STUDIA MATHEMATICA 180 (3) (2007)

On spectral properties of linear combinations of idempotents

by

HONG-KE DU (Xi'an), CHUN-YAN DENG (Xi'an), MOSTAFA MBEKHTA (Lille) and VLADIMÍR MÜLLER (Praha)

Abstract. Let P, Q be two linear idempotents on a Banach space. We show that the closedness of the range and complementarity of the kernel (range) of linear combinations of P and Q are independent of the choice of coefficients. This generalizes known results and shows that many spectral properties of linear combinations do not depend on their coefficients.

The non-singularity of the difference and sum of two idempotent matrices P and Q was first studied in [KRS]. In [BB] it was proved that the non-singularity of $P+Q$ is equivalent to the non-singularity of any linear combination c_1P+c_2Q where $c_1, c_2 \neq 0, c_1+c_2 \neq 0$. The result was further generalized [DYD] to Hilbert space operators, and in [KR1] the stability of the nullity and rank of linear combinations of idempotents was proved.

Finally, in [KR2] it was proved (for Banach space operators) that the Fredholmness and semi-Fredholmness of linear combinations of two idempotents is independent of the choice of their coefficients.

We improve these results and show that for two idempotents P, Q on a Banach space the closedness of the range of c_1P+c_2Q and the complementarity of its kernel and range are independent of the choice of the coefficients c_1, c_2. Moreover, the kernel and range are continuous in the gap topology. This implies the independence of many spectral properties of linear combinations c_1P+c_2Q from the coefficients c_1, c_2.

Let $T \in B(X)$ where $B(X)$ denotes the set of all bounded linear operators on a Banach space X. Denote by $N(T)$ and $R(T)$ the kernel and range of T, respectively.

2000 *Mathematics Subject Classification*: Primary 47B99; Secondary 47A56, 15A99.

Key words and phrases: linear combinations of idempotents, closed range, complemented subspaces.

The last author was supported by grant No. 201/06/0128 of GA ČR and by the Institutional Research Plan AV 0Z 10190503.

An operator $P \in B(X)$ is called an *idempotent* if $P^2 = P$. Note that the range of an idempotent is always closed since $R(P) = N(I - P)$, where I is the identity operator.

The main result of this paper is the following theorem:

Main Theorem. *Let $P, Q \in B(X)$ be idempotents. Let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $c_1 + c_2 \neq 0$. If $c_1 P + c_2 Q$ is invertible (left invertible, right invertible, injective, bounded below, surjective, Fredholm, upper semi-Fredholm, lower semi-Fredholm, left essentially invertible, right essentially invertible or has a generalized inverse, respectively), then $z_1 P + z_2 Q$ has the same property for all $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, $z_1 + z_2 \neq 0$.*

Let M, L be closed subspaces of a Banach space X. Let

$$\delta(M, L) = \sup\{\operatorname{dist}\{x, L\} : x \in M,\ \|x\| \leq 1\}.$$

The *gap* $\widehat{\delta}(M, L)$ between M and L is defined by

$$\widehat{\delta}(M, L) = \max\{\delta(M, L), \delta(L, M)\}.$$

The *reduced minimum modulus* of an operator $T \in B(X)$ is defined by

$$\gamma(T) = \inf\{\|Tx\| : \operatorname{dist}\{x, N(T)\} \leq 1\}.$$

The most important property of the reduced minimum modulus is that $\gamma(T) > 0$ if and only if T has closed range. For basic properties of the gap and reduced minimum modulus see [K, pp. 197–201], or [M, Sec. 10].

Let $P, Q \in B(X)$ be idempotents. It is easy to see that instead of the function $(c_1, c_2) \mapsto c_1 P + c_2 Q$ of two variables (c_1, c_2), $c_1, c_2 \neq 0, c_1 + c_2 \neq 0$, it is sufficient to study the function $z \mapsto P - zQ$ where $z \neq 0, 1$.

For $z, z' \in \mathbb{C} \setminus \{0, 1\}$ write

$$V_{z,z'} = I + \frac{z - z'}{z(z' - 1)}\, P.$$

Lemma 1. *Let $z, z' \in \mathbb{C} \setminus \{0, 1\}$. Then:*

(i) $V_{z,z'} V_{z',z} = V_{z',z} V_{z,z'} = I$;

(ii) $V_{z,z'} N(P - zQ) = N(P - z'Q)$;

(iii) $\delta(N(P - zQ), N(P - z'Q) \leq \|P\| \cdot \left|\dfrac{z - z'}{z(z' - 1)}\right|$.

Proof. (i) Clearly $V_{z,z'} V_{z',z} = V_{z',z} V_{z,z'}$ and

$$\begin{aligned} V_{z,z'} V_{z',z} &= \left(I + \frac{z - z'}{z(z' - 1)}\, P\right)\left(I + \frac{z' - z}{z'(z - 1)}\, P\right) \\ &= I + \left(\frac{z - z'}{z(z' - 1)} + \frac{z' - z}{z'(z - 1)} + \frac{(z - z')(z' - z)}{zz'(z - 1)(z' - 1)}\right) P \\ &= I + \frac{(z - z')}{zz'(z - 1)(z' - 1)}\,(z'(z - 1) - z(z' - 1) + z' - z)P = I. \end{aligned}$$

(ii) Let $x \in N(P - zQ)$, $\|x\| = 1$. Then $Qx = \frac{1}{z}PX$ and $QPx = Px$. We have

$$\begin{aligned}(P - z'Q)V_{z,z'}x &= Px + \frac{z - z'}{z(z'-1)}\,Px - \frac{z'}{z}\,Px - \frac{z'(z-z')}{z(z'-1)}\,Px \\ &= \left(\frac{zz' - z + z - z' - z'^2 + z' - z'z + z'^2}{z(z'-1)}\right)Px = 0.\end{aligned}$$

Hence

$$V_{z,z'}N(P - zQ) \subset N(P - z'Q).$$

Similarly, $V_{z',z}N(P - z'Q) \subset N(P - zQ)$ and

$$N(P - z'Q) = V_{z,z'}V_{z',z}N(P - z'Q) \subset V_{z,z'}N(P - zQ).$$

Hence $V_{z,z'}N(P - zQ) = N(P - z'Q)$.

(iii) Let $x \in N(P - zQ)$, $\|x\| = 1$. By (ii), $V_{z,z'}x \in N(P - z'Q)$, and so

$$\operatorname{dist}\{x, N(P - z'Q)\} \le \|x - V_{z,z'}x\| \le \left\|\frac{z - z'}{z(z'-1)}\,Px\right\| \le \|P\| \cdot \left|\frac{z - z'}{z(z'-1)}\right|,$$

proving (iii). ∎

Corollary 2. *The function $z \mapsto N(P - zQ)$ is continuous in the gap topology for $z \in \mathbb{C} \setminus \{0, 1\}$. Consequently, $z \mapsto \dim N(P - zQ)$ is constant for $z \in \mathbb{C} \setminus \{0, 1\}$.*

Proposition 3. *Let $P, Q \in B(X)$ be idempotents. Let $z \in \mathbb{C} \setminus \{0, 1\}$ and $0 < \varepsilon < 1/3$. Then there exists a neighbourhood U of z such that*

$$\frac{1}{1+\varepsilon}\,\gamma(P - zQ) \le \gamma(P - z'Q) \le (1+\varepsilon)\gamma(P - zQ)$$

for all $z' \in U$.

Proof. Let U be the set of all $z' \in \mathbb{C} \setminus \{0, 1\}$ such that

$$\widehat{\delta}(N(P - zQ), N(P - z'Q)) < \varepsilon/6$$

and

$$\begin{aligned}|z - z'| &< \frac{\varepsilon}{6\max\{1, \|P\|, \|Q\|\}} \\ &\quad\times \min\left\{|z(z'-1)|, |z'(z-1)|, \left|\frac{z(z'-1)}{z'}\right|, \left|\frac{z'(z-1)}{z}\right|\right\}.\end{aligned}$$

It is sufficient to show that $\gamma(P - z'Q) \le (1+\varepsilon)\gamma(P - zQ)$ for all $z' \in U$ since the conditions are symmetrical in z and z'.

Let $z' \in U$. Let (x_n) be a sequence of vectors in X satisfying

$$\operatorname{dist}\{x_n, N(P - zQ)\} = 1$$

for all n and $\|(P-zQ)x_n\| \to \gamma(P-zQ)$. Without loss of generality we may assume that $\|x_n\| \to 1$.

For each n set $x'_n = V_{z,z'}x_n$. We have

$$\begin{aligned}
&\limsup_{n\to\infty} \|(P - z'Q)x'_n\| \\
&\quad = \limsup_{n\to\infty} \left\| Px_n - z'Qx_n + \frac{z-z'}{z(z'-1)}\, Px_n - \frac{z'(z-z')}{z(z'-1)}\, QPx_n \right\| \\
&\quad = \limsup_{n\to\infty} \left\| (Px_n - zQx_n) + (z-z')Qx_n + \frac{z-z'}{z(z'-1)}\, Px_n \right. \\
&\qquad\quad \left. - \frac{z'(z-z')}{z'-1}\, Qx_n + \frac{z'(z-z')}{z(z'-1)}\, (zQx_n - QPx_n) \right\| \\
&\quad \le \gamma(P-zQ) + \|Q\| \left| \frac{z'(z-z')}{z(z'-1)} \right| \gamma(P-zQ) \\
&\qquad\quad + \left| \frac{z-z'}{z'-1} \right| \limsup_{n\to\infty} \left\| (z'-1)Qx_n + \frac{Px_n}{z} - z'Qx_n \right\| \\
&\quad \le (1+\varepsilon/6)\gamma(P-zQ) + \left| \frac{z-z'}{z(z'-1)} \right| \limsup_{n\to\infty} \|Px_n - zQx_n\| \\
&\quad \le (1+\varepsilon/3)\gamma(P-zQ).
\end{aligned}$$

We now estimate $\operatorname{dist}\{x'_n, N(P - z'Q)\}$. For all n large enough we have

$$\begin{aligned}
\operatorname{dist}\{x'_n, N(P - z'Q)\} &\ge \operatorname{dist}\{x_n, N(P - z'Q)\} - \|x_n - x'_n\| \\
&\ge \operatorname{dist}\{x_n, N(P - z'Q)\} - \varepsilon/6.
\end{aligned}$$

For each n there is a $y_n \in N(P - z'Q)$ with

$$\|x_n - y_n\| < \operatorname{dist}\{x_n, N(P - z'Q)\} + 1/n \le \|x_n\| + 1/n.$$

Hence

$$\begin{aligned}
1 = &\operatorname{dist}\{x_n, N(P - zQ)\} \le \|x_n - y_n\| + \operatorname{dist}\{y_n, N(P - zQ)\} \\
&\le \|x_n - y_n\| + \|y_n\|\delta(N(P - z'Q), N(P - zQ)) \\
&\le \operatorname{dist}\{x_n, N(P - z'Q)\} + 1/n \\
&\quad + (2\|x_n\| + 1/n)\delta(N(P - z'Q), N(P - zQ))
\end{aligned}$$

and

$$\liminf_{n\to\infty} \operatorname{dist}\{x_n, N(P - z'Q)\} \ge 1 - 2\delta(N(P - z'Q), N(P - zQ)) \ge 1 - \varepsilon/3.$$

Hence

$$\liminf_{n\to\infty} \operatorname{dist}\{x'_n, N(P - z'Q)\} \geq 1 - \varepsilon/2$$

and

$$\gamma(P - z'Q) \leq \frac{1 + \varepsilon/3}{1 - \varepsilon/2}\gamma(P - zQ) \leq (1 + \varepsilon)\gamma(P - zQ). \ ■$$

COROLLARY 4. *The function $z \mapsto \gamma(P - zQ)$ is continuous in $\mathbb{C} \setminus \{0, 1\}$. The set $\{z \in \mathbb{C} \setminus \{0, 1\} : \gamma(P - zQ) = 0\}$ is both open and closed, so it is either empty or equal to $\mathbb{C} \setminus \{0, 1\}$.*

Proof. Follows from the previous proposition and the connectivity of $\mathbb{C} \setminus \{0, 1\}$. ■

Recall that a closed subspace M of a Banach space X is called *complemented* if there exists a closed subspace $L \subset X$ such that $X = M \oplus L$. Equivalently, M is complemented if and only if there exists a bounded linear idempotent $P \in B(X)$ with $R(P) = M$.

COROLLARY 5. *Let $P, Q \in B(X)$ be idempotents. Let $z_0 \in \mathbb{C} \setminus \{0, 1\}$. Then*:

- (i) $\dim N(P - zQ) = \dim N(P - z_0Q)$ *for all* $z \in \mathbb{C} \setminus \{0, 1\}$;
- (ii) *if* $N(P - z_0Q)$ *is complemented*, *then so is* $N(P - zQ)$ *for all* $z \in \mathbb{C} \setminus \{0, 1\}$;
- (iii) *if* $R(P - z_0Q)$ *is closed then so is* $R(P - zQ)$ *for all* $z \in \mathbb{C} \setminus \{0, 1\}$. *Moreover*, *the function* $z \mapsto R(P - zQ)$ *is continuous in the gap topology. In particular*, $\operatorname{codim} R(P - zQ) = \operatorname{codim} R(P - z_0Q)$;
- (iv) *if* $R(P - z_0Q)$ *is complemented then so is* $R(P - zQ)$ *for all* $z \in \mathbb{C} \setminus \{0, 1\}$.

Proof. (i) was proved in Corollary 2.

(ii) By Lemma 1(ii), we have $N(P - zQ) = V_{z_0,z}N(P - z_0Q)$ where $V_{z_0,z}$ is an invertible operator. So $N(P - zQ)$ is complemented.

(iii) As $R(P - z_0Q)$ is closed, we have $\gamma(P - z_0Q) > 0$ and, by Corollary 4, $\gamma(P - zQ) > 0$ for all $z \in \mathbb{C}\setminus\{0, 1\}$. Hence $R(P - zQ)$ is closed. By Corollary 2 for $P^*, Q^* \in B(X^*)$, we have

$$\begin{aligned}\operatorname{codim} R(P - zQ) &= \dim N(P^* - zQ^*) = \dim N(P^* - z_0Q^*) \\ &= \operatorname{codim} R(P - z_0Q).\end{aligned}$$

Similarly, the function $z \mapsto R(P - zQ)$ is continuous in the gap topology by duality.

(iv) Let $X = R(P - z_0Q) \oplus L_0$. Then $N(P^* - z_0Q^*) = R(P - z_0Q)^{\perp}$ and $X^* = N(P^* - z_0Q^*) \oplus L_0^{\perp}$. Note that $L_0^{\perp}$ is w^*-closed. By (ii), $N(P^* - zQ^*)$

is complemented in X^*. Moreover, by the proof of (ii), $N(P^* - zQ^*) = V'N(P^* - z_0Q^*)$ where $V' = I + \frac{z_0 - z}{z_0(z-1)}P^*$ is invertible. Hence $X^* = N(P^* - zQ^*) \oplus L'$ where $L' = V'L_0^{\perp}$ and L' is w^*-closed.

Let $L = {}^{\perp}L'$. Since $R(P - zQ)^{\perp} + L^{\perp} = N(P^* - zQ^*) + L' = X^*$, which is closed, $R(P - zQ) + L$ is a closed subspace of X (see [LN, Theorem A.1.9]). We have

$$(L \cap R(P - zQ))^{\perp} = L^{\perp} + R(P - zQ)^{\perp} = L' + N(P^* - zQ^*) = X^*,$$

and so $L \cap R(P - zQ) = \{0\}$. Furthermore,

$$(L + R(P - zQ))^{\perp} = L^{\perp} \cap R(P - zQ)^{\perp} = L' \cap N(P^* - zQ^*) = \{0\},$$

and so $L + R(P - zQ) = X$.

Hence $R(P - zQ)$ is complemented. ∎

Recall that $T \in B(X)$ is *left* (*right*) *invertible* if there exists $S \in B(X)$ such that $ST = I$ ($TS = I$, respectively). It is well known that T is left (right) invertible if and only if T is injective and $R(T)$ is complemented (T is surjective and $N(T)$ is complemented, respectively). T has a *generalized inverse* if there exists $S \in B(X)$ such that $TST = T$. Equivalently, T has a generalized inverse if and only if T has closed range and both $N(T)$ and $R(T)$ are complemented.

$T \in B(X)$ is called *upper* (*lower*) *semi-Fredholm* if $R(T)$ is closed and $\dim N(T) < \infty$ ($\operatorname{codim} R(T) < \infty$, respectively). T is *left* (*right*) *essentially invertible* if there are $S, K \in B(X)$, K compact and $ST = I + K$ ($TS = I + K$, respectively). It is well known that T is left (right) essentially invertible if and only if T is upper (lower) semi-Fredholm and $R(T)$ is complemented ($N(T)$ is complemented, respectively).

The Main Theorem is now an easy consequence of Corollary 5.

References

[BB] J. K. Baksalary and O. M. Baksalary, *Nonsingularity of linear combinations of idempotent matrices*, Linear Algebra Appl. 388 (2004), 25–29.

[DYD] H. K. Du, X. Y. Yao and C. Y. Deng, *Invertibility of linear combinations of two idempotents*, Proc. Amer. Math. Soc. 134 (2006), 1451–1457.

[K] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Springer, Berlin, 1976.

[KR1] J. J. Koliha and V. Rakočevič, *The nullity and rank of linear combinations of idempotent matrices*, Linear Algebra Appl. 418 (2006), 11–14.

[KR2] —, —, *Stability theorems for linear combinations of idempotents*, Integral Equations Operator Theory, to appear.

[KRS] J. J. Koliha, V. Rakočevič and I. Straškraba, *The difference and sum of projectors*, Linear Algebra Appl. 388 (2004), 279–288.

[LN] K. B. Laursen and M. M. Neumann, *An Introduction to Local Spectral Theory*, London Math. Soc. Monogr. 20, Oxford Sci. Publ., Clarendon Press, Oxford, 2000.

[M] V. Müller, *Spectral Theory of Linear Operators*, Oper. Theory Adv. Appl. 139, Birkhäuser, Basel, 2003.

College of Mathematics
and Information Science
Shaanxi Normal University
Xi'an 710062, P.R. China
E-mail: hkdu@snnu.edu.cn
cy-deng@263.net

Université Lille 1, UFR Mathématique
59655 Villeneuve d'Ascq, France
E-mail: mbekhta@math.univ-lille1.fr

Mathematical Institute
Czech Academy of Sciences
Žitná 25
115 67 Praha 1, Czech Republic
E-mail: muller@math.cas.cz

Received September 30, 2004
Revised version May 10, 2007 (5506)