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Coorbit space theory for quasi-Banach spaces

by

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Abstract. We generalize the classical coorbit space theory developed by Feichtinger and Gröchenig to quasi-Banach spaces. As a main result we provide atomic decompositions for coorbit spaces defined with respect to quasi-Banach spaces. These atomic decompositions are used to prove fast convergence rates of best n-term approximation schemes. We apply the abstract theory to time-frequency analysis of modulation spaces $M_m^{p,q}$, $0 < p, q \leq \infty$.

1. Introduction. Coorbit space theory was originally developed by Feichtinger and Gröchenig [6–8, 11] in the late 1980's with the aim to provide a unified and group-theoretical approach to function spaces and their atomic decompositions. In particular, this theory covers the homogeneous Besov and Triebel–Lizorkin spaces and their wavelet-type atomic decompositions, as well as the modulation spaces and their Gabor-type decompositions. Recently, there has been some activity to provide generalizations to other settings than the classical one of integrable group representations [1, 9, 16].

All the approaches taken so far cover only the case of Banach spaces. In [6] it is remarked that an extension of coorbit space theory to quasi-Banach spaces would be interesting, but it seems that nothing concrete has been done since then. For instance this would allow one to describe also modulation spaces $M_m^{p,q}$ with $p < 1$ or $q < 1$, or Hardy spaces H^p with $p < 1$, as coorbit spaces. An important motivation to consider quasi-Banach spaces is an application in nonlinear approximation. Indeed, the best theoretical convergence rate of best n-term approximations is rather small if one restricts to Banach spaces, while it can get arbitrarily large for quasi-Banach spaces.

So in this paper we extend the classical coorbit space theory to quasi-Banach spaces. Our starting point is an integrable representation π of some

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locally compact group $\mathcal{G}$ on some Hilbert space $\mathcal{H}$. Associated to π is the abstract wavelet transform $V_g f(x) = \langle f, \pi(x)g\rangle$. The crucial ingredient in coorbit space theory is the reproducing formula for V_g (see (4.2)), which uses the group convolution on $\mathcal{G}$. Thus, it is essential to have convolution relations for certain quasi-Banach spaces Y on $\mathcal{G}$. Unfortunately, even for the natural choice $Y = L^p(\mathcal{G})$, $0 < p < 1$, no convolution relation is available. In order to overcome this problem we work with Wiener amalgam spaces $W(L^\infty, Y)$ with local component L^∞ instead of Y itself. Convolution relations for such spaces, where Y is allowed to be a quasi-Banach space, were shown recently by the author in [17].

Under some technical assumption on the representation, the coorbit spaces $\mathcal{C}(Y)$ are defined as a retract of the Wiener amalgam space $W(L^\infty, Y)$ via the abstract wavelet transform, i.e., $\mathcal{C}(Y) = \{f : V_g f \in W(L^\infty, Y)\}$ (see Section 4). Apart from basic properties of $\mathcal{C}(Y)$ we will provide atomic decompositions of $\mathcal{C}(Y)$ of the form $\{\pi(x_i)g\}_{i\in I}$, where $(x_i)_{i\in I}$ is a suitable point set in the group (see Section 5). Based on such decompositions we will investigate approximation rates for best n-term approximations (Section 7).

Our results are applicable to time-frequency analysis on modulation spaces $M_m^{p,q}$, $0 < p, q \le \infty$, introduced by Feichtinger [5] (see also [18, 10] for the case $p, q < 1$). Here, we improve or give alternative proofs to some of the results of Galperin and Samarah in [10] (see Section 8).

Although in this paper we restrict $\mathcal{G}$ to be an IN group for the sake of simple presentation, we remark that nevertheless the abstract theory can be developed in general and it then also applies to homogeneous (weighted) Besov spaces $\dot{B}^s_{p,q}$ and Triebel–Lizorkin spaces $\dot{F}^s_{p,q}$, $0 < p, q \le \infty$. We postpone a detailed discussion to a subsequent contribution.

2. Prerequisites. Let $\mathcal{G}$ be a locally compact group with identity e. For simplicity we always assume that $\mathcal{G}$ is an IN group. This means that there exists a compact neighborhood U of e such that $xU = Ux$. IN groups are unimodular. Integration on $\mathcal{G}$ will always be with respect to the left Haar measure. We denote by $L_x F(y) = F(x^{-1}y)$ and $R_x F(y) = F(yx)$, $x, y \in \mathcal{G}$, the left and right translation operators. For a Radon measure μ we introduce the operator $(A_x\mu)(k) = \mu(R_x k)$, $x \in \mathcal{G}$, for a continuous function k with compact support. We may identify a function $F \in L^1$ with a measure $\mu_F \in M$ by $\mu_F(k) = \int F(x)k(x)\,dx$. Then clearly $A_x F = R_{x^{-1}} F$. Further, we define the involutions $F^\vee(x) = F(x^{-1})$, $F^\nabla(x) = \overline{F(x^{-1})}$.

A quasi-norm $\|\cdot\|$ on some linear space Y is defined in the same way as a norm, with the only difference that the triangle inequality is replaced by $\|f + g\| \le C(\|f\| + \|g\|)$ with some constant $C \ge 1$. It is well-known (see e.g. [2, p. 20]) that there exists an equivalent quasi-norm $\|\cdot \mid Y\|$ on Y

and an exponent p with $0 < p \leq 1$ such that $\| \cdot \,|\, Y\|$ satisfies the p-triangle inequality, i.e., $\|f + g \,|\, Y\|^p \leq \|f \,|\, Y\|^p + \|g \,|\, Y\|^p$. We can choose $p = 1$ if and only if Y is a normed space. We always assume that such a p-norm on Y is chosen and denote it by $\| \cdot \,|\, Y\|$. If Y is complete with respect to the topology defined by the metric $d(f, g) = \|f - g \,|\, Y\|^p$ then it is called a *quasi-Banach space*.

A quasi-Banach space of measurable functions on $\mathcal{G}$ is called *solid* if $F \in Y$, G measurable and satisfying $|G(x)| \leq |F(x)|$ a.e. implies $G \in Y$ and $\|G \,|\, Y\| \leq \|F \,|\, Y\|$. The Lebesgue spaces $L^p(\mathcal{G})$, $0 < p \leq \infty$, provide natural examples of solid quasi-normed spaces on $\mathcal{G}$, and the usual quasi-norm in $L^p(\mathcal{G})$ is a p-norm if $0 < p \leq 1$. If w is some positive measurable weight function on $\mathcal{G}$ then we further define $L^p_w = \{F \text{ measurable} : Fw \in L^p\}$ with $\|F \,|\, L^p_w\| := \|Fw \,|\, L^p\|$. A continuous weight w is called *submultiplicative* if $w(xy) \leq w(x)w(y)$ for all $x, y \in \mathcal{G}$. Further, a weight m is called *w-moderate* if $m(xyz) \leq w(x)m(y)w(z)$, $x, y, z \in \mathcal{G}$. It is easy to see that L^p_m is invariant under left and right translations if m is w-moderate.

For a quasi-Banach space $(B, \| \cdot \,|\, B\|)$ we denote the quasi-norm of a bounded operator $T : B \to B$ by $|\!|\!| T \,|\, B |\!|\!|$. The symbol $A \asymp B$ indicates throughout the paper that there exist constants $C_1, C_2 > 0$ such that $C_1 A \leq B \leq C_2 A$ (independent of other quantities on which A, B might depend). The symbol C will always denote a generic constant whose precise value might differ at different occurrences.

3. Wiener amalgam spaces. Let B be one of the spaces $L^\infty(\mathcal{G}), L^1(\mathcal{G})$ or $M(\mathcal{G})$, the space of complex Radon measures. Choose some relatively compact neighborhood Q of $e \in \mathcal{G}$. We define the *control function* by

$$K(F, Q, B)(x) := \|(L_x \chi_Q)F \,|\, B\|, \quad x \in \mathcal{G}, \tag{3.1}$$

where F is locally contained in B, in symbols $F \in B_{\mathrm{loc}}$. Further, let Y be some solid quasi-Banach space of functions on $\mathcal{G}$ containing the characteristic function of any compact subset of $\mathcal{G}$. The *Wiener amalgam space* $W(B, Y)$ is then defined by

$$W(B, Y) := W(B, Y, Q) := \{F \in B_{\mathrm{loc}} : K(F, Q, B) \in Y\}$$

with quasi-norm

$$\|F \,|\, W(B, Y, Q)\| := \|K(F, Q, B) \,|\, Y\|. \tag{3.2}$$

This is indeed a p-norm with p being the exponent of the quasi-norm of Y. We denote by $W(C_0, Y)$ the closed subspace of $W(L^\infty, Y)$ consisting of continuous functions. For brevity we also write

$$\mathcal{W}(Y) := W(L^\infty, Y).$$

Definition 3.1. A discrete set $X = (x_i)_{i\in I}$ of points in $\mathcal{G}$ is called *V-well-spread* if for relatively compact neighborhoods V, W of e in $\mathcal{G}$,

(a) $\mathcal{G} = \bigcup_{i\in I} x_i V$.
(b) For all compact sets $K \subset \mathcal{G}$ there exists a constant C_K such that $\sup_{j\in I} \#\{i \in I : x_i K \cap x_j K \neq \emptyset\} \leq C_K$.

The existence of V-well-spread sets for arbitrarily small V is proven in [4] (see also [16] for a generalization). Given a well-spread family $X = (x_i)_{i\in I}$, a relatively compact neighborhood Q of $e \in \mathcal{G}$ and Y, we define the sequence space

$$Y_d := Y_d(X) := Y_d(X,Q) := \Big\{(\lambda_i)_{i\in I} : \sum_{i\in I} |\lambda_i| \chi_{x_i Q} \in Y\Big\}, \tag{3.3}$$

with natural norm $\|(\lambda_i)_{i\in I} \,|\, Y_d\| := \|\sum_{i\in I} |\lambda_i| \chi_{x_i Q} \,|\, Y\|$. Here, $\chi_{x_i Q}$ denotes the characteristic function of the set $x_i Q$. If the quasi-norm of Y is a p-norm, $0 < p \leq 1$, then also Y_d has a p-norm. For instance, if $Y = L^p$ then $Y_d = \ell^p(I)$.

We call a space of functions (measures) *left translation invariant* if all the left translations L_x, $x \in \mathcal{G}$, are bounded operators. Analogously, we define right translation invariance. We assume in the following that Y is left and right translation invariant (although one may replace this property by a slightly more general condition). It follows from results in [17] that then also $W(B,Y)$ is left and right translation invariant and we have $|\!|\!| L_y \,|\, W(B,Y) |\!|\!| \leq |\!|\!| L_y \,|\, Y |\!|\!|$, $|\!|\!| R_y \,|\, W(L^\infty,Y) |\!|\!| \leq |\!|\!| R_y \,|\, Y |\!|\!|$ and $|\!|\!| A_y \,|\, W(M,Y) |\!|\!| \leq |\!|\!| R_y \,|\, Y |\!|\!|$. Moreover, both $W(B,Y,Q) = W(B,Y)$ and $Y_d = Y_d(X,Q)$ are complete and independent of the choice of Q. Since $\mathcal{G}$ is assumed to be an IN group we further have [17]

$$\mathcal{W}(Y^\vee)^\vee = \mathcal{W}(Y). \tag{3.4}$$

Lemma 3.2 ([17]). *Let $r(x) := |\!|\!| L_{x^{-1}} \,|\, \mathcal{W}(Y) |\!|\!|$. Then $\mathcal{W}(Y)$ is continuously embedded into $L^\infty_{1/r}$.*

The main ingredient for the coorbit space theory with respect to quasi-Banach spaces will be the following convolution relations for Wiener amalgam spaces (see [17], recalling that $\mathcal{G}$ is unimodular).

Theorem 3.3. *Let $0 < p \leq 1$. Let Y be a p-normed right and left invariant solid quasi-Banach space. Set $w(x) := |\!|\!| R_x \,|\, Y |\!|\!|$. Then*

$$W(M,Y) * \mathcal{W}(L^p_w) \hookrightarrow \mathcal{W}(Y) \quad \text{and} \quad \mathcal{W}(Y) * \mathcal{W}(L^p_{w^\vee}) \hookrightarrow \mathcal{W}(Y) \tag{3.5}$$

with corresponding estimates for the quasi-norms.

Theorem 3.4. *Let w be a submultiplicative weight and $0 < p \leq 1$. Then $\mathcal{W}(L^p_w) * \mathcal{W}(L^p_w) \hookrightarrow \mathcal{W}(L^p_w)$.*

Further, we will need the following maximal function. For a relatively compact neighborhood U of $e \in \mathcal{G}$ and a function G on $\mathcal{G}$ we define the U-*oscillation* by

$$G_U^{\#}(x) := \sup_{u\in U} |G(ux) - G(x)|.$$

The following lemma on the U-oscillation is an essential tool for deriving the atomic decomposition for the coorbit spaces defined later on.

LEMMA 3.5.

(a) *If* $G \in W(C_0, Y)$ *then* $G_U^{\#} \in W(C_0, Y)$.
(b) *Let* w *be a submultiplicative weight function and* $0 < p < \infty$. *Then* $G \in W(C_0, L_w^p)$ *implies* $\lim_{U\to\{e\}} \|G_U^{\#} \mid \mathcal{W}(L_w^p)\| = 0$.

Proof. (a) The control function of $G_U^{\#}$ can be estimated as follows:

$$\begin{aligned} K(G_U^{\#}, Q, L^\infty)(x) &= \sup_{z\in xQ} G_U^{\#}(z) = \sup_{z\in xQ}\sup_{u\in U} |G(uz) - G(z)| \\ &\le \sup_{z\in xQ}\sup_{u\in U} |G(uz)| + \sup_{z\in xQ} |G(z)| = \sup_{q\in Q}\sup_{u\in U} |G(uxq)| + K(G, Q, C_0)(x). \end{aligned}$$

Clearly, we have $K(G, Q, C_0) \in Y$ by assumption on G. We further compute the function $H(x) := \sup_{q\in Q}\sup_{u\in U} |G(uxq)|$:

$$\begin{aligned} H(x) &= \sup_{q\in Q} \|\chi_U(R_{xq}G)\|_\infty = \sup_{q\in Q} \|(R_{(xq)^{-1}}\chi_U)^\vee G^\vee\|_\infty \\ &= \sup_{q\in Q} \|L_{(xq)^{-1}}\chi_{U^{-1}} G^\vee\|_\infty = \sup_{q\in Q} K(G^\vee, U^{-1}, L^\infty)^\vee(xq) \\ &= \|\chi_{xQ} K(G^\vee, U^{-1}, L^\infty)^\vee\|_\infty = K(K(G^\vee, U^{-1}, L^\infty)^\vee, Q, L^\infty)(x). \end{aligned}$$

Thus,

$$\begin{aligned} \|H \mid Y\| &= \|K(K(G^\vee, U^{-1}, L^\infty)^\vee, Q, L^\infty) \mid Y\| \\ &\le C\|G^\vee \mid \mathcal{W}(\mathcal{W}(Y)^\vee)\|. \end{aligned}$$

By (3.4), $\mathcal{W}(\mathcal{W}(Y)^\vee)^\vee = \mathcal{W}(\mathcal{W}(Y))$ and it is easy to see that $\mathcal{W}(\mathcal{W}(Y)) = \mathcal{W}(Y)$. It follows that $G_U^{\#} \in W(C_0, Y)$.

(b) By part (a), $G_U^{\#}$ is contained in $W(C_0, L_w^p)$. Let $\varepsilon > 0$. Since $U \subset U_0$ implies $G_U^{\#} \le G_{U_0}^{\#}$ we can find a compact set $V \subset \mathcal{G}$ such that

$$\int_{\mathcal{G}\setminus V} K(G_U^{\#}, Q, L^\infty)(x)^p w(x)^p \, dx \le \varepsilon/2$$

for all $U \subset U_0$. Since G is uniformly continuous on the compact set VQ we can find a neighborhood $U_1 \subset U_0$ of e such that

$$G_{U_1}^{\#}(x) \le M := \frac{\varepsilon^{1/p}}{(2|V|)^{1/p}\,\nu} \quad \text{for all } x \in VQ$$

with $\nu := \max_{x\in V} w(x)$. This implies

$$K(G^{\#}_{U_1}, Q, L^\infty)(x) = \sup_{z\in xQ} |G^{\#}_{U_1}(z)| \le M \quad \text{for all } x \in V.$$

Thus, we obtain

$$\int_V K(G^{\#}_{U_1}, Q, L^\infty)(x)^p w(x)^p \, dx \le M^p |V| \nu^p = \varepsilon/2.$$

Altogether this yields $\|G^{\#}_{U_1} \,|\, W(C_0, L^p_w)\|^p \le \varepsilon$. ∎

4. Coorbit spaces. Let π be an irreducible unitary representation of $\mathcal{G}$ on some Hilbert space $\mathcal{H}$. Then the abstract wavelet transform (*voice transform*) is defined as

$$V_g f(x) := \langle f, \pi(x) g\rangle, \quad f, g \in \mathcal{H},\ x \in \mathcal{G}.$$

The representation π is called *square-integrable* if there exists a non-zero $g \in \mathcal{H}$ (called *admissible*) such that $V_g g \in L^2(\mathcal{G})$. Since $\mathcal{G}$ is unimodular (as $\mathcal{G}$ is assumed to be an IN group) it follows from a theorem of Duflo and Moore [3] that in the case of square-integrability,

$$\|V_g f \,|\, L^2\| = \|g \,|\, \mathcal{H}\| \, \|f \,|\, \mathcal{H}\| \quad \text{for all } f \in \mathcal{H} \tag{4.1}$$

(provided the right normalization of the Haar measure is chosen), and actually all vectors $g \in \mathcal{H}$ are admissible.

As a consequence of (4.1), if g is normalized, i.e., $\|g \,|\, \mathcal{H}\| = 1$, we have the reproducing formula

$$V_g f = V_g f * V_g g. \tag{4.2}$$

In order to introduce the coorbit spaces we first need to extend the definition of the voice transform to a larger space, the "reservoir". To this end let v be some submultiplicative weight function satisfying $v \ge 1$. We define the following class of *analyzing vectors*:

$$\mathbb{A}_v := \{g \in \mathcal{H} : V_g g \in L^1_v\}.$$

We assume that $\mathbb{A}_v$ is non-trivial, i.e., π is integrable. This implies that π is also square-integrable. With some fixed $g \in \mathbb{A}_v \setminus \{0\}$ we define

$$\mathcal{H}^1_v := \{f \in \mathcal{H} : V_g f \in L^1_v\}$$

with norm $\|f \,|\, \mathcal{H}^1_v\| := \|V_g f \,|\, L^1_v\|$. It can be shown [6] that $\mathcal{H}^1_v$ is a π-invariant Banach space whose definition does not depend on the choice of g. We denote by $(\mathcal{H}^1_v)^\urcorner$ the anti-dual, i.e., the space of all bounded conjugate-linear functionals on $\mathcal{H}^1_v$. An equivalent norm on $(\mathcal{H}^1_v)^\urcorner$ is given by $\|V_g f \,|\, L^\infty_{1/v}\|$. Denoting by $\langle \cdot, \cdot \rangle$ also the dual pairing of $(\mathcal{H}^1_v, (\mathcal{H}^1_v)^\urcorner)$ we can extend the voice transform to $(\mathcal{H}^1_v)^\urcorner$ by

$$V_g f(x) = \langle f, \pi(x) g\rangle, \quad f \in (\mathcal{H}^1_v)^\urcorner,\ g \in \mathbb{A}_v.$$

Important properties of the voice transform extend to $(\mathcal{H}_v^1)^\urcorner$ as stated in the following lemma (see [6, 7]).

LEMMA 4.1. *Let $g \in \mathbb{A}_v$ with $\|g \,|\, \mathcal{H}\| = 1$.*

(a) *The reproducing formula extends to $(\mathcal{H}_v^1)^\urcorner$, i.e.,*

$$V_g f = V_g f * V_g g \quad \text{for all } f \in (\mathcal{H}_v^1)^\urcorner.$$

(b) *Conversely, if $F \in L^\infty_{1/w}$ satisfies the reproducing formula $F = F * V_g g$ then there exists a unique element $f \in (\mathcal{H}_v^1)^\urcorner$ such that $F = V_g f$.*

Let us now define a space of analyzing vectors that allows us to treat also quasi-Banach spaces. For $0 < p \le 1$ and for some submultiplicative weight function w we define

$$\mathbb{B}_w^p := \{g \in \mathcal{H} : V_g g \in \mathcal{W}(L_w^p)\}.$$

In what follows, we admit only those p and w such that $\mathbb{B}_w^p \neq \{0\}$. Then the left and right translation invariance of $\mathcal{W}(L_w^p)$ and the irreducibility of π imply that $\mathbb{B}_w^p$ is dense in $\mathcal{H}$. Now we are able to define the coorbit spaces.

DEFINITION 4.2. Let Y be a solid left and right translation invariant quasi-Banach space of functions on $\mathcal{G}$. Let $0 < p \le 1$ such that Y has a p-norm and put

$$w(x) := \max\{|||R_x \,|\, Y|||, |||R_{x^{-1}} \,|\, Y|||\}, \tag{4.3}$$

$$v(x) := \max\{1, |||L_{x^{-1}} \,|\, Y|||\}. \tag{4.4}$$

We assume that

$$\mathbb{B}(Y) := \mathbb{B}_w^p \cap \mathbb{A}_v \tag{4.5}$$

is non-trivial. Then for $g \in \mathbb{B}(Y) \setminus \{0\}$ the *coorbit space* is defined by

$$\mathcal{C}(Y) := \mathsf{Co}\,\mathcal{W}(Y) := \{f \in (\mathcal{H}_v^1)^\urcorner : V_g f \in \mathcal{W}(Y)\}$$

with quasi-norm $\|f \,|\, \mathcal{C}(Y)\| := \|V_g f \,|\, \mathcal{W}(Y)\|$.

Let us prove that the reproducing formula extends to $\mathcal{C}(Y)$, and that $\mathcal{C}(Y)$ is complete and independent of the choice of $g \in \mathbb{B}_w^p \setminus \{0\}$.

PROPOSITION 4.3. *Let $g \in \mathbb{B}(Y)$ be such that $\|g \,|\, \mathcal{H}\| = 1$. A function $F \in \mathcal{W}(Y)$ is of the form $V_g f$ for some $f \in \mathcal{C}(Y)$ if and only if F satisfies the reproducing formula $F = F * V_g g$.*

Proof. If $f \in \mathcal{C}(Y) \subset (\mathcal{H}_v^1)^\urcorner$ then $V_g f = V_g f * V_g g$ by the reproducing formula for $(\mathcal{H}_v^1)^\urcorner$ (see Lemma 4.1(a)).

Conversely, assume that $F = F * V_g g$ for some $F \in \mathcal{W}(Y)$. By Lemma 3.2, $\mathcal{W}(Y)$ is embedded into $L^\infty_{1/v}$. Thus $F \in L^\infty_{1/v}$ and by Lemma 4.1(b) we have $F = V_g f$ for some $f \in (\mathcal{H}_v^1)^\urcorner$, which is then automatically contained in $\mathcal{C}(Y)$ by assumption. ∎

Theorem 4.4.

(a) $\mathcal{C}(Y)$ *is a quasi-Banach space.*

(b) $\mathcal{C}(Y)$ *is independent of the choice of* $g \in \mathbb{B}(Y) \setminus \{0\}$.

Proof. (a) Let $g \in \mathbb{B}_w^p$ such that $\|g \,|\, \mathcal{H}\| = 1$. Assume $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}(Y)$. This means that $V_g f_n$ is a Cauchy sequence in $\mathcal{W}(Y)$. By completeness of $\mathcal{W}(Y)$ the limit $F = \lim_{n\to\infty} V_g f_n$ in $\mathcal{W}(Y)$ exists. By Theorem 3.3 the definition of the weight w implies that the operator $F \mapsto F * V_g g$ is continuous from $\mathcal{W}(Y)$ into $\mathcal{W}(Y)$. Hence, we may interchange its application with taking limits, and together with the reproducing formula (Proposition 4.3) this yields

$$F = \lim_{n\to\infty} V_g f_n = \lim_{n\to\infty} V_g f_n * V_g g = F * V_g g.$$

Using Proposition 4.3 once more we see that $F = V_g f$ for some $f \in \mathcal{C}(Y)$. Clearly, $f = \lim_{n\to\infty} f_n$ in $\mathcal{C}(Y)$, and hence $\mathcal{C}(Y)$ is complete.

(b) Let $g, g' \in \mathbb{B}_w^p \setminus \{0\}$. Without loss of generality we may assume that g, g' are normalized, i.e., $\|g\| = \|g'\| = 1$. Choose a vector $h \in \mathbb{B}_w^p$ not orthogonal to g and g'. It follows from the orthogonality relations that

$$0 \neq \langle g', h\rangle\langle h, g\rangle V_g g' = V_{g'} g' * V_h h * V_g g.$$

Since $V_g g^{\nabla} = V_g g$ and likewise for h and g', and since $w = w^{\vee}$, it follows from Theorem 3.4 that $V_g g' \in \mathcal{W}(L_w^p)$. The inversion formula for $V_{g'}$ reads $g = \int_{\mathcal{G}} V_{g'} g(y)\pi(y) g' dy$, and one easily deduces

$$V_g f = V_{g'} f * V_g g' \qquad \text{for all } f \in (\mathcal{H}_v^1)^{\urcorner}. \tag{4.6}$$

By the convolution relation in Theorem 3.3 we conclude that $V_g f \in \mathcal{W}(Y)$ if $V_{g'} f \in \mathcal{W}(Y)$. Exchanging the roles of g and g' shows the converse implication. ∎

Let us give a characterization of the space $\mathbb{B}_w^p$ of analyzing vectors.

Theorem 4.5. *Let* w *be a submultiplicative weight and* $0 < p \leq 1$. *Define* $w^{\bullet}(x) := \max\{w(x), w(x^{-1})\} \geq 1$. *Then*

$$\mathbb{B}_w^p = \mathbb{B}_{w^\bullet}^p = \mathcal{C}(L_{w^\bullet}^p).$$

Proof. Let $g \in \mathbb{B}_w^p$. It follows from $V_g g = V_g g^{\nabla}$ and (3.4) that $V_g g \in \mathcal{W}(L_{w^\bullet}^p)$, i.e., $g \in \mathbb{B}_{w^\bullet}^p$. Let g' be another element of $\mathbb{B}_w^p = \mathbb{B}_{w^\bullet}^p$. Then the previous proof shows that $V_g g', V_{g'} g \in \mathcal{W}(L_{w^\bullet}^p)$ and thus $g, g' \in \mathcal{C}(L_{w^\bullet}^p)$.

Conversely, assume that $g \in \mathcal{C}(L_{w^\bullet}^p)$. Note that $\mathcal{C}(L_{w^\bullet}^p) \subset \mathcal{H}$ so that voice transforms are well-defined. Let $g' \in \mathbb{B}_{w^\bullet}^p \setminus \{0\}$. Setting $f = g$ in (4.6) shows $V_g g = V_{g'} g * V_g g' = V_{g'} g * (V_{g'} g)^{\nabla}$. Since both $V_{g'} g$ and $(V_{g'} g)^{\nabla}$ are in $\mathcal{W}(L_{w^\bullet}^p)$ by (3.4), it follows from Theorem 3.4 that $V_g g \in \mathcal{W}(L_{w^\bullet}^p)$, i.e., $g \in \mathbb{B}_{w^\bullet}^p$. ∎

The following theorem will be useful to prove a weak version of a conjecture in [10, Conjecture 12].

THEOREM 4.6. *Let w be a submultiplicative weight function satisfying $w = w^\vee$ and assume $0 < p \leq 1$. If $V_g f \in \mathcal{W}(L^p_w)$ for $f, g \in \mathcal{H}$ then both f and g are in $\mathbb{B}^p_w = \mathcal{C}(L^p_w)$.*

Proof. It follows from (4.6) that

$$V_g g = (V_g f)^\nabla * V_g f \quad \text{and} \quad V_f f = V_g f * (V_g f)^\nabla.$$

Since $w(x) = w(x^{-1})$ we conclude by (3.4) that also $(V_g f)^\nabla$ lies in $\mathcal{W}(L^p_w)$. The convolution relation in Theorem 3.4 and application of Theorem 4.5 finally yield the assertion. ■

5. Discretizations. Our next result is concerned with atomic decompositions for coorbit spaces.

THEOREM 5.1. *Let $g \in \mathbb{B}(Y) \setminus \{0\}$. Then there exists a compact neighborhood U of e such that for any U-dense well-spread set $X = (x_i)_{i \in I}$ the family $\{\pi(x_i)g\}_{i\in I}$ forms an atomic decomposition of $\mathcal{C}(Y)$. This means that there exists a sequence $(\lambda_i)_{i \in I}$ of bounded linear functionals on $(\mathcal{H}^1_v)^\urcorner$ (not necessarily unique) such that*

(a) *$f = \sum_{i\in I} \lambda_i(f)\pi(x_i)g$ for all $f \in \mathcal{C}(Y)$ with convergence in the weak* topology of $(\mathcal{H}^1_v)^\urcorner$, and in the quasi-norm topology of $\mathcal{C}(Y)$ provided the finite sequences are dense in Y_d;*
(b) *an element $f \in (\mathcal{H}^1_v)^\urcorner$ is in $\mathcal{C}(Y)$ if and only if $(\lambda_i(f))_{i\in I} \in Y_d$ and*

$$\|(\lambda_i(f))_{i\in I} \,|\, Y_d\| \asymp \|f \,|\, \mathcal{C}(Y)\| \quad \text{for all } f \in \mathcal{C}(Y).$$

Proof. The theorem is proven analogously to the Banach space case (see [6, Sections 5, 6] or [16]). We do not give the details but rather note that the basic ingredient is a discretization of the reproducing formula (4.2). Instead of convolution relations for Y as in [6], the corresponding relations for $\mathcal{W}(Y)$ and $W(M, Y)$ stated in Theorems 3.3 and 3.4 are heavily used. Also Lemma 3.5 is needed, and the usual triangle inequality has to be replaced by the p-triangle inequality. ■

In certain situations one might be able to construct frame expansions as in (5.1) below on the level of the Hilbert space $\mathcal{H}$. The next theorem states that such expansions extend automatically from $\mathcal{H}$ to general coorbit spaces under certain assumptions. Its proof is a modification of the one in [13] again replacing convolution relations for Y by the corresponding ones for $\mathcal{W}(Y)$ and the usual triangle inequality by the p-triangle inequality.

THEOREM 5.2. *Let $g_r, \gamma_r \in \mathbb{B}(Y)$, $r = 1, \ldots, n$, and $X = (x_i)_{i\in I}$ be a well-spread set such that*

$$f = \sum_{r=1}^{n} \sum_{i\in I} \langle f, \pi(x_i)\gamma_r\rangle \pi(x_i) g_r \quad \text{for all } f \in \mathcal{H}. \tag{5.1}$$

Then expansion (5.1) *extends to all $f \in \mathcal{C}(Y)$ with norm convergence if the finite sequences are dense in Y_d and with weak* convergence in general. Moreover, $f \in (\mathcal{H}_v^1)^\urcorner$ is in $\mathcal{C}(Y)$ if and only if $(\langle f, \pi(x_i)\gamma_r\rangle)_{i\in I}$ lies in Y_d for each $r = 1, \ldots, n$, and*

$$\|((\langle f, \pi(x_i)\gamma_r\rangle)_{i\in I})_{r=1}^n \,|\, \textstyle\bigoplus_{r=1}^n Y_d\| \asymp \|f \,|\, \mathcal{C}(Y)\| \quad \text{for all } f \in \mathcal{C}(Y).$$

6. Characterizations of $\mathcal{C}(Y)$ via Y. The original definition of the coorbit spaces by Feichtinger and Gröchenig involves Y rather than $\mathcal{W}(Y)$. It is interesting to investigate what happens if we replace $\mathcal{W}(Y)$ by Y in our more general case. In order to distinguish clearly between the two spaces let us write $\mathsf{Co}\, Y = \{f \in (\mathcal{H}_v^1)^\urcorner : V_g f \in Y\}$ with natural norm $\|f \,|\, \mathsf{Co}\, Y\| = \|V_g f \,|\, Y\|$, and $\mathsf{Co}\, \mathcal{W}(Y) = \mathcal{C}(Y)$ as usual. It was already proven in [8] that in the classical Banach space case both spaces coincide:

THEOREM 6.1 (Theorem 8.3 in [8]). *Let Y be a solid Banach space of functions on $\mathcal{G}$ that is left and right translation invariant and continuously embedded into $L^1_{\mathrm{loc}}(\mathcal{G})$. Then $\mathsf{Co}\, Y = \mathsf{Co}\, \mathcal{W}(Y)$ with equivalent norms.*

In the general case of quasi-Banach spaces at least the inclusion $\mathsf{Co}\, \mathcal{W}(Y) \subset \mathsf{Co}\, Y$ holds since $\mathcal{W}(Y) \subset Y$. However, it seems doubtful that we can state results on the converse inclusion in the general abstract case. Moreover, it is even not clear whether $\mathsf{Co}\, Y$ is a complete space.

In special cases, however, one might be able to prove that $\|V_g f \,|\, \mathcal{W}(Y)\| \le C\|V_g f \,|\, Y\|$ for a very specific choice of g, by using methods that are not available in the abstract setting (like analyticity properties for instance): see e.g. Section 8. Then one may extend this inequality to more general analyzing vectors g as shown by the next result.

THEOREM 6.2. *Let Y be a left and right translation invariant solid p-normed quasi-Banach space and let v be the function defined in* (4.4)*. Assume that there exists a non-zero vector $g_0 \in \mathbb{B}(Y)$ and a constant $C > 0$ such that $V_{g_0} f \in Y$ if $V_{g_0} f \in \mathcal{W}(Y)$ and*

$$\|V_{g_0} f \,|\, \mathcal{W}(Y)\| \le C\|V_{g_0} f \,|\, Y\|$$

for all $f \in (\mathcal{H}_v^1)^\urcorner$. Let $g \in \mathbb{B}(Y) \setminus \{0\}$ be arbitrary. Then

$$\|V_g f \,|\, \mathcal{W}(Y)\| \asymp \|V_g f \,|\, Y\|$$

for all $f \in (\mathcal{H}_v^1)^\urcorner$ and $\mathsf{Co}\, \mathcal{W}(Y) = \mathsf{Co}\, Y = \{f \in (\mathcal{H}_v^1)^\urcorner, V_g f \in Y\}$. In particular, $\mathsf{Co}\, Y$ is complete.

Proof. Since $\mathcal{C}(Y)$ is independent of the choice of $g \in \mathbb{B}(Y) \setminus \{0\}$ (Theorem 4.4) we have $\|V_g f \,|\, \mathcal{W}(Y)\| \le C\|V_{g_0} f \,|\, \mathcal{W}(Y)\|$ for all $f \in (\mathcal{H}_v^1)^\urcorner$. Thus, it remains to prove that $\|V_{g_0} f \,|\, Y\| \le C\|V_g f \,|\, Y\|$ for all $f \in (\mathcal{H}_v^1)^\urcorner$. By the assumptions on g it follows from Theorem 5.1 that g_0 has a decomposition

$$g_0 = \sum_{i\in I} \lambda_i(g_0)\pi(x_i)g$$

with $(\lambda_i(g_0))_{i\in I} \in \ell_w^p = (L_w^p)_d$ and $\|(\lambda_i(g_0))_{i\in I} \,|\, \ell_w^p\| \asymp \|g_0 \,|\, \mathcal{C}(L_w^p)\|$, where w is the weight in (4.3). Hence, we obtain

$$V_{g_0} f(x) = \langle f, \pi(x)g_0\rangle = \Big\langle f, \pi(x)\sum_{i\in I} \lambda_i(g_0)\pi(x_i)g\Big\rangle = \sum_{i\in I} \overline{\lambda_i(g_0)} R_{x_i} V_g f(x).$$

By the p-triangle inequality this yields

$$\begin{aligned}\|V_{g_0} f \,|\, Y\|^p &= \Big\| \sum_{i\in I} \overline{\lambda_i(g_0)} R_{x_i} V_g f \,\Big|\, Y\Big\|^p \le \sum_{i\in I} |\lambda_i(g_0)|^p \|\!| R_{x_i} \,|\, Y \|\!|^p \|V_g f \,|\, Y\|^p \\ &\le C\|g_0 \,|\, \mathcal{C}(L_v^p)\|^p \, \|V_g f \,|\, Y\|^p\end{aligned}$$

for all $f \in (\mathcal{H}_v^1)^\urcorner$. The reverse inequality $\|V_g f \,|\, Y\| \le \|V_g f \,|\, \mathcal{W}(Y)\|$ is clear. ∎

7. Nonlinear approximation. Let us now discuss nonlinear approximation. Let $(x_i)_{i\in I}$ be some well-spread set and g such that $\{\pi(x_i)g\}_{i\in I}$ forms an atomic decomposition of the coorbit space we want to consider. We denote by

$$\sigma_n(f, \mathcal{C}(Y)) := \inf_{N\subset I,\, \#N\le n} \Big\| f - \sum_{i\in N} \lambda_i \pi(x_i) g \,\Big|\, \mathcal{C}(Y)\Big\|$$

the error of best n-term approximation in $\mathcal{C}(Y)$. Here, the infimum is also taken over all possible choices of coefficients λ_i. Our task is to find a class of elements for which this error has a certain decay when n tends to ∞.

To this end we consider coorbit spaces with respect to Lorentz spaces $L(p,\infty)$, also called weak L^p spaces. For some measurable function F on $\mathcal{G}$ let $\lambda_F(s) = |\{x : |F(x)| > s\}|$ be its distribution function, where $|\cdot|$ denotes the Haar measure of a set. Then the nonincreasing rearrangement of F is defined as $F^*(t) = \inf\{s : \lambda_F(s) \le t\}$. We let

$$\|F\|_{p,\infty}^* := \sup_{t>0} t^{1/p} F^*(t). \tag{7.1}$$

The Lorentz space $L(p,\infty)$ is defined as the collection of all F such that the quantity above is finite. If $p > 1$ then $L(p,\infty)$ is a Banach space, and for $p \le 1$ it is a quasi-Banach space. Moreover, in the latter case there exists an equivalent r-norm for any $r < p$. We note also that $L^p \subset L(p,\infty)$. For more information on Lorentz spaces we refer e.g. to [15].

By the properties of the Haar measure it is easily seen that all spaces $L(p,\infty)$ are left and right translation invariant. Thus, if m is a moderate function then also $L_m(p,\infty) = \{F \text{ measurable} : Fm \in L(p,\infty)\}$ with the quasi-norm $\|F \,|\, L_m(p,\infty)\| := \|Fm\|^*_{p,\infty}$ is left and right translation invariant. In particular, the Wiener amalgam spaces $\mathcal{W}(L_m(p,\infty))$ are well-defined. Further, if $\mathbb{B}(L_m(p,\infty))$ (see (4.5)) is nontrivial then also the coorbit space $\mathcal{C}(L_m(p,\infty))$ is well-defined.

It is not difficult to see that the sequence space $(L_m(p,\infty))_d(X)$ associated to a well-spread set $X = (x_i)_{i\in I}$ coincides with a Lorentz space $\ell_m(p,\infty)$ on the index set I. In particular, an equivalent quasi-norm on $(L_m(p,\infty))_d(X)$ is given by

$$\|(\lambda_i)_{i\in I} \,|\, \ell_m(p,\infty)\| = \sup_{n\in\mathbb{N}} n^{1/p}(\lambda m)^*(n) \tag{7.2}$$

where $(\lambda m)^*$ denotes the nonincreasing rearrangement of the sequence $(\lambda_i m(x_i))_{i\in I}$.

Theorem 7.1. *Let m be some w-moderate weight function on $\mathcal{G}$, let $0 < p < q \le \infty$ and define $\alpha = 1/p - 1/q > 0$. Let $(x_i)_{i\in I}$ be a well-spread set and $g \in \mathbb{B}(L_m(p,\infty)) \subset \mathbb{B}(L^q_m)$ such that $\{\pi(x_i)g\}_{i\in I}$ forms an atomic decomposition simultaneously of $\mathcal{C}(L_m(p,\infty))$ and $\mathcal{C}(L^q_m)$ (according to Theorem* 5.1). *Then for all $f \in \mathcal{C}(L_m(p,\infty))$,*

$$\sigma_n(f, \mathcal{C}(L^q_m)) \le C\|f \,|\, \mathcal{C}(L_m(p,\infty))\| n^{-\alpha}. \tag{7.3}$$

Proof. Let $f = \sum_{i\in I} \lambda_i(f)\pi(x_i)g$ be an expansion of $f \in \mathcal{C}(L_m(p,\infty))$ in terms of the atomic decomposition. By Theorem 5.1 we have $(\lambda m)^*_k \le C\|f \,|\, \mathcal{C}(L_m(p,\infty))\| k^{-1/p}$. Let $\tau : \mathbb{N} \to I$ be a bijection that realizes the nonincreasing rearrangement, i.e., $\lambda_{\tau(k)} m(x_{\tau(k)}) = (\lambda m)^*_k$. Moreover, $(L^q_m)_d = \ell^q_m(I)$, and $\|(\lambda_i(f))_{i\in I} \,|\, \ell^q_m(I)\|$ forms an equivalent norm on $\mathcal{C}(L^q_v)$ once again by Theorem 5.1. We obtain

$$\begin{aligned}
\sigma_n(f, \mathcal{C}(L^q_m)) &\le \Big\| f - \sum_{k=1}^{n} \lambda_{\tau(k)}\pi(x_{\tau(k)})g \,\Big|\, \mathcal{C}(L^q_m)\Big\| \\
&= \Big\| \sum_{k=n+1}^{\infty} \lambda_{\tau(k)}\pi(x_{\tau(k)})g \,\Big|\, \mathcal{C}(L^q_m)\Big\| \le C\Big(\sum_{k=n+1}^{\infty} ((\lambda m)^*_k)^q\Big)^{1/q} \\
&\le C\|f \,|\, \mathcal{C}(L_m(p,\infty))\| \Big(\sum_{k=n+1}^{\infty} k^{-q/p}\Big)^{1/q} \le C\|f \,|\, \mathcal{C}(L_m(p,\infty))\| n^{-\alpha}.
\end{aligned}$$

This completes the proof. ∎

Remark 7.2.

(a) The obvious embedding $\mathcal{C}(L^p_m) \subset \mathcal{C}(L_m(p,\infty))$ implies that we also have $\sigma_n(f, \mathcal{C}(L^q_m)) \le Cn^{1/q-1/p}$ for all $f \in \mathcal{C}(L^p_m)$ if $p < q$.

(b) In order to have a very fast decay of $\sigma_n(f, \mathcal{C}(L^q_m))$ one obviously has to take p very small in the theorem above, in particular, $p \le 1$.

Clearly, $\mathcal{C}(L_m(p,\infty))$ is no longer a Banach space in this case, but only a quasi-Banach space. So it is very natural to treat also the case of quasi-Banach spaces when dealing with problems in nonlinear approximation. This was actually one of the motivations for this paper.

8. Modulation spaces. Let $\mathbb{H}_d := \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ denote the (reduced) Heisenberg group with group law $(x,\omega,\tau)(x',\omega',\tau') = (x+x',\omega+\omega', \tau\tau' e^{\pi i(x'\cdot\omega - x\cdot\omega')})$. It is an IN group and thus unimodular. Its Haar measure is given by

$$\int_{\mathbb{H}_d} f(h)\,dh = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^1 f(x,\omega,e^{2\pi i t})\,dt\,d\omega\,dx.$$

We denote by $T_x f(t) := f(t-x)$ and $M_\omega f(t) = e^{2\pi i\omega\cdot t} f(t)$, $x,\omega,t \in \mathbb{R}^d$, the translation and modulation operators on $L^2(\mathbb{R}^d)$. Then the *Schrödinger representation* ϱ is defined by

$$\varrho(x,\omega,\tau) := \tau e^{\pi i x\cdot\omega} T_x M_\omega = \tau e^{-\pi i x\cdot\omega} M_\omega T_x.$$

It is well-known that this is an irreducible unitary and square-integrable representation of $\mathbb{H}_d$. The corresponding voice transform is essentially the short time Fourier transform:

$$\begin{aligned}(8.1)\qquad & V_g f(x,\omega,\tau)\\ &= \langle f, \varrho(x,\omega,\tau)g\rangle_{L^2(\mathbb{R}^d)} = \overline{\tau}\int_{\mathbb{R}^d} f(t)\overline{e^{-\pi i x\cdot\omega} M_\omega T_x g(t)}\,dt\\ &= \overline{\tau} e^{\pi i x\cdot\omega}\int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-2\pi i t\cdot\omega}\,dt = \overline{\tau} e^{\pi i x\cdot\omega}\,\mathrm{STFT}_g f(x,\omega).\end{aligned}$$

Let us now introduce the modulation spaces on $\mathbb{R}^d$. We consider non-negative continuous weight functions m on $\mathbb{R}^d \times \mathbb{R}^d$ that satisfy

$$m(x+y,\omega+\xi) \le C(1+|x|^2+|\omega|^2)^{a/2} m(y,\xi), \quad (x,\omega),(y,\xi) \in \mathbb{R}^d \times \mathbb{R}^d,$$

for some constants $C > 0$, $a \ge 0$. This means that m is a w-moderate function for $w(x,\omega) = (1+|x|^2+|\omega|^2)^{a/2}$ (see also [12, Chapter 11.1]). Additionally, we require m to be symmetric, i.e., $m(-x,-\omega) = m(x,\omega)$. A typical choice is $m_s(x,\omega) = (1+|\omega|)^s$, $s \in \mathbb{R}$. For $0 < p,q \le \infty$ and m as above we introduce $L_m^{p,q} := L_m^{p,q}(\mathbb{R}^{2d}) := \{F \text{ measurable} : \|F \mid L_m^{p,q}\| < \infty\}$ with quasi-norm

$$\|F \mid L_m^{p,q}\| := \Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} |F(x,\omega)|^p m(x,\omega)^p\,dx\Big)^{q/p} d\omega\Big)^{1/q}.$$

This expression is an r-norm with $r := \min\{1,p,q\}$.

Let g be some nonzero Schwartz function on $\mathbb{R}^d$. The short time Fourier transform STFT_g extends to the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions in a natural way. Given $0 < p, q \leq \infty$ and m as above, the *modulation space* is defined as

$$M_m^{p,q} := \{f \in \mathcal{S}'(\mathbb{R}^d) : \|\mathrm{STFT}_g\, f \,|\, L_m^{p,q}\| < \infty\}$$

with quasi-norm $\|f \,|\, M_m^{p,q}\| = \|\mathrm{STFT}_g\, f \,|\, L_m^{p,q}\|$. Since by (8.1), $|V_g f(x, \omega, \tau)| = |\mathrm{STFT}_g\, f(x, \omega)|$, we can identify the modulation spaces with coorbit spaces,

$$M_m^{p,q}(\mathbb{R}^d) = \mathsf{Co}\, L_m^{p,q}(\mathbb{H}_d) = \{f \in \mathcal{S}' : V_g f \in L_m^{p,q}\},$$

where m and $L_m^{p,q}$ are extended to $\mathbb{H}_d$ in an obvious way, e.g. $m(x, \omega, \tau) = m(x, \omega)$. However, at the moment we do not know yet whether $\mathsf{Co}\, L_m^{p,q}$ coincides with

$$\mathcal{C}(L_m^{p,q}) = \{f \in \mathcal{S}'(\mathbb{R}^d) : V_g f \in \mathcal{W}(L_m^{p,q})\},$$

if $p < 1$ or $q < 1$. It is not even clear yet whether $M_m^{p,q}$ is complete. We will use Theorem 6.2 and a result from [10] to clarify this problem. Let us first investigate the space $\mathbb{B}(L_m^{p,q})$ (see Definition 4.2). It is indeed not hard to see that $\mathbb{B}_w^r \subset \mathbb{B}(L_m^{p,q})$ with $r = \min\{1, p, q\}$, and Theorem 4.5 yields

$$\mathbb{B}_w^r = \mathcal{C}(L_w^r).$$

Let $g_0(t) = e^{-\pi |t|^2}$ be a Gaussian. Using the relation of STFT_{g_0} to the Bargmann transform, Galperin and Samarah proved that

$$\|V_{g_0} f \,|\, \mathcal{W}(L_m^{p,q})\| \leq C \|V_{g_0} f \,|\, L_m^{p,q}\|$$

for all $f \in M_m^{p,q}$ [10, Lemma 3.2]. Thus, it follows from Theorem 6.2 that

$$\mathcal{C}(L_m^{p,q}) = M_m^{p,q},$$

and the latter is complete. It seems that the completeness of $M_m^{p,q}$ for $p < 1$ or $q < 1$ was not stated in [10] or elsewhere in the literature although its proof is somehow hidden in [10].

The abstract discretization Theorem 5.1 yields the following result for Gabor type atomic decompositions of modulation spaces.

THEOREM 8.1. *Let $0 < p_0 \leq 1$ and w be some symmetric submultiplicative weight function on $\mathbb{R}^d \times \mathbb{R}^d$ with polynomial growth. Let $g \in M_w^{p_0}$. Then there exist constants $a, b > 0$ such that*

$$\{M_{bj} T_{ak} g : k, j \in \mathbb{Z}^d\}$$

forms an atomic decomposition for all modulation spaces $M_m^{p,q}$ with $p_0 \leq p, q \leq \infty$ and m being a w-moderate weight. This means that there exist *functionals* $\lambda_{k,j}$, $k,j \in \mathbb{Z}^d$, *on* $M^\infty_{1/w}$ $(\subset \mathcal{S}')$ *such that*

(a) *any* $f \in M^{p,q}_m$ *has the series expansion* $f = \sum_{k,j\in\mathbb{Z}^d} \lambda_{k,j}(f) M_{bj} T_{ak} g$;

(b) *a distribution* $f \in M^\infty_{1/w}$ *belongs to* $M^{p,q}_m$ *if and only if* $(\lambda_{k,j}(f))_{k,j\in\mathbb{Z}^d}$ *belongs to* $\ell^{p,q}_m(\mathbb{Z}^{2d})$, *and we have the quasi-norm equivalence*

$$\|f \mid M^{p,q}_m\| \asymp \Big(\sum_{j\in\mathbb{Z}^d}\Big(\sum_{k\in\mathbb{Z}^d} |\lambda_{k,j}(f)|^p m(ak,bj)^p\Big)^{q/p}\Big)^{1/q}$$
$$=: \|(\lambda_{k,j}(f)) \mid \ell^{p,q}_m(\mathbb{Z}^{2d})\|.$$

We remark that the abstract Theorem 5.1 allows extending the previous result also to irregular Gabor frames on $M^{p,q}_m$.

Theorem 8.1 indicates that the modulation spaces $M^{p_0}_w$ with $0 < p_0 \le 1$ are the correct window classes for time-frequency analysis on $M^{p,q}_m$. This was already conjectured in [10]. Galperin and Samarah also conjectured that whenever $V_g f \in L^p_v$ then $f \in M^p_v$ and $g \in M^p_v$ [10, Conjecture 12]. Theorem 4.6 leads to a weak version of this conjecture.

THEOREM 8.2. *Let* $f, g \in L^2(\mathbb{R}^d)$ *and* $0 < p \le 1$. *Let* v *be a symmetric submultiplicative weight function. If* $V_g f \in \mathcal{W}(L^p_v)$, *then* $g \in M^p_v$ *and* $f \in M^p_v$.

The remaining question is whether $V_g f \in L^p_v$ already implies that $V_g f \in \mathcal{W}(L^p_v)$.

Let us also apply Theorem 5.2 to our situation.

THEOREM 8.3. *Let* $g \in \mathcal{S}(\mathbb{R}^d)$ *and* $a, b > 0$ *be such that*

$$\{M_{bj} T_{ak} g : j,k \in \mathbb{Z}^d\} \tag{8.2}$$

forms a Gabor frame for $L^2(\mathbb{R}^d)$. *Then its canonical dual* γ *is also contained in* $\mathcal{S}(\mathbb{R}^d)$, *and any* $f \in M^{p,q}_m$, $0 < p, q \le \infty$, *has a decomposition*

$$f = \sum_{j,k\in\mathbb{Z}^d} \langle f, M_{bj} T_{ak} g\rangle M_{bj} T_{ak} \gamma$$

with $\|f \mid M^{p,q}_m\| \asymp \|(\langle f, M_{bj} T_{ak} g\rangle)_{j,k\in\mathbb{Z}^d} \mid \ell^{p,q}_m(\mathbb{Z}^{2d})\|$.

Proof. Since (8.2) forms a Gabor frame with dual window γ, any f in $L^2(\mathbb{R}^d)$ has a decomposition

$$f = \sum_{j,k\in\mathbb{Z}^d} \langle f, M_{bj} T_{ak} g\rangle M_{bj} T_{ak} \gamma.$$

It was shown in [12, Corollary 13.5.4] that also the dual window γ is contained in $\mathcal{S}(\mathbb{R}^d)$. Since $\mathcal{S}(\mathbb{R}^d) \subset M^{p,q}_m$ for all $0 < p, q \le \infty$ and all w-moderate weights m with w having polynomial growth, we have $g, \gamma \in \mathbb{B}(L^{p,q}_m) = M^r_w$ with $r = \min\{1, p, q\}$. Clearly, the set $\{(ak, bj) : k, j \in \mathbb{Z}^d\}$ is well-spread in $\mathbb{H}^d$. Thus, the assertion follows from Theorem 5.2. ∎

Of course, one can also apply Theorem 7.1 to best n-term approximations with Gabor frames; cf. also [14] for approximation with local Fourier bases. We leave this straightforward task to the interested reader.

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