^i_\alpha = \Big(\sum_{\beta=1}^{4} \sum_{j=1}^{n} E^{ij}_{\alpha\beta} \partial_\beta u^j \Big)^i_\alpha,$$
$$G \cdot \nabla u = (G^{ij}_\alpha \partial_\alpha u^j)^i = \Big(\sum_{\alpha=1}^{4} \sum_{j=1}^{n} G^{ij}_\alpha \partial_\alpha u^j \Big)^i;$$

- multiplication of tensor objects that is not denoted by any operator sign is standard matrix multiplication, e.g.

$$P\Delta^2 u = (P^{ij} \Delta^2 u^j)_i = \Big(\sum_{j=1}^{n} P^{ij} \Delta^2 u^j \Big)_i;$$

- tensor multiplication $(\otimes)$ denotes tensor product in $\mathbb{R}^4$ (and then, possibly, matrix multiplication in coordinates), e.g.

$$\nabla P \otimes \nabla u = \nabla P^{ij} \otimes \nabla u^j = ((\partial_\alpha P^{ij})_\alpha \otimes (\partial_\beta u^j)_\beta)^i$$
$$= \Big(\sum_{j=1}^{n} \partial_\alpha P^{ij} \partial_\beta u^j \Big)^i_{\alpha\beta}.$$

Below, we transform the system (1.4) for u_k. In the calculations that follow we omit the index k (one should not confuse this temporary notational simplification with the claim that u, a weak limit of (u_k), satisfies (1.6); proving this is the goal of our paper).

Applying P to the Laplacian on the left hand side of the system, we obtain

$$\begin{aligned}\Delta(P\Delta u)^i &= \Delta(P^{ij}\Delta u^j) = \partial_\alpha\partial_\alpha(P^{ij}\partial_\beta\partial_\beta u^j)\\ &= P^{ij}\partial_\alpha\partial_\alpha\partial_\beta\partial_\beta u^j + \partial_\beta\partial_\alpha\partial_\alpha P^{ij}\partial_\beta u^j - \partial_\beta(\partial_\alpha\partial_\alpha P^{ij}\partial_\beta u^j)\\ &\quad + 2\partial_\alpha\partial_\beta(\partial_\alpha P^{ij}\partial_\beta u^j) - 2\partial_\alpha(\partial_\beta\partial_\alpha P^{ij}\partial_\beta u^j),\end{aligned}$$

which can be rewritten briefly as

$$\begin{aligned}\Delta(P\Delta u)^i &= (P\Delta^2 u)^i + (\nabla(\Delta P)\cdot\nabla u)^i - \operatorname{div}(\Delta P\nabla u)^i\\ &\quad + 2\operatorname{div}^2(\nabla P\otimes\nabla u)^i - 2\operatorname{div}(\nabla^2 P\cdot\nabla u)^i.\end{aligned}\tag{3.3}$$

By the equation (1.4) satisfied by u_k, we have, still omitting the index k for the sake of simplicity,

$$(P\Delta^2 u)^i = P^{ij}\Delta(D\cdot\nabla u)^j + P^{ij}\operatorname{div}(E\cdot\nabla u)^j + P^{ij}(G\cdot\nabla u)^j + P^{ij}(\Delta\Omega\cdot\nabla u)^j,$$

which can be rewritten as

$$\begin{aligned}P\Delta^2 u &= \Delta(PD\cdot\nabla u) + \operatorname{div}((PE - 2\nabla PD)\cdot\nabla u)\\ &\quad + (\Delta PD + PG + P\Delta\Omega - \nabla P\cdot E)\cdot\nabla u.\end{aligned}\tag{3.4}$$

Substituting (3.4) into (3.3), after some rearranging, yields

$$\begin{aligned}\Delta(P\Delta u) &= \Delta(PD\cdot\nabla u) + 2\operatorname{div}^2(\nabla P\otimes\nabla u)\\ &\quad + \operatorname{div}\big((PE - 2\nabla PD - 2\nabla^2 P\cdot\nabla u)\cdot\nabla u - \Delta P\nabla u\big)\\ &\quad + \big(\Delta PD + PG + P\Delta\Omega + \nabla(\Delta P) - \nabla P\cdot E\big)\cdot\nabla u.\end{aligned}\tag{3.5}$$

Define

$$\begin{aligned}\tilde D^{il}_\alpha &= P^{ij}D^{jl}_\alpha,\\ \tilde E^{il}_{\alpha\beta} &= P^{ij}E^{jl}_{\alpha\beta} - 2\partial_\alpha P^{ij}D^{jl}_\beta - 2\partial_\alpha\partial_\beta P^{il} - \delta_{\alpha\beta}\Delta P^{il},\\ H &= \Delta PD + PG + P\Delta\Omega + \nabla(\Delta P) - \nabla P\cdot E - *d\Delta\xi P,\end{aligned}\tag{3.6}$$

where, as before, the Roman lowercase indices run from 1 to n, and the Greek indices from 1 to 4; $\delta_{\alpha\beta}$ denotes Kronecker's delta.

With this notation, we rewrite (3.5) as

$$\begin{aligned}\Delta(P\Delta u) &= \Delta(\tilde D\cdot\nabla u) + 2\operatorname{div}^2(\nabla P\otimes\nabla u)\\ &\quad + \operatorname{div}(\tilde E\cdot\nabla u) + H\cdot\nabla u + *d\Delta\xi\cdot P\nabla u.\end{aligned}\tag{3.7}$$

We shall need the precise form and integrability properties of the terms that appear in H. By Theorem 2.1,

$$*d\Delta\xi P = \Delta((\nabla P + P\Omega)P^{-1})P,$$

thus

$$H = \varDelta P D + P G + P\varDelta\varOmega + \nabla(\varDelta P) - \nabla P\cdot E - \varDelta((\nabla P + P\varOmega)P^{-1})P,$$

and after some simple reductions we get

$$\begin{aligned} H^{ij}_\beta = {}& \varDelta P^{il} D^{lj}_\beta + P^{il} G^{lj}_\beta - \partial_\alpha P^{il} E^{lj}_{\alpha\beta} - \varDelta P^{il}\varOmega^{lj}_\beta \\ & - 2\big(\partial_\alpha P^{il}\partial_\alpha \varOmega^{lj}_\beta + \partial_\alpha\partial_\beta P^{il}\partial_\alpha (P^{-1})^{ls}P^{sj} \\ & \qquad + \partial_\alpha P^{il}\varOmega^{lt}_\beta \partial_\alpha (P^{-1})^{ts}P^{sj} + P^{il}\partial_\alpha\varOmega^{lt}_\beta\partial_\beta (P^{-1})^{ts}P^{sj}\big) \\ & - \partial_\beta P^{il}\varDelta(P^{-1})^{ls}P^{sj} - P^{il}\varOmega^{lt}_\beta\varDelta(P^{-1})^{ts}P^{sj}, \end{aligned}$$

or, in simplified notation, without the jungle of indices,

$$\begin{aligned} H = {}& \varDelta P D + P G - \nabla P\cdot E - \varDelta P\varOmega \\ & - 2\big(\nabla P\cdot\nabla\varOmega + (\nabla^2 P\cdot\nabla(P^{-1}))P \\ & \qquad + (\nabla P\varOmega)\cdot\nabla(P^{-1})P + P(\nabla\varOmega\cdot\nabla(P^{-1}))P\big) \\ & - \nabla P\varDelta(P^{-1})P - P\varOmega\varDelta(P^{-1})P. \end{aligned} \tag{3.8}$$

Bear in mind that in fact we shall use equations (3.6)–(3.8) for each k, adding the subscript k to all the letters $H, D, E, G, P, \varOmega, \xi, \tilde{D}, \tilde{E}$.

From now on we return to using the index k where appropriate.

The bounds for D_k, E_k, G_k and $\varOmega_k$ together with the estimates for P_k and ξ_k given by Theorem 2.1 imply that $H_k \in L^{4/3}$ and

$$\|H_k\|_{L^{4/3}} \le C = C\Big(\sup_k \max(\|D_k\|_{W^{1,2}}, \|E_k\|_{L^2}, \|G_k\|_{L^{4/3}}, \|\varOmega_k\|_{W^{1,2}})\Big). \tag{3.9}$$

To check this, one just uses Hölder's inequality, and—when appropriate—the Sobolev imbedding $W^{1,2}\subset L^4$ in dimension 4; we leave the details to the reader.

Remark. If the coefficients $D_k = D(u_k,\nabla u_k)$, $E_k = E(u_k,\nabla u_k,\nabla^2 u_k)$, $G_k = G(u_k,\nabla u_k)$ and $\varOmega_k = \varOmega(u_k,\nabla u_k)$ are given by the composition of fixed smooth functions with the u_k and their derivatives, and satisfy the growth conditions (1.13), then a computation yields

$$\|H_k\|_{L^{4/3}} \lesssim \|u_k\|^{3/2}_{W^{2,2}} + \|\nabla u_k\|^3_{L^4}.$$

We do not rely on that particular estimate, though.

Let H be defined analogously to H_k, i.e. by formula (3.8) (3). The convergence of all the terms on the right hand side of the formula for H_k is such that, up to a subsequence, we can assume

$$H_k \rightharpoonup H \quad \text{in } L^q \text{ for all } 1\le q < 4/3. \tag{3.10}$$

(3) This time the subscript k in (3.8) is *really* omitted, not just for the sake of simplicity!

To see this, we just use the elementary observation: *if* $1/p + 1/q = 1/r$, $p, q, r > 1$, *and* $f_k \to f$ *in* L^p *and* $g_k \rightharpoonup g$ *in* L^q, *then* $f_k g_k \rightharpoonup fg$ *in* L^r, combining it with the imbedding $W^{2,2} \subset W^{1,4}$ and Rellich–Kondrashov's compactness theorem.

Moreover, by the estimates on H_k, we can once again choose a subsequence of H_k that is weakly convergent in $L^{4/3}$.

We write out the weak formulation of (1.4) in B_j, using its transformed form (3.7), and separating the terms into 'easy' (left hand side) and 'hard' (right hand side): the identity

$$\begin{aligned} \int P_k \Delta u_k \Delta\psi - \int (\tilde{D}_k \cdot \nabla u_k)\Delta\psi \\ + \int [2\operatorname{div}(\nabla P_k \otimes \nabla u_k) + (\tilde{E}_k \cdot \nabla u_k)] \cdot \nabla\psi \\ = \int (H_k \cdot \nabla u_k)\psi + \int {*d\Delta\xi_k} \cdot (P_k \nabla u_k)\psi \end{aligned} \tag{3.11}$$

holds for each smooth map ψ compactly supported in $B = B_j$. (Since ξ_k is only of class $W^{2,2}$, the last term has to be interpreted using one integration by parts.)

3.3. Convergence of (3.11). We consider the left hand side and the right hand side separately. (The key difficulty is to prove that the right hand sides converge to the appropriate limit.)

3.3.1. *Convergence of the left hand side of* (3.11). By assumption, $u_k \rightharpoonup u$ in $W^{2,2}$, thus, after passing to a subsequence, we may assume that

$$\begin{aligned} u_k &\to u && \text{in } L^s \text{ for all } 1 \le s < \infty, \\ \nabla u_k &\rightharpoonup \nabla u && \text{in } L^4, \\ \nabla u_k &\to \nabla u && \text{in } L^p \text{ for all } 1 \le p < 4, \\ \nabla^2 u_k &\rightharpoonup \nabla^2 u && \text{in } L^2. \end{aligned}$$

Taking into account the bounds for D_k, E_k, G_k and Ω_k, we can also assume that (again, after passing to a subsequence) there exist D, E, G and Ω such that

$$\begin{aligned} &D_k \rightharpoonup D \text{ and } \Omega_k \rightharpoonup \Omega && \text{in } W^{1,2}, \\ &E_k \rightharpoonup E && \text{in } L^p \text{ for all } 1 \le p < 2, \\ &G_k \rightharpoonup G && \text{in } L^s \text{ for all } 1 \le s < 4/3. \end{aligned}$$

Similarly, Theorem 2.1 gives uniform estimates on P_k and ξ_k, which allow us to assume (after passing to subsequences) that

$$(P_k, \xi_k) \rightharpoonup (P, \xi) \quad \text{in } W^{2,2},$$

and $\nabla P_k \to \nabla P$ and $\nabla \xi_k \to \nabla \xi$ strongly in L^s for all $1 \le s < 4$, and $P_k \to P$ in L^s for all $1 \le s < \infty$, by Rellich–Kondrashov's compactness theorem. The limits P and ξ of P_k and ξ_k satisfy the claim of Corollary 2.3 on $B = B_j$. In particular,

$$dP\,P^{-1} + P\Omega P^{-1} = *d\xi.$$

By Hölder's inequality, it follows from all these convergence assumptions that $\nabla P_k \otimes \nabla u_k \to \nabla P \otimes \nabla u$ in $\mathcal{D}'$,

$$\tilde{D}_k \cdot \nabla u_k = P_k D_k \cdot \nabla u_k \to PD \cdot \nabla u = \tilde{D} \cdot \nabla u \quad \text{in } \mathcal{D}',$$

and finally,

$$\tilde{E}_k \cdot \nabla u_k \to \tilde{E} \cdot \nabla u \quad \text{in } \mathcal{D}'$$

(to check this, see (3.6) for the relation between $\tilde{E}$ and D, E, P).

Thus, up to passing to a subsequence, for each fixed test map ψ the left hand side of (3.11) converges to

$$\int P\Delta u\Delta\psi - \int (\tilde{D} \cdot \nabla u)\Delta\psi + \int [2\operatorname{div}(\nabla P \otimes \nabla u) + \tilde{E} \cdot \nabla u] \cdot \nabla\psi.$$

3.3.2. *Convergence of the right hand side of* (3.11). Now let us concentrate on the convergence of the right hand side of (3.11), i.e. of

$$\int (H_k \cdot \nabla u_k)\psi + \int *d\Delta\xi_k \cdot (P_k \nabla u_k)\psi. \tag{3.12}$$

This is the heart of the matter. A priori, by Hölder's inequality, $H_k \cdot \nabla u_k$ is bounded just in L^1, and there is no simple means of passing to the limit, as *neither* $H_k \to H$ in $L^{4/3}$, *nor* $\nabla u_k \to \nabla u$ in L^4; the convergence in both cases is weak. The second term presents a similar problem. To circumvent this difficulty, we shall study the convergence of (3.12) using the concentration–compactness method of P.-L. Lions. To deal with the first term in (3.12), we exploit the fact that u_k has *second* order derivatives and then we use Sobolev's inequality. To cope with the second term, we proceed in a similar way, employing also the equations satisfied by P and ξ, and integration by parts.

We define the auxiliary distributions T_k by

$$\langle T_k, \psi\rangle = \int (H_k \cdot \nabla u_k)\psi + \int *d\Delta\xi_k \cdot (P_k \nabla u_k)\psi, \quad \psi \in C_0^\infty(B, \mathbb{R}^n). \tag{3.13}$$

Coordinatewise, we write $T_k = (T_k^1, \ldots, T_k^n)$, with $T_k^i \in \mathcal{D}'(\mathbb{R}^n)$ given by

$$\langle T_k^i, \phi\rangle = \int (H_k^i \cdot \nabla u_k)\phi + \int *d\Delta\xi_k^i \cdot (P_k \nabla u_k)\phi, \quad \phi \in C_0^\infty(B).$$

We define the distribution T by a formula analogous to (3.13), omitting the index k everywhere.

Let us recall again that, for $i = 1, \ldots, n$,

$$H^i = (H_\alpha^{ij}),$$
$$*d\xi^i = (*d\xi)_\alpha^{ij},$$

with $j = 1, \dots, n$ and $\alpha = 1, \dots, 4$. Hence

$$\phi H^i \cdot \nabla u = \phi H^{ij}_\alpha \partial_\alpha u^j,$$
$$\phi *d\Delta\xi^i \cdot P\nabla u = \phi(*d\Delta\xi)^{ij}_\alpha P^{jk}\partial_\alpha u^l.$$

LEMMA 3.1. *There exists a subsequence $T_{k'}$ of T_k such that*

$$T^i_{k'} \to T^i + \sum_{l\in J} a_{li}\delta_{x_{li}} \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \quad i = 1, \dots, n,$$

where J is (at most) countable, $a_{li} \in \mathbb{R}$, $x_{li} \in \mathbb{B}^4$ and $\sum_{l\in J} |a_{li}|^{4/5} < \infty$ for $i = 1, \dots, n$.

Proof. Our aim is to prove that there exist nonnegative, uniformly bounded Borel measures μ_k such that for any $\phi \in C^\infty_0(B)$ and $i = 1, \dots, n$ we have

$$|\langle T^i_k - T^i, \phi^5\rangle| \lesssim \Big(\int \phi^4\, d\mu_k\Big)^{5/4} + o(1) \quad \text{as } k \to \infty. \tag{3.14}$$

This will allow us to use the method of concentrated compactness.

We have

$$\begin{aligned}\langle T^i_k - T^i, \phi^5\rangle &= \int \phi^5(H^i_k \cdot \nabla u_k - H^i \cdot \nabla u) \\ &\quad + \int \phi^5(*d\Delta\xi^i_k \cdot P_k\nabla u_k - *d\Delta\xi^i \cdot P\nabla u) \\ &= A + B.\end{aligned} \tag{3.15}$$

Each of the integrals will be dealt with separately.

Estimate of A. We split this integral into two,

$$A = \int \phi^5 H^i_k \cdot (\nabla u_k - \nabla u) + \int \phi^5 (H^i_k - H^i)\cdot \nabla u = I + II.$$

For I we have

$$I = \int \phi^3 H^i_k \cdot \nabla(\phi^2(u_k - u)) - 2\int \phi^4 H^i_k \cdot (\nabla\phi \otimes (u_k - u)).$$

We have assumed that $u_k \to u$ in L^s for $1 \le s < \infty$ and $\nabla u_k \to \nabla u$ in L^p for $1 \le p < 4$; moreover, we know that the H_k are uniformly bounded in $L^{4/3}$. By Hölder's and Sobolev's inequalities,

$$\begin{aligned}|I| &\le \int |\phi|^3 |H_k|\, |\nabla(\phi^2(u_k - u))| + o(1) \\ &\le \Big(\int \phi^4 |H_k|^{4/3}\Big)^{3/4}\Big(\int |\nabla(\phi^2(u_k - u))|^4\Big)^{1/4} + o(1) \\ &\lesssim \Big(\int \phi^4 |H_k|^{4/3}\Big)^{3/4}\Big(\int |\nabla^2(\phi^2(u_k - u))|^2\Big)^{1/2} + o(1) \\ &\lesssim \Big(\int \phi^4 |H_k|^{4/3}\Big)^{3/4}\Big(\int \phi^4|\nabla^2(u_k - u)|^2\Big)^{1/2} + o(1) \\ &\lesssim \Big(\int \phi^4(|H_k|^{4/3} + |\nabla^2 u_k|^2 + |\nabla^2 u|^2)\Big)^{5/4} + o(1).\end{aligned} \tag{3.16}$$

With II we proceed in a similar way:

$$(3.17)\qquad \begin{aligned} II &= \int \phi^3 (H_k^i - H^i)\cdot \nabla(\phi^2 u) - 2\int \phi^4 (H_k^i - H^i)\cdot(\nabla\phi\otimes u)\\ &= II_a + II_b. \end{aligned}$$

We estimate II_a in the same way as we did with I:

$$|II_a| \lesssim \Big(\int \phi^4 (|H_k|^{4/3} + |H|^{4/3} + |\nabla^2 u|^2)\Big)^{5/4} + o(1).$$

The integral II_b converges (up to choosing a subsequence) to zero, since $u \in L^s$ for any $s \geq 1$ and $H_k \rightharpoonup H$ in L^q for $q < 4/3$.

Estimate of B. This integral is a sum of three terms,

$$(3.18)\qquad \begin{aligned} B &= \int \phi^5 \,(*d\Delta\xi_k^i \cdot P_k \nabla u_k - *d\Delta\xi^i \cdot P\nabla u)\\ &= \int \phi^5 *d\Delta\xi_k^i \cdot P_k\nabla(u_k - u) + \int \phi^5 *d\Delta\xi_k^i \cdot (P_k - P)\nabla u\\ &\quad + \int \phi^5 (*d\Delta\xi_k^i - *d\Delta\xi^i)\cdot P\nabla u\\ &= III + IV + V. \end{aligned}$$

Since, thanks to the boundary conditions on $\Delta\xi_k^i$,

$$\int *d\Delta\xi_k^i \cdot \nabla(\phi^5 P_k(u_k - u)) = 0,$$

we obtain

$$(3.19)\qquad \begin{aligned} III = &-5\int \phi^4 *d\Delta\xi_k^i \cdot (\nabla\phi \otimes P_k(u_k - u))\\ &- \int \phi^5 *d\Delta\xi_k^i \cdot (\nabla P_k)(u_k - u)\\ = &\; III_a + III_b. \end{aligned}$$

Integrating by parts, taking into account the convergence of u_k to u in any L^s and the fact that $P_k \in L^\infty$, we obtain

$$(3.20)\qquad \begin{aligned} |III_a| &= \Big|\int \Delta\xi_k^i \wedge d(\phi^4 P_k(u_k - u)) \wedge d\phi\Big|\\ &\lesssim \int \phi^4 |\nabla\phi|\,|\nabla^2\xi_k|\,|\nabla P_k|\,|u_k - u|\\ &\quad + \int \phi^4 |\nabla\phi|\,|\nabla^2\xi_k|\,|P_k|\,|\nabla u_k - \nabla u|\\ &= o(1). \end{aligned}$$

For III_b we get, integrating by parts,

$$(3.21)\qquad \begin{aligned} |III_b| &= \Big|\int \Delta\xi_k^i dP_k \wedge d(\phi^5(u_k - u))\Big|\\ &\leq \Big|\int \phi^5 \Delta\xi_k^i dP_k \wedge d(u_k - u)\Big| + \int \phi^4 |\nabla\phi|\,|\nabla^2\xi_k|\,|\nabla P_k|\,|u_k - u|\\ &\leq \Big|\int (\phi^2\Delta\xi_k^i)(\phi\, dP_k) \wedge (\phi^2\, d(u_k - u))\Big| + o(1)\\ &\leq \Big|\int (\phi^2\Delta\xi_k^i)(\phi\, dP_k) \wedge d(\phi^2(u_k - u))\Big| + o(1). \end{aligned}$$

As before, using Hölder's and Sobolev's inequalities, we estimate the last integral by the product

$$\Big(\int\phi^4|\nabla^2\xi_k|^2\Big)^{1/2}\Big(\int\phi^4|\nabla P_k|^4\Big)^{1/4}\Big(\int|\nabla^2(\phi^2(u_k-u))|^2\Big)^{1/2}.$$

This yields

$$III_b \lesssim \Big(\int\phi^4(|\nabla^2\xi_k|^2+|\nabla P_k|^4+|\nabla^2 u_k|^2+|\nabla^2 u|^2)\Big)^{5/4}+o(1). \tag{3.22}$$

The remaining integrals in B are estimated in much the same way. Integrating by parts, and then dealing as in the proof of (3.22), we obtain

$$\begin{aligned}
|IV| &= \Big|\int\phi^5 *d\Delta\xi_k^i\cdot(P_k-P)\nabla u\Big| \\
&= \Big|\int\Delta\xi_k^i d(\phi^5(P_k-P))\wedge du\Big| \\
&\le \Big|\int\phi^2\Delta\xi_k^i\, d(\phi(P_k-P))\wedge d(\phi^2 u)\Big| \\
&\quad + 4\int\phi^4|\nabla\phi|\,|\nabla^2\xi_k|\,|P_k-P|\,|\nabla u|+o(1) \\
&\lesssim \Big(\int\phi^4(|\nabla^2\xi_k|^2+|\nabla P_k|^4+|\nabla P|^4+|\nabla^2 u|^2)\Big)^{5/4}+o(1).
\end{aligned} \tag{3.23}$$

Similarly,

$$\begin{aligned}
|V| &= \Big|\int\phi^5 *d\Delta(\xi_k^i-\xi^i)\cdot P\nabla u\Big| \\
&= \Big|\int\Delta(\xi_k^i-\xi^i)d(\phi^5 P)\wedge du\Big| \\
&\le \Big|\int\phi^5\Delta(\xi_k^i-\xi^i)dP\wedge du\Big|+\Big|\int\phi^4\Delta(\xi_k^i-\xi^i)\wedge d\phi\wedge P\,du\Big| \\
&\lesssim \Big(\int\phi^4(|\nabla^2\xi_k|^2+|\nabla^2\xi|^2+|\nabla P|^4+|\nabla^2 u|^2)\Big)^{5/4}+o(1)
\end{aligned} \tag{3.24}$$

(in the calculations above we use Hölder's and Sobolev's inequalities together with the fact that $\nabla^2\xi_k \rightharpoonup \nabla^2\xi$ in L^2).

Altogether, we obtain the estimate

$$|B| \lesssim \Big(\int\phi^4 f_k\Big)^{5/4}+o(1),$$

where $f_k := |\nabla^2\xi_k|^2+|\nabla^2\xi|^2+|\nabla P_k|^4+|\nabla P|^4+|\nabla^2 u_k|^2+|\nabla^2 u|^2 \in L^1$. Putting together the estimates for A and B we get

$$|\langle T_k^i - T^i,\phi^5\rangle| \lesssim \Big(\int\phi^4\,d\mu_k\Big)^{5/4}+o(1), \tag{3.25}$$

with

$$d\mu_k = (|H_k|^{4/3}+|H|^{4/3}+f_k)\,dx. \tag{3.26}$$

Note that by our assumptions the densities of μ_k are uniformly bounded in L^1. Passing to the limit in the space of measures we obtain

$$T_k^i - T^i \to d\nu, \quad d\mu_k \to d\mu,$$

and

$$\left|\int \phi^5 \, d\nu\right| \lesssim \left(\int \phi^4 \, d\mu\right)^{5/4}.$$

Now the claim of Lemma 3.1 follows directly from the concentration–compactness lemma of P.-L. Lions (cf. [5, Lemma 1.2, p. 161]). ∎

Passing to the limit in (3.11), we obtain

$$\begin{aligned}(3.27)\qquad \int P\Delta u\Delta\psi - \int (\tilde{D}\cdot\nabla u)\Delta\psi + \int [2\operatorname{div}(\nabla P\otimes\nabla u) + (\tilde{E}\cdot\nabla u)]\cdot\nabla\psi \\ = \int (H\cdot\nabla u)\psi + \int *d\Delta\xi\cdot(P\nabla u)\psi + \langle S_j,\psi\rangle\end{aligned}$$

for each smooth test map ψ compactly supported in B_j, with H, $\tilde{D}$ and $\tilde{E}$ given by (3.6), and (P,ξ) related to $\Omega = \lim \Omega_k$ via Uhlenbeck's Theorem 2.1. The singular distribution $S_j \in \mathcal{D}'$ is a countable series of Dirac delta measures with vector-valued coefficients, $S_j = \sum_\ell a_\ell \delta_{x_\ell}$. It follows from Lemma 3.1 that the series $\sum |a_\ell|_1$ converges.

3.4. Removing the singularities. To show that the limit u of u_k satisfies the limiting system not just in $B_j \setminus A$, where $A = \{x_\ell : \ell \in J\}$ is countable, but in fact in the whole B_j, we rely on the fact that $W^{2,2}(\mathbb{R}^4)$ contains unbounded functions. More precisely, the following holds.

LEMMA 3.2. *There exists a sequence of functions* $\Phi_k \in C_0^\infty(\mathbb{R}^4)$ *such that* $0 \le \Phi_k \le 1$, $\Phi_k \equiv 1$ *on* $B(0, r_k)$ *for some* $r_k > 0$,

$$(3.28)\qquad \Phi_k \equiv 0 \quad \text{on } \mathbb{R}^n \setminus B(0,R_k), \quad R_k \to 0 \quad \text{as } k\to\infty,$$

and

$$(3.29)\qquad \|\Phi_k\|_{W^{2,2}} \to 0 \quad \text{as } k \to \infty.$$

For the reader's convenience, a proof of this lemma is sketched in the appendix.

Fix $\ell_0 \in J$ and assume for the sake of simplicity that $x_{\ell_0} = 0 \in B_j$. Testing each equation $i = 1,\dots,n$ of system (3.27) with $\psi_k = (\pm\Phi_k,\dots,\pm\Phi_k)$, the signs being equal to the signs of the coordinates of the coefficient a_{ℓ_0} in S_j, and keeping in mind that

$$P\Delta u, \tilde{D}\cdot\nabla u, \nabla P\otimes\nabla u, \Delta\xi \in L^2, \quad H\cdot\nabla u \in L^1, \quad d((P\nabla u)\psi_k) \in L^2,$$

we easily obtain

$$o(1) = o(1) + \langle S_j, \psi_k\rangle = o(1) + |a_{\ell_0}|_1 + \sum_{\ell\neq\ell_0} \langle a_\ell, \psi_k(x_l)\rangle, \quad k\to\infty.$$

By (3.28) and Lebesgue's dominated convergence theorem, the sum $\sum_{\ell\neq\ell_0}$ above tends to 0 as $k\to\infty$. Thus, upon passing to the limit, we obtain $a_\ell=0$ for $\ell=\ell_0$. It follows that $S_j=0$.

Thus, relying on the above and selecting, via the standard diagonal procedure, a subsequence $k'\to\infty$ such that $u'_k\to u$ (and all the coefficient functions converge in their appropriate spaces) in all good balls B_j simultaneously, we check that

$$\Delta^2 u=\Delta(D\cdot\nabla u)+\operatorname{div}(E\cdot\nabla u)+G\cdot\nabla u+\Delta\Omega\cdot\nabla u\quad\text{in } \mathbb{B}^4\setminus A_1, \tag{3.30}$$

where $A_1=\{x_1,\ldots,x_s\}$, $s\le N$, denotes the finite set of bad points (cf. (3.2)). To see that in fact (3.30) holds in the whole ball $\mathbb{B}^4$, we pick an arbitrary test function $\varphi\in C_0^\infty(\mathbb{B}^4)$ and write it as the sum

$$\varphi(x)=\sum_{j=1}^s\varphi(x)\Phi_k(x-x_j)+\varphi(x)\Big(1-\sum_{j=1}^s\Phi_k(x-x_j)\Big)=:\varphi_{0,k}(x)+\varphi_{1,k}(x).$$

Then $\varphi_{1,k}\to\varphi$ in $W^{2,2}$ as $k\to\infty$. Moreover, $\operatorname{supp}\varphi_{1,k}\Subset\mathbb{B}^4\setminus A_1$. It follows easily that the weak form of (3.30) holds not just for test functions $\varphi_{1,k}\in C_0^\infty(\mathbb{B}^4\setminus A_1)$, but also for an arbitrary $\varphi\in C_0^\infty(\mathbb{B}^4)$.

The whole proof of Theorem 1.1 is complete now.

4. An example. The following example shows that a system of type (1.6) satisfying assumptions (1.3) may have solutions that are in $C^1\setminus C^2$.

Set $\phi(r)=\log\log(e-\log r)$ for $r\in(0,1)$.

LEMMA 4.1. *The function $f(x)=\phi(|x|)=\log(\log(e-\log(|x|)))$ is in $W^{2,2}(\mathbb{B}^4)$.*

Proof. The function is obviously in $L^2(\mathbb{B}^4)$, therefore (e.g. by Gagliardo–Nirenberg's inequality) it is enough to check that $\nabla^2 f\in L^2(\mathbb{B}^4)$. This is done by an elementary computation which, for the sake of completeness, is sketched in the appendix. ■

LEMMA 4.2. *The functions $t\phi'(t)$, $t^2\phi''(t)$ and $t^3\phi'''(t)$ are bounded on $(0,1]$.*

Proof. A computation shows that

$$|t\phi'(t)|=\frac{1}{(e-\log t)\log(e-\log t)},$$

$$|t^2\phi''(t)|=\frac{-1+(e-1-\log t)\log(e-\log t)}{(e-\log t)^2\log^2(e-\log t)},$$

$$|t^3\phi'''(t)| = \frac{-2+3(-1+e-\log(t))\log(e-\log(t))}{(e-\log(t))^3\log(e-\log(t))^3} - \frac{\big(2-3\,e+2\,e^2+(3-4\,e)\log(t)+2\log(t)^2\big)\log(e-\log(t))^2}{(e-\log(t))^3\log(e-\log(t))^3}.$$

Clearly, the right hand sides converge to 0 as $t \to 0$. The lemma follows. ∎

Example 4.3. Define $w : \mathbb{B}^4 \to \mathbb{R}$ by

$$w(x) = |x|^2 \sin\phi(|x|). \tag{4.1}$$

One easily checks, using Lemma 4.2, that ∇w and $D^2 w$ are bounded on $\mathbb{B}^4$. However,

$$\frac{\partial^2 w}{\partial x_1^2}(0,0,0,t) = 2\sin\phi(t) + 3t\phi'(t)\cos\phi(t)$$

does not have a limit for $t \to 0$, and thus w *is not* a function of class C^2. Finally,

$$|\nabla^3 w(x)| \lesssim \frac{1}{|x|(e-\log|x|)\log(e-\log|x|)} \in L^4(\mathbb{B}^4).$$

Set $C > 0$ to be the bound on $|\nabla w|$ on $\mathbb{B}^4$ and consider

$$v(x) = w(x) + 2C\sum_{i=1}^{4} x_i, \quad a(x) = \Delta v|\nabla v|^{-2}. \tag{4.2}$$

By definition, $|\nabla v| \geq C > 0$ on $\mathbb{B}^4$, and $|\nabla v|$ and $|D^2 v|$ are bounded on $\mathbb{B}^4$. Thus, $a(x)$ is bounded on $\mathbb{B}^4$.

We also have $|D^3 v| = |D^3 w| \in L^4(\mathbb{B}^4)$, and

$$|\nabla a| \lesssim \frac{|\nabla\Delta v|\,|\nabla v|^2 + |\Delta v|\,|\nabla v|\,|D^2 v|}{|\nabla v|^4} \lesssim 1 + |D^3 v| \in L^4(\mathbb{B}^4).$$

Thus, $a \in W^{1,4}(\mathbb{B}^4) \cap L^\infty(\mathbb{B}^4)$. Consider the equation

$$\Delta(\Delta u(x)) = \Delta(a(x)|\nabla u|^2). \tag{4.3}$$

Clearly this is an equation of the type

$$\Delta^2 u = \Delta(D(u,\nabla u)\cdot\nabla u) + \text{other terms (with zero coefficients)}, \tag{4.4}$$

where $D = a(x)\nabla u$ satisfies

- $D \in W^{1,2}$ whenever $u \in W^{2,2}$,
- $|D(x)| \leq \|a\|_{L^\infty}|\nabla u(x)|$,
- $\|\nabla D\|_{L^2} \leq \|\nabla a\|_{L^4}\|\nabla u\|_{L^4} + \|a\|_{L^\infty}\|D^2 u\|_{L^2}$.

It is also obvious, by the very definition of a, that v solves (4.3). Therefore (4.4), under the conditions on D listed above, admits nonsmooth solutions.

Note, however, that the function D given above *does not* satisfy the pointwise estimate for ∇D given in (1.13).

Appendix

An unbounded function in $W^{2,2}(\mathbb{B}^4)$. As explained in Section 4, to show that $f(x) = \log(\log(e - \log|x|))$ is in $W^{2,2}$ on the unit ball $\mathbb{B}^4$, it is enough to check that $D^2 f \in L^2$. For $x \in \mathbb{B}^4$ we have (use Mathematica):

$$\begin{aligned} |\partial^2_{x_i x_j} f(x)| &= \left| - \frac{x_i x_j (1 + (1 - 2e + 2\log|x|)\log(e - \log|x|))}{|x|^4 (e - \log|x|)^2 \log^2(e - \log|x|)} \right| \\ &\lesssim \left| \frac{x_i x_j}{|x|^4 (e - \log|x|)\log(e - \log|x|)} \right| \\ &\lesssim \frac{1}{|x|^2(-\log|x|)\log(-\log|x|)}. \end{aligned}$$

Integrating the above squared over $\mathbb{B}^4$, in polar coordinates, amounts to calculating

$$\int_0^1 \frac{dr}{r\log^2 r \log^2(-\log r)} = \int_1^\infty \frac{dt}{t \log^2 t \log^2 \log t},$$

with the latter integral convergent, since the series

$$\sum \frac{1}{n \log^2 n \log^2 \log n}$$

is convergent, by Cauchy's condensation test.

Proof of Lemma 3.2. Let $f \in W^{2,2}$ be the function introduced in Lemma 4.1. Fix a function $\zeta \in C^\infty(\mathbb{R})$ with $0 \le \zeta \le 1$, $\zeta \equiv 0$ on $[1, +\infty)$ and $\zeta \equiv 1$ on $(-\infty, 1/4)$. Set

$$\Phi_k(x) = \zeta(k + 1 - f(x)), \quad x \in \mathbb{B}^4, \quad k = 1, 2, \ldots. \tag{A.1}$$

Clearly, $\Phi \in W^{2,2}(\mathbb{B}^4)$. Moreover, $\Phi_k(x) = 0$ if $f(x) \le k$. The equality $\Phi_k(x) \equiv 1$ holds whenever $f(x) \ge k + 3/4$, i.e. on a neighborhood of 0. Since $f \ge 0$ on $\mathbb{B}^4$ and f is smooth except at 0, Φ_k is smooth on $\mathbb{B}^4$, and can be extended by 0 to the whole $\mathbb{R}^4$. Finally,

$$\begin{aligned} |\nabla \Phi_k(x)| &\lesssim |\nabla f(x)| \cdot \chi_{\{f>k\}}, \\ |\nabla^2 \Phi_k(x)| &\lesssim (|\nabla^2 f(x)| + |\nabla f(x)|^2) \cdot \chi_{\{f>k\}}. \end{aligned}$$

Since $f \in W^{2,2} \cap W^{1,4}$ is radial and $f \to +\infty$ at 0, we obtain $\|\Phi_k\|_{W^{2,2}} \to 0$ as $k \to \infty$. For $x \ne 0$, we simply have $\Phi_k(x) = 0$ for all k sufficiently large. This completes the proof.

Acknowledgments. All authors were partially supported by MNiSW Grant no N N201 397737, *Nonlinear partial differential equations: geometric and variational problems.*

References

[1] A. Freire, S. Müller, and M. Struwe, *Weak compactness of wave maps and harmonic maps*, Ann. Inst. H. Poincaré Anal. Non Linéaire 15 (1998), 725–754.

[2] P. Hornung and R. Moser, *Energy identity for intrinsically biharmonic maps in four dimensions*, Anal. PDE 5 (2012), 61–80.

[3] T. Lamm and T. Rivière, *Conservation laws for fourth order systems in four dimensions*, Comm. Partial Differential Equations 33 (2008), 245–262.

[4] P. Laurain and T. Rivière, *Energy quantization for biharmonic maps*, arXiv:1112.5393.

[5] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoamer. 1 (1985), 145–201.

[6] Y. Meyer and T. Rivière, *A partial regularity result for a class of stationary Yang–Mills fields in high dimension*, Rev. Mat. Iberoamer. 19 (2003), 195–219.

[7] T. Rivière, *Conservation laws for conformally invariant variational problems*, Invent. Math. 168 (2007), 1–22.

[8] T. Rivière and Michael Struwe, *Partial regularity for harmonic maps and related problems*, Comm. Pure Appl. Math. 61 (2008), 451–463.

[9] B. Sharp and P. Topping, *Decay estimates for Rivière's equation, with applications to regularity and compactness*, Trans. Amer. Math. Soc. 365 (2013), 2317–2339.

[10] M. Struwe, *Partial regularity for biharmonic maps, revisited*, Calc. Var. Partial Differential Equations 33 (2008), 249–262.

[11] P. Strzelecki, *Gagliardo–Nirenberg inequalities with a BMO term*, Bull. London Math. Soc. 38 (2006), 294–300.

[12] P. Strzelecki and A. Zatorska-Goldstein, *A compactness theorem for weak solutions of higher-dimensional H-systems*, Duke Math. J. 121 (2004), 269–284.

[13] K. K. Uhlenbeck, *Connections with L^p bounds on curvature*, Comm. Math. Phys. 83 (1982), 31–42.

[14] C. Wang and S. Zheng, *Energy identity of approximate biharmonic maps to Riemannian manifolds and its application*, J. Funct. Anal. 263 (2012), 960–987.

[15] C. Wang and S. Zheng, *Energy identity for a class of approximate biharmonic maps into sphere in dimension four*, Discrete Contin. Dynam. Systems 33 (2013), 861–878.

Paweł Goldstein, Paweł Strzelecki
Institute of Mathematics
University of Warsaw
Banacha 2
02-097 Warszawa, Polska
E-mail: P.Goldstein@mimuw.edu.pl
P.Strzelecki@mimuw.edu.pl

Anna Zatorska-Goldstein
Institute of Applied Mathematics
and Mechanics
University of Warsaw
Banacha 2
02-097 Warszawa, Polska
E-mail: A.Zatorska-Goldstein@mimuw.edu.pl

Received June 8, 2012
Revised version March 6, 2013 (7536)