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Scattered elements of Banach algebras

by

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Abstract. A scattered element of a Banach algebra $\mathcal{A}$ is an element with at most countable spectrum. The set of all scattered elements is denoted by $\mathcal{S}(\mathcal{A})$. The scattered radical $\mathcal{R}_{\mathrm{sc}}(\mathcal{A})$ is the largest ideal consisting of scattered elements. We characterize in several ways central elements of $\mathcal{A}$ modulo the scattered radical. As a consequence, it is shown that the following conditions are equivalent: (i) $\mathcal{S}(\mathcal{A})+\mathcal{S}(\mathcal{A})\subset\mathcal{S}(\mathcal{A})$; (ii) $\mathcal{S}(\mathcal{A})\mathcal{S}(\mathcal{A})\subset\mathcal{S}(\mathcal{A})$; (iii) $[\mathcal{S}(\mathcal{A}),\mathcal{A}]\subset\mathcal{R}_{\mathrm{sc}}(\mathcal{A})$.

1. Introduction. It was proved by Z. Słodkowski, W. Wojtyński and J. Zemánek in 1977 that if the set of all quasinilpotent elements in a Banach algebra forms a subspace or a semigroup, then all quasinilpotent elements belong to the Jacobson radical. We will prove similar statements about scattered elements.

Let us introduce necessary definitions and notations. For an element a of a Banach algebra $\mathcal{A}$, let $\sigma(a)$ mean the spectrum of a, and $\rho(a)=\mathbb{C}\setminus\sigma(a)$ the resolvent set of a. The cardinality of $\sigma(a)$ is denoted by $\#\sigma(a)$; a is *quasinilpotent* if $\sigma(a)=\{0\}$. The set of all quasinilpotent elements in $\mathcal{A}$ is denoted by $Q(\mathcal{A})$. By $\mathrm{Rad}(\mathcal{A})$ we denote the *Jacobson radical* of $\mathcal{A}$. The *socle*, that is, the sum of all minimal one-sided ideals of $\mathcal{A}$, is denoted by $\mathrm{Soc}(\mathcal{A})$.

For a closed ideal $J\subset\mathcal{A}$ and an element $a\in\mathcal{A}$, we denote by a/J the coset $a+J\in\mathcal{A}/J$. In other terms, $a/J=\pi_J(a)$ where $\pi_J:\mathcal{A}\to\mathcal{A}/J$ is the standard epimorphism.

If M and N are subsets of $\mathcal{A}$, then $M+N:=\{x+y : x\in M,\ y\in N\}$. We write $x+N$ instead of $\{x\}+N$. In a similar way we define MN and so on.

It was shown in [7] that $Q(\mathcal{A})=\mathrm{Rad}(\mathcal{A})$ if and only if $Q(\mathcal{A})+Q(\mathcal{A})\subset Q(\mathcal{A})$ if and only if $Q(\mathcal{A})Q(\mathcal{A})\subset Q(\mathcal{A})$. In [4] a similar result was proved for the Lie product $[a,b]=ab-ba$: $Q(\mathcal{A})=\mathrm{Rad}(\mathcal{A})$ if and only if $[Q(\mathcal{A}),Q(\mathcal{A})]\subset Q(\mathcal{A})$. There are also some local results, for example $a+Q(\mathcal{A})\subset Q(\mathcal{A})$ if

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and only if $a \in \mathrm{Rad}(\mathcal{A})$ [8]. In [1, Theorem 5.2.1] it was shown that for an element $a \in \mathcal{A}$ the condition $\#\sigma([a,x]) = 1$ for every $x \in \mathcal{A}$ is equivalent to $[a,\mathcal{A}] \in \mathrm{Rad}(\mathcal{A})$.

Similar results hold for the finite spectrum case. Let $I(\mathcal{A})$ denote the set of all $a \in \mathcal{A}$ with $\#\sigma(a) < \infty$. If $a + I(\mathcal{A}) \subset I(\mathcal{A})$ for some $a \in \mathcal{A}$, then $aI(\mathcal{A}) \subset I(\mathcal{A})$ and $[a,\mathcal{A}] \subset I(\mathcal{A})$ by [1, Corollary 5.6.4 and Lemma 5.6.5]. Moreover, if $\mathcal{A}$ is semisimple then $[a,\mathcal{A}] \subset I(\mathcal{A})$ if and only if $[a,\mathcal{A}] \subset \mathrm{Soc}(\mathcal{A})$ if and only if every element in $[a,\mathcal{A}]$ is algebraic [2].

In this paper we will consider similar conditions for scattered elements. An element of $\mathcal{A}$ is called *scattered* if its spectrum is finite or countable. Let $\mathcal{S}(\mathcal{A})$ be the set of all scattered elements of $\mathcal{A}$. The *scattered radical* of $\mathcal{A}$ is denoted by $\mathcal{R}_{\mathrm{sc}}(\mathcal{A})$; it can be defined by several equivalent conditions [5], in particular

$$\mathcal{R}_{\mathrm{sc}}(\mathcal{A}) := \{a \in \mathcal{A} : a\mathcal{A} \subset \mathcal{S}(\mathcal{A})\}.$$

Clearly, $\mathrm{Rad}(\mathcal{A}) \subset \mathcal{R}_{\mathrm{sc}}(\mathcal{A}) \subset \mathcal{S}(\mathcal{A})$. It was proved in [6] that the map $\mathcal{A} \mapsto \mathcal{R}_{\mathrm{sc}}(\mathcal{A})$ is a hereditary topological radical on the class of Banach algebras (see the definition in [3]). In particular, the following statement holds:

Lemma 1 ([6, Section 8.2]).

(i) *$\mathcal{R}_{\mathrm{sc}}(\mathcal{A})$ is a closed (two-sided) ideal of $\mathcal{A}$.*
(ii) *$\mathcal{R}_{\mathrm{sc}}(\mathcal{A}/\mathcal{R}_{\mathrm{sc}}(\mathcal{A})) = \{0\}$.*
(iii) *$a \in \mathcal{S}(\mathcal{A})$ if and only if $a/\mathcal{R}_{\mathrm{sc}}(\mathcal{A}) \in \mathcal{S}(\mathcal{A}/\mathcal{R}_{\mathrm{sc}}(\mathcal{A}))$.*

2. Central elements modulo the scattered radical. For a unital Banach algebra $\mathcal{A}$, we set

$$\mathcal{ZS}_1(\mathcal{A}) := \{x \in \mathcal{A} : x + \mathcal{S}(\mathcal{A}) \subset \mathcal{S}(\mathcal{A})\},$$
$$\mathcal{ZS}_2(\mathcal{A}) := \{x \in \mathcal{A} : x\mathcal{S}(\mathcal{A}) \subset \mathcal{S}(\mathcal{A})\}.$$

Clearly, $\mathcal{ZS}_1(\mathcal{A}) \subset \mathcal{S}(\mathcal{A})$ and $\mathcal{ZS}_2(\mathcal{A}) \subset \mathcal{S}(\mathcal{A})$. It follows that $\mathcal{ZS}_1(\mathcal{A})$ is a linear subspace of $\mathcal{A}$, while $\mathcal{ZS}_2(\mathcal{A})$ is a multiplicative subsemigroup of $\mathcal{A}$. It follows from the Spectral Mapping Theorem that if $a \in \mathcal{S}(\mathcal{A})$ and $\alpha \in \rho(a)$ then $(a-\alpha)^{-1} \in \mathcal{S}(\mathcal{A})$.

We will use the theory of analytic multifunctions [1, Chapter VII]. Let K be an analytic multifunction from a domain $D \subset \mathbb{C}$ into $\mathbb{C}$. Then either $\{\lambda \in D : K(\lambda) \text{ is at most countable}\}$ has capacity zero, or $K(\lambda)$ is at most countable for all $\lambda \in D$ by the Scarcity Theorem [1, Theorem 7.2.8]. In the latter case, for a fixed $\eta \in \mathbb{C}$, the set $\{\lambda \in D : \eta \in K(\lambda)\}$ is either at most countable or equal to D by the Aupetit–Zemánek Theorem [1, Theorem 7.2.13].

The following two propositions show that $\mathcal{ZS}_1(\mathcal{A}) = \mathcal{ZS}_2(\mathcal{A})$.

Proposition 1. *For a unital Banach algebra $\mathcal{A}$, $\mathcal{ZS}_2(\mathcal{A}) \subset \mathcal{ZS}_1(\mathcal{A})$.*

Proof. We divide the proof into several claims.

CLAIM 1. *If $x \in \mathcal{ZS}_2(\mathcal{A})$, then $x+\lambda \in \mathcal{ZS}_2(\mathcal{A})$ for every $\lambda \in \mathbb{C}$.*

For $a \in \mathcal{S}(\mathcal{A})$ and $\mu \in \rho(\lambda a)$ we have

$$(x+\lambda)a - \mu = xa + \lambda a - \mu = (xa(\lambda a-\mu)^{-1}+1)(\lambda a - \mu).$$

So $\mu \in \sigma((x+\lambda)a)$ if and only if $-1 \in \sigma(xa(\lambda a-\mu)^{-1})$. The function $\mu \mapsto \sigma(xa(\lambda a-\mu)^{-1})$ is an at most countable analytic multifunction from $\rho(\lambda a)$ to $\mathbb{C}$ by [1, Theorem 7.1.13]. Then the set $\{\mu \in \rho(\lambda a) : -1 \in \sigma(xa(\lambda a-\mu)^{-1})\}$ is either at most countable or equal to $\rho(\lambda a)$, by [1, Theorem 7.2.13]. But in the latter case, since $(\lambda a-\mu)^{-1}$ tends to 0 as $\mu \to \infty$, we infer that

$$-1 \in \limsup_{\mu\to\infty} \sigma(xa(\lambda a-\mu)^{-1}) \subset \sigma(0),$$

a contradiction. So $(x+\lambda)a \in \mathcal{S}(\mathcal{A})$ for every $a \in \mathcal{S}(\mathcal{A})$, that is, $x+\lambda \in \mathcal{ZS}_2(\mathcal{A})$.

CLAIM 2. *If $x \in \mathcal{ZS}_2(\mathcal{A})$ and x is invertible, then $x^{-1} \in \mathcal{ZS}_2(\mathcal{A})$.*

Let $a \in \mathcal{S}(\mathcal{A})$ and $\alpha \in \rho(a)$. Then $(a-\alpha)^{-1} \in \mathcal{S}(\mathcal{A})$, whence $x(a-\alpha)^{-1} \in \mathcal{S}(\mathcal{A})$. Hence $(a-\alpha)x^{-1} \in \mathcal{S}(\mathcal{A})$ and so $x^{-1}(a-\alpha) \in \mathcal{S}(\mathcal{A})$. Now the function $\alpha \mapsto \sigma(x^{-1}(a-\alpha))$ is an analytic multifunction from $\mathbb{C}$ to $\mathbb{C}$, and it has at most countable values on $\rho(a)$. As the capacity of $\rho(a)$ is not zero, it follows that $x^{-1}(a-\alpha) \in \mathcal{S}(\mathcal{A})$ for every $\alpha \in \mathbb{C}$ by [1, Theorem 7.2.8]. For $\alpha = 0$ we obtain $x^{-1}a \in \mathcal{S}(\mathcal{A})$, so $x^{-1} \in \mathcal{ZS}_2(\mathcal{A})$.

CLAIM 3. *$\mathcal{ZS}_2(\mathcal{A}) \subset \mathcal{ZS}_1(\mathcal{A})$.*

Let $x \in \mathcal{ZS}_2(\mathcal{A})$ and $\lambda \in \rho(x)$. Then $x-\lambda \in \mathcal{ZS}_2(\mathcal{A})$ by Claim 1, and $(x-\lambda)^{-1} \in \mathcal{ZS}_2(\mathcal{A})$ by Claim 2. For every $a \in \mathcal{S}(\mathcal{A})$, we have

$$x-\lambda+a = (x-\lambda)(1+(x-\lambda)^{-1}a) \in \mathcal{S}(\mathcal{A}).$$

Changing a to $a+\lambda$, we get $x \in \mathcal{ZS}_1(\mathcal{A})$. ■

PROPOSITION 2. *For a unital Banach algebra $\mathcal{A}$, $\mathcal{ZS}_1(\mathcal{A})$ is a Lie ideal of $\mathcal{A}$ and $\mathcal{ZS}_1(\mathcal{A}) \subset \mathcal{ZS}_2(\mathcal{A})$.*

Proof. For every $x \in \mathcal{ZS}_1(\mathcal{A})$, $a \in S(\mathcal{A})$ and $b \in \mathcal{A}$, we define a function $f(\lambda)$ as follows:

$$f(\lambda) := \begin{cases} \dfrac{x - e^{\lambda b}xe^{-\lambda b}}{\lambda} + a & \text{for } \lambda \neq 0, \\ [x,b]+a & \text{for } \lambda = 0. \end{cases}$$

As

$$\frac{x - e^{\lambda b}xe^{-\lambda b}}{\lambda} + a = \frac{1}{\lambda}(x - e^{\lambda b}(x+\lambda e^{-\lambda b}ae^{\lambda b})e^{-\lambda b}),$$

$\sigma(f(\lambda))$ is at most countable for $\lambda \neq 0$. Since the function f is analytic, $\lambda \mapsto \sigma(f(\lambda))$ is an analytic multifunction on $\mathbb{C}$ by [1, Theorem 7.1.13]. Therefore

$[x,b]+a\in\mathcal{S}(\mathcal{A})$ by [1, Theorem 7.2.8]. Hence $[\mathcal{ZS}_1(\mathcal{A}),\mathcal{A}]\subset\mathcal{ZS}_1(\mathcal{A})$, that is, $\mathcal{ZS}_1(\mathcal{A})$ is a Lie ideal of $\mathcal{A}$.

Now we prove that if $x\in\mathcal{ZS}_1(\mathcal{A})$ and $a\in\mathcal{S}(\mathcal{A})$, then $xa\in\mathcal{S}(\mathcal{A})$.

For $a\in\mathcal{S}(\mathcal{A})$ and $\alpha\in\rho(a)$, we have $(a-\alpha)^{-1}\in\mathcal{S}(\mathcal{A})$. It follows that $x+\lambda(a-\alpha)^{-1}\in\mathcal{S}(A)$ for every $\lambda\in\mathbb{C}$. It is clear that $0\in\sigma(x+\lambda(a-\alpha)^{-1})$ if and only if $-\lambda\in\sigma(x(a-\alpha))$. So by [1, Theorems 7.1.13 and 7.2.13], either $\sigma(x(a-\alpha))$ is at most countable, or $0\in\sigma(x+\lambda(a-\alpha)^{-1})$ for every $\lambda\in\mathbb{C}$. But in the latter case,

$$0\in\limsup_{\lambda\to\infty}\sigma(x/\lambda+(a-\alpha)^{-1})\subset\sigma((a-\alpha)^{-1}),$$

a contradiction. Hence $x(a-\alpha)\in\mathcal{S}(\mathcal{A})$. Since $x\in\mathcal{ZS}_1(\mathcal{A})$, we get $xa\in\mathcal{S}(\mathcal{A})$. Thus $x\in\mathcal{ZS}_2(\mathcal{A})$. ∎

From now on, we can use the same notation $\mathcal{ZS}(\mathcal{A})$ for $\mathcal{ZS}_1(\mathcal{A})$ and $\mathcal{ZS}_2(\mathcal{A})$. It is a Lie ideal (by Proposition 2) and a subalgebra of $\mathcal{A}$. Now we establish another property of $\mathcal{ZS}(\mathcal{A})$.

THEOREM 1. *For a unital Banach algebra $\mathcal{A}$, $[\mathcal{ZS}(\mathcal{A}),\mathcal{A}]\subset\mathcal{R}_{\rm sc}(\mathcal{A})$.*

Proof. We divide the proof into a sequence of claims.

CLAIM 1. $[\mathcal{ZS}(\mathcal{A}),\mathcal{ZS}(\mathcal{A})]\subset\mathcal{R}_{\rm sc}(\mathcal{A})$.

For every $x,y\in\mathcal{ZS}(\mathcal{A})$ and $z\in\mathcal{A}$, we have $[x,y]z=x[y,z]+[xz,y]$. Note that $[y,z]\in\mathcal{ZS}(\mathcal{A})$ and $[xz,y]\in\mathcal{ZS}(\mathcal{A})$ by Proposition 2. Hence $[x,y]z\in\mathcal{S}(\mathcal{A})$ for every $z\in\mathcal{A}$, that is, $[x,y]\in\mathcal{R}_{\rm sc}(\mathcal{A})$. So $[\mathcal{ZS}(\mathcal{A}),\mathcal{ZS}(\mathcal{A})]\subset\mathcal{R}_{\rm sc}(\mathcal{A})$.

Let $\pi:\mathcal{A}\to B=\mathcal{A}/R_{\rm sc}(\mathcal{A})$ be the standard epimorphism.

CLAIM 2. $[\mathcal{ZS}(B),B]=\{0\}$.

For every $a\in\mathcal{ZS}(B)$ and $b\in B$, we have $[a,b]\in\mathcal{ZS}(B)$ by Proposition 2, and so $[a,[a,b]]\in\mathcal{R}_{\rm sc}(B)$ by Claim 1. But $\mathcal{R}_{\rm sc}(B)=\mathcal{R}_{\rm sc}(\mathcal{A}/\mathcal{R}_{\rm sc}(\mathcal{A}))=\{0\}$ by Lemma 1(ii). Hence $[a,[a,b]]=0$, and $\sigma([a,b])=\{0\}$ for every $b\in B$ by the Kleinecke–Shirokov Theorem. Therefore $[a,b]\in\mathrm{Rad}(B)$ by Le Page's Lemma [1, Theorem 5.2.1]. But $\mathrm{Rad}(B)\subset\mathcal{R}_{\rm sc}(B)=\{0\}$, so $[a,b]=0$ for every $a\in\mathcal{ZS}(B)$ and $b\in B$.

CLAIM 3. $\pi(\mathcal{ZS}(\mathcal{A}))\subset\mathcal{ZS}(B)$.

Note that $a/\mathcal{R}_{\rm sc}(\mathcal{A})\in\mathcal{S}(B)$ for every $a\in\mathcal{S}(\mathcal{A})$ by Lemma 1(iii). Furthermore, for each $b\in\mathcal{S}(B)$ there is $b_1\in\mathcal{A}$ such that $b=\pi(b_1)$. Then $b_1\in\mathcal{S}(\mathcal{A})$ by Lemma 1(iii). So $\pi(\mathcal{S}(\mathcal{A}))=\mathcal{S}(B)$. For every $x\in\mathcal{ZS}(\mathcal{A})$ and $b\in\mathcal{S}(B)$,

$$\pi(x)+b=\pi(x)+\pi(b_1)=\pi(x+b_1)\in\mathcal{S}(B).$$

Hence $\pi(x)\in\mathcal{ZS}(B)$.

Finally, $\pi([\mathcal{ZS}(\mathcal{A}),\mathcal{A}])=[\pi(\mathcal{ZS}(\mathcal{A})),\pi(\mathcal{A})]\subset[\mathcal{ZS}(B),B]=\{0\}$, that is, $[\mathcal{ZS}(\mathcal{A}),\mathcal{A}]\subset\mathcal{R}_{\rm sc}(\mathcal{A})$. ∎

Corollary 1. *For a unital Banach algebra $\mathcal{A}$, the following conditions are equivalent:*

(i) $\mathcal{S}(\mathcal{A})+\mathcal{S}(\mathcal{A})\subset\mathcal{S}(\mathcal{A})$;
(ii) $\mathcal{S}(\mathcal{A})\mathcal{S}(\mathcal{A})\subset\mathcal{S}(\mathcal{A})$;
(iii) $[\mathcal{S}(\mathcal{A}),\mathcal{S}(\mathcal{A})]\mathcal{S}(\mathcal{A})\subset\mathcal{S}(\mathcal{A})$;
(iv) $[\mathcal{S}(\mathcal{A}),\mathcal{S}(\mathcal{A})]+\mathcal{S}(\mathcal{A})\subset\mathcal{S}(\mathcal{A})$;
(v) $[\mathcal{S}(\mathcal{A}),\mathcal{A}]\subset\mathcal{R}_{\mathrm{sc}}(\mathcal{A})$.

Proof. Conditions (i) and (ii) are equivalent because each means that $\mathcal{ZS}(\mathcal{A})=\mathcal{S}(\mathcal{A})$. Similarly (iii) and (iv) are equivalent to $[\mathcal{S}(\mathcal{A}),\mathcal{S}(\mathcal{A})]\subset\mathcal{ZS}(\mathcal{A})$. The implication (i)⇒(v) follows from Theorem 1. Evidently (v)⇒(iii). So it remains to prove that (iii)⇒(i).

Assume that (iii) holds, that is, $[\mathcal{S}(\mathcal{A}),\mathcal{S}(\mathcal{A})]\subset\mathcal{ZS}(\mathcal{A})$. Then

$$[[\mathcal{S}(\mathcal{A}),\mathcal{S}(\mathcal{A})],\mathcal{A}]\subset\mathcal{R}_{\mathrm{sc}}(\mathcal{A})$$

by Theorem 1. Let $\pi:\mathcal{A}\to B=\mathcal{A}/\mathcal{R}_{\mathrm{sc}}(\mathcal{A})$ be the standard homomorphism. Then $[[\pi(\mathcal{S}(\mathcal{A})),\pi(\mathcal{S}(\mathcal{A}))],\pi(\mathcal{A})]=0$. Hence $[[\pi(a),\pi(b)],\pi(b)]=0$ for every $a,b\in\mathcal{S}(A)$, so $[\pi(a),\pi(b)]$ is quasinilpotent by the Kleinecke–Shirokov Theorem. This and the equality $[[\pi(a),\pi(b)],B]=0$ show by using Le Page's Lemma that $[\pi(a),\pi(b)]\in\mathrm{Rad}(B)=\{0\}$, that is, $\pi(a)$ commutes with $\pi(b)$. Since $\pi(a),\pi(b)\in\mathcal{S}(B)$, it follows that $\pi(a)+\pi(b)\in\mathcal{S}(B)$, whence $a+b\in\mathcal{S}(\mathcal{A})$, that is, (i) holds. ∎

In general, $\mathcal{ZS}(\mathcal{A})$ is not an ideal of $\mathcal{A}$. For instance, if $\mathcal{A}$ is the algebra of all bounded operators on an infinite-dimensional Hilbert space then the identity operator belongs to $\mathcal{ZS}(\mathcal{A})$, but $\mathcal{A}$ contains operators with uncountable spectrum.

Theorem 2. *For a unital Banach algebra $\mathcal{A}$,*

$$\begin{aligned}\mathcal{ZS}(\mathcal{A})&=\{x\in\mathcal{S}(\mathcal{A}):[x,\mathcal{A}]\subset\mathcal{R}_{\mathrm{sc}}(\mathcal{A})\}\\&=\{x\in\mathcal{S}(\mathcal{A}):[x,\mathcal{S}(\mathcal{A})]\subset\mathcal{R}_{\mathrm{sc}}(\mathcal{A})\}.\end{aligned}$$

Proof. Clearly $\mathcal{ZS}(\mathcal{A})\subset\{x\in\mathcal{S}(\mathcal{A}):[x,\mathcal{A}]\subset\mathcal{R}_{\mathrm{sc}}(\mathcal{A})\}$ by Theorem 1. So it suffices to show that $\{x\in\mathcal{S}(\mathcal{A}):[x,\mathcal{S}(\mathcal{A})]\subset\mathcal{R}_{\mathrm{sc}}(\mathcal{A})\}\subset\mathcal{ZS}(\mathcal{A})$.

Let $x\in\mathcal{S}(\mathcal{A})$ with $[x,\mathcal{S}(\mathcal{A})]\subset\mathcal{R}_{\mathrm{sc}}(\mathcal{A})$. Let $\pi:\mathcal{A}\to B=\mathcal{A}/\mathcal{R}_{\mathrm{sc}}(\mathcal{A})$ be the standard homomorphism. Then $[\pi(x),\pi(a)]=0$ for every $a\in\mathcal{S}(\mathcal{A})$. So we obtain $\pi(x)+\pi(a)\in\mathcal{S}(B)$. Then $x+a\in\mathcal{S}(\mathcal{A})$ for every $a\in\mathcal{S}(\mathcal{A})$, whence $x\in\mathcal{ZS}(\mathcal{A})$. ∎

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