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Closed operator ideals and limiting real interpolation

by

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Abstract. We establish interpolation properties under limiting real methods for a class of closed ideals including weakly compact operators, Banach–Saks operators, Rosenthal operators and Asplund operators. We show that they behave much better than compact operators.

1. Introduction. Limiting real interpolation methods $(A_0, A_1)_{q;K}$, $(A_0, A_1)_{q;J}$ have attracted considerable attention in recent years. See, for example, the papers by Cobos, Fernández-Cabrera, Kühn and Ullrich [5], Cobos, Fernández-Cabrera and Mastyło [7], Cobos and Kühn [9] and Cobos and Segurado [12, 13]. These methods correspond to the limit choices $\theta = 0, 1$ in the real interpolation method $(A_0, A_1)_{\theta,q}$. The K-space $(A_0, A_1)_{q;K}$ is closer to $A_0 + A_1$ than any real interpolation space $(A_0, A_1)_{\theta,q}$, and the J-space $(A_0, A_1)_{q;J}$ is very near to $A_0 \cap A_1$. For this reason, several operator properties behave worse under limiting methods than under the real method. This is the case of compactness. If $T \in \mathcal{L}(\bar{A}, \bar{B})$ is a bounded linear operator between the couples $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ such that for $j = 0$ or 1 the restriction $T : A_j \to B_j$ is compact, then the operator $T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ interpolated by the real method is also compact (see [10] and [14]). However, in the limit case, if the couples $\bar{A}$ and $\bar{B}$ are ordered, that is, $A_0 \hookrightarrow A_1$ and $B_0 \hookrightarrow B_1$, then compactness of $T : A_0 \to B_0$ is not enough to imply that $T : (A_0, A_1)_{q;K} \to (B_0, B_1)_{q;K}$ is compact, but compactness of $T : A_1 \to B_1$ does imply it (see [5, Counterexample 7.11 and Theorem 7.14]). If the couples are not ordered, then compactness of $T : A_1 \to B_1$ is not enough either. In the general case, a sufficient condition for $T : (A_0, A_1)_{q;K} \to (B_0, B_1)_{q;K}$ to be compact is that both restrictions $T : A_0 \to B_0$ and $T : A_1 \to B_1$ are compact (see [12, §5]). Compactness under limiting J-spaces behaves similarly.

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In this paper we investigate the interpolation properties under limiting methods of operator ideals $\mathcal{I}$ which are injective, surjective, closed and satisfy the Σ_q-condition (terminology is explained in Section 2). Important examples of ideals of this type are weakly compact operators, Banach–Saks operators, Rosenthal operators and Asplund operators. However, the ideal of compact operators is not of this type because it fails the Σ_q-condition. In Section 3, we establish that if $1 < q < \infty$ then a necessary and sufficient condition for $T : (A_0, A_1)_{q;K} \to (B_0, B_1)_{q;K}$ (respectively, $T : (A_0, A_1)_{q;J} \to (B_0, B_1)_{q;J}$) to belong to $\mathcal{I}$ is that $T : A_0 \cap A_1 \to B_0 + B_1$ belongs to $\mathcal{I}$. This shows that limiting interpolation properties of this type of ideals are much better that those of compact operators. In fact, they behave as in the case of the real method (see [2] and [18]). The key ingredient of the proof is the Σ_q-condition and the description of the limiting methods by the dual functional. In the second part of Section 3, we show that the Σ_q-condition is not needed if one of Banach couples reduces to a single Banach space. In this degenerate case our results work even for $q = 1$ or ∞ and apply also to compact operators, improving [12, Propositions 5.2 and 5.5].

Other interpolation properties of closed operator ideals can be found in [16]. As concerns extrapolation properties, we refer to [17].

2. Preliminaries. In what follows, the letters E, F, X, Y stand for Banach spaces. As usual, we write U_E for the closed unit ball of E and $\mathcal{L}(E, F)$ for the space of all bounded linear operators from E into F, endowed with the operator norm.

Let $\mathcal{I}$ be an operator ideal in the sense of [20] or [15]. We write $\mathcal{I}(E, F)$ for the component of $\mathcal{I}$ between E and F: $\mathcal{I}(E, F) = \mathcal{I} \cap \mathcal{L}(E, F)$. The ideal $\mathcal{I}$ is said to be *closed* if $\mathcal{I}(E, F)$ is a closed subspace of $\mathcal{L}(E, F)$ for any E, F. We say that $\mathcal{I}$ is *injective* if for every injection $P \in \mathcal{L}(F, X)$ and every $T \in \mathcal{L}(E, F)$, it follows from $PT \in \mathcal{I}(E, X)$ that $T \in \mathcal{I}(E, F)$. The ideal $\mathcal{I}$ is said to be *surjective* if for any surjection $Q \in \mathcal{L}(Y, E)$ and every $T \in \mathcal{L}(E, F)$, it follows from $TQ \in \mathcal{I}(Y, F)$ that $T \in \mathcal{I}(E, F)$. See [20, pages 70–75] for details on these properties.

Given $T \in \mathcal{L}(E, F)$, we write $\gamma_{\mathcal{I}}(T_{E,F})$ for the infimum of all $\sigma > 0$ such that $T(U_E) \subseteq \sigma U_F + R(U_X)$ for some X and some $R \in \mathcal{I}(X, F)$. We write $\beta_{\mathcal{I}}(T_{E,F})$ for the infimum of all $\sigma > 0$ such that for some Y and some $R \in \mathcal{I}(E, Y)$ the inequality $\|Tx\|_F \le \sigma \|x\|_E + \|Rx\|_Y$ holds for any $x \in E$. We refer to [1], [19], [22] and [11] for properties of the functionals $\gamma_{\mathcal{I}}$ and $\beta_{\mathcal{I}}$. In particular, the following holds:

If $\mathcal{I}$ is surjective and closed, then

$$\gamma_{\mathcal{I}}(T_{E,F}) = 0 \iff T \in \mathcal{I}(E, F), \tag{2.1}$$

and if $\mathcal{I}$ is injective and closed, then

$$\beta_{\mathcal{I}}(T_{E,F}) = 0 \ \Leftrightarrow \ T \in \mathcal{I}(E,F). \tag{2.2}$$

Let $1 \le q \le \infty$, let $(E_m)_{m\in\mathbb{Z}}$ be a sequence of Banach spaces and let $(\lambda_m)_{m\in\mathbb{Z}}$ be a sequence of non-negative numbers. We write $\ell_q(\lambda_m E_m)$ for the collection of all sequences $x = (x_m)$ such that $x_m \in E_m$ for any $m \in \mathbb{Z}$ and the norm

$$\|x\|_{\ell_q(\lambda_m E_m)} = \Big(\sum_{m=-\infty}^{\infty} (\lambda_m \|x_m\|_{E_m})^q \Big)^{1/q}$$

is finite.

For $k, r \in \mathbb{Z}$, let $Q_k : \ell_q(E_m) \to E_k$ be the projection $Q_k(x_m) = x_k$, and let $P_r : E_r \to \ell_q(E_m)$ be the injection $P_r x = (\delta_m^r x)$ where δ_m^r is the Kronecker delta.

Following [18], we say that the operator ideal $\mathcal{I}$ *satisfies the* Σ_q*-condition* if for any sequences $(E_m), (F_m)$ of Banach spaces and $T \in \mathcal{L}(\ell_q(E_m), \ell_q(F_m))$, it follows from $Q_k T P_r \in \mathcal{I}(E_r, F_k)$ that $T \in \mathcal{I}(\ell_q(E_m), \ell_q(F_m))$ for any $k, r \in \mathbb{Z}$.

It turns out that if $\mathcal{I}$ satisfies the Σ_q-condition, then $\mathcal{I}$ is closed (see [6, Lemma 3.2]). Weakly compact operators, Rosenthal operators, Banach–Saks operators and Asplund operators (also referred to as dual Radon–Nikodym operators or decomposing operators) satisfy the Σ_q-condition for $1 < q < \infty$. These ideals are also injective and surjective. The ideal of compact operators is injective, surjective and closed but it fails the Σ_q-condition.

Let $\bar{A} = (A_0, A_1)$ be a *Banach couple*, that is, A_0, A_1 are Banach spaces continuously embedded in some Hausdorff topological vector space. *Peetre's* K*- and* J*-functionals* are defined by

$$\begin{aligned} K(t,a) &= K(t,a;\bar{A}) \\ &= \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1,\ a_j \in A_j\},\ a \in A_0 + A_1, \end{aligned}$$

and

$$J(t,a) = J(t,a;\bar{A}) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, \qquad a \in A_0 \cap A_1.$$

Note that $K(1,\cdot)$ is the norm of $A_0 + A_1$ and $J(1,\cdot)$ the norm of $A_0 \cap A_1$.

We say that a Banach space A is an *intermediate space* with respect to the couple $\bar{A}$ if $A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1$. Here $\hookrightarrow$ means continuous inclusion. Put

$$\psi_A(t) = \psi_A(t;\bar{A}) = \sup\{K(t,a) : \|a\|_A = 1\}$$

and

$$\rho_A(t) = \rho_A(t;\bar{A}) = \inf\{J(t,a) : a \in A_0 \cap A_1 : \|a\|_A = 1\} \quad \text{(see [4])}.$$

Let $\bar{B} = (B_0, B_1)$ be another Banach couple. By writing $T \in \mathcal{L}(\bar{A}, \bar{B})$ we mean that T is a linear operator from $A_0 + A_1$ into $B_0 + B_1$ whose restriction to each A_j defines a bounded linear operator from A_j into B_j for $j = 0, 1$. If $A_0 = A_1 = A$ (respectively, $B_0 = B_1 = B$), we write simply $T \in \mathcal{L}(A, \bar{B})$ (respectively, $T \in \mathcal{L}(\bar{A}, B)$).

Let $\mathcal{C}$ be a family of Banach couples with $(\mathbb{K}, \mathbb{K}) \in \mathcal{C}$, where $\mathbb{K}$ is the scalar field ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). An *interpolation method* $\mathcal{F}$ in $\mathcal{C}$ is a procedure assigning to any Banach couple $\bar{A} \in \mathcal{C}$ an intermediate space $\mathcal{F}(\bar{A})$ with respect to $\bar{A}$ such that for any other Banach couple $\bar{B} \in \mathcal{C}$ and any $T \in \mathcal{L}(\bar{A}, \bar{B})$, the restriction of T to $\mathcal{F}(\bar{A})$ gives a bounded linear operator from $\mathcal{F}(\bar{A})$ into $\mathcal{F}(\bar{B})$. For $t, s > 0$, put

$$\varphi_{\mathcal{F}}(t,s) = \sup\{\|T\|_{\mathcal{F}(\bar{A}),\mathcal{F}(\bar{B})} : T \in \mathcal{L}(\bar{A}, \bar{B}) \text{ with } \|T\|_{A_0,B_0} \le t,\\ \|T\|_{A_1,B_1} \le s \text{ and } \bar{A}, \bar{B} \in \mathcal{C}\}.$$

We refer to [3] and [21] for examples of interpolation methods, including the real method. We are mainly interested here in the limiting real methods. Let $1 \le q \le \infty$ and let $\bar{A} = (A_0, A_1)$ be a Banach couple, the space $\bar{A}_{q;K} = (A_0, A_1)_{q;K}$ consists of all those $a \in A_0 + A_1$ which have a finite norm

$$\|a\|_{\bar{A}_{q;K}} = \Big(\int_0^1 K(t,a)^q \,\frac{dt}{t}\Big)^{1/q} + \Big(\int_1^\infty \Big(\frac{K(t,a)}{t}\Big)^q \frac{dt}{t}\Big)^{1/q}$$

(integrals should be replaced by suprema if $q = \infty$).

The space $\bar{A}_{q;J} = (A_0, A_1)_{q;J}$ is formed by all those $a \in A_0 + A_1$ which can be represented as

$$a = \int_0^\infty u(t)\,\frac{dt}{t} \quad \text{(convergence in } A_0 + A_1\text{)}, \tag{2.3}$$

where $u(t)$ is a strongly measurable function with values in $A_0 \cap A_1$ and such that

$$\Big(\int_0^1 \Big(\frac{J(t,u(t))}{t}\Big)^q \frac{dt}{t}\Big)^{1/q} + \Big(\int_1^\infty J(t,u(t))^q \,\frac{dt}{t}\Big)^{1/q} < \infty. \tag{2.4}$$

We set

$$\|a\|_{\bar{A}_{q;J}} = \inf\Big\{\Big(\int_0^1 \Big(\frac{J(t,u(t))}{t}\Big)^q \frac{dt}{t}\Big)^{1/q} + \Big(\int_1^\infty J(t,u(t))^q \,\frac{dt}{t}\Big)^{1/q}\Big\}$$

where the infimum is taken over all representations u satisfying (2.3) and (2.4). See [12] for properties of limiting spaces.

3. Limiting interpolation of closed ideals. Let $T \in \mathcal{L}(\bar{A}, \bar{B})$. The factorization

$$A_0 \cap A_1 \hookrightarrow \bar{A}_{q;K} \xrightarrow{T} \bar{B}_{q;K} \hookrightarrow B_0 + B_1$$

implies that if $T \in \mathcal{I}(\bar{A}_{q;K}, \bar{B}_{q;K})$ then $T \in \mathcal{I}(A_0 \cap A_1, B_0 + B_1)$. Next we show that the converse also holds under suitable assumptions on $\mathcal{I}$ and q.

THEOREM 3.1. *Let $1 < q < \infty$ and let $\mathcal{I}$ be an injective, surjective operator ideal satisfying the Σ_q-condition. Assume $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ are Banach couples and let $T \in \mathcal{L}(\bar{A}, \bar{B})$. If $T \in \mathcal{I}(A_0 \cap A_1, B_0 + B_1)$, then $T \in \mathcal{I}(\bar{A}_{q;K}, \bar{B}_{q;K})$.*

Proof. According to [12, Theorem 6.2], the space $\bar{A}_{q;K}$ is formed by all $a \in A_0 + A_1$ for which there is a representation

$$a = \int_0^\infty u(t)\,\frac{dt}{t} \quad (\text{convergence in } A_0 + A_1) \tag{3.1}$$

with $u(t)$ being a strongly measurable function with values in $A_0 \cap A_1$ and such that

$$\Big(\int_0^1 ((1 + |\log t|)J(t, u(t)))^q \,\frac{dt}{t}\Big)^{1/q} + \Big(\int_1^\infty \Big(\frac{1 + |\log t|}{t} J(t, u(t))\Big)^q \,\frac{dt}{t}\Big)^{1/q} < \infty. \tag{3.2}$$

Moreover, the infimum of the values in (3.2) over all possible representations (3.1) of a yields an equivalent norm to $\|\cdot\|_{\bar{A}_{q;K}}$. Making the discretization $t = 2^m$, $m \in \mathbb{Z}$, it is not hard to show that $\bar{A}_{q;K}$ consists of all $a \in A_0 + A_1$ for which there is a representation

$$a = \sum_{m=-\infty}^{\infty} u_m \quad (\text{convergence in } A_0 + A_1), \text{ with } (u_m) \subseteq A_0 \cap A_1 \tag{3.3}$$

and such that

$$\Big(\sum_{m=-\infty}^{0} ((1 - m)J(2^m, u_m))^q\Big)^{1/q} + \Big(\sum_{m=1}^{\infty} \Big(\frac{1 + m}{2^m} J(2^m, u_m)\Big)^q\Big)^{1/q} < \infty. \tag{3.4}$$

Furthermore, the infimum of the values (3.4) over all possible representations (3.3) of a gives a norm equivalent to $\|\cdot\|_{\bar{A}_{q;K}}$.

Form the vector-valued sequence space $\ell_q(\lambda_m G_m)$, where G_m is $A_0 \cap A_1$ normed by $J(2^m, \cdot)$ and

$$\lambda_m = \begin{cases} 1 - m & \text{if } m \le 0, \\ (1 + m)/2^m & \text{if } m > 0, \end{cases}$$

and let $Q : \ell_q(\lambda_m G_m) \to \bar{A}_{q;K}$ be the projection $Q(u_m) = \sum_{m=-\infty}^{\infty} u_m$. Since $\mathcal{I}$ is surjective, to check that $T : \bar{A}_{q;K} \to \bar{B}_{q;K}$ belongs to $\mathcal{I}$ it suffices to show that $TQ : \ell_q(\lambda_m G_m) \to \bar{B}_{q;K}$ belongs to $\mathcal{I}$.

On the other hand, the discretization $t = 2^m$, $m \in \mathbb{Z}$, implies that the norm of $\bar{B}_{q;K}$ is equivalent to

$$\|b\|_{q;K} = \Big(\sum_{m=-\infty}^{0} K(2^m, b)^q\Big)^{1/q} + \Big(\sum_{m=1}^{\infty}\Big(\frac{K(2^m, b)}{2^m}\Big)^q\Big)^{1/q}.$$

Let

$$\tau_m = \begin{cases} 1 & \text{if } m \le 0, \\ 2^{-m} & \text{if } m > 0 \end{cases}$$

and let F_m be $B_0 + B_1$ normed by $K(2^m, \cdot)$. Then the function $P : \bar{B}_{q;K} \to \ell_q(\tau_m F_m)$ defined by $P(b) = \{\dots, b, b, b, \dots\}$ is an injection. Hence, if we show that $PTQ : \ell_q(\lambda_m G_m) \to \ell_q(\tau_m F_m)$ belongs to $\mathcal{I}$, then injectivity of $\mathcal{I}$ will imply the desired result. To this end, we shall use the fact that $\mathcal{I}$ satisfies the Σ_q-condition.

Given any $k, r \in \mathbb{Z}$, we have

$$\begin{aligned}(Q_k PTQP_r)u &= (Q_k PTQ)(\delta_m^r u) = (Q_k PT)u \\ &= (Q_k P)(Tu) = Q_k(\dots, Tu, Tu, Tu, \dots) = Tu.\end{aligned}$$

The assumption $T \in \mathcal{I}(A_0 \cap A_1, B_0 + B_1)$ implies that $Q_k PTQP_r \in \mathcal{I}(G_r, F_k)$ for any $k, r \in \mathbb{Z}$. Consequently, by the Σ_q-condition,

$$PTQ \in \mathcal{I}(\ell_q(\lambda_m G_m), \ell_q(\tau_m F_m)).$$

This completes the proof. ∎

Since the limiting J-space can be described as a K-space (see [8, Theorem 3.10]), using similar arguments we obtain the following.

Theorem 3.2. *Let $1 < q < \infty$ and let $\mathcal{I}$ be an injective and surjective operator ideal satisfying the Σ_q-condition. Assume that $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ are Banach couples and let $T \in \mathcal{L}(\bar{A}, \bar{B})$. If $T \in \mathcal{I}(A_0 \cap A_1, B_0 + B_1)$, then $T \in \mathcal{I}(\bar{A}_{q;J}, \bar{B}_{q;J})$.*

Corollary 3.3. *Let $1 < q < \infty$, let $\mathcal{I}$ be an injective and surjective operator ideal satisfying the Σ_q-condition and let $\bar{A} = (A_0, A_1)$ be a Banach couple. Then a necessary and sufficient condition for the identity operator in $\bar{A}_{q;K}$ (respectively, $\bar{A}_{q;J}$) to belong to $\mathcal{I}$ is that the embedding $A_0 \cap A_1 \hookrightarrow A_0 + A_1$ belongs to $\mathcal{I}$.*

The previous three results apply to weakly compact operators, Rosenthal operators, Banach–Saks operators and Asplund operators but they do not work for compact operators because this ideal fails the Σ_q-condition. In the rest of this section we shall show that if one of the couples $\bar{A}$, $\bar{B}$ reduces to a single Banach space, then we can get rid of the assumption on the

Σ_q-condition. We start by comparing the functions ψ, ρ and φ. The result is similar to [6, Lemma 2.1].

LEMMA 3.4. *Let $\mathcal{F}$ be an interpolation method in $\mathcal{C}$. There are positive constants c_1, c_2 such that for any $\bar{A} \in \mathcal{C}$ and any $t > 0$ we have*

$$\psi_{\mathcal{F}(\bar{A})}(t) \leq c_1 \varphi_{\mathcal{F}}(1,t), \qquad \frac{1}{\varphi_{\mathcal{F}}(1,1/t)} \leq c_2 \rho_{\mathcal{F}(\bar{A})}(t).$$

Proof. Clearly $\mathcal{F}(\mathbb{K},\mathbb{K}) = \mathbb{K}$ with equivalence of norms. Let $c_1, c_2 > 0$ be constants such that

$$|\lambda| \leq c_1 \|\lambda\|_{\mathcal{F}(\mathbb{K},\mathbb{K})}, \qquad \|\lambda\|_{\mathcal{F}(\mathbb{K},\mathbb{K})} \leq c_2 |\lambda|, \qquad \lambda \in \mathbb{K}.$$

Given any $a \in \mathcal{F}(\bar{A})$ and any $t > 0$, applying the Hahn–Banach theorem to the space $(A_0 + A_1, K(t,\cdot))$, we can find a functional f in its dual with norm 1 and such that $f(a) = K(t,a)$. Since for $j = 1, 0$ the embedding $A_j \hookrightarrow (A_0 + A_1, K(t,\cdot))$ has norm less than or equal than t^j, we see that $\|f\|_{A_0,\mathbb{K}} \leq 1$ and $\|f\|_{A_1,\mathbb{K}} \leq t$. Hence

$$\begin{aligned} K(t,a) = f(a) &\leq c_1 \|f(a)\|_{\mathcal{F}(\mathbb{K},\mathbb{K})} \leq c_1 \|f\|_{\mathcal{F}(\bar{A}),\mathcal{F}(\mathbb{K},\mathbb{K})} \|a\|_{\mathcal{F}(\bar{A})} \\ &\leq c_1 \varphi_{\mathcal{F}}(1,t) \|a\|_{\mathcal{F}(\bar{A})}. \end{aligned}$$

Hence

$$\psi_{\mathcal{F}(\bar{A})}(t) = \sup\{K(t,a) : \|a\|_{\mathcal{F}(\bar{A})} = 1\} \leq c_1 \varphi_{\mathcal{F}}(1,t).$$

On the other hand, given any $a \in A_0 \cap A_1$ and $u, v > 0$, consider the operator $T(\lambda) = \min(u,v)\lambda a$. Clearly $T : \mathbb{K} \to A_j$ is bounded for $j = 0, 1$, with $\|T\|_{\mathbb{K},A_0} \leq u\|a\|_{A_0}$ and $\|T\|_{\mathbb{K},A_1} \leq v\|a\|_{A_1}$. Therefore

$$\min(u,v)\|a\|_{\mathcal{F}(\bar{A})} \leq c_2 \|T\|_{\mathcal{F}(\mathbb{K},\mathbb{K}),\mathcal{F}(\bar{A})} \leq c_2 \varphi_{\mathcal{F}}(u\|a\|_{A_0}, v\|a\|_{A_1}).$$

Taking $u = 1/\|a\|_{A_0}$ and $v = 1/t\|a\|_{A_1}$, it follows that

$$\|a\|_{\mathcal{F}(\bar{A})}/J(t,a) = \min(1/\|a\|_{A_0}, 1/t\|a\|_{A_1})\|a\|_{\mathcal{F}(\bar{A})} \leq c_2 \varphi_{\mathcal{F}}(1,1/t).$$

This yields

$$1/\varphi_{\mathcal{F}}(1,1/t) \leq c_2 \inf\{J(t,a)/\|a\|_{\mathcal{F}(\bar{A})} : a \in A_0 \cap A_1, a \neq 0\} \leq c_2 \rho_{\mathcal{F}(\bar{A})}(t)$$

and completes the proof. ∎

THEOREM 3.5. *Let $\mathcal{I}$ be a surjective, closed operator ideal and let $\mathcal{F}$ be an interpolation method in $\mathcal{C}$. Assume that $\bar{A} = (A_0, A_1) \in \mathcal{C}$, let B be a Banach space and let $T \in \mathcal{L}(\bar{A}, B)$.*

(a) *If $T \in \mathcal{I}(A_0, B)$ and $\lim_{s\to\infty} \varphi_{\mathcal{F}}(1,s)/s = 0$, then $T \in \mathcal{I}(\mathcal{F}(\bar{A}), B)$.*
(b) *If $T \in \mathcal{I}(A_1, B)$ and $\lim_{s\to 0} \varphi_{\mathcal{F}}(1,s) = 0$, then $T \in \mathcal{I}(\mathcal{F}(\bar{A}), B)$.*

Proof. Under the assumptions of (a), it follows from [4, Theorem 4.1(a)] and Lemma 3.4 that

$$\gamma_{\mathcal{I}}(T_{\mathcal{F}(\bar{A}),B}) \leq \gamma_{\mathcal{I}}(T_{A_1,B}) \lim_{s\to\infty} \frac{\psi_{\mathcal{F}(\bar{A})}(s)}{s} \leq c_1 \gamma_{\mathcal{I}}(T_{A_1,B}) \lim_{s\to\infty} \frac{\varphi_{\mathcal{F}}(1,s)}{s} = 0.$$

Hence, (2.1) yields $T \in \mathcal{I}(\mathcal{F}(\bar{A}), B)$. The proof of (b) is similar by using now [4, Theorem 4.1(b)]. ∎

The result for injective ideals reads as follows.

THEOREM 3.6. *Let $\mathcal{I}$ be an injective, closed operator ideal and let $\mathcal{F}$ be an interpolation method in $\mathcal{C}$. Assume that $\bar{B} = (B_0, B_1) \in \mathcal{C}$, let A be a Banach space and let $T \in \mathcal{L}(A, \bar{B})$.*

(a) *If $T \in \mathcal{I}(A, B_0)$ and $\lim_{s\to\infty} \varphi_{\mathcal{F}}(1,s)/s = 0$, then $T \in \mathcal{I}(A, \mathcal{F}(\bar{B}))$.*
(b) *If $T \in \mathcal{I}(A, B_1)$ and $\lim_{s\to 0} \varphi_{\mathcal{F}}(1,s) = 0$, then $T \in \mathcal{I}(A, \mathcal{F}(\bar{B}))$.*

Proof. Combine [4, Theorem 4.2] and Lemma 3.4. ∎

In the family of all Banach couples, the $(q;K)$- and $(q;J)$-methods satisfy poor norm estimates for interpolated operators (see [12, Counterexample 3.6] and [13, Propositions 4.2 and 4.3]). However, in the family $\mathcal{C}$ of all Banach couples $\bar{A} = (A_0, A_1)$ with $A_0 \hookrightarrow A_1$, for $1 \le q < \infty$ it follows from [5, Theorem 7.9] that

$$\varphi_{(q;K)}(t,s) \le cs(1 + \max\{0, \log(t/s)\}). \tag{3.5}$$

Hence $\lim_{s\to 0} \varphi_{(q;K)}(1,s) = 0$. For the $(q;J)$-method in $\mathcal{C}$ and $1 < q \le \infty$, we have

$$\varphi_{(q;J)}(t,s) \le ct(1 + \max\{0, \log(t/s)\}) \tag{3.6}$$

(see [5, Theorem 4.9]). So $\lim_{s\to\infty} \varphi_{(q;J)}(1,s)/s = 0$. Next we write down Theorems 3.5 and 3.6 in those cases.

COROLLARY 3.7. *Let $\mathcal{I}$ be a surjective, closed operator ideal, let $\bar{A} = (A_0, A_1)$ be a Banach couple with $A_0 \hookrightarrow A_1$, let B be a Banach space and $T \in \mathcal{L}(\bar{A}, B)$. If $1 < q \le \infty$ and $T \in \mathcal{I}(A_0, B)$, then $T \in \mathcal{I}(\bar{A}_{q;J}, B)$.*

COROLLARY 3.8. *Let $\mathcal{I}$ be an injective, closed operator ideal, let $\bar{B} = (B_0, B_1)$ be a Banach couple with $B_0 \hookrightarrow B_1$, let A be a Banach space and $T \in \mathcal{L}(A, \bar{B})$. If $1 \le q < \infty$ and $T \in \mathcal{I}(A, B_1)$, then $T \in \mathcal{I}(A, \bar{B}_{q;K})$.*

Other interpolation methods working in $\mathcal{C}$ and satisfying similar estimates to (3.5) and (3.6) can be found in [7] and [9].

Next we extend these results to the case of general couples.

COROLLARY 3.9. *Let $\mathcal{I}$ be a surjective and closed operator ideal. Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let B be a Banach space and let $T \in \mathcal{L}(\bar{A}, B)$. If $1 \le q \le \infty$ and $T \in \mathcal{I}(A_0 \cap A_1, B)$, then $T \in \mathcal{I}(\bar{A}_{q;J}, B)$.*

Proof. By [12, Lemma 4.2], we have $\bar{A}_{1;J} = A_0 \cap A_1$. So if $q = 1$ the result is trivial. If $1 < q \le \infty$, according to [12, Lemma 4.3] we have

$$(A_0, A_1)_{q;J} = (A_0 \cap A_1, A_0 + A_1)_{q;J}.$$

Therefore the result follows by applying Corollary 3.7 to the ordered couple $(A_0 \cap A_1, A_0 + A_1)$. ∎

The corresponding result for the $(q;K)$-method is a consequence of the equality $(B_0, B_1)_{q;K} = (B_0 \cap B_1, B_0 + B_1)_{q;K}$ (see [12, Lemma 3.5]) and Corollary 3.8. It reads as follows.

COROLLARY 3.10. *Let $\mathcal{I}$ be an injective and closed operator ideal. Let A be a Banach space, let $\bar{B} = (B_0, B_1)$ be a Banach couple and let $T \in \mathcal{L}(A, \bar{B})$. If $1 \le q \le \infty$ and $T \in \mathcal{I}(A, B_0 + B_1)$, then $T \in \mathcal{I}(A, \bar{B}_{q;K})$.*

Writing down Corollaries 3.9 and 3.10 for the case of compact operators, we obtain results which improve [12, Propositions 5.2 and 5.5].

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