STUDIA MATHEMATICA 220 (2) (2014)

On Hamel bases in Banach spaces

by

Juan Carlos Ferrando (Elche)

Abstract. It is shown that no infinite-dimensional Banach space can have a weakly K-analytic Hamel basis. As consequences, (i) no infinite-dimensional weakly analytic separable Banach space E has a Hamel basis C-embedded in E(weak), and (ii) no infinite-dimensional Banach space has a weakly pseudocompact Hamel basis. Among other results, it is also shown that there exist noncomplete normed barrelled spaces with closed discrete Hamel bases of arbitrarily large cardinality.

1. Preliminaries. Bartoszyński et al. proved in [3, Theorem 3.10] that no infinite-dimensional separable Banach space has an analytic Hamel basis. In this paper this result is extended by showing that no infinite-dimensional Banach E space has a weakly K-analytic Hamel basis, i.e. a Hamel basis which is a K-analytic topological space under the relative weak topology of E. In the proof of this result an indirect approach, based on some techniques of C_{p}-theory, is used. Hence no infinite-dimensional weakly analytic separable Banach space E has a Hamel basis C-embedded in E(weak), and no infinite-dimensional Banach space has a weakly pseudocompact Hamel basis. Some properties of bounded Hamel bases are investigated. We also show that there are noncomplete normed barrelled spaces with an arbitrarily large closed discrete Hamel basis. Other results are (i) no infinite-dimensional separable Banach space has a Hamel basis which is covered by an ordered family $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of weakly compact sets, and (ii) if a locally convex space E has a Hamel basis X covered by an ordered family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ consisting of weakly topologically bounded sets, then υX (the realcompactification of X) is countably K-determined. For the definitions not included in the paper we refer the reader to [4, 7, 8, 12].

2. On weakly K-analytic Hamel bases. If X is a completely regular Hausdorff space, we denote by $C(X)$ the real linear space of all real-valued

2010 *Mathematics Subject Classification*: 46B20, 46A03, 54C35.
Key words and phrases: Hamel basis, Banach space, K-analytic space, countably K-determined space, barrelled space.

DOI: 10.4064/sm220-2-5

functions defined on X, and by $C_{\mathrm{p}}(X)$ the space $C(X)$ equipped with the pointwise convergence topology. The topological dual of $C_{\mathrm{p}}(X)$ is denoted by $L(X)$, or by $L_{\mathrm{p}}(X)$ when provided with the weak* topology. The linear space $C(X)$ equipped with the compact-open topology will be denoted by $C_{\mathrm{c}}(X)$. All topological spaces we use are supposed to be Hausdorff. If E is a real linear space and X a Hamel basis of E, we shall frequently identify E with $L(X)$ algebraically by means of the mapping $x \mapsto \delta_x$, where $\delta_x : C(X) \to \mathbb{R}$ denotes the evaluation map at $x \in X$ given by $\delta_x(f) = f(x)$, extended by linearity to E.

LEMMA 2.1. *Let (E, τ) be a real locally convex space which is the locally convex hull of a family of locally convex Baire spaces. If there exists in E a weakly K-analytic Hamel basis, then the weak* dual $(E', \sigma(E', E))$ of E is a C_{p}-space.*

Proof. Assume that X is a weakly K-analytic Hamel basis of E and consider $F = C(X)$ as the topological dual of $L_{\mathrm{p}}(X)$. If $u \in E'$, let $f := u|_X \in C(X)$. If u_f stands for the linear extension (so called *linearization*) of f to the whole of $L(X) = E$ defined by $\langle \sum_{i=1}^n \alpha_i \delta_{x_i}, u_f \rangle = \sum_{i=1}^n \alpha_i f(x_i)$, then clearly $u_f = u$. Since u_f is a continuous linear functional on $L_{\mathrm{p}}(X)$ (see [2, Proposition 0.5.11]), we see that $u \in F$. Hence $E' \subseteq F$.

Now consider the identity map $\mathrm{id}_E : (E, \tau) \to (E, \sigma(E, F))$. Given that obviously $\mathrm{id}_E : (E, \tau) \to (E, \sigma(E, E'))$ is continuous and $\sigma(E, E') \leq \sigma(E, F)$, we conclude that id_E has closed graph in $(E, \tau) \times (E, \sigma(E, F))$. Furthermore, since X is K-analytic under the relative weak topology of E, it follows from [2, Proposition 0.5.14] that $(E, \sigma(E, F)) = L_{\mathrm{p}}(X)$ is also a K-analytic space. So, bearing in mind that (E, τ) is the locally convex hull of a family of locally convex Baire spaces, an application of [12, I.4.3.(17)] shows that id_E is $(\tau, \sigma(E, F))$-continuous, thus weakly continuous. This implies that $F \subseteq E'$. Thus we conclude that $E' = F$, which ensures that $C_{\mathrm{p}}(X) = (E', \sigma(E', E))$, as stated. ∎

LEMMA 2.2. *If X is a completely regular μ-space, then the compact-open topology τ_{c} on $C(X)$ coincides with the strong topology $\beta(C(X), L(X))$.*

Proof. If E stands for the topological dual of $C_{\mathrm{c}}(X)$, clearly $\beta(C(X), E)$ is a locally convex topology on $C(X)$ stronger than $\beta(C(X), L(X))$. Moreover, since X is a μ-space, $C_{\mathrm{c}}(X)$ is barrelled by the Nachbin–Shirota theorem. Consequently, the compact-open topology coincides with $\beta(C(X), E)$. So the compact-open topology τ_{c} on $C(X)$ is stronger than $\beta(C(X), L(X))$. On the other hand, if U is a neighborhood of the origin in τ_{c}, there are a compact set K in X and $\epsilon > 0$ such that $\{f \in C(X) : \sup_{x \in K} |f(x)| \leq \epsilon\} \subseteq U$, i.e. $\epsilon \delta(K)^0 \subseteq U$, where $\delta : X \to L(X)$ is the canonical embedding map defined by $\delta(x) = \delta_x$ and the polar $\delta(K)^0$ of $\delta(X) \subseteq L(X)$ is with respect to the dual pair $\langle L(X), C(X)\rangle$. Since K is a compact subset of X, $\delta(K)$ is a compact (hence bounded) subset of the locally convex space $L_{\mathrm{p}}(X)$, which shows that the strong topology $\beta(C(X), L(X))$ is stronger than the compact-open topology τ_{c}, so that both topologies coincide. ∎

THEOREM 2.3. *No infinite-dimensional Banach space has a weakly K-analytic Hamel basis.*

Proof. Notice that the theorem need only be proved for real Banach spaces. So, assume for contradiction that there exists an infinite-dimensional real Banach space E with a weakly K-analytic Hamel basis X. According to Lemma 2.1, the weak* dual $(E', \sigma(E', E))$ of E can be identified with the function space $C_{\mathrm{p}}(X)$. Then, since X is a μ-space with the relative weak topology of E, Lemma 2.2 applies to show that the strong dual $(E', \beta(E', E))$ of E coincides with $C_{\mathrm{c}}(X)$. This means that $C_{\mathrm{c}}(X)$ is a Banach space, a fact that requires X to be a compact subset of E(weak) [1, Theorem 13]. In fact, if $B_{E'}$ denotes the closed unit ball of the Banach space $(E', \beta(E', E))$ and K is a compact subset of X such that

$$\Big\{f \in C(X) : \sup_{x \in K} |f(x)| < \epsilon\Big\} \subseteq B_{E'}$$

for some $\epsilon > 0$, the complete regularity of X along with the boundedness of $B_{E'}$ easily yield $X = K$. Given that E is a Banach space, the Krein–Šmulian theorem [9, 2.8.14 Theorem] ensures that the closed absolutely convex cover $Q = \overline{\mathrm{abx}(X)}$ of X is also weakly compact. Since $E = \bigcup_{n=1}^{\infty} nQ$ and E with its original topology is a locally convex Baire space, if B_E stands for the closed unit ball of E there is $0 < \epsilon < 1$ such that $\epsilon B_E \subseteq Q$, which implies that the ball B_E is weakly compact. Therefore E is reflexive and consequently $C_{\mathrm{c}}(X) = (E', \beta(E', E))$ is also a reflexive Banach space. Given that E is infinite-dimensional, it turns out that $C_{\mathrm{c}}(X)$ must be an infinite-dimensional Banach space as well. The latter guarantees that $C_{\mathrm{c}}(X)$ contains a copy of c_0, which contradicts the reflexivity of $C_{\mathrm{c}}(X)$. ∎

COROLLARY 2.4. *No infinite-dimensional weakly analytic separable Banach space E has a Hamel basis which is C-embedded in E*(weak).

Proof. Assume that E is an infinite-dimensional separable Banach space E with a Hamel basis X which is C-embedded in E(weak). Since the cardinality of E is $\mathfrak{c}$ and E(weak) is submetrizable, every topological subspace of E(weak) is realcompact. Since X is assumed to be C-embedded in E(weak), it turns out that $\overline{X}^{\upsilon E(\mathrm{weak})}$, where the closure is in the Hewitt realcompactification υE(weak) of E(weak), coincides with υX, the realcompactification of X equipped with the relative weak topology (see [6, 8.10(a)]). Since X and E(weak) are realcompact, it follows that X is closed in E(weak). Consequently, X is weakly analytic, which contradicts Theorem 2.3. ∎

Corollary 2.5. *No infinite-dimensional Banach space has a weakly pseudocompact Hamel basis. In particular, no infinite-dimensional Banach space has a weakly countably compact Hamel basis.*

Proof. If X is a pseudocompact set of a Banach space E in its weak topology, then $X(\text{weak})$ is an Eberlein compact set [2, IV.5.6 Corollary]. Consequently, according to Theorem 2.3, no infinite-dimensional Banach space E can have a weakly pseudocompact Hamel basis X. ∎

Proposition 2.6. *Let E be a normed space. If either E is not weakly K-analytic or the strong dual $(E', \beta(E', E))$ of E is not weakly K-analytic, then E has no weakly K-analytic Hamel basis.*

Proof. We may assume that E is a real normed space. If X is a weakly K-analytic Hamel basis of the normed space E and $n \in \mathbb{N}$, the continuity of the mapping $\varphi_n : X^n \times \mathbb{R}^n \to (E, \sigma(E, E'))$ given by $\varphi_n(x_1, \ldots, x_n, a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i\, x_i$ along with the fact that $E = \bigcup_{n=1}^{\infty} \varphi_n(X^n \times \mathbb{R}^n)$ ensures that E is weakly K-analytic.

For the second statement assume that there is in E a weakly K-analytic Hamel basis X. Reasoning as in the first part of the proof of the theorem above, use Lemma 2.1 to deduce that $(E', \sigma(E', E)) = C_{\mathrm{p}}(X)$ and Lemma 2.2 to show that $(E', \beta(E', E)) = C_{\mathrm{c}}(X)$. These facts ensure that X is a weakly compact subset of the normed space E, hence an Eberlein compact set. In this case, a classical result of Talagrand that asserts that if X is Eberlein compact then the Banach space $C_{\mathrm{c}}(X)$ is weakly K-analytic [10] applies to show that the space $(E', \sigma(E', E''))$ is K-analytic, contradicting the hypotheses. ∎

Example 2.7. *The converse of the first statement of the previous proposition fails.* In fact every infinite-dimensional weakly compactly generated Banach space E is weakly K-analytic [10], but by Theorem 2.3 no Hamel basis of E is weakly K-analytic. In particular, no infinite-dimensional Hilbert space has a weakly closed Hamel basis.

3. On bounded Hamel bases. If X is completely regular, we denote by $C^*(X)$ the real linear subspace of $C(X)$ consisting of all bounded functions. It becomes a Banach space when equipped with the norm $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. On the other hand, we shall denote by $L_{\mathrm{p}^*}(X)$ the space $L(X)$ provided with the weak locally convex topology $\sigma(L(X), C^*(X))$. Although $L_{\mathrm{p}}(X)$ is K-analytic if and only if X is, and $L_{\mathrm{p}^*}(X)$ is K-analytic if X is, X need not be K-analytic if $L_{\mathrm{p}^*}(X)$ is K-analytic.

Theorem 3.1. *Let X be a bounded Hamel basis of a real Banach space E, equipped with the relative weak topology. If either*

- *$L_{\mathrm{p}^*}(X)$ is K-analytic, or*

- $L_{p^*}(X)$ *is sequentially complete,*

then E *is* (*isomorphic to*) *a predual of* $C^*(X)$.

Proof. Identifying E and $L(X)$ in the usual way and keeping in mind that, by hypothesis, the restriction $u|_X$ of each continuous linear functional u on E is bounded on X, we see that $E' \subseteq C^*(X) \subseteq C(X)$. Denoting by B_E the closed unit ball of the original Banach space $(E, \|\cdot\|)$, we may assume without loss of generality that $X \subseteq B_E$.

Put $F := C(X)$ and $G = C^*(X)$ and consider the absolutely convex set

$$P := \overline{\mathrm{abx}}^{\sigma(E,G)}(X),$$

i.e. the $\sigma(E,G)$-closure of the absolutely convex hull of X. Observe that $E = \bigcup_{n=1}^{\infty} nP$. If $z = \sum_{i=1}^{n} a_i\, x_i \in \mathrm{abx}(X)$ and $f \in C^*(X)$, then

$$|\langle z, u_f\rangle| = \Big|\Big\langle \sum_{i=1}^{n} a_i\, x_i, u_f\Big\rangle\Big| \leq \sum_{i=1}^{n} |a_i|\, |f(x_i)| \leq \|f\|_\infty,$$

which implies that P is a bounded set in $(E, \sigma(E,G))$. If H stands for the topological dual of $(C^*(X), \|\cdot\|_\infty)$, the fact that

$$P^0 = X^0 = \Big\{f \in C^*(X) : \sup_{x\in X} |f(x)| \leq 1\Big\} = \{f \in C^*(X) : \|f\|_\infty \leq 1\}$$

under the dual pair $\langle E, G\rangle$ tells us that the strong topology $\beta(G,E)$ coincides with the norm-topology $\beta(G,H)$ of $C^*(X)$.

For the proof of the first statement note that the identity map $\mathrm{id}_E : E \to L_{p^*}(X)$ has closed graph, which implies that $E' = C^*(X) = G$. Hence the strong dual $(E', \beta(E',E))$ of $(E, \|\cdot\|)$ coincides with $(C^*(X), \|\cdot\|_\infty)$.

Concerning the second statement denote by $\|\cdot\|_P$ the Minkowski functional of P. We claim that $(E, \|\cdot\|_P)$ is a Banach space. Indeed, if $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(E, \|\cdot\|_P)$, given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$z_m - z_n \in \epsilon P \tag{3.1}$$

for $m, n \geq n_0$. Since P is $\sigma(E,G)$-bounded, the topology on E induced by the Minkowski norm $\|\cdot\|_P$ is stronger than the topology $\sigma(E,G)$ and $\{z_n\}_{n=1}^{\infty}$ is a $\sigma(E,G)$-Cauchy sequence. Given that $L_{p^*}(X)$ is assumed to be sequentially complete, $(E, \sigma(E,G))$ is a sequentially complete locally convex space. Consequently, there exists $z \in E$ such that $z_n \to z$ in E in the weak topology $\sigma(E,G)$. Letting $m \to \infty$ in $\sigma(E,G)$ in (3.1) we get $z - z_n \in \epsilon P$ for all $n \geq n_0$, which shows that $z_n \to z$ in the norm $\|\cdot\|_P$. So $(E, \|\cdot\|_P)$ is a Banach space.

Since $(E, \|\cdot\|)' = E' \subseteq G \subseteq (E, \|\cdot\|_P)'$, the identity map from $(E, \|\cdot\|)$ onto $(E, \|\cdot\|_P)$ has closed graph, which allows us to conclude that $E' = C^*(X)$. So we may reason as in the final part of the proof of the first statement to deduce that $(E, \|\cdot\|)$ is a predual of $(C^*(X), \|\cdot\|_\infty)$. ∎

Corollary 3.2. *No infinite-dimensional reflexive real Banach space E has a bounded Hamel basis X such that $L_{\mathrm{p}^*}(X)$ is K-analytic or sequentially complete when X is endowed with the relative weak topology. In particular, no infinite-dimensional real Banach space E has a Hamel basis X such that $(C^*(X), \mu(C^*(X), L(X)))$ is barrelled.*

Proof. If E is reflexive and X is a bounded Hamel basis such that $L_{\mathrm{p}^*}(X)$ is K-analytic or sequentially complete, Theorem 3.1 implies that E coincides with the topological dual of the Banach space $C^*(X)$, where X is equipped with the relative weak topology. This means that $L(X)$ is algebraically isomorphic to the space $\mathrm{rca}(\Sigma)$ of all regular countably additive real-valued measures (Radon measures) on the Borel σ-algebra Σ of the Stone–Čech realcompactification βX of X(weak). So every Radon measure μ on βX must be a linear combination of Dirac measures concentrated at points of X. This forces X(weak) to be compact, so Theorem 2.3 applies to show that E must be finite-dimensional.

Regarding the second statement, if $(C^*(X), \mu(C^*(X), L(X)))$ is barrelled then $L_{\mathrm{p}^*}(X)$ is sequentially complete; but, as a consequence of the closed graph theorem, E is reflexive. ∎

4. Barrelled normed spaces with closed Hamel bases. It is shown in [3, Theorem 3.8] that there are Hilbert spaces of arbitrarily large cardinality that have a discrete and closed (necessarily not weakly closed) Hamel basis. Next we shall see that there are also noncomplete normed barrelled spaces with closed Hamel bases of arbitrarily large cardinality.

Let Ω be a nonempty set and Σ a σ-algebra of subsets of Ω. Let us denote by $\ell_0^\infty(\Sigma)$ the linear space of all real-valued Σ-simple functions f defined on Ω equipped with the supremum norm

$$\|f\|_\infty = \sup\{|f(\omega)| : \omega \in \Omega\}.$$

As is well known, $\ell_0^\infty(\Sigma)$ is a nonultrabornological barrelled space, which, for Σ infinite, is not complete. Moreover, setting $Z = \{\chi_A : A \in \Sigma\}$ one has $\ell_0^\infty(\Sigma) = \mathrm{span}(Z)$.

Lemma 4.1. *Z is a closed subset of $\ell_0^\infty(\Sigma)$ in the norm topology.*

Proof. Let $y \in \overline{Z}$, where the closure is in the norm of $\ell_0^\infty(\Sigma)$. Given $0 < \epsilon < 1/2$, choose $x \in Z$ such that $\|x - y\|_\infty < \epsilon$.

Suppose that $y = \sum_{i=1}^n a_i \chi_{A_i}$, where $\{A_i : 1 \le i \le n\}$ is a partition of Ω into elements of Σ with $a_i \ne a_j$ whenever $i \ne j$, and $x = \chi_B$ with $B \in \Sigma$. Given a fixed $i \in \{1, \dots, n\}$ we claim that either $B \cap A_i = \emptyset$ or $A_i \subseteq B$. In fact, otherwise there are $\omega \in A_i \setminus B$ and $\varepsilon \in A_i \cap B$ and hence

$$1/2 > \epsilon > \|x-y\|_\infty \geq |x(\omega) - y(\omega)| = |y(\omega)|,$$
$$1/2 > \epsilon > \|x-y\|_\infty \geq |x(\varepsilon), -y(\varepsilon)| \geq 1 - |y(\varepsilon)|$$

which yields the contradiction

$$1/2 > |y(\omega)| = |y(\varepsilon)| > 1 - 1/2 = 1/2.$$

This shows that there are $A_{i_1}, \ldots, A_{i_m}$ such that $B = \bigcup_{j=1}^m A_{i_j}$ whereas $B \cap A_j = \emptyset$ if $j \notin \{i_1, \ldots, i_m\}$.

Let $0 < \epsilon_1 < 1/2$ be such that $2\epsilon_1 < \min\{|a_i - a_j| : 1 \leq i, j \leq n,\ i \neq j\}$ and select $z_1 = \chi_E$ with $E \in \Sigma$ so that $\|y - z_1\|_\infty < \epsilon_1$. Note that if there are $i, j \in \{1, \ldots, n\}$ with $A_i, A_j \subseteq E$ and $i \neq j$, then choosing $\omega_i \in A_i$ and $\omega_j \in A_j$ we get the contradiction

$$|a_i - a_j| = |y(\omega_i) - y(\omega_j)| = |y(\omega_i) - z_1(\omega_i)| + |z_1(\omega_j) - y(\omega_j)| < |a_i - a_j|$$

since $z_1(\omega_i) = z_1(\omega_j)$. Thus there is a unique $k \in \{1, \ldots, n\}$ with $A_k = E$ and $A_j \cap E = \emptyset$ if $j \neq k$, which implies that $|1 - a_k| < 1/2$ and $|a_i| < 1/2$ if $i \neq k$.

Now if $0 < \epsilon_2 < 1/4$ and $\|y - z_2\|_\infty < \epsilon_2$ with $z_2 = \chi_F$, it follows that $F = E$. Indeed, according to the previous argument, there is $j \in \{1, \ldots, n\}$ with $A_j = F$ and $A_i \cap F = \emptyset$ if $i \neq j$. But necessarily $j = k$ since otherwise on the one hand $|a_j| < \epsilon_1 < 1/2$ and on the other hand $|1 - a_j| < \epsilon_2 < 1/4$, a contradiction. Consequently, $|1 - a_k| < 1/4$ and $|a_i| < 1/4$ if $i \neq k$.

Proceeding by recurrence we get $y = \sum_{i \neq k} a_i \chi_{A_i} + a_k \chi_E$ with $|1 - a_k| < 1/2^l$ and $|a_i| < 1/2^l$ if $i \neq k$ for all $l \in \mathbb{N}$, which means that $y = \chi_E \in Z$. ∎

THEOREM 4.2. *There exists a closed and discrete Hamel basis in $\ell_0^\infty(\Sigma)$.*

Proof. By Zorn's lemma there is a subset X of Z that is a Hamel basis of $\ell_0^\infty(\Sigma)$. We claim that X is a closed and discrete subset of $\ell_0^\infty(\Sigma)$.

In fact, according to Lemma 4.1, in order to prove closedness we need only show that X is a closed set of Z. But if $\chi_B \in \overline{X}^Z$, there is $\chi_A \in X$ such that $\|\chi_B - \chi_A\| < 1/2$, which implies that $A = B$. Therefore X is closed in $\ell_0^\infty(\Sigma)$. On the other hand, if $0 < \epsilon < 1$ for each $\chi_A \in X$ the ball $B(\chi_A, \epsilon)$ of $\ell_0^\infty(\Sigma)$ with centre χ_A and radius ϵ contains no other function of X. Consequently, the set X is a discrete topological subspace of $\ell_0^\infty(\Sigma)$ in the norm topology. ∎

If Ω is an infinite set and Σ coincides with the σ-algebra 2^Ω of all subsets of Ω, by the previous theorem there exists a subfamily Λ of Σ such that $X = \{\chi_A : A \in \Lambda\}$ is a closed discrete Hamel basis of $\ell_0^\infty(\Sigma)$.

PROPOSITION 4.3. *If Ω is an infinite set and Σ coincides with the σ-algebra 2^Ω of all subsets of Ω, then the Hamel basis $X = \{\chi_A : A \in \Lambda\}$ of the real space $\ell_0^\infty(\Sigma)$ is closed and discrete but not weakly K-analytic.*

Proof. If X is weakly K-analytic, by Proposition 2.6 the space $\ell_0^\infty(2^\Omega)$ is weakly K-analytic as well. Since $\ell_0^\infty(2^\Omega)$ is a dense subspace of the Banach space $\ell_\infty(\Omega) = C(\beta\Omega)$, where $\beta\Omega$ stands for the Stone–Čech compactification of Ω equipped with the discrete topology, it follows that $\beta\Omega$ embeds in the weak* dual $\mathrm{rca}(\beta\Omega)(\mathrm{weak}^*)$ of $C(\beta\Omega)$, where $\mathrm{rca}(\beta\Omega)$ denotes the Banach space of all regular countably additive (real-valued) measures on the Borel σ-algebra of $\beta\Omega$. Since $\beta\Omega = \overline{\Omega}^{\beta\Omega}$ and no $\mu \in \beta\Omega \setminus \Omega$ is the limit of a sequence in Ω, then $\beta\Omega$ is not a Fréchet–Urysohn space, so $\beta\Omega$ is not Corson compact. Consequently, $\beta\Omega$ is homeomorphic to some non-Talagrand compact subset of the weak* dual of $\ell_0^\infty(2^\Omega)$. This is a contradiction, since we have seen that $\ell_0^\infty(2^\Omega)$ is weakly K-analytic, and it is well-known that every compact set of the weak* dual of a K-analytic space is a Talagrand compact. ∎

5. Further results. In this section we shall frequently use reference [5], from which we recall some definitions. A covering $\{A_\alpha : \alpha \in \Sigma\}$ of a set X indexed by a subset Σ of $\mathbb{N}^\mathbb{N}$ is called a *Σ-covering* of X. An $\mathbb{N}^\mathbb{N}$-covering $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of X with the additional property that $A_\alpha \subseteq A_\beta$ if $\alpha \le \beta$ pointwise, i.e. $\alpha(i) \le \beta(i)$ for every $i \in \mathbb{N}$, is called a *resolution* of X. Each K-analytic space X has a resolution consisting of compact sets [11]. If a locally convex space E has a Hamel basis X which admits a resolution consisting of weakly compact sets, then [5, Lemma 29] ensures that υX is K-analytic.

PROPOSITION 5.1. *No infinite-dimensional separable Banach space E has a Hamel basis with a resolution consisting of weakly compact sets.*

Proof. Since E is separable, every subset of E and in particular every Hamel basis of E is weakly realcompact. Hence, if X were a Hamel basis with a resolution as in the statement, the preceding considerations would ensure that X is weakly K-analytic (in fact, weakly analytic), contradicting Theorem 2.3. ∎

A Σ-covering $\{A_\alpha : \alpha \in \Sigma\}$ of a locally convex space E is said to have *limited envelope* if $\bigcup\{A_{\alpha_n} : n \in \mathbb{N}\}$ is a bounded set of E whenever $\{\alpha_n\}$ is a convergent sequence in Σ, the latter considered as a topological subspace of the product space $\mathbb{N}^\mathbb{N}$ where $\mathbb{N}$ is endowed with the discrete topology [5, Proposition 9]. On the other hand, a subset A of a topological space X is called *topologically bounded* if $f(A)$ is always a bounded subset of $\mathbb{R}$ for every $f \in C(X)$ (see [2, Chapter 0]). In particular, every compact subset K of a topological space X is topologically bounded. The following result examines what happens if a locally convex space has a weak Hamel basis with a resolution of topologically bounded sets.

THEOREM 5.2. *Let E be a locally convex space and X be a Hamel basis of E. If $X(\text{weak})$ has a resolution consisting of topologically bounded sets then its realcompactification υX is countably K-determined.*

Proof. Assume that E is real and let $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a resolution of $X(\text{weak})$ made up of topologically bounded sets. If δ is the canonical embedding of $X(\text{weak})$ in $L_{\mathrm{p}}(X)$, then $\{\delta(A_\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a covering of $\delta(X)$ in $L_{\mathrm{p}}(X)$ made up of precompact, hence (linearly) bounded, sets. For $\alpha \in \mathbb{N}^{\mathbb{N}}$ put $\alpha'(i) := \alpha(i+1)$ for all $i \in \mathbb{N}$ and define

$$B_\alpha = \Big\{\sum_{i=1}^{n} a_i\delta_{x_i} : x_i \in A_{\alpha'},\ \sum_{i=1}^{n} |a_i| \le \alpha(1),\ \forall a_i \in \mathbb{R},\ 1 \le i \le n,\ \forall n \in \mathbb{N}\Big\}.$$

One can easily check that $\bigcup\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\} = L(X)$ and, since $A_\alpha \subseteq A_\beta$ if $\alpha \le \beta$, that $B_\alpha \subseteq B_\beta$ if $\alpha \le \beta$. Moreover, each B_α is a bounded subset of the locally convex space $L_{\mathrm{p}}(X)$. Indeed, if $f \in C(X)$ and u_f stands as usual for the linearization of f then

$$\Big|u_f\Big(\sum_{i=1}^{n} a_i\,\delta_{x_i}\Big)\Big| \le \sum_{i=1}^{n} |a_i|\,|f(x_i)| \le \alpha(1) \sup_{x\in A_\alpha} |f(x)|.$$

Consequently, $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution of $L_{\mathrm{p}}(X)$ consisting of bounded sets. According to [5, Proposition 13] this implies that $L_{\mathrm{p}}(X)$ has an $\mathbb{N}^{\mathbb{N}}$-covering with limited envelope. So [5, Lemma 2] applies again to get a countably K-determined space Z such that $C_{\mathrm{p}}(X) \subseteq Z \subseteq \mathbb{R}^{L(X)}$, which allows using [5, Proposition 10] to deduce that $C_{\mathrm{p}}(X)$ has a Σ-covering with limited envelope. Then [5, Theorem 3] shows that υX is countably K-determined. ∎

Acknowledgements. This research was partially supported by the Conselleria d'Educació, Cultura i Esport of the Generalitat Valenciana, Grant PROMETEO/2013/058.

References

[1] R. F. Arens, *A topology for spaces of transformations*, Ann. of Math. 47 (1946), 480–495.

[2] A. V. Arkhangel'skiĭ, *Topological Function Spaces*, Math. Appl. 78, Kluwer, Dordrecht, 1992.

[3] T. Bartoszyński, M. Džamonja, L. Halbeisen, E. Murtinová and A. Plichko, *On bases in Banach spaces*, Studia Math. 170 (2005), 147–171.

[4] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.

[5] J. C. Ferrando, *Some characterizations for υX to be Lindelöf Σ or K-analytic in terms of $C_{\mathrm{p}}(X)$*, Topology Appl. 156 (2009), 823–830.

[6] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, NJ, 1960.

[7] J. Kąkol, W. Kubiś and M. López Pellicer, *Descriptive Topology in Selected Topics of Functional Analysis*, Springer, 2011.

[8] G. Köthe, *Topological Vector Spaces I*, Springer, Berlin, 1969.
[9] R. E. Megginson, *An Introduction to Banach Space Theory*, Grad. Texts in Math. 183, Springer, New York, 1998.
[10] M. Talagrand, *Sur une conjecture de H. H. Corson*, Bull. Sci. Math. 99 (1975), 211–212.
[11] M. Talagrand, *Espaces de Banach faiblement $\mathcal{K}$-analytiques*, Ann. of Math. 110 (1979), 407–438.
[12] M. Valdivia, *Topics in Locally Convex Spaces*, North-Holland, Amsterdam, 1982.

Juan Carlos Ferrando
Centro de Investigación Operativa
Edificio Torretamarit, Avda de la Universidad s/n
Universidad Miguel Hernández
E-03202 Elche (Alicante), Spain
E-mail: jc.ferrando@umh.es

Received July 19, 2013 (7821)