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Convergence of Taylor series in Fock spaces

by

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Abstract. It is well known that the Taylor series of every function in the Fock space F^p_α converges in norm when $1 < p < \infty$. It is also known that this is no longer true when $p = 1$. In this note we consider the case $0 < p < 1$ and show that the Taylor series of functions in F^p_α do not necessarily converge "in norm".

1. Introduction. For $\alpha > 0$ we consider the Gaussian probability measure

$$d\lambda_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2}\, dA(z),$$

where dA is the Euclidean area measure on the complex plane $\mathbb{C}$. For $0 < p \le \infty$ we introduce the space L^p_α consisting of all Lebesgue measurable functions f such that the function $f(z)e^{-\alpha|z|^2/2}$ is in $L^p(\mathbb{C}, dA)$. When $0 < p < \infty$, we write

$$\|f\|_{p,\alpha} = \left\{ \frac{p\alpha}{2\pi} \int_{\mathbb{C}} |f(z)e^{-\alpha|z|^2/2}|^p\, dA(z) \right\}^{1/p}$$

for $f \in L^p_\alpha$. For $f \in L^\infty_\alpha$ we write

$$\|f\|_{\infty,\alpha} = \operatorname{ess\,sup}\{|f(z)|e^{-\alpha|z|^2/2} : z \in \mathbb{C}\}.$$

It is clear that for $0 < p < \infty$ we have

$$L^p_\alpha = L^p(\mathbb{C}, d\lambda_{p\alpha/2}).$$

But $L^\infty_\alpha \ne L^\infty(\mathbb{C})$.

Let $H(\mathbb{C})$ be the family of entire functions on $\mathbb{C}$. For $0 < p \le \infty$ we define the spaces $F^p_\alpha = H(\mathbb{C}) \cap L^p_\alpha$. These are called *Fock spaces*. We mention here that the polynomials are dense in F^p_α when $0 < p < \infty$ and each F^p_α is closed in L^p_α. When $1 \le p \le \infty$, L^p_α is a Banach space with the norm $\|f\|_{p,\alpha}$. When $0 < p < 1$, L^p_α is a complete metric space with the distance $d(f, g) = \|f - g\|^p_{p,\alpha}$. Therefore, F^p_α is a Banach space when $1 \le p \le \infty$, and it is an F-space under $d(f, g) = \|f - g\|^p_{p,\alpha}$ when $0 < p < 1$ (see [4, 7, 12]).

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Let f_α^∞ denote the space of entire functions $f(z)$ such that

$$\lim_{z\to\infty} f(z)e^{-\alpha|z|^2/2} = 0.$$

Obviously, f_α^∞ is a closed subspace of F_α^∞, so f_α^∞ is a Banach space. In particular, the polynomials are dense in f_α^∞ (see [12]).

There is another definition of Fock spaces using weights that do not depend on p. For $0 < p < \infty$, denote $\mathcal{L}_\alpha^p = L^p(e^{-\alpha|z|^2}dA(z))$ and let $\mathcal{F}_\alpha^p$ be the subspace of $\mathcal{L}_\alpha^p$ consisting of entire functions. This seems to be the less common usage; it was adopted e.g. in Zhu [10] and Garling and Wojtaszczyk [2]. The norm of $f \in \mathcal{L}_\alpha^p$ is

$$\|f\| = \Big\{ \int_{\mathbb{C}} |f(z)|^p e^{-\alpha|z|^2}\, dA(z)\Big\}^{1/p}.$$

We note that the two definitions, of L_α^p and $\mathcal{L}_\alpha^p$, are the same if and only if $p = 2$. The two notions can be converted into one another by

$$\mathcal{L}_\alpha^p = L_{2\alpha/p}^p, \qquad L_\alpha^p = \mathcal{L}_{\alpha p/2}^p,$$

so that theorems in terms of one definition can be stated in terms of the other.

Recall that for $0 < p < \infty$ the *Hardy space* H^p consists of analytic functions f on the open unit disk $\mathbb{D}$ such that

$$\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta < \infty.$$

For $\gamma > -1$ and $0 < p < \infty$ the *weighted Bergman space* $A_\gamma^p = A_\gamma^p(\mathbb{D}, dA_\gamma)$ is the subspace of $L^p(\mathbb{D}, dA_\gamma)$ consisting of analytic functions, where

$$dA_\gamma(z) = (\gamma+1)(1-|z|^2)^\gamma dA(z),$$

and $dA(z)$ is the normalized area measure on $\mathbb{D}$. Again, H^p and A_γ^p are Banach spaces for $1 \le p < \infty$. When $0 < p < 1$, H^p and A_γ^p are F-spaces (see [3, 8]).

The problem of norm convergence of Taylor series for each function in H^p spaces is classical: The Taylor series for each function in H^p converges in norm if and only if $p > 1$. This result is equivalent to the boundedness of the Szegö projection on L^p of the circle when $p > 1$ and the unboundedness of the Szegö projection on L^1 of the circle. By use of polar coordinates and the result in H^p, if $1 < p < \infty$ and $\gamma > -1$, then the Taylor series of every function in the weighted Bergman space A_γ^p converges in norm. More details can be seen in [3, 9].

Although every function f in F_α^p, $0 < p < \infty$, can be approximated by a sequence of polynomials, it is not necessarily true that a function in F_α^p can be approximated by its Taylor polynomials $\{f_n\}$ in norm. So it is natural to ask when $\{f_n\}$ converges to f in the norm topology of F^p_α. The same question can be asked for f^∞_α. The case $1 < p < \infty$ follows from the corresponding result in the theory of Hardy spaces. This is included in Section 2 for completeness, along with some consequences. The case $p = 1$ was discussed in [2]. In Section 3, we will consider the case $0 < p < 1$ and basically reduce it to the case $p = 1$ with the help of duality.

2. The case $1 < p < \infty$. Let X be a linear space of analytic functions in the unit disk $\mathbb{D}$ or in the complex plane $\mathbb{C}$. Given f in X, let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

be the Taylor expansion of f. For any integer $n \ge 1$ let

$$f_n(z) = \sum_{k=0}^{n} a_k z^k$$

be the nth Taylor polynomial of f and define a linear operator S_n by $S_n f = f_n$.

The following proposition can be found in [1, 9]. It is stated for analytic functions in the unit disk, but the proof works for entire functions in the complex plane $\mathbb{C}$ as well (see [7]).

PROPOSITION 2.1. *Suppose X is a Banach space of analytic functions in the unit disk $\mathbb{D}$ or in the complex plane $\mathbb{C}$ with the property that the polynomials are dense in X. Then $\|f_n - f\| \to 0$ as $n \to \infty$ for each $f \in X$ if and only if there is a positive constant $C > 0$ such that $\|S_n\| \le C$ for all $n \ge 1$.*

THEOREM 2.2. *If $1 < p < \infty$ and $f \in F^p_\alpha$, then the Taylor series of f converges to f in norm.*

Proof. There exists a positive constant C such that

$$\int_0^{2\pi} |S_n g(e^{i\theta})|^p \, d\theta \le C \int_0^{2\pi} |g(e^{i\theta})|^p \, d\theta$$

for all $g \in H^p$ and $n \ge 1$. Thus for any $f \in F^p_\alpha$ we have

$$\begin{aligned}\|S_n f\|^p_{p,\alpha} &= \frac{\alpha p}{2\pi} \int_0^\infty \int_0^{2\pi} |S_n f(re^{i\theta})|^p \, d\theta \, e^{-\alpha p r^2/2} r \, dr \\ &\le \frac{C\alpha p}{2\pi} \int_0^\infty \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \, e^{-\alpha p r^2/2} r \, dr = C\|f\|^p_{p,\alpha}.\end{aligned}$$

This shows that $\|S_n\|_{p,\alpha} \le C$ for all $n \ge 1$. The desired result then follows from Proposition 2.1. ∎

3. The case $0 < p \le 1$. Garling and Wojtaszczyk asked in [2] whether Theorem 2.2 remains true for $\mathcal{F}_1^1$. This was answered negatively by Lusky in [5].

PROPOSITION 3.1. *There exists a function in F_α^1 whose Taylor series does not converge in norm.*

To settle the case $0 < p < 1$, we need the following result concerning the duality of Fock spaces (see [12]).

PROPOSITION 3.2. *If $0 < p \le 1$, then the dual space of F_α^p can be identified with F_α^∞ under the integral pairing*

$$\langle f, g\rangle_\alpha = \frac{\alpha}{\pi}\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-\alpha|z|^2}\,dA(z).$$

Furthermore, the dual space of f_α^∞ can be identified with F_α^1 under the same integral pairing above.

LEMMA 3.3. *Under the duality pairing $\langle f, g\rangle_\alpha$ we have $S_n^* = S_n$.*

Proof. If

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

then it is easy to see that

$$\langle f, g\rangle_\alpha = \sum_{k=0}^{\infty} \frac{k!}{\alpha^k} a_k \overline{b}_k.$$

From this we deduce that

$$\langle S_n f, g\rangle_\alpha = \langle f_n, g_n\rangle = \langle f, S_n g\rangle_\alpha. \ \blacksquare$$

LEMMA 3.4. *We have*

$$\sup_n \|S_n\|_{f_\alpha^\infty} = \sup_n \|S_n\|_{F_\alpha^\infty} = \infty.$$

Proof. Recall from functional analysis that for any bounded linear operator T on a Banach space X we have $\|T\|_X = \|T^*\|_{X^*}$. By Propositions 2.1 and 3.1, we have

$$\sup_n \|S_n\|_{F_\alpha^1} = \infty.$$

This together with Proposition 3.2 and Lemma 3.3 shows that

$$\sup_n \|S_n\|_{f_\alpha^\infty} = \infty.$$

Since $\|S_n\|_{F_\alpha^\infty} \ge \|S_n\|_{f_\alpha^\infty}$, we must also have

$$\sup_n \|S_n\|_{F_\alpha^\infty} = \infty. \ \blacksquare$$

Corollary 3.5. *There exists a function in f_α^∞ whose Taylor series does not converge in norm.*

Note that the space F_α^∞ is not separable. In particular, the polynomials are not dense in F_α^∞. So generally speaking, the Taylor series of a function in F_α^∞ does not converge in norm. However, we cannot use Proposition 2.1 to conclude that

$$\sup_n \|S_n\|_{F_\alpha^\infty} = \infty,$$

because the main assumption in Proposition 2.1 is that the polynomials are dense.

To complete our discussion about the convergence of Taylor series in Fock spaces, we need to generalize Proposition 2.1 to the spaces F_α^p when $0 < p < \infty$. This is accomplished with the help of the following two results in [6].

Lemma 3.6. *Suppose X and Y are topological vector spaces, Γ is an equicontinuous collection of linear mappings from X into Y, and E is a bounded subset of X. Then Y has a bounded subset F such that $\Lambda(E) \subset F$ for every $\Lambda \in \Gamma$.*

Lemma 3.7. *If Γ is a collection of continuous mappings from an F-space X into a topological vector space, and if the set $\Gamma(x) = \{\Lambda(x) : \Lambda \in \Gamma\}$ is bounded in Y for every $x \in X$, then Γ is equicontinuous.*

The following uniform boundedness principle in F-spaces easily follows from Lemmas 3.6 and 3.7.

Proposition 3.8. *Let Γ be a collection of continuous mappings from an F-space X into X. If the set $\Gamma(x) = \{\Lambda(x) : \Lambda \in \Gamma\}$ is bounded in X for every $x \in X$, then Γ is uniformly bounded.*

Lemma 3.9. *Suppose $0 < p < 1$. Then the following two conditions are equivalent.*

(a) *For every $f \in F_\alpha^p$ we have $\|f_n - f\|_{p,\alpha} \to 0$ as $n \to \infty$.*
(b) *There exists a positive constant C such that $\|S_n\|_{F_\alpha^p} \le C$ for all $n \ge 1$.*

Proof. Basically, the spaces F_α^p are not Banach spaces when $0 < p < 1$, so Proposition 2.1 cannot be applied directly. However, a careful examination of the proof of [9, Proposition 2.1] reveals that the result still holds for F_α^p when $0 < p < 1$. All we need is Proposition 3.8, the uniform boundedness principle for F-spaces. ∎

Theorem 3.10. *For any $0 < p < 1$ there exists a function f in the Fock space F_α^p such that $\|f_n - f\|_{p,\alpha} \nrightarrow 0$ as $n \to \infty$.*

Proof. Let $\|S_n\|_{p,\alpha}$ denote the norm of S_n on F^p_α, including the case when $p = \infty$. It is well known that each S_n is a bounded linear operator on F^p_α. By Lemma 3.9, we just need to show that the sequence $\{\|S_n\|_{p,\alpha}\}$ is unbounded.

If T is a bounded linear operator on a Banach space X, then $T^* : X^* \to X^*$ is a bounded linear operator with $\|T\| = \|T^*\|$. The proof of this identity depends on the Hahn–Banach extension theorem, and the proof of the latter requires the space X to be at least locally convex. The complication here is that the spaces F^p_α are not locally convex when $0 < p < 1$. Therefore, the duality $(F^p_\alpha)^* = F^\infty_\alpha$ under the integral pairing $\langle f, g\rangle_\alpha$ does not automatically guarantee that $\|S_n\|_{p,\alpha}$ is comparable to $\|S_n\|_{\infty,\alpha}$. Fortunately, we only need one half of this argument, and this part can be obtained directly.

More specifically, there exist positive constants C and C' (independent of f, g, and n below) such that for any $f \in F^\infty_\alpha = (F^p_\alpha)^*$, we have

$$\begin{aligned}\|S_n f\|_{\infty,\alpha} &\le C \sup\{|\langle S_n f, g\rangle_\alpha| : g \in F^p_\alpha,\ \|g\|_{p,\alpha} \le 1\}\\ &= C \sup\{|\langle f, S_n g\rangle_\alpha| : g \in F^p_\alpha,\ \|g\|_{p,\alpha} \le 1\}\\ &\le C'\|f\|_{\infty,\alpha} \sup\{\|S_n g\|_{p,\alpha} : g \in F^p_\alpha,\ \|g\|_{p,\alpha} \le 1\}\\ &= C'\|S_n\|_{p,\alpha}\|f\|_{\infty,\alpha}.\end{aligned}$$

Taking the supremum over all $\|f\|_{\infty,\alpha} \le 1$, we obtain $\|S_n\|_{\infty,\alpha} \le C'\|S_n\|_{p,\alpha}$ for all $n \ge 1$. Combining this with Lemma 3.4, we conclude that the sequence $\{\|S_n\|_{p,\alpha}\}$ is unbounded. ∎

The arguments above can be applied to the Bergman and Hardy spaces as well. In fact, if $\mathcal{B}$ denotes the Bloch space of analytic functions f in the unit disk $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup\{(1 - |z|^2)|f'(z)| : z \in \mathbb{D}\} < \infty,$$

then the dual space of A^p_γ, where $0 < p \le 1$ and $\gamma > -1$, can be identified with $\mathcal{B}$ under the integral pairing

$$\langle f, g\rangle_\beta = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA_\beta(z),$$

where $\beta = [(2 + \gamma)/p] - 2$. Similarly, the dual space of the Hardy space H^p, where $0 < p < 1$ (note that the duality for $p = 1$ must be treated differently using the space BMOA), can be identified with $\mathcal{B}$ under the integral pairing above with $\beta = 1/p - 2$ (see [11]). It was shown in [9] that the Taylor series of a function in A^1_γ does not necessarily converge in norm. It follows that the operators S_n are not uniformly bounded on A^1_γ. By duality of Banach spaces, the operators S_n are not uniformly bounded on the Bloch space $\mathcal{B}$. By the proof of Theorem 3.10 and the duality $(H^p)^* = \mathcal{B}$, the operators S_n are not uniformly bounded on H^p. Also, Lemma 3.9 remains valid for Bergman and Hardy spaces. Consequently, we have the following result.

THEOREM 3.11. *Let $0 < p < 1$ and $\gamma > -1$. There exists a function $f \in A^p_\gamma$ such that*

$$\|f_n - f\|_{A^p_\gamma} \nrightarrow 0, \qquad n \to \infty.$$

Similarly, there exists a function $g \in H^p$ such that

$$\|f_n - g\|_{H^p} \nrightarrow 0, \qquad n \to \infty.$$

The above theorem was stated as an exercise in [8], but no proof was ever given explicitly.

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