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Reducibility and unitary equivalence for a class of multiplication operators on the Dirichlet space

by

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Abstract. We consider the reducibility and unitary equivalence of multiplication operators on the Dirichlet space. We first characterize reducibility of a multiplication operator induced by a finite Blaschke product and, as an application, we show that a multiplication operator induced by a Blaschke product with two zeros is reducible only in an obvious case. Also, we prove that a multiplication operator induced by a multiplier ϕ is unitarily equivalent to a weighted shift of multiplicity 2 if and only if $\phi = \lambda z^2$ for some unimodular constant λ.

1. Introduction. Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, and $\mathbb{T}$ be the unit circle. The *Dirichlet space* $\mathscr{D}$ consists of all analytic functions f on $\mathbb{D}$ for which

$$\int_{\mathbb{D}} |f'|^2 \, dA < \infty$$

where dA is the normalized area measure on $\mathbb{D}$. Note that $\mathscr{D} \subset H^2$ where H^2 is the well known Hardy space consisting of analytic functions f on $\mathbb{D}$ such that

$$\sup_{0 \le r < 1} \int_{\mathbb{T}} |f(r\zeta)|^2 \frac{|d\zeta|}{2\pi} < \infty.$$

It is known that the Dirichlet space $\mathscr{D}$ is a Hilbert space with the norm

$$\|f\| = \left(\int_{\mathbb{T}} |f|^2 \frac{|d\zeta|}{2\pi} + \int_{\mathbb{D}} |f'|^2 \, dA \right)^{1/2}$$

and the inner product

$$\langle f, g \rangle = \int_{\mathbb{T}} f\overline{g} \frac{|d\zeta|}{2\pi} + \int_{\mathbb{D}} f' \, \overline{g'} \, dA$$

for $f, g \in \mathscr{D}$. See [R] for more information on the Dirichlet space.

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We say that a function ϕ on $\mathbb{D}$ is a *multiplier* on $\mathscr{D}$ if $\phi\mathscr{D} \subset \mathscr{D}$. By the closed graph theorem, each multiplier ϕ induces a bounded multiplication operator M_ϕ on $\mathscr{D}$ defined by $M_\phi f = \phi f$. Let $\mathcal{M}(\mathscr{D})$ be the set of all multipliers on $\mathscr{D}$. Recall that each multiplier is a bounded analytic function on $\mathbb{D}$ (see [S] for details).

For a bounded linear operator T on a Hilbert space $\mathcal{H}$, a closed subspace A in $\mathcal{H}$ is said to be *invariant* under T, or an *invariant subspace* of T, if $TA \subset A$. Also, we say that A is a *reducing subspace* of T if A is invariant under both T and its adjoint T^*. We also say an operator T on $\mathcal{H}$ is *reducible* if T has a nontrivial reducing subspace.

The problem of characterizing reducing subspaces of certain multiplication operators has been well studied on the Hardy space and Bergman spaces. For examples, see [C], [DPW], [DSZ], [GH], [HSXY], [SW], [T1], [T2], [Zhu1] and references therein.

In [SZ], Stessin and Zhu studied the problem of characterizing reducing subspaces on certain weighted Hilbert spaces of analytic functions and obtained a complete description of reducing subspaces for weighted unilateral shift operators of finite multiplicity. As a consequence of their result, we know that for a given positive integer N, M_{z^N} has exactly $2^N - 2$ nontrivial reducing subspaces on the Dirichlet space $\mathscr{D}$. Later, Zhao [Zh1] studied the same problem on a Dirichlet space equipped with a norm smaller than $\| \ \|$. To be more precise, let $\mathscr{D}_0$ be the space of all analytic functions f on $\mathbb{D}$ for which

$$\|f\|_0 = \Big(|f(0)|^2 + \int_{\mathbb{D}} |f'|^2 \, dA\Big)^{1/2} < \infty.$$

Also, given a point $a \in \mathbb{D}$, let

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}, \qquad z \in \mathbb{D},$$

be the Möbius transformation. For finitely many points $a_1, \ldots, a_N \in \mathbb{D}$, we call $\varphi_{a_1} \cdots \varphi_{a_N}$ a *finite Blaschke product* with zeros $a_1, \ldots, a_N$. Note each finite Blaschke product is a multiplier for $\mathscr{D}$.

Zhao [Zh1] considered multiplication operators $M_{\varphi_a\varphi_b}$ induced by Blaschke products with two zeros a, b and proved that $M_{\varphi_a\varphi_b}$ has a nontrivial reducing subspace on $\mathscr{D}_0$ if and only if $a + b = 0$. Moreover, if $a + b = 0$, he shows that $M_{\varphi_a\varphi_b}$ has only two nontrivial reducing subspaces on $\mathscr{D}_0$.

Motivated by the result of Zhao, we consider the same problem on the Dirichlet space $\mathscr{D}$ for multiplication operators induced by finite Blaschke products. In Section 2, we first give a characterization of reducibility of multiplication operators induced by general finite Blaschke products on $\mathscr{D}$ (Proposition 2.2). As an application, we show that $M_{\varphi_a\varphi_b}$ is reducible on $\mathscr{D}$ if and only if $a + b = 0$ (Theorem 2.5). Also, for a multiplication operator M_ϕ induced by a general finite Blaschke product ϕ, we point out that nonreducibility of M_ϕ on $\mathscr{D}$ implies the same on $\mathscr{D}_0$. As an immediate consequence, we recover the result of Zhao [Zh1] mentioned above.

In Section 3, we consider the problem of when a multiplication operator induced by a multiplier on $\mathscr{D}$ is unitarily equivalent to a weighted unilateral shift of finite multiplicity. The corresponding problem on the Bergman space has been studied in [GZ] and [SZZ]. Also, the same problem on $\mathscr{D}_0$ has been studied by Zhao [Zh2].

We first characterize multipliers on $\mathscr{D}$ for which the corresponding multiplication operator is unitarily equivalent to a weighted unilateral shift of finite multiplicity (Proposition 2.5). In particular, we show that for a multiplier $\phi \in M(\mathscr{D})$, M_ϕ is unitarily equivalent to M_{z^2} on $\mathscr{D}$ if and only if $\phi = \lambda z^2$ for some unimodular constant λ (Theorem 3.2).

2. Reducibility of multiplication operators.

In this section, we give a characterization of multiplication operators induced by finite Blaschke products and having a reducing subspace. We start with an observation on analytic branches of ϕ^{-1} for a finite Blaschke product ϕ.

Let ϕ be a finite Blaschke product with N zeros and let $\{\beta_1, \ldots, \beta_N\}$ be N analytic branches of ϕ^{-1} on a simply connected set $\mathbb{D} \setminus L$ where L is a curve connecting all finite points in $\{\phi(z) : z \in \mathbb{D},\ \phi'(z) = 0\}$ and a point ξ_0 in $\mathbb{T}$. For $k = 1, \ldots, N$, put $D_k = \beta_k(\mathbb{D} \setminus L)$ and $T_k = \beta_k(\mathbb{T} \setminus \{\xi_0\})$. Then $D_j \cap D_k = \emptyset$ and $T_j \cap T_k = \emptyset$ for all $j \neq k$. Also, we have

$$\mathbb{D} \setminus \phi^{-1}(L) = \bigcup_{k=1}^{N} D_k, \qquad \mathbb{T} \setminus \phi^{-1}(\{\xi_0\}) = \bigcup_{k=1}^{N} T_k.$$

Note that $\phi \circ \beta_k(\lambda) = \lambda$ for all $\lambda \in \mathbb{D} \setminus L$ and $\phi \circ \beta_k(\xi) = \xi$ for all $\lambda \in \mathbb{T} \setminus \{\xi_0\}$. Also, $\beta_k \circ \phi|_{D_k}(\lambda) = \lambda$ for all $\lambda \in D_k$ and $\beta_k \circ \phi|_{T_k}(\xi) = \xi$ for all $\lambda \in T_k$. For the details one is referred to [DPW], [DSZ] or [GH].

For a function $f \in \mathscr{D}$, the function

$$\big(f(\beta_1(z)), \ldots, f(\beta_N(z))\big), \qquad z \in \mathbb{D} \setminus L,$$

is a vector-valued analytic function on $\mathbb{D} \setminus L$. Since $\mathscr{D} \subset H^2$ and each function in H^2 admits a nontangential limit at almost all points in $\mathbb{T}$, we see that to each $f \in \mathscr{D}$ corresponds a unique vector-valued function

$$\big(f(\beta_1(\xi)), \ldots, f(\beta_N(\xi))\big)$$

for almost all points $\xi \in \mathbb{T}$. These vector-valued functions will play an important role in our characterizations of reducibility and unitary equivalence.

Let $\mathcal{H} = \mathscr{D}$ or $\mathcal{H} = H^2$. For $\psi \in \mathcal{H}$ and a multiplier ϕ on $\mathcal{H}$, we shall denote by $[\psi]_{\phi,\mathcal{H}}$ the smallest invariant subspace of M_ϕ generated by ψ in $\mathcal{H}$.

Namely, $[\psi]_{\phi,\mathcal{H}}$ is the closure of the set of linear combinations of functions of the form $\phi^k\psi$ for $k = 0, 1, 2, \ldots$.

The following result generalizes the arguments for local inverse in [GH] on the Bergman space, and provides a practical way towards solutions for the reducing subspace problem. It also shows that there is a useful connection between the smallest invariant subspaces on the Dirichlet space and Hardy space for the multiplication operator induced by a finite Blaschke product. Note that

$$\langle f, g\rangle = \int_{\mathbb{D}} (zf)'\,\overline{(zg)'}\,dA \tag{2.1}$$

for all $f, g \in \mathscr{D}$.

Proposition 2.1. *Let ϕ be a finite Blaschke product with N zeros. Suppose $f, g \in \mathscr{D} \ominus \phi\mathscr{D}$ and $f \perp g$. Then the following conditions are equivalent:*

- (a) $[f]_{\phi,\mathscr{D}} \perp [g]_{\phi,\mathscr{D}}$.
- (b) $[\xi\phi' f]_{\phi,H^2} \perp [g]_{\phi,H^2}$.
- (c) $\big(f(\beta_1(\xi)), \ldots, f(\beta_N(\xi))\big) \perp \big(g(\beta_1(\xi)), \ldots, g(\beta_N(\xi))\big)$ *for almost all* $\xi \in \mathbb{T}$.

Proof. Put $F = zf$ and $G = zg$ for simplicity. Note that F and G are analytic on the closed unit disk $\overline{\mathbb{D}}$ (see (2.3) below). By (2.1) and Stokes' theorem, we first note that

$$\begin{aligned}
\langle \phi^m f, \phi^n g\rangle &= \int_{\mathbb{D}} (\phi^m F)'\,\overline{(\phi^n G)'}\,dA \\
&= \frac{-1}{2\pi i}\int_{\mathbb{D}} \frac{\partial}{\partial \bar z}\big((\phi^m F)'\,\overline{\phi^n G}\big)\,dz \wedge d\bar z = \frac{1}{2\pi i}\int_{\mathbb{T}} (\phi^m F)'\,\overline{\phi^n G}\,d\xi \\
&= \int_{\mathbb{T}} \xi F'\phi^{m-n}\overline{G}\,\frac{|d\xi|}{2\pi} + m\int_{\mathbb{T}} \xi F\phi'\phi^{m-n-1}\,\overline{G}\,\frac{|d\xi|}{2\pi} \\
&= \int_{\mathbb{T}} \xi F'\phi^{m-n}\overline{G}\,\frac{|d\xi|}{2\pi} + m\int_{\mathbb{T}} \xi f\phi'\phi^{m-n-1}\overline{g}\,\frac{|d\xi|}{2\pi}
\end{aligned} \tag{2.2}$$

for any integers $m, n \geq 0$.

First suppose (a). By (2.2), we have

$$\int_{\mathbb{T}} \xi F'\phi^{m-n}\overline{G}\,\frac{|d\xi|}{2\pi} = -m\int_{\mathbb{T}} \xi f\phi'\phi^{m-n-1}\overline{g}\,\frac{|d\xi|}{2\pi}$$

for any integers $m, n \geq 0$. Thus, given integers t and $\ell > 0$ with $\ell > t$, the above equation shows that

$$\int_{\mathbb{T}} \xi f\phi'\phi^{t-1}\overline{g}\,\frac{|d\xi|}{2\pi} = \frac{-1}{\ell}\int_{\mathbb{T}} \xi F'\phi^{t}\overline{G}\,\frac{|d\xi|}{2\pi}.$$

Now fixing an integer t and letting $\ell \to \infty$, we see that

$$\int_{\mathbb{T}} \xi f \phi' \phi^{t-1} \overline{g} \, d\sigma(\xi) = 0$$

for all integers t, which implies (b).

Now we prove the equivalence of (b) and (c). For each integer k, we note that

$$\begin{aligned}\int_{\mathbb{T}} \xi \phi' \overline{\phi} \phi^k f \overline{g} \, \frac{|d\xi|}{2\pi} &= \sum_{j=1}^{N} \int_{T_j} \frac{\xi \phi' \phi^k f \overline{g}}{\phi} \, \frac{|d\xi|}{2\pi} = \sum_{j=1}^{N} \int_{\mathbb{T}} \eta^k f(\beta_j(\eta)) \, \overline{g(\beta_j(\eta))} \, \frac{|d\eta|}{2\pi} \\ &= \int_{\mathbb{T}} \eta^k \sum_{j=1}^{N} f(\beta_j(\eta)) \, \overline{g(\beta_j(\eta))} \, \frac{|d\eta|}{2\pi}\end{aligned}$$

where we used the change of variables $\eta = \phi(\xi)$, $\xi \in \mathbb{T}$, with

$$|d\eta| = \frac{\xi \phi'(\xi)}{\phi(\xi)} |d\xi|,$$

which gives the equivalence of (b) and (c).

Finally, we assume (b) and prove (a). By (2.2), it suffices to show that

$$I(t) := \int_{\mathbb{T}} \xi F' \phi^t \overline{G} \, \frac{|d\xi|}{2\pi} = 0$$

for every integer t. For $t = 0$, (2.2) shows that

$$I(0) = \int_{\mathbb{T}} \xi F' \overline{G} \, \frac{|d\xi|}{2\pi} = \langle f, g \rangle = 0$$

because $f \perp g$ by the assumption. For $t < 0$, we note that

$$\overline{I(t)} = \int_{\mathbb{T}} \phi^{-t} G \, \overline{\xi F'} \, \frac{|d\xi|}{2\pi}.$$

Since ϕ is a finite Blaschke product with N zeros, we may write

$$\phi = z^M \varphi_{a_1}^{M_1} \cdots \varphi_{a_k}^{M_k}$$

where $M + M_1 + \cdots + M_k = N$, $a_j \neq 0$ and $a_i \neq a_j$ for all $i \neq j$. Then it is not hard to see that the space $z(\mathscr{D} \ominus \phi \mathscr{D})$ is spanned by the functions

$$z, z^2, \ldots, z^M, \log \frac{1}{1 - \overline{a_i} z}, \frac{z^j}{(1 - \overline{a_i} z)^j}, \qquad j = 1, \ldots, M_i - 1,$$

for $i = 1, \ldots, k$. Since $F = zf \in z(\mathscr{D} \ominus \phi \mathscr{D})$, we can write

$$F(z) = \sum_{j=1}^{M} a_j z^j + \sum_{j=1}^{k} b_j \log \frac{1}{1 - \overline{a_j} z} + \sum_{i=1}^{k} \sum_{j=1}^{M_i - 1} c_{ij} \frac{z^j}{(1 - \overline{a_i} z)^j} \tag{2.3}$$

for some constants a_j, b_j and c_{ij}. Then

$$zF'(z) = \sum_{j=1}^{M} ja_j z^j + \sum_{j=1}^{k} \frac{b_j \overline{a_j} z}{1-\overline{a_j}z} + \sum_{i,j} jc_{ij}\left[\frac{z^j}{(1-\overline{a_i}z)^j} + \frac{\overline{a_i}z^{j+1}}{(1-\overline{a_i}z)^{j+1}}\right] \tag{2.4}$$

for all $z \in \mathbb{D}$. Recalling $\phi^{-t}G$ has a zero of order at least $M+1$ at 0, we have

$$\int_{\mathbb{T}} \phi^{-t} G \overline{\xi^j}\, \frac{|d\xi|}{2\pi} = \frac{(\phi^{-t}G)^{(j)}(0)}{j!} = 0 \tag{2.5}$$

for all $j = 1, \ldots, M$. Also, since $\phi(a_j) = 0$ for $j = 1, \ldots, k$, it follows from the reproducing property of the Hardy space that

$$\begin{aligned}\int_{\mathbb{T}} \frac{\phi^{-t}(\xi)G(\xi)a_j\overline{\xi}}{1-a_j\overline{\xi}}\, \frac{|d\xi|}{2\pi} &= \int_{\mathbb{T}} \phi^{-t}(\xi)G(\xi)\left(\frac{1}{1-a_j\overline{\xi}} - 1\right)\frac{|d\xi|}{2\pi} \\ &= \phi^{-t}(a_j)G(a_j) - \phi^{-t}(0)G(0) = 0\end{aligned} \tag{2.6}$$

for all $j = 1, \ldots, k$. Also, since $\xi/(1-\overline{a}\xi) = 1/\overline{\xi - a_j}$ for each $a \in \mathbb{D}$ and $\xi \in \mathbb{T}$, we have, by the Cauchy integral formula,

$$\begin{aligned}\int_{\mathbb{T}} \frac{\phi^{-t}(\xi)G(\xi)\overline{\xi^j}}{(1-a_i\overline{\xi})^j}\, \frac{|d\xi|}{2\pi} &= \int_{\mathbb{T}} \frac{\phi^{-t}(\xi)G(\xi)}{(\xi - a_i)^j}\, \frac{|d\xi|}{2\pi} = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\phi^{-t}(\xi)G(\xi)}{\xi(\xi-a_i)^j}\, d\xi \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\phi^{-t}(\xi)g(\xi)}{(\xi-a_i)^j}\, d\xi = \frac{1}{(j-1)!}(\phi^{-t}g)^{(j-1)}(a_i)\end{aligned} \tag{2.7}$$

for all $i = 1, \ldots, k$ and $j = 2, \ldots, M_i$. But, since $\phi = z^M \varphi_{a_1}^{M_1} \cdots \varphi_{a_k}^{M_k}$, we can easily see that $(\phi^{-t}g)^{(j-1)}(a_i) = 0$ for all $i = 1, \ldots, k$ and $j = 2, \ldots, M_i$. Now, combining (2.5)–(2.7) with (2.4), we see that $I(t) = 0$ for $t < 0$.

Finally, consider the case $t > 0$. Using integration by parts and condition (b), we have

$$\begin{aligned}I(t) &= \frac{1}{2\pi i}\int_0^{2\pi} \phi^t(e^{i\theta})\,\overline{G(e^{i\theta})}\, dF(e^{i\theta}) = \frac{-1}{2\pi i}\int_0^{2\pi} F(e^{i\theta})\frac{d}{d\theta}[\phi^t(e^{i\theta})\,\overline{G(e^{i\theta})}]\, d\theta \\ &= \frac{-t}{2\pi i}\int_0^{2\pi} F(e^{i\theta})\phi^{t-1}(e^{i\theta})\phi'(e^{i\theta})ie^{i\theta}\,\overline{G(e^{i\theta})}\, d\theta \\ &\quad - \frac{1}{2\pi i}\int_0^{2\pi} F(e^{i\theta})\phi^t(e^{i\theta})\,\overline{G'(e^{i\theta})ie^{i\theta}}\, d\theta \\ &= -t\int_{\mathbb{T}} \xi f\phi'\phi^{t-1}\overline{g}\, \frac{|d\xi|}{2\pi} + \int_{\mathbb{T}} F\phi^t\,\overline{\xi G'}\, \frac{|d\xi|}{2\pi} = \int_{\mathbb{T}} F\phi^t\,\overline{\xi G'}\, \frac{|d\xi|}{2\pi}.\end{aligned}$$

Now, using the same argument as in the case $t < 0$, we can see that $I(t) = 0$ for $t > 0$, as desired. The proof is complete. ■

A theorem of Richter [Ric, Theorem 1] says that if S is a two-isometry operator on $\mathscr{D}$ which satisfies $\bigcap_{n>0} S^n\mathscr{D} = \{0\}$, then any invariant subspace $\mathcal{N}$ of S must be of the form

$$\mathcal{N} = \bigvee_{n\geq 0} S^n(\mathcal{N} \ominus S\mathcal{N}).$$

Also, it is known that a multiplier ψ on $\mathscr{D}$ is a two-isometry if and only if ψ is an inner function (see [RS, Theorem 4.2] for details). So, if ϕ is a finite Blaschke product, we have

$$\mathscr{D} = \bigvee_{n\geq 0} \phi^n(\mathscr{D} \ominus \phi\mathscr{D}). \tag{2.8}$$

In addition, if $\mathcal{M}$ is a reducing subspace for M_ϕ in $\mathscr{D}$, we have

$$\mathcal{M} = \bigvee_{n\geq 0} \phi^n(\mathcal{M} \ominus \phi\mathcal{M}), \qquad \mathcal{M}^\perp = \bigvee_{n\geq 0} \phi^n(\mathcal{M}^\perp \ominus \phi\mathcal{M}^\perp). \tag{2.9}$$

Now we are ready to prove the main result of this section which characterizes reducibility of multiplication operators induced by finite Blaschke products.

THEOREM 2.2. *Let ϕ be a finite Blaschke product with N zeros. Then the following conditions are equivalent:*

(a) *M_ϕ is reducible on $\mathscr{D}$.*
(b) *There exist nonempty sets $E_1 \neq \{0\}$ and $E_2 \neq \{0\}$ in $\mathscr{D}$ such that:*
  (b1) *$E_1 \perp E_2$.*
  (b2) *$\mathscr{D} \ominus \phi\mathscr{D} = \operatorname{span}(E_1 \cup E_2)$.*
  (b3) *For any $f \in E_1$ and $g \in E_2$, we have*

$$\big(f(\beta_1(\xi)), \ldots, f(\beta_N(\xi))\big) \perp \big(g(\beta_1(\xi)), \ldots, g(\beta_N(\xi))\big)$$

for almost all $\xi \in \mathbb{T}$.

Proof. Suppose (a) and let $\mathcal{M}$ be a nontrivial reducing subspace of M_ϕ. Let $E_1 = \mathcal{M} \ominus \phi\mathcal{M}$ and $E_2 = \mathcal{M}^\perp \ominus \phi\mathcal{M}^\perp$. Since E_1, E_2 are all contained in $\mathscr{D} \ominus \phi\mathscr{D}$ and $\mathscr{D} \ominus \phi\mathscr{D} = E_1 \oplus E_2$, we see that (b1) and (b2) hold. Also, for any $f \in E_1$ and $g \in E_2$, we see that $[f]_{\phi,\mathscr{D}} \subset \mathcal{M}$ and $[g]_{\phi,\mathscr{D}} \subset \mathcal{M}^\perp$ by (2.9). So, (b3) is a consequence of Proposition 2.1.

Now, to prove (b)⇒(a), put

$$\mathcal{M} = \bigvee_{n\geq 0} \phi^n \operatorname{span}(E_1), \qquad \mathcal{N} = \bigvee_{n\geq 0} \phi^n \operatorname{span}(E_2).$$

Then $\mathcal{M}$ and $\mathcal{N}$ are invariant under M_ϕ and $\mathcal{M} \perp \mathcal{N}$ by Proposition 2.1.

On the other hand, (2.8) implies

$$\bigvee_{n\geq 0} \phi^n \operatorname{span}(E_1 \cup E_2) = \bigvee_{n\geq 0} \phi^n(\mathscr{D} \ominus \phi\mathscr{D}) = \mathscr{D}.$$

It follows easily that $\mathcal{N} = \mathcal{M}^{\perp}$ and $\mathcal{M}$ is a nontrivial reducing subspace for M_ϕ, thus M_ϕ is reducible. ■

As an application of Theorem 2.2, we characterize the reducibility of multiplication operators induced by finite Blaschke products with two zeros. First we consider the case when the zeros are the same.

PROPOSITION 2.3. *Let $\phi = \varphi_a^2$ where $a \in \mathbb{D}$. Then M_ϕ is reducible on $\mathscr{D}$ if and only if $a = 0$.*

Proof. First suppose M_ϕ is reducible on $\mathscr{D}$ and $a \neq 0$. By Theorem 2.2, we can choose nonzero $f, g \in \mathscr{D} \ominus \phi\mathscr{D}$ such that $[f]_{\phi,\mathscr{D}} \perp [g]_{\phi,\mathscr{D}}$. Put $F = zf$ and $G = zg$. Since $F, G \in z(\mathscr{D} \ominus \phi\mathscr{D})$ and

$$z(\mathscr{D} \ominus \phi\mathscr{D}) = \operatorname{span}\left\{\log\frac{1}{1-\overline{a}z}, \frac{z}{1-\overline{a}z}\right\},$$

we can write

$$F = c_1 \log\frac{1}{1-\overline{a}z} + c_2\frac{z}{1-\overline{a}z}, \qquad G = d_1 \log\frac{1}{1-\overline{a}z} + d_2\frac{z}{1-\overline{a}z}$$

for some constants c_1, c_2, d_1 and d_2.Then, by Proposition 2.1 and (2.2),

$$J(t) := \int_{\mathbb{T}} \xi F' \phi^t \overline{G}\, \frac{|d\xi|}{2\pi} = 0$$

for all integers t. By the reproducing property for the Hardy space, we note

$$\begin{aligned}\int_{\mathbb{T}} \xi F'(\xi)\phi^t(\xi)\frac{\overline{\xi}}{1-a\overline{\xi}}\,\frac{|d\xi|}{2\pi} &= \frac{1}{a}\int_{\mathbb{T}} \xi F'(\xi)\phi^t(\xi)\left(\frac{1}{1-a\overline{\xi}} - 1\right)\frac{|d\xi|}{2\pi}\\ &= \frac{1}{a}[(zF'\phi^t)(a) - (zF'\phi^t)(0)] = 0\end{aligned}$$

for every $t > 0$. It follows that

$$J(t) = \overline{d_1}\int_{\mathbb{T}} \xi F'\phi^t\, \overline{\log\frac{1}{1-\overline{a}\xi}}\,\frac{|d\xi|}{2\pi} = 0 \tag{2.10}$$

for all integers t. In particular, since $J(1) = 0$, we have

$$\overline{d_1}\int_{\mathbb{T}} \xi F'(\xi)\phi(\xi)\, \overline{\log\frac{1}{1-\overline{a}\xi}}\,\frac{|d\xi|}{2\pi} = 0. \tag{2.11}$$

Note $\phi = \varphi_a^2$ and

$$F' = \frac{c_1\overline{a}}{1-\overline{a}z} + \frac{c_2}{(1-\overline{a}z)^2}.$$

Thus, by the Cauchy integral formula,

$$\begin{aligned}\int_{\mathbb{T}} \log \frac{1}{1-\overline{a}\xi}\, \overline{\xi F'(\xi)\phi(\xi)}\, \frac{|d\xi|}{2\pi} &= \int_{\mathbb{T}} \overline{\xi} \log \frac{1}{1-\overline{a}\xi} \overline{\left(\frac{\overline{a}c_1(a-\xi)^2}{(1-\overline{a}\xi)^3}+\frac{c_2(a-\xi)^2}{(1-\overline{a}\xi)^4}\right)} \frac{|d\xi|}{2\pi}\\ &= \int_{\mathbb{T}} \log \frac{1}{1-\overline{a}\xi} \left(\frac{a\overline{c_1}(1-\overline{a}\xi)^2}{\xi(\xi-a)^3}+\frac{\overline{c_2}(1-\overline{a}\xi)^2}{(\xi-a)^4}\right) \frac{d\xi}{2\pi i}\\ &= \frac{1}{2}\overline{c_1}\frac{2\log(1-|a|^2)+2|a|^2+|a|^4}{a^2}-\frac{1}{3}\overline{c_2}\frac{\overline{a}^3}{1-|a|^2}.\end{aligned}$$

Thus, if $d_1 \neq 0$, from (2.11) we have

$$c_1\left[6\log(1-|a|^2)+6|a|^2+3|a|^4\right] = c_2\frac{2a|a|^4}{1-|a|^2}. \tag{2.12}$$

Also, since $J(2)=0$, from (2.10) we have again

$$d_1 \int_{\mathbb{T}} \log \frac{1}{1-\overline{a}\xi}\, \overline{\xi F'\phi^2}\, \frac{|d\xi|}{2\pi} = 0.$$

On the other hand, one can see as before that

$$\begin{aligned}&\int_{\mathbb{T}} \log \frac{1}{1-\overline{a}\xi}\, \overline{\xi F'\phi^2}\, \frac{|d\xi|}{2\pi}\\ &\quad = \frac{2}{4!}\overline{c_1}\frac{12\log(1-|a|^2)+12|a|^2+6|a|^4+4|a|^6+3|a|^8}{a^4} - \frac{24}{5!}\overline{c_2}\frac{\overline{a}^5}{1-|a|^2}.\end{aligned}$$

Thus, if $d_1 \neq 0$ we get

$$5c_1\left[12\log(1-|a|^2)+12|a|^2+6|a|^4+4|a|^6+3|a|^8\right] = c_2\frac{12a|a|^8}{1-|a|^2}.$$

Hence, if $d_1 \neq 0$, $c_1 \neq 0$ and $c_2 \neq 0$, we see using (2.12) that

$$(72x^2-60)\log(1-x) = 60x+30x^2-52x^3-21x^4$$

where $x=|a|^2$, which is impossible because $x \neq 0$.

If $d_1 \neq 0$, $c_1 \neq 0$ and $c_2 = 0$, (2.12) implies

$$2\log(1-x)+2x+x^2 = 0,$$

which is impossible too because $x = |a|^2 \neq 0$.

Also, if $d_1 \neq 0$, $c_2 \neq 0$ and $c_1 = 0$, we have by (2.12), $a|a|^4 = 0$ and then $a = 0$, which is a contradiction. Hence d_1 must be 0. Also, by replacing the roles of F and G in the proof above, we see that c_1 must be 0 too. It follows that

$$F = \frac{c_2 z}{1-\overline{a}z}, \qquad G = \frac{d_2 z}{1-\overline{a}z}.$$

Since $\langle f, g\rangle = 0$, by (2.1) and the reproducing property of the Bergman space [Zhu2, Chapter 4], we have

$$0 = \int_{\mathbb{D}} F'\overline{G'}\, dA = c_2\overline{d_2}\frac{1}{(1-|a|^2)^2},$$

which gives $c_2 = 0$ or $d_2 = 0$, so in turn $F = 0$ or $G = 0$, which is a contradiction. Therefore we conclude that $a = 0$.

The converse implication follows from the result of Stessin and Zhu [SZ] as mentioned at the introduction. ∎

We say that two Blaschke products ϕ_1 and ϕ_2 are *equivalent* if there exists $\lambda \in \mathbb{D}$ such that

$$\phi_1 = \varphi_\lambda \circ \phi_2.$$

If two Blaschke products ϕ_1 and ϕ_2 are equivalent, it turns out that M_{ϕ_1} and M_{ϕ_1} share reducing subspaces (see Lemmas 2.1 and 2.2 in [Zh1], for example).

For a function ϕ analytic on $\mathbb{D}$, we recall that $c \in \mathbb{D}$ is a *critical point* of ϕ if $\phi'(c) = 0$. Bochner's theorem [W1, W2] says that every Blaschke product with N zeros has exactly $N-1$ critical points in $\mathbb{D}$.

Now suppose the Blaschke product $\phi = \varphi_a\varphi_b$ has two zeros $a, b \in \mathbb{D}$. Then ϕ has only one critical point c in $\mathbb{D}$. If we let $\phi_1 = \varphi_{\phi(c)} \circ \phi$, then it is easy to see that $\phi_1 = \lambda\varphi_c^2$ for some unimodular constant λ and hence ϕ is equivalent to φ_c^2. Note that $(\varphi_a\varphi_b)'(z) = 0$ if and only if

$$\big[\overline{b}(|a|^2-1) + \overline{a}(|b|^2-1)\big]z^2 + 2(1-|a|^2|b|^2)z + b(|a|^2-1) + a(|b|^2-1) = 0.$$

It follows that 0 is the solution of the equation $(\varphi_a\varphi_b)'(z) = 0$ if and only if $a + b = 0$. Also, if ϕ is equivalent to z^2, we see the only critical point of ϕ is 0. So the observation above gives the following lemma.

LEMMA 2.4. *Let $\phi = \varphi_a\varphi_b$ where $a, b \in \mathbb{D}$. Then ϕ is equivalent to φ_c^2 where c is the critical point of ϕ. In particular, ϕ is equivalent to z^2 if and only if $a + b = 0$.*

Now we characterize the reducibility of M_ϕ with $\phi = \varphi_a\varphi_b$.

THEOREM 2.5. *Let $\phi = \varphi_a\varphi_b$ where $a, b \in \mathbb{D}$. Then M_ϕ is reducible on $\mathscr{D}$ if and only if $a+b = 0$. In this case, M_ϕ has exactly two nontrivial reducing subspaces.*

Proof. Let c be the critical point of ϕ. By Lemma 2.4, ϕ is equivalent to φ_c^2. By Proposition 2.3, M_ϕ is reducible if and only if ϕ is equivalent to z^2, which is in turn equivalent to $a+b = 0$ by Lemma 2.4 again. The remaining assertion follows from the result of Stessin–Zhu [SZ]. ∎

Recall that $\mathscr{D}_0$ is the space of all functions f analytic on $\mathbb{D}$ for which

$$\|f\|_0 = \Big(|f(0)|^2 + \int_{\mathbb{D}} |f'|^2\, dA\Big)^{1/2} < \infty.$$

In the rest of this section, we find a relation between reducibility of multiplication operators on the spaces $\mathscr{D}$ and $\mathscr{D}_0$. To do this, we need the following useful lemma, taken from [Zh2]. For $a \in \mathbb{D}$, we let $P_a(\xi) = \frac{1-|a|^2}{|\xi-a|^2}$ be the Poisson kernel on $\mathbb{D}$. Also, we let $\langle f, g\rangle_0 = f(0)\overline{g(0)} + \langle f, g\rangle_*$ where

$$\langle f, g\rangle_* = \int_{\mathbb{D}} f'\overline{g'}\, dA$$

for $f, g \in \mathscr{D}_0$.

Lemma 2.6. *Let $\phi = \varphi_{a_1}\cdots\varphi_{a_N}$ where $a_1, \dots, a_N \in \mathbb{D}$. Then*

$$\langle \phi^m f, \phi^m g\rangle_0 = m\int_{\mathbb{T}}\sum_{j=1}^{N} f\overline{g}P_{a_j}\frac{|d\xi|}{2\pi} + \langle f, g\rangle_0 + (|\phi(0)|^{2m} - 1)f(0)\overline{g(0)}$$

for all $f, g \in \mathscr{D}_0$ and integers $m > 0$.

In the following result, we let $\widetilde{\mathscr{D}}$ be the space of all f analytic on $\mathbb{D}$ for which $f(0) = 0$ and

$$\int_{\mathbb{D}} |f'|^2\, dA < \infty.$$

Note that $\widetilde{\mathscr{D}}$ is a Hilbert space with respect to the inner product $\langle\ ,\ \rangle_*$.

Proposition 2.7. *Let ϕ be a finite Blaschke product. If $\mathcal{M}$ is a nontrivial reducing subspace of M_ϕ on $\mathscr{D}_0$, then $(1/z)\mathcal{M}$ or $(1/z)\mathcal{M}^\perp$ is a nontrivial reducing subspace of M_ϕ on $\mathscr{D}$. Hence, the reducibility of M_ϕ on $\mathscr{D}_0$ implies the same on $\mathscr{D}$.*

Proof. Put $\phi = \varphi_{\lambda_1}\cdots\varphi_{\lambda_N}$ where $\lambda_1, \dots, \lambda_N \in \mathbb{D}$. Let $\mathcal{M}$ be a nontrivial reducing subspace of M_ϕ on $\mathscr{D}_0$. Let $f \in \mathcal{M}$ and $g \in \mathcal{M}^\perp$. Since $\langle \phi^m f, \phi^m g\rangle_0 = 0$, it follows from Lemma 2.6 that

$$\int_{\mathbb{T}}\sum_{k=1}^{N} f\overline{g}P_{\lambda_k}\frac{|d\xi|}{2\pi} = -\frac{1}{m}(|\phi(0)|^{2m} - 1)f(0)\overline{g(0)}$$

for every integer $m > 0$. Taking $m \to \infty$ in the above, we see that

$$\int_{\mathbb{T}}\sum_{k=1}^{N} f\overline{g}P_{\lambda_k}\frac{|d\xi|}{2\pi} = 0$$

and hence $(|\phi(0)|^{2m} - 1)f(0)\overline{g(0)} = 0$ for all integers $m > 0$. Thus either $f(0) = 0$ or $g(0) = 0$. Decompose $1 = \epsilon + \zeta$ where $\epsilon \in \mathcal{M}$ and $\zeta \in \mathcal{M}^\perp$. By the observation above, $\epsilon(0) = 0$ or $\zeta(0) = 0$. If $\epsilon(0) = 0$, then

$$\langle \epsilon, \epsilon\rangle_0 = \langle \epsilon, 1 - \zeta\rangle_0 = \langle \epsilon, 1\rangle_0 = \epsilon(0) = 0.$$

Hence $\epsilon = 0$ and so $1 \in \mathcal{M}^\perp$. Similarly, $\zeta(0) = 0$ implies $1 \in \mathcal{M}$. Thus either $1 \in \mathcal{M}$ or $1 \in \mathcal{M}^\perp$.

Assume $1 \in \mathcal{M}$. Since $\langle f, 1\rangle_0 = f(0)$, we see $f(0) = 0$ for any $f \in \mathcal{M}^\perp$ and hence $\mathcal{M}^\perp \subset \widetilde{\mathscr{D}}$. Let $\mathcal{N} = \{f - f(0) : f \in \mathcal{M}\}$. Since $1 \in \mathcal{M}$, we see that $\mathcal{N} = \{f \in \mathcal{M} : f(0) = 0\}$ and $\mathscr{D}_0 = \mathbb{C} \oplus \mathcal{N} \oplus \mathcal{M}^\perp$. Note $\phi\mathcal{N} \subset \mathcal{N}$, $\phi\mathcal{M}^\perp \subset \mathcal{M}^\perp$ and $\mathcal{N} \perp \mathcal{M}^\perp$. Thus $\mathcal{M}^\perp$ is a nontrivial reducing subspace for M_ϕ on $\widetilde{\mathscr{D}}$. Consider the multiplication operator $M_z : \mathscr{D} \to \widetilde{\mathscr{D}}$. By (2.1), it is easy to see that M_z is a unitary operator. Moreover, M_ϕ on $\mathscr{D}$ is unitarily equivalent to M_ϕ on $\widetilde{\mathscr{D}}$ via M_z. Thus $M_z^{-1}\mathcal{M}^\perp = \frac{1}{z}\mathcal{M}^\perp$ is a nontrivial reducing subspace for M_ϕ on $\mathscr{D}$.

Similarly, if $1 \in \mathcal{M}^\perp$, we see that $(1/z)\mathcal{M}$ is a nontrivial reducing subspace of M_ϕ on $\mathscr{D}$. ■

The above result says that nonreducibility of M_ϕ on $\mathscr{D}$ implies the same on $\mathscr{D}_0$. Thus, by Theorem 2.5, together with Lemma 2.4 and the result of Stessin–Zhu [SZ], we see that $M_{\varphi_a\varphi_b}$ is reducible on $\mathscr{D}_0$ if and only if $a + b = 0$, which is already noticed in [Zh1].

3. Unitary equivalence to M_{z^2}.

In this section, we characterize multipliers on $\mathscr{D}$ for which the corresponding multiplication operator is unitarily equivalent to a weighted shift of multiplicity 2. We first characterize finite Blaschke products for which the corresponding multiplication operator on $\mathscr{D}$ is unitarily equivalent to a weighted unilateral shift of finite multiplicity:

Proposition 3.1. *Let ϕ be a finite Blaschke product with N zeros. Then the following are equivalent:*

- (a) *M_ϕ on $\mathscr{D}$ is unitarily equivalent to a weighted unilateral shift of finite multiplicity N.*
- (b) *There exists an orthogonal set $\{f_1, \dots, f_N\} \subset \mathscr{D} \ominus \phi\mathscr{D}$ for which the $N \times N$ matrix*

$$U(\xi) := \begin{pmatrix} f_1(\beta_1(\xi)) & f_2(\beta_1(\xi)) & \cdots & f_N(\beta_1(\xi)) \\ f_1(\beta_2(\xi)) & f_2(\beta_2(\xi)) & \cdots & f_N(\beta_2(\xi)) \\ \vdots & \vdots & \cdots & \vdots \\ f_1(\beta_N(\xi)) & f_2(\beta_N(\xi)) & \cdots & f_N(\beta_N(\xi)) \end{pmatrix}$$

is unitary for almost all $\xi \in \mathbb{T}$.

Proof. First assume (a). Then there is an orthogonal set $\{g_1, \dots, g_N\}$ in $\mathscr{D} \ominus \phi\mathscr{D}$ such that

$$\langle \phi^m g_j, \phi^n g_k\rangle = 0 \tag{3.1}$$

for $m, n \geq 0$ and $j \neq k$. Also,

$$\langle \phi^m g_k, \phi^n g_k\rangle = 0 \tag{3.2}$$

for $m \neq n$ and $k = 1, \ldots, N$. By Proposition 2.1, (3.1) is equivalent to

$$\sum_{l=1}^{N} g_j(\beta_l(\xi))\, \overline{g_k(\beta_l(\xi))} = 0, \qquad j \neq k, \tag{3.3}$$

for almost all $\xi \in \mathbb{T}$. Also, by a similar argument to that for Proposition 2.1, we see that (3.2) is equivalent to

$$\sum_{l=1}^{N} |g_k(\beta_l(\xi))|^2 = c_k, \qquad k = 1, \ldots, N, \tag{3.4}$$

for almost all $\xi \in \mathbb{T}$ and some constants c_k. Now, letting $f_k = g_k/\sqrt{c_k}$ for $k = 1, \ldots, N$, we see from (3.3) and (3.4) that the matrix $U(\xi)$ is unitary for almost all $\xi \in \mathbb{T}$, so we have (a)⇒(b).

For (b)⇒(a), it is obvious that (b) implies (3.3) and (3.4), and in turn (3.1) and (3.2) hold. Then it is easy to see that M_ϕ is unitarily equivalent to a weighted unilateral shift of finite multiplicity N, so (a) holds. ■

Now we characterize multipliers on $\mathscr{D}$ which are unitarily equivalent to a weighted shift of multiplicity 2.

THEOREM 3.2. *Let $\phi \in \mathcal{M}(\mathscr{D})$. Then M_ϕ is unitarily equivalent to M_{z^2} if and only if $\phi = \lambda z^2$ for some unimodular constant λ.*

Proof. First suppose M_ϕ is unitarily equivalent to M_{z^2}. By Lemma 2.1 of [Zh2], we see that $\phi = \lambda \varphi_a \varphi_b$ for some $a, b \in \mathbb{D}$ and unimodular constant λ. Notice that M_{z^2} is reducible, and hence an application of Theorem 2.5 implies $\phi = \lambda \varphi_a \varphi_{-a}$.

Now we show $a = 0$. If $a \neq 0$, it follows that there exist nonzero $g_1, g_2 \in \mathscr{D} \ominus \phi\mathscr{D}$ such that

$$\langle \phi^m g_1, \phi^n g_2 \rangle = 0, \qquad m, n \geq 0, \tag{3.5}$$

$$\langle \phi^m g_1, \phi^n g_1 \rangle = \langle \phi^m g_2, \phi^n g_2 \rangle = 0, \qquad m \neq n. \tag{3.6}$$

Since the set $z(\mathscr{D} \ominus \phi\mathscr{D})$ is spanned by the functions $-\log(1 - \overline{a}z)$ and $-\log(1 + \overline{a}z)$, we write

$$\begin{aligned} F &:= zg_1 = -c_1 \log(1 - \overline{a}z) - c_2 \log(1 + \overline{a}z), \\ G &:= zg_2 = -d_1 \log(1 - \overline{a}z) - d_2 \log(1 + \overline{a}z) \end{aligned}$$

for some constants c_1, c_2, d_1 and d_2 and then

$$F' = \frac{c_1\overline{a}}{1 - \overline{a}z} - \frac{c_2\overline{a}}{1 + \overline{a}z}.$$

On the other hand, since $0 = \langle g_1, g_2 \rangle = \langle F, G \rangle_*$, it follows that

$$(\overline{c_1}d_1 + \overline{c_2}d_2) \log(1 - |a|^2) = -(\overline{c_1}d_2 + \overline{c_2}d_1) \log(1 + |a|^2). \tag{3.7}$$

Since $0<|a|<1$, the above shows that there are two cases to consider:

$$\text{Case 1:}\quad \overline{c_1}d_1+\overline{c_2}d_2\neq 0 \quad\text{and}\quad \overline{c_1}d_2+\overline{c_2}d_1\neq 0.$$
$$\text{Case 2:}\quad \overline{c_1}d_1+\overline{c_2}d_2= 0 \quad\text{and}\quad \overline{c_1}d_2+\overline{c_2}d_1= 0.$$

First consider Case 1. By (3.5) and (2.2), together with Proposition 2.1,

$$\int_{\mathbb{T}} \xi F'\phi\overline{G}\,\frac{|d\xi|}{2\pi}=0.$$

Using a similar argument to the proof of Proposition 2.3, we see that the above equation yields

$$\begin{aligned}\overline{c_1}d_1\int_{\mathbb{T}}\frac{(1-\overline{a}^2\xi^2)\log(1-\overline{a}\xi)}{(\xi-a)^2(\xi+a)\xi}\frac{d\xi}{2\pi i}+\overline{c_1}d_2\int_{\mathbb{T}}\frac{(1-\overline{a}^2\xi^2)\log(1+\overline{a}\xi)}{(\xi-a)^2(\xi+a)\xi}\frac{d\xi}{2\pi i}\\ =\overline{c_2}d_1\int_{\mathbb{T}}\frac{(1-\overline{a}^2\xi^2)\log(1-\overline{a}\xi)}{(\xi+a)^2(\xi-a)\xi}\frac{d\xi}{2\pi i}+\overline{c_2}d_2\int_{\mathbb{T}}\frac{(1-\overline{a}^2\xi^2)\log(1+\overline{a}\xi)}{(\xi+a)^2(\xi-a)\xi}\frac{d\xi}{2\pi i}.\end{aligned}$$

By applying the Cauchy integral formula in the above equation, we see that

$$\begin{aligned}&(\overline{c_1}d_1+\overline{c_2}d_2)[(1-x^2)\log(1+x)+(x^2+3)\log(1-x)+2x(1+x)]\\ &\quad=-(\overline{c_1}d_2+\overline{c_2}d_1)[(1-x^2)\log(1-x)+(x^2+3)\log(1+x)-2x(1-x)]\end{aligned}$$

where $x=|a|^2\in(0,1)$. It follows from (3.7) that

$$\frac{(1-x^2)\log(1-x)-2x(1-x)}{(1-x^2)\log(1+x)+2x(1+x)}=\frac{\log(1+x)}{\log(1-x)}.$$

Now, we prove the above is impossible. To do this, we define

$$\begin{aligned}h(t)&=[(1-t^2)\log(1-t)-2t(1-t)]\log(1-t)\\ &\quad-[(1-t^2)\log(1+t)+2t(1+t)]\log(1+t),\qquad t\in(0,1).\end{aligned}$$

Since $h(0)=0$, it suffices to prove $h'(t)<0$ for all $t\in(0,1)$. Since

$$\begin{aligned}h'(t)&=2t[\log^2(1+t)-\log^2(1-t)]\\ &\quad+2t[\log(1-t)-\log(1+t)]-4[\log(1-t)+\log(1+t)],\end{aligned}$$

we see by simple calculations that

$$h'(t)=4\sum_{j=1}^{\infty}\left(\frac{j}{(j+1)(2j+1)}-\sum_{k+m=2j+1}\frac{1}{km}\right)t^{2j+2},$$

which is negative because

$$\begin{aligned}\sum_{k+m=2j+1}\frac{1}{km}&=2\left[\frac{1}{2j}+\frac{1}{2(2j-1)}+\cdots+\frac{1}{j(j+1)}\right]\\ &>\frac{2j}{(j+1)(2j+1)}>\frac{j}{(j+1)(2j+1)}\end{aligned}$$

for all j. Thus Case 1 cannot happen.

Now, consider Case 2. Since $(c_1, c_2) \neq (0,0)$, we have either $(c_1, d_1) = (c_2, -d_2)$ or $(c_1, d_1) = (-c_2, d_2)$. Without loss of generality, we may assume $(c_1, d_1) = (-c_2, d_2) = (1,1)$ and then

$$F = -\log(1-\overline{a}z) + \log(1+\overline{a}z).$$

Note

$$F' = \frac{\overline{a}}{1-\overline{a}z} + \frac{\overline{a}}{1+\overline{a}z}.$$

By (3.6) and (2.2), together with the proof of Proposition 2.1,

$$\int\limits_{\mathbb{T}} \xi F' \phi \overline{F} \, \frac{|d\xi|}{2\pi} = 0.$$

By the same arguments as in Case 1, the above equation gives

$$\begin{aligned}(1-x^2)\log(1+x) &+ (x^2+3)\log(1-x) + 2x(1+x) \\ &= (1-x^2)\log(1-x) + (x^2+3)\log(1+x) - 2x(1-x)\end{aligned}$$

where $x = |a|^2$. Thus we have

$$(1+x^2)[\log(1-x) - \log(1+x)] + 2x = 0.$$

But a simple calculation shows that the above is impossible for all $x \in (0,1)$, which gives a contradiction. Hence $a = 0$ and so $\phi = \lambda z^2$, as desired.

The converse implication is clear. ∎

We close this paper with a remark that for $\phi = \varphi_a \varphi_{-a}$ with $a \neq 0$, the proof of Theorem 3.2 also shows that two nontrivial reducing subspaces of M_ϕ are $[f]_{\phi,\mathscr{D}}$ and $[g]_{\phi,\mathscr{D}}$ where

$$f = \frac{1}{z}\left(\log\frac{1}{1-\overline{a}z} - \log\frac{1}{1+\overline{a}z}\right), \quad g = \frac{1}{z}\left(\log\frac{1}{1-\overline{a}z} + \log\frac{1}{1+\overline{a}z}\right).$$

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