STUDIA MATHEMATICA 220 (2) (2014)

Lineability and algebrability of the set of holomorphic functions with a given domain of existence

by

THIAGO R. ALVES (Campinas)

Abstract. We show that if U is a domain of existence in a separable Banach space, then the set of holomorphic functions on U whose domain of existence is U is lineable and algebrable.

1. Introduction. Let U be an open subset of a complex Banach space E. Let $\mathcal{H}(U)$ denote the algebra of all holomorphic functions on U, and let $\mathcal{E}(U)$ denote the set of all $f \in \mathcal{H}(U)$ such that U is the domain of existence of f. Informally U is the domain of existence of f if f cannot be holomorphically extended beyond the boundary of U. The precise definition of domain of existence will be provided in the next section.

In this paper we first show that, if E is separable and U is a domain of existence, then $\mathcal{E}(U)$ is *lineable*, that is, there is an infinite-dimensional subspace $\mathcal{F}$ of $\mathcal{H}(U)$ such that $\mathcal{F} \subset \mathcal{E}(U) \cup \{0\}$. Next we show that, under the same hypotheses, $\mathcal{E}(U)$ is $\mathfrak{c}$-*lineable*, that is, there is a $\mathfrak{c}$-dimensional subspace $\mathcal{F}$ of $\mathcal{H}(U)$ such that $\mathcal{F} \subset \mathcal{E}(U) \cup \{0\}$. Here $\mathfrak{c}$ denotes the cardinality of the continuum. Finally we show that, under the same hypotheses, $\mathcal{E}(U)$ is *algebrable*, that is, there is a subalgebra $\mathcal{A}$ of $\mathcal{H}(U)$, generated by an infinite algebraically independent set, such that $\mathcal{A} \subset \mathcal{E}(U) \cup \{0\}$.

The notion of lineable set appeared for the first time in [1], and many authors have devoted their attention to the study of lineable sets and algebrable sets during the last decade. We refer the reader to [2] for a survey of this recent trend in functional analysis.

2. Lineability and $\mathfrak{c}$-lineability of $\mathcal{E}(U)$. We recall that U is the *domain of existence* of a function $f \in \mathcal{H}(U)$ if there are no open sets V and W in E and no function $\widetilde{f} \in \mathcal{H}(V)$ such that

2010 *Mathematics Subject Classification*: Primary 46G20, 46J10; Secondary 15A03.
Key words and phrases: Banach space, holomorphic function, domain of existence, lineable set, algebrable set.

DOI: 10.4064/sm220-2-4

(a) V is connected and $V \not\subset U$;
(b) $\emptyset \neq W \subset U \cap V$;
(c) $\widetilde{f} = f$ on W.

Given $A \subset U$, let

$$\widehat{A}_{\mathcal{H}(U)} = \Big\{ x \in U : |f(x)| \leq \sup_A |f| \text{ for all } f \in \mathcal{H}(U) \Big\}.$$

Given $x \in U$, let $d_U(x)$ denote the distance from x to the boundary of U, and let $B(x)$ denote the ball $B(x) = B(x; d_U(x))$. For $A \subset U$, let $d_U(A) = \inf_{x \in A} d_U(x)$.

Before proving our first theorem, we need two preparatory lemmas.

LEMMA 2.1 ([4, Theorem 11.4]). *Let E be a separable Banach space and U be an open subset of E. Then U is a domain of existence if and only if U is the union of an increasing sequence of open sets A_j such that $d_U((\widehat{A_j})_{\mathcal{H}(U)}) > 0$ for every j.*

LEMMA 2.2. *Let E be a Banach space, U be an open subset in E and $(c_j)_{j=0}^{\infty}$ be a sequence of positive numbers. Let $(A_j)_{j=1}^{\infty}$ be a sequence of open subsets of U and let $(y_j)_{j=1}^{\infty}$ be a sequence of points of U such that*

$$U = \bigcup_{j=1}^{\infty} A_j, \qquad A_j \subset A_{j+1}, \qquad y_j \notin (\widehat{A_j})_{\mathcal{H}(U)} \quad \textit{and} \quad y_j \in (\widehat{A_{j+1}})_{\mathcal{H}(U)}$$

for each $j \in \mathbb{N}$. Then there exists a sequence $(f_j)_{j=1}^{\infty}$ of functions in $\mathcal{H}(U)$ such that

$$f = \sum_{j=1}^{\infty} f_j \in \mathcal{H}(U), \qquad \sup_{B_j} |f_j| \leq 2^{-j} c_0 \quad \textit{and} \quad |f(y_j)| \geq c_j$$

for each $j \in \mathbb{N}$, where $B_j := (\widehat{A_j})_{\mathcal{H}(U)}$.

Proof. First of all, notice that $B_j = (\widehat{B_j})_{\mathcal{H}(U)}$ for each $j \in \mathbb{N}$. Indeed, we will just prove the nontrivial inclusion $(\widehat{B_j})_{\mathcal{H}(U)} \subset B_j$. Let $z \in (\widehat{B_j})_{\mathcal{H}(U)}$. Then

$$|f(z)| \leq \sup_{w \in B_j} |f(w)| \leq \sup_{w \in B_j} \sup_{A_j} |f| = \sup_{A_j} |f|$$

for all $f \in \mathcal{H}(U)$. Hence $z \in (\widehat{A_j})_{\mathcal{H}(U)} = B_j$, as desired.

Since $y_j \notin B_j = (\widehat{B_j})_{\mathcal{H}(U)}$ for each $j \in \mathbb{N}$, we can find a sequence $(\varphi_j)_{j=1}^{\infty}$ in $\mathcal{H}(U)$ and a sequence $(b_j)_{j=1}^{\infty}$ in $\mathbb{R}$ such that

$$\sup_{B_j} |\varphi_j| < b_j < |\varphi_j(y_j)|$$

for each $j \in \mathbb{N}$. Therefore, if we set $\psi_j := \varphi_j/b_j$ for each $j \in \mathbb{N}$, then

$$\sup_{B_j} |\psi_j| < 1 < |\psi_j(y_j)|$$

for each $j \in \mathbb{N}$. It follows that

$$\lim_{n\to\infty} \sup_{B_j} |\psi_j|^n = 0 \quad \text{and} \quad \lim_{n\to\infty} |\psi_j(y_j)|^n = \infty \tag{2.1}$$

for each $j \in \mathbb{N}$. Thus, it follows from (2.1) that there exists $n_1 \in \mathbb{N}$ so that $f_1 := \psi_1^{n_1}$ satisfies

$$\sup_{B_1} |f_1| \le 2^{-1}c_0 \quad \text{and} \quad |f_1(y_1)| \ge c_1 + c_0.$$

By applying (2.1) again, we can find $n_2 \in \mathbb{N}$ such that $f_2 := \psi_2^{n_2}$ satisfies

$$\sup_{B_2} |f_2| \le 2^{-2}c_0 \quad \text{and} \quad |f_2(y_2)| \ge c_2 + c_0 + |f_1(y_2)|.$$

Now, if we apply (2.1) again, we obtain $n_3 \in \mathbb{N}$ so that $f_3 := \psi_3^{n_3}$ satisfies

$$\sup_{B_3} |f_3| \le 2^{-3}c_0 \quad \text{and} \quad |f_3(y_3)| \ge c_3 + c_0 + \sum_{i<3} |f_i(y_3)|.$$

Inductively, we find a sequence $(f_j)_{j=1}^{\infty}$ in $\mathcal{H}(U)$ such that

$$\sup_{B_j} |f_j| \le 2^{-j}c_0 \quad \text{and} \quad |f_j(y_j)| \ge c_j + c_0 + \sum_{i<j} |f_i(y_j)| \tag{2.2}$$

for each $j \in \mathbb{N}$. It follows that the series $\sum_{j=1}^{\infty} f_j$ converges uniformly on each B_j to a function $f \in \mathcal{H}(U)$. Furthermore,

$$|f(y_j)| = \Big|\sum_{i=1}^{\infty} f_i(y_j)\Big| \ge |f_j(y_j)| - \sum_{i<j} |f_i(y_j)| - c_0 \tag{2.3}$$

for each $j \in \mathbb{N}$. Now (2.2) and (2.3) imply $|f(y_j)| \ge c_j$ for each $j \in \mathbb{N}$. ∎

The preceding proof is based on [4, Theorem 11.4].

Theorem 2.3. *Let E be a separable Banach space and U be a domain of existence in E. If D is a countable dense subset of U, then the set*

$$\mathcal{F}(U) := \Big\{ f \in \mathcal{H}(U) : \sup_{z\in B(x)} |f(z)| = \infty \text{ for all } x \in D \Big\}$$

is lineable.

Proof. The proof consists in the construction of a linearly independent sequence $(f_k)_{k=1}^{\infty}$ in $\mathcal{F}(U)$ such that $\mathcal{F}(U) \cup \{0\}$ contains the subspace spanned by $(f_k)_{k=1}^{\infty}$. Let $(x_j)_{j=1}^{\infty}$ be a sequence in D such that each point of D appears in $(x_j)_{j=1}^{\infty}$ infinitely many times. Since U is a domain of existence in E, Lemma 2.1 shows that U is the union of an increasing sequence of open sets A_j such that $d_U((\widehat{A_j})_{\mathcal{H}(U)}) > 0$ for every $j \in \mathbb{N}$. Set $B_j := (\widehat{A_j})_{\mathcal{H}(U)}$ for each $j \in \mathbb{N}$, and recall that $B(x) := B(x; d_U(x))$. Notice that $B(x) \not\subset B_j$

for each $x \in D$ and $j \in \mathbb{N}$. Thus, after replacing (B_j) by a suitable subsequence, we can find a sequence $(y_j)_{j=1}^{\infty}$ in U such that $y_j \in B(x_j)$, $y_j \notin B_j$ and $y_j \in B_{j+1}$ for each $j \in \mathbb{N}$. Accordingly, by Lemma 2.2 we can find $f_1 \in \mathcal{H}(U)$ so that $|f_1(y_j)| \geq j$ for all $j \in \mathbb{N}$. Applying Lemma 2.2 again, we find $f_2 \in \mathcal{H}(U)$ such that $|f_2(y_j)| \geq j(1 + |f_1(y_j)|)$ for each $j \in \mathbb{N}$. Again by Lemma 2.2, we obtain $f_3 \in \mathcal{H}(U)$ so that $|f_3(y_j)| \geq j(1 + \sum_{i<3} |f_i(y_j)|)$ for each $j \in \mathbb{N}$. Continuing, we construct a sequence $(f_k)_{k=1}^{\infty}$ in $\mathcal{H}(U)$ such that

$$|f_1(y_j)| \geq j \quad \text{and} \quad |f_k(y_j)| \geq j\Big(1 + \sum_{i<k} |f_i(y_j)|\Big) \tag{2.4}$$

for each $k \in \mathbb{N} \setminus \{1\}$ and $j \in \mathbb{N}$.

We assert that $(f_k)_{k=1}^{\infty}$ is as desired. Indeed, first let us prove that $\mathcal{F}(U) \cup \{0\}$ contains the subspace spanned by $(f_k)_{k=1}^{\infty}$: Let $f := \lambda_1 f_1 + \cdots + \lambda_n f_n$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $\lambda_n \neq 0$ and $n \in \mathbb{N}$. We recall that each point of D appears in the sequence $(x_j)_{j=1}^{\infty}$ infinitely many times. Since $y_j \in B(x_j)$ for each $j \in \mathbb{N}$, $\lim_{j\to\infty} |f(y_j)| = \infty$ implies that f is an unbounded function on $B(x)$ for each $x \in D$. Therefore it is sufficient to verify that $\lim_{j\to\infty} |f(y_j)| = \infty$. Indeed,

$$|f(y_j)| = |\lambda_1 f_1(y_j) + \cdots + \lambda_n f_n(y_j)| \geq |\lambda_n f_n(y_j)| - \sum_{i<n} |\lambda_i f_i(y_j)| \tag{2.5}$$

for each $j \in \mathbb{N}$. Thus by (2.4) and (2.5) we easily obtain

$$\begin{aligned} |f(y_j)| &\geq j|\lambda_n|\Big(1 + \sum_{i<n} |f_i(y_j)|\Big) - \sum_{i<n} |\lambda_i f_i(y_j)| \\ &= j|\lambda_n| + \sum_{i<n} (j|\lambda_n| - |\lambda_i|)|f_i(y_j)| \end{aligned}$$

for each $j \in \mathbb{N}$, and so $|f(y_j)| \geq j|\lambda_n|$ for j large enough. Hence $\lim_{j\to\infty} |f(y_j)| = \infty$.

Now we prove that the sequence $(f_k)_{k=1}^{\infty}$ is linearly independent. Let $\lambda_1 f_1 + \cdots + \lambda_n f_n = 0$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $n \in \mathbb{N}$. We suppose that $\lambda_n \neq 0$. Repeating the argument of the previous paragraph, we obtain

$$0 = |\lambda_1 f_1(y_j) + \cdots + \lambda_n f_n(y_j)| \geq j|\lambda_n|$$

for j large enough, a contradiction. It follows that $\lambda_n = 0$. Now suppose that $\lambda_{n-1} \neq 0$, so using the same argument we obtain

$$0 = |\lambda_1 f_1(y_j) + \cdots + \lambda_{n-1} f_{n-1}(y_j)| \geq j|\lambda_{n-1}|$$

for j large enough, a contradiction again. Hence we get $\lambda_{n-1} = 0$. Continuing, we conclude that $\lambda_1 = \cdots = \lambda_n = 0$. ∎

The next lemma is well known and can be found in [4, Theorem 11.4].

LEMMA 2.4. *Let E be a Banach space, U be an open subset in E, and D be a dense subset of U. If $f \in \mathcal{H}(U)$ is an unbounded function on $B(x) = B(x; d_U(x))$ for each $x \in D$, then U is the domain of existence of f.*

Proof. Suppose that U is not the domain of existence of f. Thus we can find subsets V and W in U satisfying (a)–(c) of the definition of domain of existence. Without loss of generality we may assume that W is a connected component of $U \cap V$. Consider a point $a \in V \cap \partial U \cap \partial W$, and let $r > 0$ be such that $B(a; 2r) \subset V$. Take a point $x \in D \cap W \cap B(a; r)$. Since $x \in B(a; r)$ and $a \in \partial U$ we obtain $d_U(x) < r$, and therefore $B(x) \subset B(a; 2r) \subset V$. Hence $B(x) \subset U \cap V$ and $x \in W$. Since $B(x)$ is connected, it follows that $B(x) \subset W$. Moreover, since $f = \tilde{f}$ on W and f is unbounded on $B(x)$, it follows that $\tilde{f}$ is unbounded on $B(x) \subset B(a; 2r)$. Hence $\tilde{f}$ is not locally bounded at a, as $r > 0$ can be taken arbitrarily small. ■

THEOREM 2.5. *Let E be a separable Banach space and U be a domain of existence in E. Then the set $\mathcal{E}(U)$ is lineable.*

Proof. By Lemma 2.4, $\mathcal{F}(U) \subset \mathcal{E}(U)$, and therefore Theorem 2.5 follows from Theorem 2.3. ■

We finish this section with two theorems which tell us that $\mathcal{E}(U)$ is $\mathfrak{c}$-lineable.

THEOREM 2.6. *Let E be a separable Banach space and U be a domain of existence in E. If $(x_j)_{j=1}^{\infty}$ is a dense sequence in U, then the set*

$$\mathcal{F}(U) := \Big\{ g \in \mathcal{H}(U) : \sup_{z \in B(x_j)} |g(z)| = \infty \text{ for all } j \in \mathbb{N} \Big\}$$

is $\mathfrak{c}$-lineable.

Proof. By a result of [5], $\ell_2 \setminus \ell_1$ is $\mathfrak{c}$-lineable. Let Λ be a $\mathfrak{c}$-dimensional subspace of ℓ_2 such that $\Lambda \subset (\ell_2 \setminus \ell_1) \cup \{0\}$. The proof will be based on the construction of a sequence $(g_k)_{k=1}^{\infty}$ in $\mathcal{H}(U)$ such that the set

$$H := \Big\{ \sum_{k=1}^{\infty} \lambda_k g_k : (\lambda_k)_{k=1}^{\infty} \in \Lambda \Big\}$$

satisfies the following conditions:

(i) H is a $\mathfrak{c}$-dimensional subspace of $\mathcal{H}(U)$.
(ii) $H \subset \mathcal{F}(U) \cup \{0\}$.

We begin by constructing the sequence $(g_k)_{k=1}^{\infty}$. Since U is a domain of existence in E, Lemma 2.1 shows that U is the union of an increasing sequence of open sets A_j such that $d_U((\widehat{A_j})_{\mathcal{H}(U)}) > 0$ for every $j \in \mathbb{N}$. Set $B_j := (\widehat{A_j})_{\mathcal{H}(U)}$ for each $j \in \mathbb{N}$. Since $B(x_j) \not\subset B_k$ for each $j \in \mathbb{N}$ and $k \in \mathbb{N}$, we can find a subsequence $(B_{1,j})_{j=1}^{\infty}$ of $(B_j)_{j=1}^{\infty}$ and a sequence $(y_{1,j})_{j=1}^{\infty}$ in U such that

$$y_{1,j} \in B(x_j), \quad y_{1,j} \notin B_{1,j} \quad \text{and} \quad y_{1,j} \in B_{1,j+1}$$

for each $j \in \mathbb{N}$. Applying Lemma 2.2 we can find a sequence $(g_{1,j})_{j=1}^{\infty}$ in $\mathcal{H}(U)$ such that

$$g_1 := \sum_{j=1}^{\infty} g_{1,j} \in \mathcal{H}(U), \quad \sup_{B_{1,j}} |g_{1,j}| \leq 2^{-j-1} \quad \text{and} \quad |g_1(y_{1,j})| \geq 1$$

for every $j \in \mathbb{N}$. Likewise, since $B(x_j) \not\subset B_{1,k}$ for every $j \in \mathbb{N}$ and $k \in \mathbb{N}$, we can find a subsequence $(B_{2,j})_{j=1}^{\infty}$ of $(B_{1,j})_{j=1}^{\infty}$ and a sequence $(y_{2,j})_{j=1}^{\infty}$ in U such that

$$B_{1,2} \subset B_{2,1}, \quad y_{2,j} \in B(x_j), \quad y_{2,j} \notin B_{2,j} \quad \text{and} \quad y_{2,j} \in B_{2,j+1}$$

for each $j \in \mathbb{N}$. Therefore, by Lemma 2.2 again, we obtain a sequence $(g_{2,j})_{j=1}^{\infty}$ in $\mathcal{H}(U)$ such that

$$g_2 := \sum_{j=1}^{\infty} g_{2,j} \in \mathcal{H}(U), \quad \sup_{B_{2,j}} |g_{2,j}| \leq 2^{-j-2} \quad \text{and} \quad |g_2(y_{2,j})| \geq 2^2(1+|g_1(y_{2,j})|)$$

for every $j \in \mathbb{N}$. Repeating this argument we inductively construct a subsequence $(B_{k,j})_{j=1}^{\infty}$ of $(B_{k-1,j})_{j=1}^{\infty}$, a sequence $(y_{k,j})_{j=1}^{\infty}$ in U, and a sequence $(g_{k,j})_{j=1}^{\infty}$ in $\mathcal{H}(U)$ such that

$$B_{k-1,k} \subset B_{k,1}, \quad y_{k,j} \in B(x_j), \quad y_{k,j} \notin B_{k,j}, \quad y_{k,j} \in B_{k,j+1}, \tag{2.6}$$

$$g_k := \sum_{j=1}^{\infty} g_{k,j} \in \mathcal{H}(U), \quad \sup_{B_{k,j}} |g_{k,j}| \leq 2^{-j-k}, \tag{2.7}$$

and

$$|g_k(y_{k,j})| \geq k^2\Big(1 + \sum_{i<k} |g_i(y_{k,j})|\Big) \tag{2.8}$$

for every $k \in \mathbb{N} \setminus \{1\}$ and $j \in \mathbb{N}$.

We assert that $(g_k)_{k=1}^{\infty}$ is as desired. Indeed, first $\sum_{k=1}^{\infty} \lambda_k g_k \in \mathcal{H}(U)$ whenever $(\lambda_k)_{k=1}^{\infty} \in \Lambda$. Since $(B_{k,1})_{k=1}^{\infty}$ is a subsequence of $(B_j)_{j=1}^{\infty}$, it is sufficient to prove that $(\sum_{k=1}^{n} \lambda_k g_k)_{n=1}^{\infty}$ converges uniformly on $B_{p,1}$ for each $p \in \mathbb{N}$. In fact, if $z \in B_{p,1}$, then

$$|\lambda_i g_i(z)| \leq \sum_{j=1}^{\infty} |\lambda_i g_{i,j}(z)| \leq \sum_{j=1}^{\infty} |\lambda_i| 2^{-i-j} = |\lambda_i| 2^{-i}$$

for each $i \geq p$, and applying the Weierstrass M-test we obtain the desired result. Hence H is a subspace of $\mathcal{H}(U)$.

Next we show that $\mathcal{F}(U) \cup \{0\}$ contains H. Let $g := \sum_{k=1}^{\infty} \alpha_k g_k$, where $(\alpha_k)_{k=1}^{\infty} \in \Lambda \subset \ell_2 \setminus \ell_1$. Since $(\alpha_m)_{m=1}^{\infty} \notin \ell_1$ and $(1/m^2)_{m=1}^{\infty} \in \ell_1$, we have

$(\alpha_m m^2)_{m=1}^\infty \notin \ell_\infty$. Therefore there exists a subsequence $(\alpha_{m_i})_{i=1}^\infty$ of $(\alpha_m)_{m=1}^\infty$ such that

$$\lim_{i\to\infty} |\alpha_{m_i}| m_i^2 = \infty. \tag{2.9}$$

Fix $j \in \mathbb{N}$. Notice that $B_{m,j} \subset B_{m,m+1} \subset B_{m+1,1}$ for every $m > j$. Take the subsequence $(\alpha_{m_i})_{i=1}^\infty$ given in (2.9), and consider $i_0 \in \mathbb{N}$ such that $m_i > j$ whenever $i > i_0$. Then, using (2.6) and (2.7), we have

$$\begin{aligned}
|g(y_{m_i,j})| &\geq |\alpha_{m_i} g_{m_i}(y_{m_i,j})| - \sum_{k<m_i} |\alpha_k g_k(y_{m_i,j})| - \sum_{k>m_i} |\alpha_k g_k(y_{m_i,j})| \\
&\geq |\alpha_{m_i} g_{m_i}(y_{m_i,j})| - \sum_{k<m_i} |\alpha_k g_k(y_{m_i,j})| - \sum_{k>m_i} \Big(|\alpha_k| \sum_{t=1}^\infty |g_{k,t}(y_{m_i,j})|\Big) \\
&\geq |\alpha_{m_i} g_{m_i}(y_{m_i,j})| - \sum_{k<m_i} |\alpha_k g_k(y_{m_i,j})| - \sum_{k>m_i} \Big(|\alpha_k| \sum_{t=1}^\infty 2^{-k-t}\Big) \\
&= |\alpha_{m_i} g_{m_i}(y_{m_i,j})| - \sum_{k<m_i} |\alpha_k g_k(y_{m_i,j})| - \sum_{k>m_i} 2^{-k}|\alpha_k| \\
&\geq |\alpha_{m_i} g_{m_i}(y_{m_i,j})| - \sum_{k<m_i} |\alpha_k g_k(y_{m_i,j})| - \|(2^{-k}\alpha_k)_{k=1}^\infty\|_1
\end{aligned}$$

for each $i > i_0$, and therefore

$$\begin{aligned}
|g(y_{m_i,j})| &\geq |\alpha_{m_i}| m_i^2 \Big(1 + \sum_{k<m_i} |g_k(y_{m_i,j})|\Big) - \sum_{k<m_i} |\alpha_k g_k(y_{m_i,j})| - \|(2^{-k}\alpha_k)_{k=1}^\infty\|_1 \\
&\geq |\alpha_{m_i}| m_i^2 \Big(1 + \sum_{k<m_i} |g_k(y_{m_i,j})|\Big) - \sup_k |\alpha_k| \sum_{k<m_i} |g_k(y_{m_i,j})| - \|(2^{-k}\alpha_k)_{k=1}^\infty\|_1 \\
&= |\alpha_{m_i}| m_i^2 + (|\alpha_{m_i}| m_i^2 - \sup_k |\alpha_k|) \sum_{k<m_i} |g_k(y_{m_i,j})| - \|(2^{-k}\alpha_k)_{k=1}^\infty\|_1
\end{aligned}$$

for each $i > i_0$, where the first inequality follows from (2.8). Thus (2.9) shows that $|g(y_{m_i,j})| \geq |\alpha_{m_i}| m_i^2 - \|(2^{-k}\alpha_k)_{k=1}^\infty\|_1$ for i large enough. Therefore, applying (2.9) again, we obtain $\lim_{i\to\infty} |g(y_{m_i,j})| = \infty$. Since $y_{m_i,j} \in B(x_j)$ for every i, we conclude that g is unbounded on $B(x_j)$. Since $j \in \mathbb{N}$ is arbitrary, it follows that $g \in \mathcal{F}(U)$.

The subspace H is $\mathfrak{c}$-dimensional, since the linear transformation $T : \Lambda \to \mathcal{H}(U)$ defined by $T((\lambda_k)_{k=1}^\infty) := \sum_{k=1}^\infty \lambda_k g_k$ is injective and $T(\Lambda) = H$. Indeed, we verify the first assertion: Let $(\alpha_{1,k})_{k=1}^\infty \neq (\alpha_{2,k})_{k=1}^\infty$ in Λ. Suppose that $T((\alpha_{1,k})_{k=1}^\infty) = T((\alpha_{2,k})_{k=1}^\infty)$. Since $(\alpha_{1,k} - \alpha_{2,k})_{k=1}^\infty \in \ell_2 \setminus \ell_1$, we can find a subsequence $(\alpha_{1,m_i} - \alpha_{2,m_i})_{i=1}^\infty$ such that

$$\lim_{i\to\infty} |\alpha_{1,m_i} - \alpha_{2,m_i}| m_i^2 = \infty.$$

Thus repeating the argument of the previous paragraph, we obtain

$$\begin{aligned}|T((\alpha_{1,k})_{k=1}^{\infty})(y_{m_i,1}) - T((\alpha_{2,k})_{k=1}^{\infty})(y_{m_i,1})| \\ \geq |\alpha_{1,m_i} - \alpha_{2,m_i}|m_i^2 - \|(2^{-k}(\alpha_{1,k} - \alpha_{2,k}))_{k=1}^{\infty}\|_1\end{aligned}$$

for i large enough, a contradiction. Therefore T is injective. ∎

THEOREM 2.7. *Let E be a separable Banach space and U be a domain of existence in E. Then the set $\mathcal{E}(U)$ is $\mathfrak{c}$-lineable.*

Proof. By Lemma 2.4, $\mathcal{F}(U) \subset \mathcal{E}(U)$, and therefore Theorem 2.7 follows from Theorem 2.6. ∎

It is clear that Theorem 2.5 follows from Theorem 2.7, and we could have omitted the proof of Theorem 2.5. However, we have decided to give both proofs because the proof of Theorem 2.5 is much simpler and the ideas involved help to understand better the proof of Theorem 2.7.

3. Algebrability of $\mathcal{E}(U)$

THEOREM 3.1. *Let E be a separable Banach space, and let U be a domain of existence in E. If D is a countable dense subset of U, then the set*

$$\mathcal{F}(U) := \Big\{ f \in \mathcal{H}(U) : \sup_{z \in B(x)} |f(z)| = \infty \text{ for all } x \in D \Big\}$$

is algebrable. In particular, $\mathcal{F}(U)$ is lineable.

Proof. We shall construct an algebraically independent sequence $(f_k)_{k=1}^{\infty}$ in $\mathcal{F}(U)$ such that $\mathcal{F}(U) \cup \{0\}$ contains the subalgebra generated by $(f_k)_{k=1}^{\infty}$. We begin by repeating an argument of the proof of Theorem 2.3. Let $(x_j)_{j=1}^{\infty}$ be a sequence in D such that each point of D appears in $(x_j)_{j=1}^{\infty}$ infinitely many times. Since U is a domain of existence in E, it follows from Lemma 2.1 that U is the union of an increasing sequence of open sets A_j such that $d_U((\widehat{A_j})_{\mathcal{H}(U)}) > 0$ for every $j \in \mathbb{N}$. Set $B_j := (\widehat{A_j})_{\mathcal{H}(U)}$ for each $j \in \mathbb{N}$, and notice that $B(x) \not\subset B_j$ for each $x \in D$ and $j \in \mathbb{N}$. Thus, after replacing $(B_j)_{j=1}^{\infty}$ by a suitable subsequence, we can find a sequence $(y_j)_{j=1}^{\infty}$ in U such that $y_j \in B(x_j)$, $y_j \notin B_j$ and $y_j \in B_{j+1}$ for each $j \in \mathbb{N}$. Then, by applying Lemma 2.2, we can inductively construct a sequence $(f_k)_{k=1}^{\infty}$ in $\mathcal{H}(U)$ such that

$$|f_1(y_j)| \geq j \quad \text{and} \quad |f_k(y_j)| \geq \prod_{i<k} |f_i(y_j)|^j \tag{3.1}$$

for each $k \in \mathbb{N} \setminus \{1\}$ and $j \in \mathbb{N}$.

Now let us show that $(f_k)_{k=1}^{\infty}$ is as desired. First we shall see that $\mathcal{F}(U) \cup \{0\}$ contains the subalgebra $\mathcal{A}$ generated by $(f_k)_{k=1}^{\infty}$, i.e. the set of all functions of the form $P(f_1, \ldots, f_n)$, where $n \in \mathbb{N}$ and P is a polynomial in n variables *without constant term.* Observe that $\mathcal{A} \setminus \{0\} = \bigcup_{N=1}^{\infty} \mathcal{A}_N$, where

$$\mathcal{A}_N := \Big\{ \sum_{n=1}^{N} \lambda_n \prod_{s\in S_n} f_s^{p_{n,s}} : \lambda_n \in \mathbb{C}\setminus\{0\},\, \emptyset \neq S_n \subset \mathbb{N} \text{ finite and } p_{n,s} \in \mathbb{N} \Big\}$$

for every $N \in \mathbb{N}$. For each $\sum_{n=1}^{N} \lambda_n \prod_{s\in S_n} f_s^{p_{n,s}} \in \mathcal{A}_N$, we can suppose that if $n \neq m$ then there does not exist a constant $\alpha \in \mathbb{C}$ such that $\lambda_n \prod_{s\in S_n} f_s^{p_{n,s}} = \alpha\lambda_m \prod_{s\in S_m} f_s^{p_{m,s}}$.

We shall prove that $\mathcal{A}_N \subset \mathcal{F}(U)$ for each $N \in \mathbb{N}$, so that $\mathcal{A} \subset \mathcal{F}(U) \cup \{0\}$. Since $y_j \in B(x_j)$ for each $j \in \mathbb{N}$, $\lim_{j\to\infty} |f(y_j)| = \infty$ implies that f is unbounded on $B(x)$ for each $x \in D$. Thus we just need to verify the following assertion:

$$N \in \mathbb{N},\ f \in \mathcal{A}_N \ \Rightarrow\ \lim_{j\to\infty} |f(y_j)| = \infty. \tag{3.2}$$

We will prove it by induction on N. It is clearly true for $N = 1$. We suppose it is true for all $J \leq N$, and we take

$$f := \sum_{n=1}^{N+1} \lambda_n \prod_{s\in S_n} f_s^{p_{n,s}} \in \mathcal{A}_{N+1}. \tag{3.3}$$

Set $m_0 := \max \bigcup_{k=1}^{N+1} S_k$. First we consider the case where f_{m_0} does not appear in some term in the summation (3.3). Without loss of generality we can assume that f_{m_0} does not appear in the first M terms in (3.3), where $M \in \{1, \ldots, N\}$. Thus we can write

$$f = \sum_{n=1}^{M} \Big(\lambda_n \prod_{s\in S_n} f_s^{p_{n,s}}\Big) + h f_{m_0},$$

where

$$h := \sum_{n=M+1}^{N+1} \Big[\lambda_n \prod_{s\in S_n\setminus\{m_0\}} \big(f_s^{p_{n,s}} f_{m_0}^{p_{n,m_0}-1}\big)\Big].$$

Notice that either $h \in \mathbb{C}\setminus\{0\}$ or $h = h_1 + \alpha$, where $h_1 \in \mathcal{A}_{N-M} \cup \mathcal{A}_{N+1-M}$ and $\alpha \in \mathbb{C}$. In the second case, by the induction hypothesis we obtain $\lim_{j\to\infty} |h_1(y_j)| = \infty$, and so $\lim_{j\to\infty} |h(y_j)| = \infty$. Thus, we can always find $\delta > 0$ and $j_0 \in \mathbb{N}$ such that $|h(y_j)| \geq \delta$ whenever $j \geq j_0$. Furthermore,

$$\begin{aligned} |f(y_j)| &= \Big| \sum_{n=1}^{M} \Big(\lambda_n \prod_{s\in S_n} f_s^{p_{n,s}}(y_j)\Big) + h(y_j) f_{m_0}(y_j) \Big| \\ &\geq |h(y_j) f_{m_0}(y_j)| - \Big| \sum_{n=1}^{M} \Big(\lambda_n \prod_{s\in S_n} f_s^{p_{n,s}}(y_j)\Big) \Big| \end{aligned}$$

for each $j \in \mathbb{N}$. Accordingly, by using (3.1) we obtain

$$
\begin{aligned}
|f(y_j)| &\geq |h(y_j)| \prod_{i<m_0} |f_i(y_j)|^j - \sum_{n=1}^{M} \Big(|\lambda_n| \prod_{s \in S_n} |f_s(y_j)|^{p_{n,s}} \Big) \\
&= \sum_{n=1}^{M} \bigg[\bigg(\frac{|h(y_j)|}{M} \prod_{i<m_0} |f_i(y_j)|^{j-p_{n,i}} - |\lambda_n| \bigg) \Big(\prod_{s \in S_n} |f_s(y_j)|^{p_{n,s}} \Big) \bigg] \\
&\geq \sum_{n=1}^{M} \bigg[\bigg(\frac{\delta}{M} \prod_{i<m_0} |f_i(y_j)|^{j-p_{n,i}} - |\lambda_n| \bigg) \Big(\prod_{s \in S_n} |f_s(y_j)|^{p_{n,s}} \Big) \bigg]
\end{aligned}
$$

for every $j \geq j_0 + \max\{p_{n,s} : n = 1, \ldots, M \text{ and } s \in S_n\}$; we assume $p_{n,i} = 0$ whenever $i \notin S_n$. Since

$$
\lim_{j \to \infty} \prod_{i<m_0} |f_i(y_j)|^{j-p_{n,i}} = \lim_{j \to \infty} \prod_{s \in S_n} |f_s(y_j)|^{p_{n,s}} = \infty
$$

for every $n = 1, \ldots, M$, it follows that $\lim_{j\to\infty} |f(y_j)| = \infty$.

Now we assume that f_{m_0} appears in all terms in the summation (3.3). In this case we can write

$$
f = g_1 f_{m_0},
$$

where

$$
g_1 := \sum_{n=1}^{N+1} \Big[\lambda_n \prod_{s \in S_n \setminus \{m_0\}} (f_s^{p_{n,s}} f_{m_0}^{p_{n,m_0}-1}) \Big].
$$

Thus either $g_1 \in \mathcal{A}_{N+1}$ or $g_1 = h_1 + \alpha_1$, where $h_1 \in \mathcal{A}_N$ and $\alpha_1 \in \mathbb{C} \setminus \{0\}$. In the second case, the result follows by induction hypothesis. Otherwise, we set $m_1 := \max\{k : f_k \text{ appears in some term of } g_1\}$. If f_{m_1} does not appear in some term of g_1, then $\lim_{j\to\infty} |g_1(y_j)| = \infty$ by the argument of the previous paragraph. Since $|f(y_j)| \geq |g_1(y_j)|$ for each $j \in \mathbb{N}$, we deduce that $\lim_{j\to\infty} |f(y_j)| = \infty$. On the other hand, if f_{m_1} appears in all terms of g_1, we can write

$$
f = g_2 f_{m_1} f_{m_0},
$$

where

$$
g_2 := \sum_{n=1}^{N+1} \Big[\lambda_n \prod_{s \in S_n \setminus \{m_0, m_1\}} \big(f_s^{p_{n,s}} f_{m_0}^{p_{n,m_0}-1} f_{m_1}^{p_{n,m_1}-1} \big) \Big].
$$

Therefore either $g_2 \in \mathcal{A}_{N+1}$ or $g_2 = h_2 + \alpha_2$, where $h_2 \in \mathcal{A}_N$ and $\alpha_2 \in \mathbb{C} \setminus \{0\}$. In the second case the result follows by induction hypothesis. Otherwise, we set $m_2 := \max\{k : f_k \text{ appears in some term of } g_2\}$. If f_{m_2} does not appear in some term of g_2, then again $\lim_{j\to\infty} |g_2(y_j)| = \infty$. Hence $\lim_{j\to\infty} |f(y_j)| = \infty$. Repeating this argument finitely many times we deduce the desired result, and thus the proof by induction is complete.

We assert that the sequence $(f_k)_{k=1}^{\infty}$ is algebraically independent. Indeed, as we have seen above,

$$(3.4) \qquad f := \sum_{n=1}^{N} \lambda_n \prod_{s \in S_n} f_s^{p_{n,s}} \in \mathcal{A}_N \ \Rightarrow \ \lim_{j\to\infty} |f(y_j)| = \infty.$$

Therefore, we cannot have $\sum_{n=1}^{N} \lambda_n g_n = 0$ with $\lambda_n \neq 0$ and g_n's distinct generators of the subalgebra generated by $(f_k)_{k=1}^{\infty}$. ∎

THEOREM 3.2. *Let E be a separable Banach space and U be a domain of existence in E. Then the set $\mathcal{E}(U)$ is algebrable. In particular, $\mathcal{E}(U)$ is lineable.*

Proof. By Lemma 2.4, $\mathcal{F}(U) \subset \mathcal{E}(U)$, and therefore Theorem 3.2 follows from Theorem 3.1. ∎

In Theorem 3.1 we have shown that the set of functions in $\mathcal{H}(U)$ which are unbounded on each ball $B(x)$, with $x \in D$, is algebrable. This resembles a result of J. López-Salazar [3], which asserts that $\mathcal{H}(E)\backslash\mathcal{H}_b(E)$ is algebrable whenever E is an infinite-dimensional Banach space.

Acknowledgments. This research was partly supported by CAPES and CNPq. This paper is based on part of the author's doctoral dissertation at UNICAMP, written under the supervision of Professor Jorge Mujica. The author thanks his advisor for helpful suggestions concerning this work. The author also thanks Professor José M. Ansemil and the referee for some useful suggestions that have helped to improve the presentation.

References

[1] R. M. Aron, V. I. Gurariy and J. B. Seoane-Sepúlveda, *Lineability and spaceability of sets of functions on* $\mathbb{R}$, Proc. Amer. Math. Soc. 133 (2005), 795–803.

[2] L. Bernal-González, D. Pellegrino and J. B. Seoane-Sepúlveda, *Linear subsets of nonlinear sets in topological vector spaces*, Bull. Amer. Math. Soc. 51 (2014), 71–130.

[3] J. López-Salazar Codes, *Vector spaces of entire functions of unbounded type*, Proc. Amer. Math. Soc. 139 (2011), 1347–1360.

[4] J. Mujica, *Complex Analysis in Banach Spaces*, Dover Publ., New York, 2010.

[5] G. A. Muñoz-Fernández, N. Palmberg, D. Puglisi and J. B. Seoane-Sepúlveda, *Lineability in subsets of measure and function spaces*, Linear Algebra Appl. 428 (2008), 2805–2812.

Thiago R. Alves
IMECC - UNICAMP
Rua Sérgio Buarque de Holanda, 651
13083-859 Campinas, SP, Brazil
E-mail: tralves.math@gmail.com

Received May 24, 2013
Revised version September 16, 2013 (7793)