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New fixed point free nonexpansive maps on weakly compact, convex subsets of $L^1[0,1]$

by

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Abstract. We show that every subset of $L^1[0,1]$ that contains the nontrivial intersection of an order interval and finitely many hyperplanes fails to have the fixed point property for nonexpansive mappings.

1. Introduction and preliminaries. In 1981, D. Alspach [1] gave the first example of a weakly compact, convex subset of a Banach space that fails the fixed point property for nonexpansive mappings. A modification of Alspach's example by R. Sine [11] was used in [5] to show that every closed, bounded, convex subset of $L^1[0,1]$ that contains a nontrivial order interval fails the fixed point property for nonexpansive mappings. In this paper, Alspach's example is used to show that every subset of $L^1[0,1]$ that contains the nontrivial intersection of an order interval and finitely many hyperplanes fails to have the fixed point property for nonexpansive mappings. This generalizes the result in [5] and, unlike the previous theorem, also includes Alspach's example.

As usual, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{Z}$ is the set of all integers and $\mathbb{R}$ denotes the set of all real numbers. A set K has the *fixed point property for nonexpansive mappings* if every nonexpansive map of K into itself has a fixed point. We refer the reader to the texts of Diestel [3] and Goebel and Kirk [6] for any unexplained terminology.

2. A new fixed point free mapping theorem in $L^1[0,1]$. Recall Alspach's construction. Let $C := \{f \in L^1[0,1] : 0 \le f(t) \le 1$ for all $t \in [0,1]\}$. Now define $T : C \to C$ by setting, for all $f \in C$,

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$$(Tf)(t) := \begin{cases} 2f(2t) \wedge 1 & \text{if } 0 \le t < 1/2, \\ (2f(2t-1)-1) \vee 0 & \text{if } 1/2 \le t < 1. \end{cases}$$

Alspach showed that the mapping T is an isometry on C which has only two fixed points: 0 and $\chi_{[0,1]}$. Alspach also showed that T maps the closed convex subset $C_0 := \{f \in C : \int_{[0,1]} f\,dm = 1/2\}$ of C into itself. This proves that T is a fixed point free isometry on the weakly compact, convex set C_0. Here, m denotes Lebesgue measure on the σ-algebra $\mathcal{M}$ of Lebesgue measurable subsets of $[0,1]$.

Sine [11] considered a modification of Alspach's example by defining $S : C \to C$ by $S(f) := \chi_{[0,1]} - Tf$ for all $f \in C$. Note that S is nonexpansive on C, and that the two fixed points of T in C, namely 0 and $\chi_{[0,1]}$, are not fixed points of S. In fact, Sine proved that S has no fixed points in the set C.

For our purpose, it will be useful to construct a somewhat different fixed point free nonexpansive mapping on C. To this end, we define a mapping $\varDelta$ on C by setting, for all $f \in C$,

$$(\varDelta f)(t) := \begin{cases} f(2t) & \text{if } 0 \le t < 1/2, \\ 1 - f(2t-1) & \text{if } 1/2 \le t < 1. \end{cases}$$

Two key properties of $\varDelta$ that are easy to check are that $\varDelta$ maps C into C_0 and $\varDelta$ is an $L^1[0,1]$-isometry on C.

Lemma 1. *The function $T\varDelta : C \to C_0 \subseteq C$ is fixed point free on C.*

Proof. A straightforward calculation shows that for all $f \in C$,

$$(T\varDelta f)(t) := \begin{cases} 2f(4t) \wedge 1 & \text{if } 0 \le t < 1/4, \\ 2(1 - f(4t-1)) \wedge 1 & \text{if } 1/4 \le t < 1/2, \\ (2f(4t-2)-1) \vee 0 & \text{if } 1/2 \le t < 3/4, \\ (1 - 2f(4t-3)) \vee 0 & \text{if } 3/4 \le t < 1. \end{cases}$$

Now, assume, to get a contradiction, that there exists $f \in C$ such that $T\varDelta(f) = f$. Then clearly, $f = T\varDelta(f) \in C_0$. Define sets

$$A := [f = 0] = \{t \in [0,1] : f(t) = 0\}, \quad B := [f = 1],$$
$$D := [0 < f < 1/2], \quad E := [1/2 < f < 1], \quad F := [f = 1/2].$$

Then, with $\dot\cup$ denoting disjoint union and all set equalities modulo sets of measure zero, it follows that

$$A = \frac{1}{4}A \mathbin{\dot\cup} \left(\frac{1}{4} + \frac{1}{4}B\right) \mathbin{\dot\cup} \left(\frac{1}{2} + \frac{1}{4}A\right) \mathbin{\dot\cup} \left(\frac{1}{2} + \frac{1}{4}D\right)$$
$$\mathbin{\dot\cup} \left(\frac{1}{2} + \frac{1}{4}F\right) \mathbin{\dot\cup} \left(\frac{3}{4} + \frac{1}{4}B\right) \mathbin{\dot\cup} \left(\frac{3}{4} + \frac{1}{4}E\right) \mathbin{\dot\cup} \left(\frac{3}{4} + \frac{1}{4}F\right)$$

and

$$B = \frac{1}{4}\,B \mathbin{\dot\cup} \frac{1}{4}\,E \mathbin{\dot\cup} \frac{1}{4}\,F \mathbin{\dot\cup} \left(\frac{1}{4} + \frac{1}{4}\,A\right) \mathbin{\dot\cup} \left(\frac{1}{4} + \frac{1}{4}\,D\right)$$
$$\mathbin{\dot\cup} \left(\frac{1}{4} + \frac{1}{4}\,F\right) \mathbin{\dot\cup} \left(\frac{1}{2} + \frac{1}{4}\,B\right) \mathbin{\dot\cup} \left(\frac{3}{4} + \frac{1}{4}\,A\right).$$

From these two equalities, it follows that

$$2(m(A) - m(B)) = m(D) + m(E) + 2m(F) = 2(m(B) - m(A)).$$

Thus, $m(A) = m(B)$ and $m(D) = m(E) = m(F) = 0$ and

$$f = \chi_B \text{ a.e.} \quad \text{and} \quad m(B) = 1/2.$$

Moreover, ignoring sets of measure zero and substituting the above equations for A and B into the right-hand side of the equation for B shows, upon iteration, that for all $\nu \in \mathbb{N}$ and all $j \in \{0, \ldots, 4^\nu - 1\}$,

$$B \cap \left[\frac{j}{4^\nu}, \frac{j+1}{4^\nu}\right) = \frac{j}{4^\nu} + \frac{1}{4^\nu}\,S_j^\nu,$$

where each $S_j^\nu \in \{A, B\}$. It follows that

$$m\left(B \cap \left[\frac{j}{4^\nu}, \frac{j+1}{4^\nu}\right)\right) = \frac{1}{4^\nu}\,m(S_j^\nu) = \frac{1}{2 \cdot 4^\nu}.$$

Lebesgue point ideas (see, for example, [10, Theorem 7.10]) imply that the sequence of functions $(f_N)_{N\in\mathbb{N}}$ on $[0, 1]$ given by

$$f_N(t) := \sum_{j=0}^{N-1} \Big(N \int\limits_{j/N}^{(j+1)/N} f\,dm\Big) \chi_{[j/N,(j+1)/N)}(t)$$

converges to $f(t) \in \{0, 1\}$ for almost all $t \in [0, 1]$. However, for each $t \in [0, 1)$ and for all $\nu \in \mathbb{N}$, letting $N := 4^\nu$ gives

$$f_N(t) = \sum_{j=0}^{N-1} Nm\left(B \cap \left[\frac{j}{N}, \frac{j+1}{N}\right)\right) \chi_{[j/N,(j+1)/N)}(t)$$
$$= \frac{1}{2N}\,N \sum_{j=0}^{N-1} \chi_{[j/N,(j+1)/N)}(t) = \frac{1}{2}.$$

This contradicts the fact that $(f_N)_{N\in\mathbb{N}}$ converges to a $\{0, 1\}$-valued function. ∎

Our main result shows how Alspach's map, in conjunction with the map Δ, can be used to give new examples of fixed point free nonexpansive self-maps of weakly compact convex sets in $L^1[0, 1]$. In particular, we show that every subset of $L^1[0, 1]$ that contains the (nontrivial) intersection of an order interval and finitely many hyperplanes fails to have the fixed point property for nonexpansive mappings. To facilitate the proof, we introduce some notation.

Denote by $\mathcal{I}_w$ the linear functional on $L^1[0,1]$ determined by $w \in L^\infty[0,1]$; that is, $\mathcal{I}_w(f) := \int_{[0,1]} fw\, dm$ for all $f \in L^1[0,1]$. Also, define the order interval $[h,g]$ in $L^1[0,1]$ by

$$[h,g] := \{f \in L^1[0,1] : h \le f \le g \text{ a.e.}\}$$

for all $h, g \in L^1[0,1]$ with $h \le g$ a.e. Given an order interval $[h,g]$ in $L^1[0,1]$, a natural number n, and bounded, measurable functions $w_1, \ldots, w_n$, the $\mathbb{R}^n$-valued set function $\mathcal{G} : \mathcal{M} \to \mathbb{R}^n$ defined by

$$\mathcal{G}(A) = \Big(\int (g\chi_A + h\chi_{[0,1]\setminus A})w_1\, dm, \ldots, \int (g\chi_A + h\chi_{[0,1]\setminus A})w_n\, dm\Big)$$

will be called the set function determined by the order interval $[h,g]$ and the bounded, measurable functions $w_1, \ldots, w_n$. It is useful to note that $\mathcal{G}$ is the translate of the vector-valued measure $(\int_A (g-h)w_j\, dm)_{j=1}^n$ by the fixed vector $(\int hw_j\, dm)_{j=1}^n$. We remark that, using Lyapunov's theorem and Lemma IX.3(c) in [4], it is straightforward to check that $\mathcal{G}(\mathcal{M}) = \{(\beta_1, \ldots, \beta_n) \in \mathbb{R}^n : [h,g] \cap \bigcap_{j=1}^n \mathcal{I}_{w_j}^{-1}(\beta_j) \ne \emptyset\}$.

The following theorem is an extension of [5, Theorem 3.2] that includes Alspach's example.

Theorem 2. *Let $[h,g]$ be a nontrivial order interval in $L^1[0,1]$; let $n \in \mathbb{N}$; and let $w_1, \ldots, w_n$ be in $L^\infty[0,1]$. Let $\mathcal{G}$ be the set function determined by $[h,g]$ and $w_1, \ldots, w_n$. If $(\alpha_1, \ldots, \alpha_n)$ is a point in the interior $\mathcal{G}(\mathcal{M})^\circ$ of the range of $\mathcal{G}$ and K is a closed, bounded, convex subset of $L^1[0,1]$ satisfying*

$$K \supseteq [h,g] \cap \mathcal{I}_{w_1}^{-1}(\alpha_1) \cap \cdots \cap \mathcal{I}_{w_n}^{-1}(\alpha_n),$$

then K fails the fixed point property for nonexpansive mappings.

In the simpler setting where $n = 1$ and $w_1 \ge 0$, Theorem 2 becomes:

Theorem 3. *Let $\alpha \in \mathbb{R}$ and let h and g be functions in $L^1[0,1]$ satisfying: $h \le g$ a.e., h is not equivalent to g, and $\mathcal{I}_w(h) < \alpha < \mathcal{I}_w(g)$ for some nonnegative, bounded, measurable function w. Let K be a nonempty, closed, bounded, convex subset of $L^1[0,1]$ such that*

$$K \supseteq [h,g] \cap \mathcal{I}_w^{-1}(\alpha).$$

Then K fails the fixed point property for nonexpansive mappings.

Since the appearance of the set function $\mathcal{G}$ in Theorem 2 tends to complicate the appearance of the theorem, before giving its proof, we provide a brief sketch of the ideas that will be used. To begin with, consider the simpler setting in Theorem 3. Assuming that $h = 0$, we can choose $c > 0$ and a subset E of $[0,1]$ with positive measure such that $c\chi_E \le g$. For convenience, assume that $c = 1$. If $f \in K$, it would be nice if we could nonexpansively project f into $[0,\chi_E]$ and then use an analogue of Alspach's map on E to get a fixed point free map from K into itself. The map $R(f)=|f|\wedge\chi_E$ is a nonexpansive retract of K onto $[0,\chi_E]$ but, unfortunately, $R(f)$ may fail to lie in an analogue of Alspach's set C_0 or even in the set K.

In order to account for these failures, we take an analogue $\widetilde{T}$ of Alspach's map and an analogue $\widetilde{\Delta}$ of the map Δ defined on functions that are measurable with respect to a certain σ-algebra $\mathcal{A}$ of subsets of E. We then find that $\int_E \widetilde{T}\widetilde{\Delta}E^{\mathcal{A}}R(f)w\,dm=\frac{1}{2}\int_E w\,dm$ where $E^{\mathcal{A}}$ is the conditional expectation operator with respect to $\mathcal{A}$. However, the value $\frac{1}{2}\int_E w\,dm$ may fail to equal α and we want a map whose range lies in $[0,g]\cap\mathcal{I}_w^{-1}(\alpha)$ and, hence, in K. In order to modify the map $\widetilde{T}\widetilde{\Delta}E^{\mathcal{A}}R$ so that its range is in $[0,g]\cap\mathcal{I}_w^{-1}(\alpha)$, we take the set E small enough so that $\frac{1}{2}\int_E w\,dm<\alpha$ and $\int_{[0,1]\setminus E} gw\,dm>\alpha-\frac{1}{2}\int_E w\,dm$. We can then choose a measurable set $G\subseteq[0,1]\setminus E$ so that $\int_G gw\,dm=\alpha-\frac{1}{2}\int_E w\,dm$, the difference between the value α that we want and the value $\frac{1}{2}\int_E w\,dm$ that the integration provided. The map $U:K\to[0,g]\cap\mathcal{I}_w^{-1}(\alpha)$ defined by $U(f)=g\chi_G+\widetilde{T}\widetilde{\Delta}E^{\mathcal{A}}R(f)$ will then be a nonexpansive map of K into itself and we will see that if U has a fixed point, so will $T\Delta$, a contradiction to Lemma 1. Thus K will fail to have the fixed point property.

For the general case when $n>1$, the set function $\mathcal{G}$, in conjunction with Lyapunov's theorem, is used to find a set E and then a single set G that simultaneously makes up the shortfalls $\alpha_j-\frac{1}{2}\int_E w_j\,dm$, $j=1,\dots,n$, needed to ensure that the resulting map sends K into K.

Proof of Theorem 2. By translating by $-h$, relabeling $K-h$ as K, $g-h$ as g and $\alpha_j-\mathcal{I}_{w_j}(h)$ as α_j for $j=1,\dots,n$, there is no loss of generality in assuming that $h=0$ and $g\ge 0$ a.e. with g nontrivial. Then $\mathcal{G}:\mathcal{M}\to\mathbb{R}^n$ is the m-continuous, countably additive vector measure defined by $\mathcal{G}(A)=(\int_A gw_1\,dm,\dots,\int_A gw_n\,dm)$. Since $(\alpha_1,\dots,\alpha_n)$ is in the range of $\mathcal{G}$,

$$K\supseteq[0,g]\cap\mathcal{I}_{w_1}^{-1}(\alpha_1)\cap\dots\cap\mathcal{I}_{w_n}^{-1}(\alpha_n)\ne\emptyset.$$

Moreover, since $(\alpha_1,\dots,\alpha_n)$ lies in the interior of the range of $\mathcal{G}$, it is easy to see that the set $\{w_1,\dots,w_n\}$ is a linearly independent set of functions in $L^\infty[0,1]$. Thus $w_1\ne 0$. In fact, w_1 is not identically 0 on the set $[g>0]$. Indeed, if w_1 vanishes almost everywhere on $[g>0]$, then, for every $A\in\mathcal{M}$, $\mathcal{G}(A)=(0,\int_A gw_2\,dm,\dots,\int_A gw_n\,dm)$ and $\mathcal{G}(\mathcal{M})^\circ=\emptyset$, a contradiction.

Let E_1' be a measurable subset of $[g>0]$ such that $m(E_1')>0$ and, without loss of generality, $w_1(t)>0$ for each $t\in E_1'$. Then there exists a real number $c>0$ and a measurable set $E_1''\subseteq E_1'$ with $m(E_1'')>0$ such that $g\ge c\chi_{E_1''}$. By rescaling K and $\alpha_1,\dots,\alpha_n$ by $1/c$, there is no loss of generality in assuming that $c=1$. Let E_1 denote this rescaled E_1''. Thus E_1 is a subset of $[g\ge 1]$ with $m(E_1)>0$ and $w_1(t)>0$ for each $t\in E_1$.

Now consider w_2. If w_2 vanishes almost everywhere on E_1, set $E_2 := E_1$. Otherwise there exists a subset E_2 of E_1 with $m(E_2) > 0$ and either $w_2(t) > 0$ for each $t \in E_2$ or $w_2(t) < 0$ for each $t \in E_2$. Repeating this process for $j = 3, \dots, n$ gives a decreasing sequence of sets $E_1 \supseteq \dots \supseteq E_n$ such that, if $E' := E_n$, then $w_1(t) > 0$ for each $t \in E'$ and, for each $j = 2, \dots, n$, one of the following holds: $w_j(t) = 0$ for each $t \in E'$; $w_j(t) > 0$ for each $t \in E'$; or $w_j(t) < 0$ for each $t \in E'$.

Let $J := \{j \in \{1, \dots, n\} : w_j(t) > 0 \text{ for each } t \in E' \text{ or } w_j(t) < 0 \text{ for each } t \in E'\}$. Obviously $1 \in J$.

Since $(\alpha_1, \dots, \alpha_n) \in \mathcal{G}(\mathcal{M})^\circ$, there exists $0 < \delta < 1/2$ so that the closed ball $B((\alpha_1, \dots, \alpha_n), \delta) \subseteq \mathcal{G}(\mathcal{M})$. By compactness of the sphere $S := S((\alpha_1, \dots, \alpha_n), \delta/2)$, there exists a finite set $\{y_1, \dots, y_N\}$ in S such that $\{y_1, \dots, y_N\}$ is a $\delta^2/4$-net for S. Since this set lies in the range of $\mathcal{G}$, for each $i = 1, \dots, N$ there exists a measurable set A_i in $[0,1]$ with $y_i = \mathcal{G}(A_i)$.

Let $M := \sup_{j=1,\dots,n} \|w_j\|_\infty$. Choose a measurable subset E of E' satisfying $0 < m(E) < \delta/2M\sqrt{n}$ and $|\mathcal{G}|(E) < \delta^2/4$. This is possible since the measure space is nonatomic and the variation $|\mathcal{G}|$ of the vector measure $\mathcal{G}$ is m-continuous.

For $i = 1, \dots, N$, define $z_i = \mathcal{G}(A_i \setminus E)$ in $(\mathbb{R}^n, \|\cdot\|_2)$. Then, for $i = 1, \dots, N$,

$$\|y_i - z_i\|_2 = \|\mathcal{G}(A_i) - \mathcal{G}(A_i \setminus E)\|_2 = \|\mathcal{G}(A_i \cap E)\|_2 \le |\mathcal{G}|(E) < \delta^2/4.$$

A quick application of the triangle inequality shows that

$$(1) \qquad S = S((\alpha_1, \dots, \alpha_n), \delta/2) \subseteq \bigcup_{i=1}^{N} \big(z_i + B((0, \dots, 0), \delta^2/2)\big).$$

Claim. $B((\alpha_1, \dots, \alpha_n), \delta/4) \subseteq \operatorname{co}\{z_1, \dots, z_N\}$.

An argument proving this claim can be found in [7, Lemma 2.2]. Since these notes are unpublished, we include a short proof. Taking convex hulls in (1) yields

$$B((\alpha_1, \dots, \alpha_n), \delta/2) \subseteq \operatorname{co}\{z_1, \dots, z_N\} + B((0, \dots, 0), \delta^2/2)$$

or, equivalently, with $\vec{\alpha} := (\alpha_1, \dots, \alpha_n)$ and $\vec{0} := (0, \dots, 0)$,

$$B(\vec{0}, \delta/2) \subseteq \operatorname{co}\{z_1 - \vec{\alpha}, \dots, z_N - \vec{\alpha}\} + \delta \cdot B(\vec{0}, \delta/2).$$

Then, if $u_0 \in B(\vec{0}, \delta/2)$, there exist $b_k \in \operatorname{co}\{z_1 - \vec{\alpha}, \dots, z_N - \vec{\alpha}\}$ and $u_k \in B(\vec{0}, \delta/2)$ such that

$$u_{k-1} = b_k + \delta u_k$$

for all $k \in \mathbb{N}$. Note that $\delta^k u_k \to 0$,

$$\frac{1-\delta}{1-\delta^l} \sum_{k=1}^{l} \delta^{k-1} b_k = \sum_{k=1}^{l} \frac{\delta^{k-1}}{1 + \delta + \dots + \delta^{l-1}}\, b_k \in \operatorname{co}\{z_1 - \vec{\alpha}, \dots, z_N - \vec{\alpha}\},$$

and $\mathrm{co}\{z_1 - \vec{\alpha}, \ldots, z_N - \vec{\alpha}\}$ is closed. It follows that

$$u_0 = \sum_{k=1}^{\infty} \delta^{k-1} b_k \in \frac{1}{1-\delta}\, \mathrm{co}\{z_1 - \vec{\alpha}, \ldots, z_N - \vec{\alpha}\}.$$

Therefore $(1-\delta)B(\vec{0}, \delta/2) \subseteq \mathrm{co}\{z_1 - \vec{\alpha}, \ldots, z_N - \vec{\alpha}\}$ or, equivalently,

$$B(\vec{\alpha}, (1-\delta)\delta/2) \subseteq \mathrm{co}\{z_1, \ldots, z_N\}.$$

Since $0 < \delta < 1/2$, the Claim follows.

Note also that the vector $\big(\alpha_1 - \frac{1}{2}\int_E w_1\, dm, \ldots, \alpha_n - \frac{1}{2}\int_E w_n\, dm\big)$ is in $B((\alpha_1, \ldots, \alpha_n), \delta/4)$. This follows from the choice of E and

$$\bigg\|\bigg(\frac{1}{2}\int_E w_1\, dm, \ldots, \frac{1}{2}\int_E w_n\, dm\bigg)\bigg\|_2 \le \frac{1}{2} M\sqrt{n}\, m(E) \le \frac{\delta}{4}.$$

Then, by the Claim, the choice of the z_j's in the range $\mathcal{G}(\mathcal{M}_{[0,1]\setminus E})$ of the nonatomic, countably additive vector measure $\mathcal{G}$, and Lyapunov's theorem [4, p. 264], it follows that

$$\bigg(\alpha_1 - \frac{1}{2}\int_E w_1\, dm, \ldots, \alpha_n - \frac{1}{2}\int_E w_n\, dm\bigg) \in \mathcal{G}(\mathcal{M}_{[0,1]\setminus E})$$

where $\mathcal{M}_{[0,1]\setminus E}$ is the σ-algebra of Lebesgue measurable subsets of $[0,1] \setminus E$. Therefore there exists a measurable set $G \subseteq [0,1] \setminus E$ such that

$$\mathcal{G}(G) = \bigg(\alpha_1 - \frac{1}{2}\int_E w_1\, dm, \ldots, \alpha_n - \frac{1}{2}\int_E w_n\, dm\bigg). \tag{2}$$

Now, consider the sets E and G as chosen above, and let $\nu_j(A) := \int_A w_j\, dm$ for $j = 1, \ldots, n$.

Claim. *There exist measurable subsets $(E_{i,k})_{i\in\mathbb{N}\cup\{0\},\, k=1,\ldots,2^i}$ of E which are "dyadic sets" for the measures m and ν_j for all $j = 1, \ldots, n$; that is, $E_{0,1} = E$, $E_{i,k} = E_{i+1,2k-1} \cup E_{i+1,2k}$, $E_{i,k} \cap E_{i,l} = \emptyset$ if $k \ne l$, $m(E_{i,k}) = m(E)/2^i$, and $\nu_j(E_{i,k}) = \nu_j(E)/2^i$ for $j = 1, \ldots, n$.*

To prove the Claim, define a vector measure $\mathcal{F} : \mathcal{M}_E \to \mathbb{R}^{n+1}$ by

$$\mathcal{F}(A) := (m(A), \nu_1(A), \ldots, \nu_n(A))$$

for each $A \in \mathcal{M}_E$. Lyapunov's theorem implies that $\mathcal{F}(\mathcal{M}_E)$ is a convex subset of $\mathbb{R}^{n+1}$. Therefore, since

$$\mathcal{F}(\emptyset) = (0, \ldots, 0) \quad \text{and} \quad \mathcal{F}(E) = (m(E), \nu_1(E), \ldots, \nu_n(E)),$$

there is a measurable subset $E_{1,1}$ of E such that

$$\mathcal{F}(E_{1,1}) = \big(\tfrac{1}{2}m(E), \tfrac{1}{2}\nu_1(E), \ldots, \tfrac{1}{2}\nu_n(E)\big).$$

Letting $E_{1,2} = E \setminus E_{1,1}$, we easily see that

$$\mathcal{F}(E_{1,2}) = \big(\tfrac{1}{2}m(E), \tfrac{1}{2}\nu_1(E), \ldots, \tfrac{1}{2}\nu_n(E)\big).$$

Using Lyapunov's theorem again on the sets $E_{1,1}$ and $E_{1,2}$, we get pairwise disjoint sets $E_{2,1}, E_{2,2}, E_{2,3}, E_{2,4}$ such that

$$E_{1,1} = E_{2,1} \cup E_{2,2}, \qquad E_{1,2} = E_{2,3} \cup E_{2,4},$$

and

$$\mathcal{F}(E_{2,l}) = \big(\tfrac{1}{4} m(E), \tfrac{1}{4}\nu_1(E), \ldots, \tfrac{1}{4}\nu_n(E)\big)$$

for all $l = 1, 2, 3, 4$.

Continuing this process inductively completes the proof of the Claim.

Let $\mathcal{A}$ denote the σ-algebra of subsets of E generated by the sets $(E_{i,k})$. Then, with the measure μ defined on the $\mathcal{A}$-measurable subsets of E by $\mu := m/m(E) = \nu_j/\nu_j(E)$ for $j \in J$, it follows that the Banach space $L^1(E, \mathcal{A}, \mu)$ is isometrically isomorphic to $L^1([0,1], m) = L^1[0,1]$ via the mapping Z defined as follows:

$$Z(\chi_{E_{i,k}}) := \chi_{[(k-1)/2^i, k/2^i)}$$

for each $\chi_{E_{i,k}}$. The mapping Z is extended to the linear span L of the functions $\chi_{E_{i,k}}$ in the usual way. Of course, Z is an isometry on L. Finally, since L is dense in $L^1(E, \mathcal{A}, \mu)$ and $Z(L)$ is dense in $L^1[0,1]$, Z extends to a linear isometry from $L^1(E, \mathcal{A}, \mu)$ onto $L^1[0,1]$.

Analogues of Alspach's map T and the map Δ will now be defined directly on E using the dyadic sets $E_{i,k}$. For an $\mathcal{A}$-measurable function $f = \sum_{k=1}^{2^i} \alpha_k \chi_{E_{i,k}}$ in $[0, \chi_E]$, define

$$\widetilde{\Delta}(f) = \sum_{k=1}^{2^i} (\alpha_k \chi_{E_{i+1,k}} + (1-\alpha_k)\chi_{E_{i+1,2^i+k}}).$$

Since the functions of the form $\sum_{k=1}^{2^i} \alpha_k \chi_{E_{i,k}}$ are dense in $L^1(E, \mathcal{A}, \mu)$, the definition can be extended to define a map $\widetilde{\Delta} : [0, \chi_E] \cap L^1(E, \mathcal{A}, \mu) \to [0, \chi_E] \cap L^1(E, \mathcal{A}, \mu)$. Also, since Δ and Z are isometries and $\widetilde{\Delta} = Z^{-1} \Delta Z$, $\widetilde{\Delta}$ is an $L^1(E, \mathcal{A}, \mu)$-isometry on $[0, \chi_E] \cap L^1(E, \mathcal{A}, \mu)$.

An easy computation shows that, if f is an $\mathcal{A}$-measurable function in $[0, \chi_E]$, then

$$\int_E \widetilde{\Delta}(f) w_j \, dm = \frac{1}{2} \nu_j(E) = \frac{1}{2} \int_E w_j \, dm \tag{3}$$

for $j = 1, \ldots, n$. Indeed, it suffices to check the equality for functions f of the form $\sum_{k=1}^{2^i} \alpha_k \chi_{E_{i,k}}$. So, if $f = \sum_{k=1}^{2^i} \alpha_k \chi_{E_{i,k}}$ is in the order interval $[0, \chi_E]$, then

$$\int_E \widetilde{\Delta}(f) w_j \, dm = \sum_{k=1}^{2^i} (\alpha_k \nu_j(E_{i+1,k}) + (1-\alpha_k)\nu_j(E_{i+1,2^i+k}))$$
$$= \sum_{k=1}^{2^i} \left(\alpha_k \frac{1}{2^{i+1}} \nu_j(E) + (1-\alpha_k) \frac{1}{2^{i+1}} \nu_j(E) \right)$$
$$= \sum_{k=1}^{2^i} \frac{1}{2^{i+1}} \nu_j(E) = \frac{1}{2} \nu_j(E).$$

Similarly, an analogue of Alspach's map can be defined directly on E. With $f = \sum_{k=1}^{2^i} \alpha_k \chi_{E_{i,k}}$, define

$$\widetilde{T}(f) = \sum_{k=1}^{2^i} ((2\alpha_k \wedge 1)\chi_{E_{i+1,k}} + ((2\alpha_k - 1) \vee 0)\chi_{E_{i+1,2^i+k}}).$$

As before, $\widetilde{T}$ extends to a map from $[0, \chi_E] \cap L^1(E, \mathcal{A}, \mu)$ into itself. Using the Alspach map T on $L^1[0,1]$ and the fact that $\widetilde{T} = Z^{-1}TZ$, we see that $\widetilde{T}$ is an isometry sending $[0, \chi_E] \cap L^1(E, \mathcal{A}, \mu)$ into itself. Note also that, for $f = \sum_{k=1}^{2^i} \alpha_k \chi_{E_{i,k}}$ in $[0, \chi_E] \cap L^1(E, \mathcal{A}, \mu)$ and $j = 1, \ldots, n$,

$$\text{(4)} \quad \int_E \widetilde{T}(f) w_j \, dm$$
$$= \sum_{k=1}^{2^i} ((2\alpha_k \wedge 1)\nu_j(E_{i+1,k}) + ((2\alpha_k - 1) \vee 0)\nu_j(E_{i+1,2^i+k}))$$
$$= \sum_{k=1}^{2^i} ((2\alpha_k \wedge 1) + ((2\alpha_k - 1) \vee 0)) \frac{1}{2^{i+1}} \nu_j(E)$$
$$= \sum_{k=1}^{2^i} \alpha_k \frac{1}{2^i} \nu_j(E) = \sum_{k=1}^{2^i} \alpha_k \nu_j(E_{i,k}) = \int_E f w_j \, dm.$$

By the density of such f in $[0, \chi_E] \cap L^1(E, \mathcal{A}, \mu)$, this equality holds for all f in $[0, \chi_E] \cap L^1(E, \mathcal{A}, \mu)$.

In order to define a nonexpansive self-map of K without a fixed point, it is useful to define a few more preliminary maps. Let $E^{\mathcal{A}}$ be the conditional expectation operator with respect to the σ-algebra $\mathcal{A}$ and let $R : K \to [0, \chi_E]$ be a restriction map defined by

$$R(f) := |f| \wedge \chi_E \quad \text{for all } f \in K.$$

Then $E^{\mathcal{A}}$ is a nonexpansive map from $L^1(E, \mathcal{M}_E, \mu)$ onto $L^1(E, \mathcal{A}, \mu)$. (For a proof of this, see [4, p. 122].) The map R is $L^1[0,1]$-nonexpansive and R equals the identity on $[0, \chi_E] \cap \mathcal{I}_{w_1}^{-1}(\alpha_1) \cap \cdots \cap \mathcal{I}_{w_n}^{-1}(\alpha_n) \subseteq K$. Note that, for any $f \in K$, (3) and (4) imply that

$$\widetilde{T}\widetilde{\Delta}E^{\mathcal{A}}R(f) \in [0,\chi_E] \cap L^1(E,\mathcal{A},\mu) \cap \bigcap_{j=1}^{n} \mathcal{I}_{w_j}^{-1}\Big(\frac{1}{2}\int_E w_j\,dm\Big). \tag{5}$$

It is now time to reconsider the set $H := [0,g] \cap \mathcal{I}_{w_1}^{-1}(\alpha_1) \cap \cdots \cap \mathcal{I}_{w_n}^{-1}(\alpha_n) \subseteq K$. Note that the function $\widetilde{T}\widetilde{\Delta}E^{\mathcal{A}}R(f)$ lies in the order interval $[0,g]$, has support in E, but there is no guarantee that it lies on any of the hyperplanes $\mathcal{I}_{w_j}^{-1}(\alpha_j)$. Thus, for any $f \in K$, $\widetilde{T}\widetilde{\Delta}E^{\mathcal{A}}R(f)$ may fail to be a member of H. Recall also the choice of the set G defined in (2), and define the function $\phi := g\chi_G$ and the set $H_0 \subseteq H$ by

$$H_0 := \Big\{\phi + u : u \in [0,\chi_E],\ u \text{ is } \mathcal{A}\text{-measurable}, \text{ and } \int_E uw_j\,dm = \frac{1}{2}\int_E w_j\,dm \text{ for } j = 1,\ldots,n\Big\}.$$

It is easy to check that $H_0 = \phi + Z^{-1}(C_0)$. Indeed, if $f \in C_0$ then $f \in C = [0,\chi_{[0,1]}]$, and so $Z^{-1}(f)$ is an $\mathcal{A}$-measurable function in $[0,\chi_E]$. Furthermore, for $j = 1,\ldots,n$,

$$\begin{aligned}
&\int_E Z^{-1}(f)w_j\,dm \\
&\quad= \int_E Z^{-1}(f)\,d\nu_j = \nu_j(E)\int_E Z^{-1}(f)\,d\mu \\
&\quad= \nu_j(E)\int_{[0,1]} f\,dm \qquad \text{since } Z \text{ is an isometry and } f, Z^{-1}(f) \ge 0 \\
&\quad= \nu_j(E)\cdot\frac{1}{2} \qquad \text{since } f \in C_0 \\
&\quad= \frac{1}{2}\int_E w_j\,dm.
\end{aligned}$$

So, $\phi + Z^{-1}(C_0) \subseteq H_0$. The reverse set inclusion is proven similarly. In particular, H_0 is nonempty.

Moreover, for any $u \in Z^{-1}(C_0)$, the maps u and $\phi = g\chi_G$ are disjointly supported. Thus

$$0 \le \phi + u \le g\chi_{[0,1]\setminus E} + g\chi_E = g$$

and, by the definition of G in equation (2),

$$\mathcal{I}_{w_j}(\phi + u) = \int_{[0,1]} \phi w_j\,dm + \int_{[0,1]} uw_j\,dm = \int_G gw_j\,dm + \frac{1}{2}\int_E w_j\,dm = \alpha_j$$

for $j = 1,\ldots,n$. Thus $H_0 \subseteq H \subseteq K$.

Finally, define the mapping $U : K \to H_0 \subseteq K$ by

$$U(f) := \phi + \widetilde{T}\widetilde{\Delta}E^{\mathcal{A}}R(f) \quad \text{for all } f \in K$$

By (5), U maps into H_0. It remains to show that U is a nonexpansive map without a fixed point.

In order to see that U is nonexpansive, take $f_1, f_2 \in K$. Then

$$\begin{aligned}
&\|U(f_1) - U(f_2)\|_{L^1[0,1]} \\
&= \int_{[0,1]} |\widetilde{T}\widetilde{\Delta}E^{\mathcal{A}}R(f_1) - \widetilde{T}\widetilde{\Delta}E^{\mathcal{A}}R(f_2)|\, dm \\
&= m(E)\int_E |\widetilde{T}\widetilde{\Delta}E^{\mathcal{A}}R(f_1) - \widetilde{T}\widetilde{\Delta}E^{\mathcal{A}}R(f_2)|\, d\mu \\
&= m(E)\int_E |E^{\mathcal{A}}R(f_1) - E^{\mathcal{A}}R(f_2)|\, d\mu \quad \text{since } \widetilde{T}, \widetilde{\Delta} \text{ are } L^1(E,\mathcal{A},\mu)\text{-isometries} \\
&\le m(E)\int_E |R(f_1) - R(f_2)|\, d\mu \qquad \text{since } E^{\mathcal{A}} \text{ is nonexpansive} \\
&= \int_E |R(f_1) - R(f_2)|\, dm = \|R(f_1) - R(f_2)\|_{L^1[0,1]} \\
&\le \|f_1 - f_2\|_{L^1[0,1]} \qquad \text{since } R \text{ is nonexpansive.}
\end{aligned}$$

It follows that $U : K \to K$ is $L^1[0,1]$-nonexpansive.

To see that U is fixed point free on K, suppose, for the sake of a contradiction, that $f \in K$ is a fixed point of U. Then

$$f = U(f) = \phi + \widetilde{T}\widetilde{\Delta}E^{\mathcal{A}}R(f).$$

But this implies that $f = U(f) \in H_0$. Thus $f = g\chi_G + u$, where $u = \widetilde{T}\widetilde{\Delta}E^{\mathcal{A}}R(f)$ is an $\mathcal{A}$-measurable function in $[0, \chi_E]$ and G and E are disjoint. Consequently, $R(f) = (g\chi_G + u) \wedge \chi_E = u$ and, since u is $\mathcal{A}$-measurable, $u = E^{\mathcal{A}}R(f)$. Hence,

$$E^{\mathcal{A}}R(f) = R(f) = u = \widetilde{T}\widetilde{\Delta}E^{\mathcal{A}}R(f).$$

Thus the function $E^{\mathcal{A}}R(f)$ is a fixed point of $\widetilde{T}\widetilde{\Delta}$. Since $\widetilde{T} = Z^{-1}TZ$ and $\widetilde{\Delta} = Z^{-1}\Delta Z$, this implies that

$$Z^{-1}T\Delta Z(E^{\mathcal{A}}R(f)) = E^{\mathcal{A}}R(f),$$

or

$$T\Delta(ZE^{\mathcal{A}}R(f)) = ZE^{\mathcal{A}}R(f).$$

Since $ZE^{\mathcal{A}}R(f)$ lies in C for each f in K, this contradicts the conclusion of Lemma 1 that $T\Delta$ is fixed point free on C, and the proof is complete. ∎

Remark. Note that the proof of Theorem 2 still works if we everywhere replace m by any nontrivial, finite, positive, purely nonatomic measure μ on a measurable space.

It should also be noted that some restriction on the location of the point $(\alpha_1, \ldots, \alpha_n)$ in Theorem 2 is necessary, although the condition used in Theorem 2, that the point lies in the interior of the range of the set function $\mathcal{G}$, is not the most general possible. It is easy to give examples in the setting of Theorem 3 where K is a singleton and hence has the fixed point property if α were equal to either $\mathcal{I}_w(h)$ or $\mathcal{I}_w(g)$.

It is perhaps more instructive to consider examples when $n = 2$ in Theorem 2. If $h \equiv 0$, $g \equiv 1$, $w_1(t) = t$, and $w_2(t) = 1 - t$, the set function $\mathcal{G}$ is defined by $\mathcal{G}(A) = (\int_A w_1\, dm, \int_A w_2\, dm)$ and its range $\mathcal{G}(\mathcal{M})$ is the region in the plane bounded by the graphs of $y = \sqrt{2x} - x$ and $x = \sqrt{2y} - y$. If (α_1, α_2) is a point on the boundary of $\mathcal{G}(\mathcal{M})$, the set $[h, g] \cap \mathcal{I}_{w_1}^{-1}(\alpha_1) \cap \mathcal{I}_{w_2}^{-1}(\alpha_2)$ is a singleton. As a second example, with $h \equiv 0$, $g \equiv 1$, $w_1(t) = \chi_{[0,1/2]}$, and $w_2(t) = \chi_{[1/2,1]}$, it is easy to check that $\mathcal{G}(\mathcal{M})$ is the square $[0, 1/2] \times [0, 1/2]$ and the sets $[h, g] \cap \mathcal{I}_{w_1}^{-1}(\alpha_1) \cap \mathcal{I}_{w_2}^{-1}(\alpha_2)$ fail the fixed point property for all points (α_1, α_2) in the closed square other than the four corner points. It is therefore reasonable to ask if the interior point hypothesis in Theorem 2 can be relaxed so that the conclusion of the theorem holds whenever the point $(\alpha_1, \ldots, \alpha_n)$ is not an extreme point of $\mathcal{G}(\mathcal{M})$.

We finish with some consequences of Theorem 2.

COROLLARY 4. *Let X be a subspace of $(L^1[0,1], \|\cdot\|_1)$ of codimension $n \in \mathbb{N}$. Let $g \in L^1[0,1]$ with $g \geq 0$ and $g \neq 0$, and let K be a closed, bounded, convex subset of X satisfying $K \supseteq [-g, g] \cap X$. Then K fails the fixed point property for nonexpansive mappings.*

Proof. By our hypotheses, there exist linearly independent functions $w_1, \ldots, w_n \in L^\infty[0,1]$ such that

$$X = \bigcap_{j=1}^{n} \mathcal{I}_{w_j}^{-1}(0) = \Big\{ f \in L^1[0,1] : \int_{[0,1]} f w_j\, dm = 0 \text{ for all } j = 1, \ldots, n \Big\}.$$

First, consider the special case where $g > 0$ a.e. on $[0,1]$. As in Theorem 2, consider the set function $\mathcal{G} : \mathcal{M} \to \mathbb{R}^n$ given by

$$\mathcal{G}(A) := \Big(\int_A g w_j\, dm - \int_{A^c} g w_j\, dm \Big)_{j=1}^{n} \quad \text{for all } A \in \mathcal{M}.$$

CLAIM. *The linear span of $\mathcal{G}(\mathcal{M})$ is $\mathbb{R}^n$.*

To prove the Claim, assume the contrary. Then there exist $a_1, \ldots, a_n \in \mathbb{R}$, not all zero, with $\mathcal{G}(\mathcal{M}) \subseteq \{(x_1, \ldots, x_n) \in \mathbb{R}^n : a_1 x_1 + \cdots + a_n x_n = 0\}$. Thus, for all $A \in \mathcal{M}$,

$$\sum_{j=1}^{n} a_j \Big(\int_A g w_j\, dm - \int_{A^c} g w_j\, dm \Big) = 0;$$

therefore

$$2\sum_{j=1}^{n} a_j \int_A g w_j\, dm = \sum_{j=1}^{n} a_j \int_{[0,1]} g w_j\, dm =: y.$$

Letting $A = \emptyset$ shows that $y = 0$, and so for all $A \in \mathcal{M}$, $\int_A (\sum_{j=1}^n a_j g w_j)\, dm = 0$. Hence, $g \sum_{j=1}^n a_j w_j = 0$ almost everywhere on $[0,1]$. Since, in this special case, $g > 0$ a.e., we get $\sum_{j=1}^n a_j w_j = 0$. This contradicts the linear independence of $\{w_1, \ldots, w_n\}$ and the Claim is proven.

Next, note that $\mathcal{G}([0,1]) = -\mathcal{G}(\emptyset)$. Since $\mathcal{G}$ is a translate of a vector measure, by Lyapunov's theorem, $\vec{0} = (\mathcal{G}(\emptyset) + \mathcal{G}([0,1]))/2 \in \mathcal{G}(\mathcal{M})$. Further, for all $\gamma = \mathcal{G}(A) \in \mathcal{G}(\mathcal{M})$, it follows that $-\gamma = \mathcal{G}(A^{\mathrm{c}}) \in \mathcal{G}(\mathcal{M})$. Hence, as observed by E. Bolker [2], $\mathcal{G}(\mathcal{M})$ is symmetric about $\vec{0}$, and so, by the Claim, $\vec{0}$ is an internal point of $\mathcal{G}(\mathcal{M})$ in $\mathbb{R}^n$. (For the definition of an internal point of a set, see [9, p. 239].) Since internal points of convex sets in finite-dimensional spaces are interior points, $\vec{0}$ is an interior point of $\mathcal{G}(\mathcal{M})$ [9, p. 243, Problem 44(b)]. Indeed, from the Claim, there exist n linearly independent vectors $\vec{q}_1, \ldots, \vec{q}_n \in \mathcal{G}(\mathcal{M})$. Define the subset P of $\mathbb{R}^n$ by $P := \{s_1\vec{q}_1 + \cdots + s_n\vec{q}_n : |s_1| + \cdots + |s_n| \le 1\}$. Clearly, P is a convex, balanced, absorbing subset of $\mathbb{R}^n$, and so the Minkowski functional μ_P of P is a norm on $\mathbb{R}^n$ equivalent to the usual $\|\cdot\|_2$ norm. Thus, $\vec{0}$ is an interior point of P and, by Lyapunov's theorem, $P \subseteq \mathcal{G}(\mathcal{M})$. Hence, $\vec{0}$ is an interior point of $\mathcal{G}(\mathcal{M})$.

Because $K \supseteq [-g,g] \cap X = [-g,g] \cap \mathcal{I}_{w_1}^{-1}(0) \cap \cdots \cap \mathcal{I}_{w_n}^{-1}(0)$, Theorem 2 implies that K fails the fixed point property for nonexpansive mappings.

Finally, consider the general case: $g \ge 0$ and $g \ne 0$. Define $\Omega := [g > 0]$; note that $m(\Omega) > 0$. Consider

$$\begin{aligned}
&[-g,g] \cap X \\
&= \Big\{ f \in L^1[0,1] : -g \le f \le g \text{ and } \int_{[0,1]} f w_j\, dm = 0 \text{ for all } j = 1, \ldots, n \Big\} \\
&= \Big\{ f \in L^1[0,1] : -g \le f \le g \text{ on } \Omega,\ f = 0 \text{ on } \Omega^{\mathrm{c}}, \text{ and each } \int_{\Omega} f w_j\, dm = 0 \Big\}.
\end{aligned}$$

We will denote by $L^1(\Omega)$ the space of all Lebesgue integrable functions from Ω into $\mathbb{R}$. We identify $L^1(\Omega)$ with a subspace of $L^1[0,1]$ by extending each $f \in L^1(\Omega)$ to be identically zero on Ω^{c}. Note that $g \in L^1(\Omega)$. Then $[-g,g] \cap X = [-g,g] \cap Y$, where $Y := \{f \in L^1(\Omega) : \int_\Omega f w_j\, dm = 0$ for all $j = 1, \ldots, n\}$ is a subspace of $L^1(\Omega)$ of codimension at most n. By Theorem 2 and the remark following its proof, the general case reduces to the special case proven above. ■

Corollary 5. *If X is a subspace of $(L^1[0,1], \|\cdot\|_1)$ of codimension n, then X contains a nonempty, weakly compact, convex set which fails the fixed point property for nonexpansive mappings.*

Stated more succinctly, subspaces of codimension n in $L^1[0,1]$ fail the weak fixed point property. This also follows from the proof of Theorem 2, which shows that all subspaces of finite codimension in $L^1[0,1]$ actually contain an isometric copy of $L^1[0,1]$. This latter fact is already known. Indeed, it follows from Theorem 1 in Section 10 of Plichko and Popov [8].

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