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# Hypercontractivity of simple random variables

by

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**Abstract.** The optimal hypercontractivity constant for a natural operator semigroup acting on a discrete finite probability space is established up to a universal factor. The two-point spaces are proved to be the extremal case. The constants obtained are also optimal in the related moment inequalities for sums of independent random variables.

**1. Notation and preliminary results.** Let $\Omega$ be an arbitrary finite set and let $\mu$ be a probability measure on the measurable space $(\Omega, 2^{\Omega})$. For any function $f\colon \Omega \to \mathbb{R}$, $E_\mu f$ denotes the expected value of $f$ with respect to $\mu$. To avoid trivialities, we assume that $\mu$ has at least two atoms with non-zero measure.

We shall consider operators acting on the finite-dimensional space $L_2(\mu)$ (for abbreviation, we write $L_p(\mu)$ instead of $L_p(\Omega, \mu)$) over the real scalar field. Let $L$ be the orthogonal projection in $L_2(\mu)$ onto the subspace of functions with zero mean, i.e. $L = \mathrm{Id} - E_\mu$. We define the semigroup $T_t = e^{-tL}$ $(t \geq 0)$ of self-adjoint operators. Throughout the paper, we shall use the explicit formula

$$T_t = E_\mu + e^{-t}L = (1 - e^{-t})E_\mu + e^{-t}\,\mathrm{Id},$$

which implies that $(T_t)_{t\geq 0}$ is a non-negativity preserving semigroup of contractions from $L_p(\mu)$ to $L_p(\mu)$ for $1 \leq p \leq \infty$.

DEFINITION 1.1. The semigroup $(T_t)_{t\geq 0}$ is $(p,q)$-*hypercontractive* (for $1 < q < p < \infty$) if there exists $t_0 \geq 0$ such that for all $t \geq t_0$ and $f \in L_q(\mu)$,

$$\|T_t f\|_{L_p(\mu)} \leq \|f\|_{L_q(\mu)}. \tag{1.1}$$

If such a $t_0$ exists and is the least possible (the minimum exists, since obviously $T_t$ is a strongly continuous semigroup), then $\sigma_{p,q}(\mu) := e^{-t_0}$ is the

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$(p,q)$-*hypercontractivity constant* for the measure $\mu$. (Non-triviality of $\mu$ gives $t_0 > 0$, i.e., $\sigma_{p,q}(\mu) \in (0,1)$.)

Note that hypercontractivity of the semigroup $(T_t)_{t\geq 0}$ does not depend on the set $\Omega$ itself, but only on the weights of the atoms of the measure $\mu$. In particular, if $\#\Omega = 2$, then we may assume that $\mu = \beta\delta_{-\alpha} + \alpha\delta_\beta$ for some $\alpha \in (0, 1/2]$ with $\beta = 1-\alpha$. Therefore we shall use the notation

$$\sigma_{p,q}(\alpha) := \sigma_{p,q}(\beta\delta_{-\alpha} + \alpha\delta_\beta).$$

REMARK 1.1. The standard duality argument yields $\sigma_{p,q}(\mu) = \sigma_{q',p'}(\mu)$, where $p' = p/(p-1)$ and $q' = q/(q-1)$. Indeed, self-adjointness of $T_t$ implies that $\|T_t\|_{L_q(\mu)\to L_p(\mu)} = \|T_t\|_{L_{p'}(\mu)\to L_{q'}(\mu)}$.

REMARK 1.2. For $f = a + g$, where $a \in \mathbb{R}$ and $E_\mu g = 0$, we have $T_t f = a + e^{-t} g$. Since $T_t$ preserves non-negativity, we have $T_t|f| \geq T_t f$ and $T_t|-f| \geq -T_t f$, so $T_t|f| \geq |T_t f|$. Therefore in order to check (1.1) we can consider only $f = a + g \geq 0$. Then, except the trivial case $g \equiv 0$, we must have $a > 0$ and, by homogeneity, we can assume $a = 1$. Therefore

$$\sigma_{p,q}(\mu) = \max\{\sigma \in (0,1) : \|1+\sigma g\|_{L_p(\mu)} \leq \|1+g\|_{L_q(\mu)} \text{ whenever } E_\mu g = 0 \text{ and } 1+g \geq 0\} \tag{1.2}$$

and in the case of the two-point measure $\mu = \beta\delta_{-\alpha} + \alpha\delta_\beta$ every $g$ is of the form $g(x) = xu$ for $u \in [-1/\beta, 1/\alpha]$.

The hypercontractivity of semigroups has been transferred to the context of random variables. Let us recall the following definition introduced by Krakowiak and Szulga [5] (see also [6] and [7, Ch. 3]).

DEFINITION 1.2. Let $0 < q < p < \infty$, let $(\mathbf{F}, \|\cdot\|)$ be a separable Banach space and let $X$ be a random vector with values in $\mathbf{F}$ such that $E\|X\|^p < \infty$. Then we say that $X$ is $(p,q)$-*hypercontractive* with constant $\sigma \in (0,1)$ if

$$\forall_{v\in\mathbf{F}} \quad \|v + \sigma X\|_p \leq \|v + X\|_q.$$

(The notation $\|Y\|_p$ means $(E\|Y\|^p)^{1/p}$.)

REMARK 1.3. Let $X$ be a zero-mean $\mathbf{F}$-valued random vector which is simple (i.e. takes on finitely many values) and $\mu$ be its distribution. A standard reasoning shows that if the semigroup $T_t = e^{-t(\mathrm{Id} - E_\mu)}$ is $(p,q)$-hypercontractive, then $X$ is $(p,q)$-hypercontractive with constant $\sigma_{p,q}(\mu)$. Indeed, taking $e^{-t_0} = \sigma := \sigma_{p,q}(\mu)$ we have $\|T_{t_0}\|_{L_q(\mu)\to L_p(\mu)} \leq 1$. Considering, for a fixed $v \in \mathbf{F}$, the convex function $f(x) = \|v + x\|$ and using the Jensen inequality we obtain

$$(T_{t_0} f)(X) = (1-\sigma)Ef(X) + \sigma f(X) \geq (1-\sigma)f(0) + \sigma f(X) \geq f(\sigma X),$$

so $\|v + \sigma X\|_p = \|f(\sigma X)\|_p \leq \|(T_{t_0} f)(X)\|_p \leq \|f(X)\|_q = \|v + X\|_q$.

Let us recall the following result [6, 7]:

PROPOSITION 1.1. *Let $0 < q < p < \infty$. If $(X_k)_{k=1}^n$ is a sequence of independent random vectors with values in a separable Banach space $\mathbf{F}$ and each $X_k$ is $(p,q)$-hypercontractive with the same constant $\sigma$, then*

$$\forall_{v\in\mathbf{F}} \quad \|v+\sigma S\|_p \le \|v+S\|_q$$

*where $S=\sum_{k=1}^n X_k$. In particular,*

$$\|S\|_p \le \sigma^{-1}\|S\|_q.$$

We shall use the following notation. For $a,b\in\mathbb{R}$, $a\vee b=\max(a,b)$, $a\wedge b=\min(a,b)$. If $x,y\ge 0$ and $c\ge 1$ are such that $c^{-1}x\le y\le cx$, then we shall write $x\overset{c}{\sim} y$. We shall often omit the constant $c$ and write $x\sim y$ whenever $c$ is some universal (numerical) constant. For $x,y\ge 0$, $x\lesssim y$ means that $x\le cy$, where $c>0$ is a universal constant.

In Section 2 we formulate known and new results on the constants $\sigma_{p,q}(\alpha)$. Proofs of the latter are presented in Section 4. In Section 3 we show that $\sigma_{p,q}(\mu)$ are comparable with $\sigma_{p,q}(\alpha_*)$, where $\alpha_*$ is the mass of the least atom of $\mu$. This solves a problem posed by Janson [4].

**2. Hypercontractivity constants for two-point measures.** It is a classical result [2, 1] that for $1<q<p<\infty$,

$$\sigma_{p,q}(1/2)=\sqrt{\frac{q-1}{p-1}}. \tag{2.1}$$

The values of $\sigma_{2,q}(\alpha)$ and $\sigma_{p,2}(\alpha)$ for $\alpha\in(0,1/2)$ and $1<q<2<p<\infty$ were established by Oleszkiewicz [10]:

$$\sigma_{2,q}(\alpha)=\left(\frac{\beta^{2-2/q}-\alpha^{2-2/q}}{\alpha^{1-2/q}\beta-\beta^{1-2/q}\alpha}\right)^{1/2}, \qquad \sigma_{p,2}(\alpha)=\left(\frac{\beta^{2/p}-\alpha^{2/p}}{\alpha^{2/p-1}\beta-\beta^{2/p-1}\alpha}\right)^{1/2}$$

where $\beta=1-\alpha$. (One can easily check that $\lim_{\alpha\to 1/2^-}\sigma_{2,q}(\alpha)=\sigma_{2,q}(1/2)$ and $\lim_{\alpha\to 1/2^-}\sigma_{p,2}(\alpha)=\sigma_{p,2}(1/2)$.) The above results imply that $\sigma_{p,q}(\alpha)$ is well-defined for all $1<q<p<\infty$ (i.e. the related semigroup is $(p,q)$-hypercontractive), since $\sigma_{p,q}(\alpha)\ge\sigma_{2,q}(\alpha)$ for $p\le 2$, $\sigma_{p,q}(\alpha)\ge\sigma_{p,2}(\alpha)$ for $q\ge 2$, and $\sigma_{p,q}(\alpha)\ge\sigma_{2,q}(\alpha)\sigma_{p,2}(\alpha)$ for $q<2<p$. Elementary calculations show the asymptotic behaviour of $\sigma_{2,q}(\alpha)$ (and $\sigma_{p,2}(\alpha)$) as $\alpha\to 0$ or $q\to 1$ ($p\to\infty$) (for more details see [10, proof of Theorem 2.1]):

$$\sigma_{2,q}(\alpha)\sim\begin{cases}\alpha^{1/q-1/2} & \text{for } \dfrac{1}{q-1}\le\ln\dfrac{1}{\alpha},\\[2ex] \sqrt{(q-1)\alpha\ln(1/\alpha)} & \text{for } \dfrac{1}{q-1}>\ln\dfrac{1}{\alpha},\end{cases}$$

$$\sigma_{p,2}(\alpha)\sim\begin{cases}\alpha^{1/2-1/p} & \text{for } p\le\ln\dfrac{1}{\alpha},\\[2ex] \sqrt{(1/p)\alpha\ln(1/\alpha)} & \text{for } p>\ln\dfrac{1}{\alpha}.\end{cases}$$

In Section 3 we will need the following

LEMMA 2.1. *For* $1 < q < 2 < p < \infty$, *the functions* $(0,1/2] \ni \alpha \mapsto \sigma_{2,q}(\alpha)$ *and* $(0,1/2] \ni \alpha \mapsto \sigma_{p,2}(\alpha)$ *are strictly increasing.*

We postpone the proof of Lemma 2.1 and of other results stated in this section until Section 4.

The main result of this section deals with estimates of $\sigma_{p,q}(\alpha)$ for all combinations of parameters $\alpha, p, q$ except the case $q < 2 < p$.

THEOREM 2.1. *For all* $1 < q < p \le 2$ *and* $\alpha \in (0,1/2]$ *we have* $\sigma_{p,q}(\alpha) \sim \widetilde{\sigma}_{p,q}(\alpha)$, *where*

$$\widetilde{\sigma}_{p,q}(\alpha) = \begin{cases} \alpha^{1/q-1/p} & \text{for } \alpha \in I_1 = \left\{\alpha\colon \dfrac{e^{1/(q-1)}}{q-1} \le \dfrac{1}{\alpha}\ln\dfrac{1}{\alpha}\right\}, \\ ((q-1)\ln(1/\alpha))^{1/p}\alpha^{1-1/p} & \\ & \text{for } \alpha \in I_2 = \left\{\alpha\colon \dfrac{e^{1/(p-1)}}{q-1} \le \dfrac{1}{\alpha}\ln\dfrac{1}{\alpha} < \dfrac{e^{1/(q-1)}}{q-1}\right\}, \\ \dfrac{q-1}{p-1}\,\dfrac{\ln(1/\alpha)}{1+\ln\left(\frac{q-1}{p-1}\frac{1}{\alpha}\ln\frac{1}{\alpha}\right)} & \\ & \text{for } \alpha \in I_3 = \left\{\alpha\colon \dfrac{p-1}{q-1} \le \dfrac{1}{\alpha}\ln\dfrac{1}{\alpha} < \dfrac{e^{1/(p-1)}}{q-1}\right\}, \\ \sqrt{\dfrac{q-1}{p-1}\,\alpha\ln(1/\alpha)} & \text{for } \alpha \in I_4 = \left\{\alpha\colon \dfrac{1}{\alpha}\ln\dfrac{1}{\alpha} < \dfrac{p-1}{q-1}\right\}. \end{cases}$$

As a corollary we obtain

THEOREM 2.2. *For all* $2 \le q < p < \infty$ *and* $\alpha \in (0,1/2]$ *we have* $\sigma_{p,q}(\alpha) \sim \widetilde{\sigma}_{p,q}(\alpha)$, *where*

$$\widetilde{\sigma}_{p,q}(\alpha) = \begin{cases} \alpha^{1/q-1/p} & \text{for } \alpha \in I_1' = \left\{\alpha\colon pe^p \le \dfrac{1}{\alpha}\ln\dfrac{1}{\alpha}\right\}, \\ \left(\dfrac{\ln(1/\alpha)}{p}\right)^{1-1/q}\alpha^{1/q} & \text{for } \alpha \in I_2' = \left\{\alpha\colon pe^q \le \dfrac{1}{\alpha}\ln\dfrac{1}{\alpha} < pe^p\right\}, \\ \dfrac{q}{p}\,\dfrac{\ln(1/\alpha)}{1+\ln\left(\frac{q}{p}\frac{1}{\alpha}\ln\frac{1}{\alpha}\right)} & \text{for } \alpha \in I_3' = \left\{\alpha\colon \dfrac{p}{q} \le \dfrac{1}{\alpha}\ln\dfrac{1}{\alpha} < pe^q\right\}, \\ \sqrt{\dfrac{q}{p}\,\alpha\ln(1/\alpha)} & \text{for } \alpha \in I_4' = \left\{\alpha\colon \dfrac{1}{\alpha}\ln\dfrac{1}{\alpha} < \dfrac{p}{q}\right\}. \end{cases}$$

As in the case of the constants $\sigma_{2,q}(\alpha)$ and $\sigma_{p,2}(\alpha)$, in Section 3 we will use the following

LEMMA 2.2. *For all* $p, q$ *such that* $1 < q < p \le 2$ *or* $2 \le q < p < \infty$ *we have* $\sigma_{p,q}(\alpha_1) \lesssim \sigma_{p,q}(\alpha_2)$ *for all* $0 < \alpha_1 < \alpha_2 \le 1/2$.

By applying results from [8], we may observe that up to a universal factor, the constants $\widetilde{\sigma}_{p,q}(\alpha)$ appear as a ratio of the $q$th and $p$th moments of certain sums of independent two-point random variables.

THEOREM 2.3. *Let $(X_k)_{k=1}^{\infty}$ be a sequence of independent random variables with a common distribution $\beta\delta_{-\alpha}+\alpha\delta_{\beta}$, where $\alpha+\beta=1$, $\alpha\in(0,1/2]$. Set $n(\alpha,p)=\lceil p/\ln(1/\alpha)\rceil$. Then for all $2\le q<p<\infty$ and for $S=\sum_{k=1}^{n(\alpha,p)}X_k$ we have*

$$\|S\|_q\sim\widetilde{\sigma}_{p,q}(\alpha)\|S\|_p.$$

On the other hand, by Theorem 2.2, Remark 1.3 and Proposition 1.1, for any sum $S=\sum_{k=1}^{n}v_kX_k$ where $(v_k)_{k=1}^{n}$ is a sequence in any Banach space $\mathbf{F}$, we have

$$\|S\|_q\gtrsim\widetilde{\sigma}_{p,q}(\alpha)\|S\|_p.$$

Thus we see that the case $n=\lceil p/\ln(1/\alpha)\rceil$ and $v_1=\cdots=v_n\neq 0$ (or even just $\mathbf{F}=\mathbb{R}$ and $v_1=\cdots=v_n=1$) is extremal up to some universal constant. It is interesting that $n$ depends only on $\alpha$ and $p$ and does not depend on $q$. Moreover, $n$ can be defined as the greatest integer for which $\|\sum_{k=1}^{n}X_k\|_p\overset{c}{\sim}\|\sum_{k=1}^{n}X_k\|_\infty$ with some absolute constant $c>0$ (see the proof of Theorem 2.3).

## 3. Hypercontractivity constants for discrete measures

THEOREM 3.1. *Let $\mu$ be a discrete measure with the mass of the least atom equal to $\alpha_*>0$. Then for all $1<q<p<\infty$ we have*

$$\inf_{\alpha\in[\alpha_*,1/2]}\sigma_{p,q}(\alpha)\le\sigma_{p,q}(\mu)\le\sigma_{p,q}(\alpha_*).$$

*Proof.* The second inequality is quite obvious (e.g. see (1.2)). From now on, we consider only the case $p\ge 2$; the case $1<q<p<2$ follows from the previous one and a duality argument (see Remark 1.1). In order to prove the first inequality we shall prove that for all $t\ge 0$ we have

$$\|T_t\|_{L_q(\mu)\to L_p(\mu)}\le\sup_{\alpha\in[\alpha_*,1/2]}\|S_t^{(\alpha)}\|_{L_q(\nu^{(\alpha)})\to L_p(\nu^{(\alpha)})}\tag{3.1}$$

where $(T_t)_{t\ge 0}$ is the semigroup attached to $\mu$ and $(S_t^{(\alpha)})_{t\ge 0}$ is the semigroup associated to the two-point measure $\nu^{(\alpha)}=\beta\delta_{-\alpha}+\alpha\delta_{\beta}$ $(\alpha+\beta=1)$. The argument used below is similar to one presented by Diaconis and Saloff-Coste [3] and originates in Rothaus [11]. The main difference is the context—we deal with hypercontractivity, whereas in the cited papers this argument was used to prove logarithmic Sobolev inequalities.

For fixed $t\ge 0$, set

$$A=\|T_t\|_{L_q(\mu)\to L_p(\mu)}=\sup\{\|T_tf\|_{L_p(\mu)}\colon f\in L_q(\mu),\ \|f\|_{L_q(\mu)}=1\}.$$

Clearly, if $A = 1$ then (3.1) is trivial. So assume $A > 1$. The fact that $L_q(\mu)$ has finite dimension implies that $A < \infty$ and there exists $f_0$ for which the above supremum is attained. Defining the functionals $I(f) = \|T_t f\|^p_{L_p(\mu)}$ and $J(f) = \|f\|^q_{L_q(\mu)}$ on $L_q(\mu)$, we see that $f_0$ maximizes $I$ subject to the constraint $J(f) = 1$. The functionals $I$ and $J$ are both $C^1$ and $DJ(f_0) \neq 0$, since $J(f_0) = 1$. Therefore for some constant $\lambda$ we have

$$DI(f_0) = \lambda DJ(f_0). \tag{3.2}$$

Since $T_t$ preserves non-negativity, we may choose $f_0$ to be non-negative. The assumption $A > 1$ implies that $f_0$ cannot be a constant function. We shall show that $f_0$ takes at most two values. The computations of the derivatives from (3.2) are as follows ($\langle f, g\rangle_\mu$ stands for $E_\mu fg$):

$$\begin{aligned} DJ(f_0) &= q\langle f_0^{q-1}, \cdot\rangle_\mu, \\ DI(f_0) &= D\|\cdot\|^p_{L_p(\mu)}(T_t f_0) \circ DT_t(f_0) = p\langle (T_t f_0)^{p-1}, \cdot\rangle_\mu \circ T_t \\ &= p\langle (T_t f_0)^{p-1}, T_t(\cdot)\rangle_\mu = p\langle T_t((T_t f_0)^{p-1}), \cdot\rangle_\mu \end{aligned}$$

(the last equality follows from the fact that $T_t$ is a self-adjoint operator). Thus by (3.2) we have

$$T_t((T_t f_0)^{p-1}) = \widetilde{\lambda} f_0^{q-1}$$

for some constant $\widetilde{\lambda}$. Since $f_0 \geq 0$, also $g_0 := T_t f_0 \geq 0$. We have $f_0 = (T_t)^{-1} g_0 = e^t g_0 + (1 - e^t) E_\mu g_0$, so

$$(e^{-t} g_0^{p-1} + (1 - e^{-t})\|g_0\|^{p-1}_{L_{p-1}(\mu)})^{1/(q-1)} = \widetilde{\lambda}(e^t g_0 + (1 - e^t) E_\mu g_0).$$

Fixing an argument $x$ of $g_0$ and putting $u := g_0(x) \in [0, \infty)$, we obtain

$$((au^{p-1} + bv^{p-1})^{1/(p-1)})^{(p-1)/(q-1)} = cu - d \tag{3.3}$$

for some constants $a > 0$, $b, v \geq 0$ and $c, d \in \mathbb{R}$ which do not depend on the choice of $x$. The right hand side of (3.3) is a linear function of $u$, whereas the left hand side of (3.3) is a strictly convex function of $u$. Indeed, it is the composition of the convex and strictly increasing function $[0, \infty) \ni u \mapsto (au^{p-1} + bv^{p-1})^{1/(p-1)}$ (this expression is just some $L_{p-1}$-norm because we have assumed $p \geq 2$) and the function $[0, \infty) \ni z \mapsto z^{(p-1)/(q-1)}$, which is strictly convex and strictly increasing. Therefore the equation (3.3) has at most two solutions $u \in [0, \infty)$, so $g_0$ (and also $f_0$) takes on at most two values.

Therefore $f_0$ takes on exactly two different values. Denote them by $u_1$, $u_2$ so that $\mu(f_0^{-1}(u_1)) = \alpha$ and $\mu(f_0^{-1}(u_2)) = 1 - \alpha$ for some $\alpha \in (0, 1/2]$. Clearly, $\alpha \in [\alpha_*, 1/2]$ and $\|T_t\|_{L_q(\mu) \to L_p(\mu)} = \|S_t^{(\alpha)}\|_{L_q(\nu^{(\alpha)}) \to L_p(\nu^{(\alpha)})}$, which proves (3.1). ∎

COROLLARY 3.1. *Let $\mu$ be a discrete measure with the mass of the least atom equal to $\alpha \in (0, 1/2)$. Then for all $1 < q < 2 < p < \infty$ we have*

$$\sigma_{2,q}(\mu) = \sigma_{2,q}(\alpha) \quad \text{and} \quad \sigma_{p,2}(\mu) = \sigma_{p,2}(\alpha),$$

*and for all $1 < q < p < \infty$ satisfying $q, p \le 2$ or $q, p \ge 2$ we have*

$$\sigma_{p,q}(\mu) \sim \sigma_{p,q}(\alpha).$$

*Proof.* The first part follows from Theorem 3.1 and Lemma 2.1, whereas the second part follows from Theorem 3.1 and Lemma 2.2. ∎

It is worth mentioning that Theorem 3.1 allows us to generalize some Sobolev-type inequality considered by Latała and Oleszkiewicz [9, Remark 2] from a two-point space to any finite space.

COROLLARY 3.2. *Let $\mu$ be a discrete measure with the mass of the least atom equal to $\alpha \in (0, 1/2)$, $\beta = 1 - \alpha$, $1 < q < 2$. Define the quadratic form $\mathcal{E}(f) = E_\mu f L f$, where $L = \mathrm{Id} - E_\mu$. Then for every function $f \in L_2(\mu)$,*

$$E_\mu f^2 - (E_\mu |f|^q)^{2/q} \le C_q(\alpha)\mathcal{E}(f) \tag{3.4}$$

*where*

$$C_q(\alpha) = \frac{\alpha^{1-2/q} - \beta^{1-2/q}}{\alpha^{1-2/q}\beta - \beta^{1-2/q}\alpha}.$$

*Proof.* Consider the self-adjoint semigroup $T_t = e^{-tL}$. Corollary 3.1 implies that if $t_0$ satisfies $e^{-t_0} = \sigma_{2,q}(\alpha)$ then $\|T_{t_0}\|_{L_q(\mu)\to L_2(\mu)} \le 1$, hence

$$\begin{aligned}\|f\|^2_{L^q(\mu)} &\ge \|T_{t_0} f\|^2_{L^2(\mu)} = \langle T_{t_0} f, T_{t_0} f\rangle_\mu = \langle f, T_{2t_0} f\rangle_\mu \\ &= e^{-2t_0} E_\mu f^2 + (1 - e^{-2t_0})(E_\mu f)^2 = E_\mu f^2 - (1 - e^{-2t_0})\mathcal{E}(f) \\ &= E_\mu f^2 - C_q(\alpha)\mathcal{E}(f),\end{aligned}$$

because, as one may check, $C_q(\alpha) = 1 - \sigma^2_{2,q}(\alpha)$. ∎

Letting $q \to 2$ in the inequality (3.4) one can obtain the logarithmic Sobolev inequality considered in [3, Theorem A.1]:

$$E_\mu f^2 \ln f^2 - E_\mu f^2 \ln E_\mu f^2 \le \frac{\ln\beta - \ln\alpha}{\beta - \alpha}\,\mathcal{E}(f).$$

By Corollary 3.1, Remark 1.3 and Proposition 1.1 we finally get

COROLLARY 3.3. *Let $(X_k)_{k=1}^n$ be a sequence of independent simple random vectors with values in a Banach space $\mathbf{F}$, and $S = \sum_{k=1}^n X_k$. Let $\alpha = \min\{P(X_k = v) \ne 0 \colon v \in \mathbf{F}\}$. Then for every $2 \le q < p < \infty$,*

$$\|S\|_p \le C\sigma^{-1}_{p,q}(\alpha)\|S\|_q$$

*where $C > 0$ is a universal constant, which may be 1 if $q = 2$.*

Moreover, by Theorem 2.2 we clearly get $\|S\|_p \lesssim \widetilde{\sigma}_{p,q}^{-1}(\alpha)\|S\|_q$, whereas Theorem 2.3 implies that for fixed $\alpha, p, q$ the constants in the above moment inequalities are optimal up to some universal factor.

## 4. Proofs

*Proof of Lemma 2.1.* Due to Remark 1.1 it suffices to prove that $f(\alpha) = \sigma_{2,q}^2(\alpha)$ is increasing for $\alpha \in (0, 1/2)$. For $x, y > 0$, $x \neq y$ and $t \in (0,1)$, we define

$$\varphi(x,y) = \frac{x^{1-t} - y^{1-t}}{xy^{-t} - yx^{-t}}.$$

Clearly, $\varphi(x,y) = \varphi(y,x)$ and $\varphi(x,y) = \varphi(\lambda x, \lambda y)$ for $\lambda > 0$. Putting $t = 2/q - 1$ we have $f(\alpha) = \varphi(1-\alpha, \alpha)$, thus

$$\begin{aligned} f'(\alpha) &= \partial_y \varphi(1-\alpha,\alpha) - \partial_x \varphi(1-\alpha,\alpha) \\ &= \frac{1}{\alpha}\,\partial_x \varphi\Big(\frac{1-\alpha}{\alpha}, 1\Big) - \frac{1}{1-\alpha}\,\partial_x \varphi\Big(\frac{\alpha}{1-\alpha}, 1\Big). \end{aligned}$$

Therefore the proof will be completed by showing that

$$\partial_x \varphi(x,1) > 0 \quad \text{for } x \in (0,1) \quad \text{and} \quad \partial_x \varphi(x,1) < 0 \quad \text{for } x \in (1,\infty).$$

Putting

$$\psi(x,s) = \frac{1}{s}\,(x^s - x^{-s}) = \frac{\ln x}{s} \int_0^s (x^u + x^{-u})\,du$$

for $x > 0$ and $s \in (0,1]$, we have

$$\begin{aligned} \partial_x \varphi(x,1) &= \frac{1}{(x - x^{-t})^2}\,((1-t)x^{-t}(x - x^{-t}) - (x^{-t+1} - 1)(1 + tx^{-t-1})) \\ &= \frac{tx^{-t}}{(x - x^{-t})^2}\,(\psi(x,t) - \psi(x,1)). \end{aligned}$$

Now it suffices to show that $\psi(x,s)$ is increasing as a function of $s \in (0,1]$ for $x > 1$ and decreasing for $x \in (0,1)$. The former statement implies the latter, because $\psi(x,s) = -\psi(1/x, s)$. Therefore we consider the case $x > 1$. Fix $0 < s_1 < s_2 \leq 1$ and put $\lambda = s_1/s_2 < 1$. Using the inequality $x^{\lambda u} + x^{-\lambda u} < x^u + x^{-u}$, which holds by the convexity of $u \mapsto x^u$, we obtain

$$\begin{aligned} \psi(x,s_1) &= \frac{\ln x}{s_1} \int_0^{s_2} (x^{\lambda u} + x^{-\lambda u})\lambda\,du \\ &< \frac{\ln x}{s_2} \int_0^{s_2} (x^u + x^{-u})\,du = \psi(x,s_2). \ \blacksquare \end{aligned}$$

Now, we turn to the proof of Theorem 2.1. First, note that Remark 1.2 implies that $\sigma_{p,q}(\alpha)$ is the greatest $\sigma$ such that

$$(\beta|1-\sigma\alpha u|^p+\alpha|1+\sigma\beta u|^p)^{1/p}\le(\beta|1-\alpha u|^q+\alpha|1+\beta u|^q)^{1/q}\quad(\beta=1-\alpha)$$

for all $u\in[-1/\beta,1/\alpha]$ (or, equivalently, for all $u\in\mathbb{R}$).

The following lemmas will be used in the proof of Theorem 2.1.

LEMMA 4.1. *For all $1<q<p<\infty$ and $0<\alpha_1<\alpha_2\le 1/2$ with $\beta_i=1-\alpha_i$ $(i=1,2)$ we have $\sigma_{p,q}(\alpha_1)\overset{c}{\sim}\sigma_{p,q}(\alpha_2)$, where $c=\alpha_2\beta_1/(\alpha_1\beta_2)$.*

*Proof.* First, note that for any convex function $\varphi\colon\mathbb{R}\to\mathbb{R}$ we have

$$\beta_1\varphi(-\alpha_1 y)+\alpha_1\varphi(\beta_1 y)\le\beta_2\varphi\Big(-\alpha_2\frac{\beta_1}{\beta_2}y\Big)+\alpha_2\varphi\Big(\beta_2\frac{\beta_1}{\beta_2}y\Big),\tag{4.1}$$

$$\beta_2\varphi(-\alpha_2 y)+\alpha_2\varphi(\beta_2 y)\le\beta_1\varphi\Big(-\alpha_1\frac{\alpha_2}{\alpha_1}y\Big)+\alpha_1\varphi\Big(\beta_1\frac{\alpha_2}{\alpha_1}y\Big),\tag{4.2}$$

for every $y\in\mathbb{R}$. For any fixed $u\in\mathbb{R}$ and $r\ge 1$ we consider the convex function $\varphi_r(x)=|1+xu|^r$. Putting $\sigma=\sigma_{p,q}(\alpha_2)$ and using (4.1), the definition of $\sigma_{p,q}$ and (4.2) we get

$$\begin{aligned}(\beta_1\varphi_p(-\alpha_1\sigma)+\alpha_1\varphi_p(\beta_1\sigma))^{1/p}&\le\Big(\beta_2\varphi_p\Big(-\alpha_2\frac{\beta_1}{\beta_2}\sigma\Big)+\alpha_2\varphi_p\Big(\beta_2\frac{\beta_1}{\beta_2}\sigma\Big)\Big)^{1/p}\\&\le\Big(\beta_2\varphi_q\Big(-\alpha_2\frac{\beta_1}{\beta_2}\Big)+\alpha_2\varphi_q\Big(\beta_2\frac{\beta_1}{\beta_2}\Big)\Big)^{1/q}\le(\beta_1\varphi_q(-c\alpha_1)+\alpha_1\varphi_q(c\beta_1))^{1/q},\end{aligned}$$

which proves $\sigma_{p,q}(\alpha_2)\le c\sigma_{p,q}(\alpha_1)$, since $u\in\mathbb{R}$ can be chosen arbitrarily. Putting $\sigma=\sigma_{p,q}(\alpha_1)$ and using (4.2), the definition of $\sigma_{p,q}$ and (4.1) we have

$$\begin{aligned}(\beta_2\varphi_p(-\alpha_2\sigma)+\alpha_2\varphi_p(\beta_2\sigma))^{1/p}&\le\Big(\beta_1\varphi_p\Big(-\alpha_1\frac{\alpha_2}{\alpha_1}\sigma\Big)+\alpha_1\varphi_p\Big(\beta_1\frac{\alpha_2}{\alpha_1}\sigma\Big)\Big)^{1/p}\\&\le\Big(\beta_1\varphi_q\Big(-\alpha_1\frac{\alpha_2}{\alpha_1}\Big)+\alpha_1\varphi_q\Big(\beta_1\frac{\alpha_2}{\alpha_1}\Big)\Big)^{1/q}\le(\beta_2\varphi_q(-c\alpha_2)+\alpha_2\varphi_q(c\beta_2))^{1/q},\end{aligned}$$

which yields $\sigma_{p,q}(\alpha_1)\le c\sigma_{p,q}(\alpha_2)$. ∎

LEMMA 4.2. *For each $C>1$ there exists $D>1$ such that for all $1<q<p\le 2$ and $\alpha_1,\alpha_2\in(0,1/2]$, if $\alpha_1\overset{C}{\sim}\alpha_2$, then $\widetilde{\sigma}_{p,q}(\alpha_1)\overset{D}{\sim}\widetilde{\sigma}_{p,q}(\alpha_2)$.*

*Proof.* For $\alpha_1,\alpha_2\in I_j$ $(j=1,\ldots,4)$ this is quite obvious—it suffices to observe that $\ln(1/\alpha_1)\sim\ln(1/\alpha_2)$ and in case $\alpha_1,\alpha_2\in I_3$ also that

$$1+\ln\Big(\frac{q-1}{p-1}\,\frac{1}{\alpha_1}\ln\frac{1}{\alpha_1}\Big)\sim 1+\ln\Big(\frac{q-1}{p-1}\,\frac{1}{\alpha_2}\ln\frac{1}{\alpha_2}\Big).$$

To see the assertion for any pair $\alpha_1,\alpha_2$ satisfying the assumptions of the lemma, it suffices to show that $\widetilde{\sigma}_{p,q}(\alpha)\sim\lim_{\alpha'\to\alpha^+}\widetilde{\sigma}_{p,q}(\alpha')$ for $\alpha=\sup I_j$

$(j = 1, 2, 3)$. For $j = 1$, we have

$$\lim_{\alpha' \to \alpha^+} \widetilde{\sigma}_{p,q}(\alpha') = \alpha^{1-1/p} \sim \alpha^{1/q-1/p} = \widetilde{\sigma}_{p,q}(\alpha),$$

since $\alpha^{1/q-1} = e^{1/q} \sim 1$. For $j = 2$,

$$\begin{aligned} \lim_{\alpha' \to \alpha^+} \widetilde{\sigma}_{p,q}(\alpha') &= \frac{q-1}{p-1} \frac{\ln(1/\alpha)}{1 + \ln(1/(p-1)) + 1/(p-1)} \\ &\sim (q-1)\ln(1/\alpha), \\ \widetilde{\sigma}_{p,q}(\alpha) &= (q-1)\ln(1/\alpha)\left((q-1)\frac{1}{\alpha}\ln\frac{1}{\alpha}\right)^{1/p-1} \\ &= (q-1)\ln(1/\alpha)e^{-1/p}, \end{aligned} \tag{4.3}$$

and finally, for $j = 3$ we have $\widetilde{\sigma}_{p,q}(\alpha) = \alpha = \lim_{\alpha' \to \alpha^+} \widetilde{\sigma}_{p,q}(\alpha')$. ∎

In the next two lemmas and in the proof of Theorem 2.1 we shall use the following notation. Let $1 < q < p \le 2$, $\alpha \in (0, 1/2]$ with $\beta = 1 - \alpha$. For $x \in [-1/\beta, 1/\alpha]$, we define

$$\begin{aligned} L_{\alpha,p}(x) &= \beta(1-\alpha x)^p + \alpha(1+\beta x)^p - 1, \\ R_{\alpha,q,p}(x) &= (\beta(1-\alpha x)^q + \alpha(1+\beta x)^q)^{p/q} - 1. \end{aligned}$$

If $\alpha \in (0, e^{-2}]$, we set

$$\widetilde{L}_{\alpha,p}(x) = \begin{cases} (p-1)\alpha x^2 & \text{for } x \in [0, e], \\ e(p-1)\alpha x \ln x & \text{for } x \in (e, e^{1/(p-1)} \wedge 1/\alpha], \\ \alpha x^p & \text{for } x \in (e^{1/(p-1)} \wedge 1/\alpha, 1/\alpha], \end{cases}$$

and, when additionally $1/\alpha \notin (e^{1/(q-1)}, e^{q/(q-1)}]$,

$$\widetilde{R}_{\alpha,q,p}(x) = \begin{cases} (q-1)\alpha x^2 & \text{for } x \in [0, e], \\ e(q-1)\alpha x \ln x & \text{for } x \in (e, 1/\alpha], \end{cases} \qquad \text{if } 1/\alpha \le e^{1/(q-1)},$$

and

$$\widetilde{R}_{\alpha,q,p}(x) = \begin{cases} (q-1)\alpha x^2 & \text{for } x \in [0, e], \\ e(q-1)\alpha x \ln x & \text{for } x \in (e, e^{1/(q-1)}], \\ \alpha x^q & \text{for } x \in (e^{1/(q-1)}, (1/\alpha)^{1/q}], \\ \alpha^{p/q} x^p & \text{for } x \in ((1/\alpha)^{1/q}, 1/\alpha], \end{cases} \qquad \text{if } 1/\alpha > e^{q/(q-1)}.$$

Note that $\widetilde{L}_{\alpha,p}$ and $\widetilde{R}_{\alpha,q,p}$ are continuous, differentiable (except the case $1/\alpha > e^{q/(q-1)}$ when $\widetilde{R}_{\alpha,q,p}$ is not differentiable at $(1/\alpha)^{1/q}$) and increasing.

Additionally, the following approximate formulas will be used: for every $\varepsilon > 0$ there exist $c_1, c_2, c_3 > 0$ (each depends only on $\varepsilon$) such that

$$e^t - 1 \overset{c_1}{\sim} t \quad \text{for } 0 \le t \le \varepsilon, \tag{4.4}$$

$$e^t - 1 \overset{c_2}{\approx} e^t \quad \text{for } t \ge \varepsilon, \tag{4.5}$$

$$\ln(1+t) \overset{c_3}{\sim} t \quad \text{for } 0 \le t \le \varepsilon. \tag{4.6}$$

LEMMA 4.3. *For $\alpha \le e^{-2}$ and $x \in [-1/\beta, 1/\alpha]$, $\widetilde{L}_{\alpha,p}(|x|) \sim L_{\alpha,p}(x)$.*

*Proof.* Obviously, for all $x_1, x_2 \in [0, 1/\alpha]$ such that $x_1 \overset{C}{\sim} x_2$ we have $\widetilde{L}_{\alpha,p}(x_1) \overset{D}{\sim} \widetilde{L}_{\alpha,p}(x_2)$, where $D > 0$ depends only on $C$. Moreover, $L_{\alpha,p}$ is convex and $C^\infty$ on $(-1/\beta, 1/\alpha)$, hence $L_{\alpha,p}$ is decreasing on $[-1/\beta, 0]$ and increasing on $[0, 1/\alpha]$, since $L'_{\alpha,p}(0) = 0$. Therefore it is sufficient to prove the assertion only for $x \in \{-1/\beta\} \cup [-1/(2\beta), 1/(2\beta)] \cup [e^2, 1/\alpha]$.

CASE $x \in [-1/(2\beta), 1/(2\beta)]$. It suffices to prove that

$$L''_{\alpha,p}(x) \sim (p-1)\alpha. \tag{4.7}$$

Indeed, by integrating we get $L'_{\alpha,p}(x) \sim (p-1)\alpha x$, now, $L'_{\alpha,p}(0) = 0$, and by integrating once again we get $L_{\alpha,p}(x) \sim (p-1)\alpha x^2$, since $L_{\alpha,p}(0) = 0$. Now,

$$L''_{\alpha,p}(x) = p(p-1)\alpha\beta(\alpha(1-\alpha x)^{p-2} + \beta(1+\beta x)^{p-2}).$$

Clearly, we have $1 - \alpha x \sim 1$ and $1 + \beta x \sim 1$. Since $p - 2 \in (-1, 0]$, we also have $(1-\alpha x)^{p-2} \sim 1$ and $(1+\beta x)^{p-2} \sim 1$, which implies (4.7).

CASE $x = -1/\beta$. From (4.4) and (4.6) we have

$$\begin{aligned} L_{\alpha,p}(-1/\beta) &= \beta(1+\alpha/\beta)^p - 1 = (1+\alpha/\beta)^{p-1} - 1 = e^{(p-1)\ln(1+\alpha/\beta)} - 1 \\ &\sim (p-1)\ln(1+\alpha/\beta) \sim (p-1)\alpha/\beta \sim \widetilde{L}_{\alpha,p}(1/\beta), \end{aligned}$$

since $(p-1)\ln(1+\alpha/\beta) \le (p-1)\alpha/\beta < 1$ and $\alpha/\beta < 1$.

CASE $x \in [e^2, 1/\alpha]$. We shall approximate

$$\begin{aligned} L_{\alpha,p}(x) &= \beta(1-\alpha x)((1-\alpha x)^{p-1} - 1) + \alpha(1+\beta x)((1+\beta x)^{p-1} - 1) \\ &= M_1 + M_2. \end{aligned}$$

The first term is equal to zero if $x = 1/\alpha$, and is negative otherwise. In the latter case we have the following estimate:

$$\begin{aligned} M_1 &= \beta(1-\alpha x)(e^{(p-1)\ln(1-\alpha x)} - 1) \ge \beta(1-\alpha x)(p-1)\ln(1-\alpha x) \\ &\ge \beta(p-1)(-\alpha x) > -(p-1)\alpha x \end{aligned}$$

where the second inequality follows from the inequality $\ln(1+y) \ge y/(1+y)$, which holds for $y > -1$. For the second term we have

$$\begin{aligned} M_2 &= \alpha(1+\beta x)(e^{(p-1)\ln(1+\beta x)} - 1) \ge \alpha(1+\beta x)(p-1)\ln(1+\beta x) \\ &\ge \alpha x(p-1)\ln x \ge 2(p-1)\alpha x, \end{aligned}$$

since $1+\beta x \geq x$ and $\ln x \geq 2$. Therefore $M_2 \geq M_1 + M_2 = (M_1 + M_2/2) + M_2/2 \geq M_2/2$, hence $L_{\alpha,p}(x) \sim M_2$ and further we deal with $M_2$. Since $1+\beta x \geq x \geq e^2$, we have $1+\beta x \sim x$, $\ln(1+\beta x) \sim \ln x$ and $(1+\beta x)^{p-1} \sim x^{p-1}$. Thus from (4.4) and (4.5) we obtain

$$M_2 \sim \alpha x(e^{(p-1)\ln(1+\beta x)} - 1) \sim \alpha x \begin{cases} (p-1)\ln x & \text{if } x \leq e^{1/(p-1)}, \\ x^{p-1} & \text{if } x > e^{1/(p-1)}. \end{cases} \blacksquare$$

LEMMA 4.4. *For all $x \in [-1/\beta, 1/\alpha]$, $\widetilde{R}_{\alpha,q,p}(|x|) \sim R_{\alpha,q,p}(x)$, whenever $\widetilde{R}_{\alpha,q,p}$ is defined.*

*Proof.* Since $p/q \in (1,2)$, we have $(1+t)^{p/q} - 1 \sim (1+t)^{p/q} \sim t^{p/q}$ for $t \geq 1$, and $(1+t)^{p/q} - 1 \sim t$ for $t \in [0,1]$. Therefore

$$R_{\alpha,q,p}(x) = (1 + L_{\alpha,q}(x))^{p/q} - 1 \sim L_{\alpha,q}(x) \vee L_{\alpha,q}(x)^{p/q},$$

while an easy computation shows that $\widetilde{R}_{\alpha,q,p}(x) \sim \widetilde{L}_{\alpha,q}(x) \vee \widetilde{L}_{\alpha,q}(x)^{p/q}$. Then Lemma 4.3 completes the proof. $\blacksquare$

*Proof of Theorem 2.1.* To prove optimality of the constants $\widetilde{\sigma}_{p,q}(\alpha)$ we may proceed as follows. Let $p' = p/(p-1)$, $q' = q/(q-1)$. Theorem 2.3 yields $\|S\|_{q'}/\|S\|_{p'} \sim \widetilde{\sigma}^{-1}_{q',p'}(\alpha)$. Transforming $(q',p')$ to $(p,q)$ in the formula for $\widetilde{\sigma}_{q',p'}(\alpha)$ from Theorem 2.2, together with Lemma 4.2, shows that $\widetilde{\sigma}_{q',p'}(\alpha) \sim \widetilde{\sigma}_{p,q}(\alpha)$. On the other hand, Proposition 1.1 implies that $\|S\|_{q'}/\|S\|_{p'} \leq \sigma^{-1}_{q',p'}(\alpha) = \sigma^{-1}_{p,q}(\alpha)$ (the last equality follows from Remark 1.1). Thus we get $\widetilde{\sigma}_{p,q}(\alpha) \gtrsim \sigma_{p,q}(\alpha)$.

We proceed to show the converse inequality, i.e. $\widetilde{\sigma}_{p,q}(\alpha) \lesssim \sigma_{p,q}(\alpha)$. Lemmas 4.1 and 4.2 imply that we may neglect some values of $\alpha$, therefore we assume $\alpha \notin (e^{-2}, 1/2]$ and $1/\alpha \notin (e^{1/(q-1)}, e^{q/(q-1)}]$, since $e^{-2} \sim 1/2$ and $e^{1/(q-1)} \sim e^{q/(q-1)}$. By Remark 1.2, it suffices to prove that

$$L_{\alpha,p}(\sigma x) \leq R_{\alpha,q,p}(x) \quad \text{for all } x \in [-1/\beta, 1/\alpha], \tag{4.8}$$

where $\sigma \sim \widetilde{\sigma} := \widetilde{\sigma}_{p,q}(\alpha)$. This, in fact, can be reduced to showing that for some universal constant $C_1 > 0$,

$$\widetilde{L}_{\alpha,p}(\widetilde{\sigma} x) \leq C_1 \widetilde{R}_{\alpha,q,p}(x) \quad \text{for all } x \in [0, 1/\alpha]. \tag{4.9}$$

Indeed, combining (4.9) with Lemmas 4.3 and 4.4, we get

$$L_{\alpha,p}(\widetilde{\sigma} x) \leq C_2 R_{\alpha,q,p}(x) \quad \text{for all } x \in [-1/\beta, 1/\alpha], \tag{4.10}$$

where $C_2$ is some universal constant. Clearly, we may assume $C_2 \geq 1$. Since the left hand side of (4.10) is a convex function of $x$, taking the zero value at $x = 0$, we have $L_{\alpha,p}(C_2^{-1}\widetilde{\sigma} x) \leq C_2^{-1} L_{\alpha,p}(\widetilde{\sigma} x)$, which combined with (4.10) gives (4.8) with $\sigma = C_2^{-1}\widetilde{\sigma}$.

We now claim that the continuous function $(0, 1/\alpha) \ni x \mapsto \frac{\widetilde{L}_{\alpha,p}(\widetilde{\sigma} x)}{\widetilde{R}_{\alpha,q,p}(x)}$ is non-decreasing. Since it is differentiable (except at the point $(1/\alpha)^{1/q}$ in case

$1/\alpha > e^{q/(q-1)}$), it suffices to show that its first derivative is non-negative, or, equivalently, that

$$\widehat{L}_{\alpha,p}(\widetilde{\sigma}x) \geq \widehat{R}_{\alpha,q,p}(x) \quad \text{for all } x \in (0, 1/\alpha), \tag{4.11}$$

(excluding $x = (1/\alpha)^{1/q}$ when $1/\alpha > e^{q/(q-1)}$), where $\widehat{L}_{\alpha,p}(x) = x\frac{\widetilde{L}'_{\alpha,p}(x)}{\widetilde{L}_{\alpha,p}(x)}$ and $\widehat{R}_{\alpha,q,p}(x) = x\frac{\widetilde{R}'_{\alpha,q,p}(x)}{\widetilde{R}_{\alpha,q,p}(x)}$. Simple computations give

$$\widehat{L}_{\alpha,p}(x) = \begin{cases} 2 & \text{for } x \in (0, e], \\ 1 + 1/\ln x & \text{for } x \in (e, e^{1/(p-1)} \wedge 1/\alpha), \\ p & \text{for } x \in [e^{1/(p-1)} \wedge 1/\alpha, 1/\alpha), \end{cases}$$

$$\widehat{R}_{\alpha,q,p}(x) = \widehat{L}_{\alpha,p}(x) \qquad \text{if } 1/\alpha \leq e^{1/(q-1)},$$

$$\widehat{R}_{\alpha,q,p}(x) = \begin{cases} 2 & \text{for } x \in (0, e], \\ 1 + 1/\ln x & \text{for } x \in (e, e^{1/(q-1)}], \\ q & \text{for } x \in (e^{1/(q-1)}, (1/\alpha)^{1/q}), \\ p & \text{for } x \in ((1/\alpha)^{1/q}, 1/\alpha), \end{cases} \qquad \text{if } 1/\alpha > e^{q/(q-1)}.$$

$\widehat{L}_{\alpha,p}$ is non-increasing, thus $\widehat{L}_{\alpha,p}(\widetilde{\sigma}x) \geq \widehat{L}_{\alpha,p}(x)$, since $\widetilde{\sigma} \leq 1$ (this can be checked in many ways, e.g. see the proof of Lemma 2.2, where monotonicity of $\alpha \mapsto \widetilde{\sigma}_{p,q}(\alpha)$ is investigated). Moreover, $\widehat{L}_{\alpha,p}(x) = \widehat{R}_{\alpha,q,p}(x)$ for $x \in (0, e^{1/(p-1)} \wedge 1/\alpha)$ and $\widehat{L}_{\alpha,p}(x) = p \geq \widehat{R}_{\alpha,q,p}(x)$ for $x \in [e^{1/(p-1)} \wedge 1/\alpha, 1/\alpha)$ (excluding $x = (1/\alpha)^{1/q}$ when $1/\alpha > e^{q/(q-1)}$), which proves (4.11).

To complete the proof, it remains to check (4.9) for $x = 1/\alpha$. We consider four cases (as in the formula for $\widetilde{\sigma}_{p,q}(\alpha)$).

Case $\alpha \in I_1$. Clearly $\widetilde{\sigma}/\alpha = (1/\alpha)^{1-1/q+1/p} > (1/\alpha)^{1/p}$. Since we assumed that $1/\alpha \notin (e^{1/(q-1)}, e^{q/(q-1)}]$, we have $1/\alpha > e^{q/(q-1)} > e^{p/(p-1)}$, and therefore $\widetilde{\sigma}/\alpha > e^{1/(p-1)}$, hence $\widetilde{L}_{\alpha,p}(\widetilde{\sigma}/\alpha) = \alpha(\widetilde{\sigma}/\alpha)^p = (1/\alpha)^{p-p/q} = \widetilde{R}_{\alpha,q,p}(1/\alpha)$.

Case $\alpha \in I_2$. Clearly, $\widetilde{R}_{\alpha,q,p}(1/\alpha) = e(q-1)\ln(1/\alpha)$. If $\widetilde{\sigma}/\alpha > e^{1/(p-1)}$, then $\widetilde{L}_{\alpha,p}(\widetilde{\sigma}/\alpha) = \alpha(\widetilde{\sigma}/\alpha)^p = (q-1)\ln(1/\alpha)$, and if $\widetilde{\sigma}/\alpha \leq e^{1/(p-1)}$, then by the monotonicity of $\widetilde{L}_{\alpha,p}(x)$ and $x \mapsto x\ln x$ we have

$$\widetilde{L}_{\alpha,p}(\widetilde{\sigma}/\alpha) \leq e(p-1)\alpha e^{1/(p-1)}\ln(e^{1/(p-1)}) = \alpha e^{p/(p-1)} \leq (q-1)\ln(1/\alpha),$$

where the last inequality is due to the fact that $\alpha \in I_2$.

Case $\alpha \in I_3$. $\widetilde{R}_{\alpha,q,p}(1/\alpha)$ is the same as in the previous case. We now show that $\widetilde{\sigma}/\alpha \leq e^{1/(p-1)}$. Setting

$$u_0 := \ln\left(\frac{q-1}{p-1}\,\frac{1}{\alpha}\ln\frac{1}{\alpha}\right),$$

we get $\widetilde{\sigma}/\alpha = e^{u_0}/(1+u_0)$. Since $\alpha \in I_3$, we have

$$0 \le u_0 \le \frac{1}{p-1} + \ln\frac{1}{p-1}.$$

The function $h(u)=e^u/(1+u)$ is increasing for $u\ge 0$ ($h'(u)=ue^u/(1+u)^2$), hence

$$\widetilde{\sigma}/\alpha \le h\left(\frac{1}{p-1} + \ln\frac{1}{p-1}\right) < e^{1/(p-1)}.$$

We next estimate the fraction $\widetilde{L}_{\alpha,p}(\widetilde{\sigma}/\alpha)/\widetilde{R}_{\alpha,q,p}(1/\alpha)$. If $\widetilde{\sigma}/\alpha > e$, then $\widetilde{L}_{\alpha,p}(\widetilde{\sigma}/\alpha) = e(p-1)\widetilde{\sigma}\ln(\widetilde{\sigma}/\alpha)$, otherwise, by the monotonicity of $\widetilde{L}_{\alpha,p}(x)$, $\widetilde{L}_{\alpha,p}(\widetilde{\sigma}/\alpha) \le (p-1)\alpha e^2$. In the first case, the fraction above is equal to

$$\frac{e(p-1)\widetilde{\sigma}\ln(\widetilde{\sigma}/\alpha)}{e(q-1)\ln(1/\alpha)} = \frac{u_0 + \ln\frac{1}{1+u_0}}{1+u_0} < 1.$$

In the second case the same fraction is equal to

$$\frac{(p-1)\alpha e^2}{e(q-1)\ln(1/\alpha)} \le e, \quad \text{since} \quad \frac{p-1}{q-1} \le \frac{1}{\alpha}\ln\frac{1}{\alpha}$$

(because $\alpha \in I_3$).

Case $\alpha \in I_4$. $\widetilde{R}_{\alpha,q,p}(1/\alpha)$ is as in the previous case. Clearly, $\widetilde{\sigma}/\alpha \in (0,1)$, thus

$$\frac{\widetilde{L}_{\alpha,p}(\widetilde{\sigma}/\alpha)}{\widetilde{R}_{\alpha,q,p}(1/\alpha)} = \frac{(p-1)\widetilde{\sigma}^2/\alpha}{e(q-1)\ln(1/\alpha)} = e^{-1}.$$

This gives (4.9) with the constant $C_1 = e$, which completes the proof. ∎

*Proof of Theorem 2.2.* At the beginning of the proof of Theorem 2.1 we stated that $\widetilde{\sigma}_{p,q}(\alpha) \sim \widetilde{\sigma}_{q',p'}(\alpha)$ for $p' = p/(p-1)$, $q' = q/(q-1)$, whereas Theorem 2.1 and Remark 1.1 imply that $\widetilde{\sigma}_{q',p'}(\alpha) \sim \sigma_{q',p'}(\alpha) = \sigma_{p,q}(\alpha)$. ∎

*Proof of Lemma 2.2.* By Remark 1.1, we can assume $1 < q < p \le 2$. Theorem 2.1 implies that it suffices to prove $\widetilde{\sigma}_{p,q}(\alpha_1) \lesssim \widetilde{\sigma}_{p,q}(\alpha_2)$, and Lemma 4.2 shows that it suffices to do it only for $\alpha_1, \alpha_2 \in I_j$, $j = 1, \ldots, 4$.

Case $\alpha_1, \alpha_2 \in I_1$. The function $\alpha \mapsto \alpha^{1/q-1/p}$ is increasing, since $1/q - 1/p > 0$.

Case $\alpha_1, \alpha_2 \in I_2$. We check that the function $h(\alpha) = (\ln(1/\alpha))^{1/p}\alpha^{1-1/p}$ is increasing for $\alpha \le e^{-1/(p-1)}$, so in particular for $\alpha \in I_2$. Indeed,

$$\begin{aligned} h'(\alpha) &= -\frac{1}{p}(\alpha\ln(1/\alpha))^{-1}h(\alpha) + \left(1 - \frac{1}{p}\right)\alpha^{-1}h(\alpha) \\ &= \alpha^{-1}h(\alpha)\left(1 - \frac{1}{p}(1 + (\ln(1/\alpha))^{-1})\right), \end{aligned}$$

which is non-negative, since $1 + (\ln(1/\alpha))^{-1} \le p$.

CASE $\alpha_1, \alpha_2 \in I_3$. We check that the function

$$h(\alpha) = \frac{h_1(\alpha)}{1 + \ln h_2(\alpha)} = \frac{\ln(1/\alpha)}{1 + \ln\big(\frac{q-1}{p-1}\frac{1}{\alpha}\ln\frac{1}{\alpha}\big)}$$

is increasing if $\ln(1/\alpha) \le (p-1)/(q-1)$ and decreasing otherwise. Indeed, the sign of its first derivative is the same as the sign of

$$\begin{aligned} h_1'(\alpha)(1 + \ln h_2(\alpha)) - h_1(\alpha)\,\frac{h_2'(\alpha)}{h_2(\alpha)} \\ &= -\alpha^{-1}(1 + \ln h_2(\alpha)) + h_1(\alpha)\,\frac{1 + \ln(1/\alpha)}{\alpha\ln(1/\alpha)} \\ &= -\alpha^{-1}(\ln h_2(\alpha) - \ln(1/\alpha)) = -\alpha^{-1}\ln\bigg(\frac{q-1}{p-1}\ln\frac{1}{\alpha}\bigg). \end{aligned}$$

Thus if $\{\alpha\colon \ln(1/\alpha) > (p-1)/(q-1)\} \cap I_3 = \emptyset$, then this case is done. Assume the opposite. Then we have the non-empty interval $(\underline{\alpha}, \overline{\alpha}]$ (where $\underline{\alpha} = \inf I_3$ and $\ln(1/\overline{\alpha}) = (p-1)/(q-1)$) on which $h(\alpha)$ is decreasing. So $\underline{\alpha} < \overline{\alpha}$, which implies that

$$\frac{p-1}{q-1}\,e^{(p-1)/(q-1)} < \frac{1}{q-1}\,e^{1/(p-1)},$$

so

$$\frac{1}{q-1} < \frac{1}{(p-1)^2} + \frac{1}{p-1}\ln\frac{1}{p-1} < \frac{2}{(p-1)^2},$$

hence

$$\ln\frac{1}{q-1} < \ln 2 + 2\ln\frac{1}{p-1} < \frac{3}{p-1}.$$

It suffices to show that for some universal constant $C \ge 1$ we have $\lim_{\alpha\to\underline{\alpha}^+}\widetilde{\sigma}_{p,q}(\alpha) \le C\widetilde{\sigma}_{p,q}(\overline{\alpha})$. Clearly,

$$\widetilde{\sigma}_{p,q}(\overline{\alpha}) = \frac{1}{1 + \frac{p-1}{q-1}} \sim \frac{q-1}{p-1}$$

and from (4.3) we have

$$\lim_{\alpha\to\underline{\alpha}^+}\widetilde{\sigma}_{p,q}(\alpha) \sim (q-1)\ln\frac{1}{\underline{\alpha}} \sim (q-1)\bigg(\frac{1}{p-1} + \ln\frac{1}{q-1}\bigg),$$

since

$$\ln\frac{1}{\underline{\alpha}} \sim \ln\frac{e^{1/(p-1)}}{q-1}.$$

Hence

$$\lim_{\alpha\to\underline{\alpha}^+}\widetilde{\sigma}_{p,q}(\alpha) \sim \frac{q-1}{p-1}, \quad \text{since} \quad \ln\frac{1}{q-1} \lesssim \frac{1}{p-1}.$$

CASE $\alpha_1, \alpha_2 \in I_4$. We check that the function $\alpha \mapsto \alpha\ln(1/\alpha)$ is increasing for $\alpha \le e^{-1}$. Indeed, its first derivative is $\ln(1/\alpha) - 1$. By Lemma 4.2 we have $\widetilde{\sigma}_{p,q}(\alpha) \sim \widetilde{\sigma}_{p,q}(e^{-1})$ for $\alpha \in (e^{-1}, 1/2]$, so this case is done. ∎

*Proof of Theorem 2.3.* Let $(X_k')_{k=1}^{\infty}$ be an independent copy of the sequence $(X_k)_{k=1}^{\infty}$. We set $Y_k = X_k - X_k'$, $S' = \sum_{k=1}^{n(\alpha,p)} X_k'$ and $S'' = S - S'$ and $X = X_1$, $Y = Y_1$. Each random variable $Y_k$ is symmetric and its distribution is equal to $\alpha\beta\delta_{-1} + (1-2\alpha\beta)\delta_0 + \alpha\beta\delta_1$. Moreover, for any $r \geq 1$ we have $\|X\|_r \leq \|Y\|_r \leq 2\|X\|_r$ (the first inequality comes from the Jensen inequality and the fact that $EX = 0$, whereas the second is just the triangle inequality for the $L_r$-norm). Clearly, the same applies to $S$ and $S''$. Therefore we proceed to show that $\|S''\|_q/\|S''\|_p \sim \widetilde{\sigma}_{p,q}(\alpha)$.

First, note that if $p \leq \ln(1/\alpha)$ (equivalently $n(\alpha,p) = 1$), then $\alpha \in I_1'$ and we have $\|S''\|_q = \|Y\|_q = (2\alpha\beta)^{1/q} \sim \alpha^{1/q}$ and $\|S''\|_p \sim \alpha^{1/p}$.

Further we assume that $p > \ln(1/\alpha)$ (equivalently, $n := n(\alpha,p) \geq 2$). It is a simple matter to approximate $\|S''\|_p$. Indeed, we have $n/2 < p/\ln(1/\alpha)$, thus $e^{-4p} < \alpha^{2n} < (\alpha\beta)^n$, so $P(S'' = n) > e^{-4p}$. Hence

$$n = \|S''\|_\infty \geq \|S''\|_p \geq nP(S'' = n)^{1/p} > ne^{-4},$$

so $\|S''\|_p \sim n \sim p/\ln(1/\alpha)$. We now want to approximate $\|S''\|_q$. Originally the author just used an explicit approximate formula due to Rafał Latała [unpublished]. We may also apply [8, Corollary 2] which states that for $q \geq 2$ and i.i.d. symmetric random variables $Z, Z_1, \ldots, Z_n$,

$$\|Z_1 + \cdots + Z_n\|_q \sim \sup\left\{\frac{q}{s}\left(\frac{n}{q}\right)^{1/s}\|Z\|_s \colon 2 \vee \frac{q}{n} \leq s \leq q\right\}.$$

In our situation, we have

$$\frac{\|S''\|_q}{\|S''\|_p} \sim \bar{f} := \sup\left\{f(s) = \frac{1}{s}\left(\frac{q}{p}\ln\frac{1}{\alpha}\right)^{1-1/s}\alpha^{1/s} \colon 2 \vee \frac{q}{\lceil \frac{p}{\ln(1/\alpha)} \rceil} \leq s \leq q\right\},$$

since $n^{1/s} \sim (p/\ln(1/\alpha))^{1/s}$. Note that $f$ is differentiable for $s \in (0,\infty)$ and

$$f'(s) = \left(\frac{q}{p}\ln\frac{1}{\alpha}\right)\left(\frac{1}{s}\left(\frac{q}{p}\frac{1}{\alpha}\ln\frac{1}{\alpha}\right)^{-1/s}\right)' = \frac{1}{s^2}f(s)\left(\ln\left(\frac{q}{p}\frac{1}{\alpha}\ln\frac{1}{\alpha}\right) - s\right),$$

hence $f(s)$ is increasing for $0 < s < s_0$ and decreasing for $s_0 < s < \infty$, where $s_0 = \ln\left(\frac{q}{p}\frac{1}{\alpha}\ln\frac{1}{\alpha}\right)$. Let us consider three cases:

Case $(p/q)e^q \leq (1/\alpha)\ln(1/\alpha) < pe^p$. Clearly, $s_0 \geq q$, hence

$$\bar{f} = f(q) = q^{-1/q}\left(\frac{\ln(1/\alpha)}{p}\right)^{1-1/q}\alpha^{1/q} \sim \left(\frac{\ln(1/\alpha)}{p}\right)^{1-1/q}\alpha^{1/q}.$$

If $pe^q \leq (1/\alpha)\ln(1/\alpha) < pe^p$ (i.e. $\alpha \in I_2'$), then we have $\bar{f} \sim \widetilde{\sigma}_{p,q}(\alpha)$. If $(p/q)e^q \leq (1/\alpha)\ln(1/\alpha) < pe^q$ (hence $\alpha \in I_3'$), then we see that

$$\frac{\widetilde{\sigma}_{p,q}(\alpha)}{\left(\frac{\ln(1/\alpha)}{p}\right)^{1-1/q}\alpha^{1/q}} = \frac{q}{1 + \ln\left(\frac{q}{p}\frac{1}{\alpha}\ln\frac{1}{\alpha}\right)}\left(\frac{1}{p}\frac{1}{\alpha}\ln\frac{1}{\alpha}\right)^{1/q} \sim 1,$$

since the first term is $\sim 1$ because $q \le \ln\left(\frac{q}{p}\frac{1}{\alpha}\ln\frac{1}{\alpha}\right) < q + \ln q$, as also is the second, because $q^{-1}e^q \le \frac{1}{p}\frac{1}{\alpha}\ln\frac{1}{\alpha} < e^q$, so this case is done.

CASE $(p/q)e^2 \le (1/\alpha)\ln(1/\alpha) < (p/q)e^q$. First, we show that

$$2 \vee \frac{q}{\lceil p/\ln(1/\alpha)\rceil} \le s_0 < q.$$

It is clear that $2 \le s_0 < q$. If $s_0 < q/\lceil p/\ln(1/\alpha)\rceil$, we would have $s_0 < (q/p)\ln(1/\alpha)$, hence by monotonicity of ln we would obtain

$$\ln\frac{1}{\alpha} < \ln\frac{2}{\alpha} < \ln\left(\frac{q}{p}\frac{1}{\alpha}\ln\frac{1}{\alpha}\right) = s_0 < \frac{q}{p}\ln\frac{1}{\alpha},$$

which is a contradiction, since $q < p$. Therefore $\bar{f} = f(s_0)$. We now show that $f(s_0) \sim \widetilde{\sigma}_{p,q}(\alpha)$. Since $s_0 \ge 2$, we have $s_0 \sim 1 + s_0$, so

$$f(s_0) \sim \frac{q}{p}\frac{\ln(1/\alpha)}{1+s_0}\left(\frac{q}{p}\frac{1}{\alpha}\ln\frac{1}{\alpha}\right)^{-1/s_0} = \widetilde{\sigma}_{p,q}(\alpha)(e^{s_0})^{-1/s_0} \sim \widetilde{\sigma}_{p,q}(\alpha).$$

CASE $(1/\alpha)\ln(1/\alpha) < (p/q)e^2$. In this case $s_0 < 2$ and

$$\frac{q}{\lceil p/\ln(1/\alpha)\rceil} \le \frac{q}{p}\ln\frac{1}{\alpha} < \alpha e^2.$$

If $\alpha e^2 \le 2$, then $q/\lceil p/\ln(1/\alpha)\rceil < 2$. Hence in the case $\alpha \le 2e^{-2}$ or $\alpha \in (2e^{-2}, 1/2]$ and $q/\lceil p/\ln(1/\alpha)\rceil \le 2$ we have $\bar{f} = f(2) = \frac{1}{2}\sqrt{(q/p)\alpha\ln(1/\alpha)}$, which is $\sim \widetilde{\sigma}_{p,q}(\alpha)$ if $\alpha \in I_4'$. If $\alpha \in I_3'$, i.e. $p/q \le (1/\alpha)\ln(1/\alpha) < (p/q)e^2$, then by Lemma 4.2 (more precisely: by an analogous fact which deals with the dual case $2 \le q < p < \infty$) we get $\bar{f} \sim \alpha \sim \widetilde{\sigma}_{p,q}(\alpha)$. There remains the case when $\alpha \in (2e^{-2}, 1/2]$ and $s_1 := q/\lceil p/\ln(1/\alpha)\rceil > 2$. Then $\bar{f} = f(s_1)$ and $2 < s_1 \le (q/p)\ln(1/\alpha) < \alpha e^2 \le \frac{1}{2}e^2$. Thus $\bar{f} \sim 1$ and $q/p \sim 1$, since $\ln(1/\alpha) \sim 1$. The latter implies that $\widetilde{\sigma}_{p,q}(1/2) \sim 1$ and from Lemma 4.2 we know that $\widetilde{\sigma}_{p,q}(\alpha) \sim \widetilde{\sigma}_{p,q}(1/2)$. ∎

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