Ergodic averages with generalized weights

by

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Abstract. Two types of weighted ergodic averages are studied. It is shown that if $F = \{F_n\}$ is an admissible superadditive process relative to a measure preserving transformation, then a Wiener–Wintner type result holds for F. Using this result new good classes of weights generated by such processes are obtained. We also introduce another class of weights via the group of unitary functions, and study the convergence of the corresponding weighted averages. The limits of such weighted averages are also identified.

1. Introduction. This article is inspired by the celebrated Wiener–Wintner theorem which has been instrumental in studying various properties of weighted and subsequential averages as well as constructing some nontrivial classes of good sequences. It turns out that, utilizing the tools developed in [Ç, ÇF], one can obtain a version of this theorem in the setting of a class of superadditive processes. In turn, this superadditive version of Wiener–Wintner theorem leads to new classes of weights. In Sections 2–4 of this article, besides proving these results, we will also study convergence of sequences of weights defined by such superadditive processes.

The study of the behavior of the ergodic averages modulated by means of Besicovitch's sequences was initiated by C. Ryll-Nardzewski [Ry] (for $p = 1$), and A. Tempelman [T] (for $p > 1$ and in the context of Besicovitch functions on LCA groups). Later, various generalizations of this result were obtained [BeL, JO, LO, LOT]. Recently, another such attempt was made in [Li] in the non-commutative setting. The tools utilized in [Li] have some interesting ramifications in the commutative case as well. In Sections 5 and 6 of this article, following an observation made on the Wiener–Wintner theorem and adapting some of the techniques of [Li], we arrive at two rather distinct weighted ergodic theorems in the spirit of [Ry]. First we define a generalized Besicovitch sequence associated with a subset of the group of unimodular

functions in L_∞, and then study convergence of the corresponding weighted averages. In the case of numerical Besicovitch sequences, which are discussed in [Ry], the role of such a subset is played by the unit circle in $\mathbb{C}$. This type of weights (generated by the unimodular group) is fundamentally different from the ones studied in [LO]. Section 7 is devoted to the identification of the limits of the weighted averages along generalized Besicovitch sequences.

Let (X, Σ, μ) be a probability space, and $T : X \to X$ be an invertible measure preserving transformation (MPT). Often, we will write $T^i f$ instead of $f \circ T^i$. A sequence $\mathbf{n} = \{n_k\}$ of positive integers (or a weight $\mathbf{a} = \{a_i\}$) is called *good in the p-mean for T* if the limit of the corresponding averages $N^{-1} \sum_{i=0}^{N-1} f(T^{n_i} x)$ (or $N^{-1} \sum_{i=0}^{N-1} a_i T^i f(x)$) exists in L_p-norm for all $f \in L_p$, and is called *good a.e. in L_p for T* if the limit of the averages exists a.e. for all $f \in L_p$. If $\mathbf{n} = \{n_k\}$ (or $\mathbf{a} = \{a_i\}$) is good a.e. (good in the p-mean) in L_p for all MPTs, it is called *good a.e. in L_p* (*good in the p-mean*).

A family $F = \{f_n\}_{n \geq 0} \subset L_p$ is called a *T-superadditive process* if the sequence $\{F_n\}$ of partial sums, where $F_0 = 0$ and $F_n = \sum_{i=0}^{n-1} f_i$, satisfies $F_{n+m} \geq F_n + T^n F_m$ for all $n, m \geq 0$. If the equality holds, it is called *T-additive,* and if the reverse inequality holds, it is called *T-subadditive.* T-additive processes are necessarily of the form $F_n = \sum_{i=0}^{n-1} T^i f_0$. If $\sup_{n \geq 1} n^{-1} \|F_n\|_p < \infty$, then the process $F \subset L_p$ is called *p-bounded* (or simply *bounded* when $p = 1$). For a T-superadditive process F, $F_n \geq \sum_{i=0}^{n-1} T^i f_0$ for all $n \geq 1$, hence $F_n' = F_n - \sum_{i=0}^{n-1} T^i f_0$ is a positive superadditive process (i.e. $F_n' \geq 0$ for all n) and is necessarily increasing. It follows that, if a result is valid for additive processes, then the same holds for F if and only if it holds for F'.

Given a sequence $\mathbf{n} = \{n_k\}$ or a weight $\mathbf{a} = \{a_i\}$ and a T-superadditive process F, we will define corresponding subsequential averages as $N^{-1} \sum_{i=0}^{N-1} f_{n_i}$ and the weighted averages as $N^{-1} \sum_{i=0}^{N-1} a_i f_i$. If F is a T-superadditive process, a sequence $\mathbf{n} = \{n_k\}$ or a weight $\mathbf{a} = \{a_i\}$ is called *good in the p-mean (a.e.) for F* if the limit of the associated averages exists in L_p-norm (a.e.).

2. Wiener–Wintner theorem for admissible processes. The particular class of superadditive processes we will study are the admissible processes. A family $\{f_n\}_{n \geq 0} \subset L_p$ is said to be *T-admissible* (or simply *admissible*) if $Tf_i \leq f_{i+1}$ for $i \geq 0$. If $F = \{f_n\}$ is a T-admissible family, then the associated T-superadditive process $\{F_n\}_{n \geq 1}$, where $F_n = \sum_{i=0}^{n-1} f_i$, is called *$T$-admissible.* An admissible process F is called *strongly p-bounded* (or simply *strongly bounded* when $p = 1$) if $\sup_n \|f_n\|_p < \infty$.

When $p = 1$, the boundedness and admissibility of the process F implies that it is strongly bounded [ÇF]. Therefore, strong boundedness and bound-

edness are the same for admissible processes in L_1. Clearly, any strongly p-bounded process is p-bounded, however, when $p > 1$, the converse is not the case. Indeed, as the following example shows, one can have an admissible 2-bounded process $F \subset L_2$ which is not strongly 2-bounded.

EXAMPLE. Let bounded positive functions g_n, $n = 0, 1, 2, \ldots$, be given on some probability space, and suppose that a measure preserving point transformation T is given such that the entire doubly-indexed family $\{T^n g_m\}_{n,m=0,1,2,\ldots}$ is independent, considered as a family of random variables. For example, we could take T to be the product of countably many shifts on countably many infinite product spaces, and choose g_n as a function of the first coordinate of the nth product space. Consider the case in which $g_0 = 0$ and $\int g_n = 1/2^n$, $\int g_n^2 = 1/\sqrt{n}$. Define f_n, $n = 0, 1, 2, \ldots$, by

$$f_n = T^n g_0 + T^{n-1} g_1 + \cdots + T g_{n-1} + g_n.$$

As usual, define $F_n = f_0 + f_1 + \cdots + f_{n-1}$. It is easy to check that the sequence $\{f_n\}$ is T-admissible. Clearly $\int f_n \leq 1$ for all n. Also, since $\mathrm{Var}(g_n) \leq \int g_n^2$, independence yields

$$\int f_n^2 \leq \left(\int f_n \right)^2 + \int g_0^2 + \int g_1^2 + \cdots + \int g_n^2,$$

and therefore,

$$\int f_n^2 \leq 1 + c_1 \sqrt{n}.$$

The independence assumption implies that $\{f_i\}$ is an independent family. It follows that

$$\int F_n^2 \leq \left(\int F_n \right)^2 + \int f_0^2 + \cdots + \int f_{n-1}^2 \leq n^2 + c_2 n \sqrt{n},$$

and so $\sup_n n^{-1} \|F_n\|_2 < \infty$. But

$$\int f_n^2 \geq \int g_0^2 + \int g_1^2 + \cdots + \int g_n^2 \geq c_3 \sqrt{n}$$

and hence F is not strongly 2-bounded.

It is well known that every bounded superadditive process $F \subset L_1$ has an *exact dominant* [AS], that is, there exists a function $\delta \in L_1^+$ such that $F_n \leq \sum_{i=0}^{n-1} T^i \delta$ for all $n \geq 1$ and $\int \delta = \sup_{n \geq 1} n^{-1} \|F_n\|_1$. It turns out that more is true for strongly p-bounded admissible processes $F \subset L_p$, $1 \leq p < \infty$.

PROPOSITION 2.1. *For any positive, strongly p-bounded T-admissible process $F = \{f_n\} \subset L_p, 1 \leq p < \infty$, there exists $\delta \in L_p^+$ such that $f_n \leq T^n \delta$ for all $n \geq 0$ and $\|\delta\|_p = \sup_{n \geq 1} \|f_n\|_p = \lim_n \|f_n\|_p$.*

Proof. Define a sequence $\{v_n\} \subset L_p^+$ by $v_n = T^{-n} f_n, n \geq 0$. From the T-admissibility of F, we see that $v_n \leq v_{n+1}$. Thus, since F is strongly

p-bounded, by the monotone convergence theorem, $\delta := \lim_n v_n \in L_p^+$ and $\|\delta\|_p = \lim_n \|v_n\|_p$. Clearly, for $n \geq 0, v_n \leq \delta$, and hence $f_n \leq T^n\delta$. $\blacksquare$

REMARKS. 1. The function $\delta \in L_p^+$ is called a *dominant for F*. Clearly, if $p = 1$, then δ is an exact dominant. Furthermore, if $b_k = \delta - v_k$, then $b_k \in L_p$ for each $k \geq 1$, and since $\|T^k\delta - f_k\|_p \to 0$, we have $\|b_k\|_p \to 0$ as $k \to \infty$, for all $1 \leq p < \infty$.

2. If the admissible process $F = \{f_n\}$ is not necessarily positive then the process $F' = \{f'_n\}$, where $f'_n = f_n - T^n f_0$, as defined in Section 1, is positive. Hence, by Proposition 2.1 there exists $\delta' \in L_p^+$ such that $f'_n \leq T^n\delta'$ for all $n \geq 0$ and $\|\delta'\|_p = \lim_n \|f'_n\|_p$. Therefore, the function $\delta = \delta' + f_0$ satisfies $f_n \leq T^n\delta$ for all $n \geq 0$ and $\|\delta\|_p = \lim_n \|f_n\|_p$, which extends Proposition 2.1 to all strongly p-bounded admissible processes.

In order to study norm and a.e. convergence of the averages weighted by means of sequences induced by admissible processes, one should check whether such sequences have the right properties, like being Hartman almost periodic sequences to start with. Now we state and prove the Wiener–Wintner type theorem for admissible processes, which shows that sequences induced by admissible processes indeed define Hartman almost periodic sequences.

THEOREM 2.2. *Let $F = \{f_n\} \subset L_1$ be a bounded T-admissible process. There exists a set $N \in \Sigma$ with $\mu(N) = 0$ such that for $x \in X \setminus N$,*

$$\frac{1}{n}\sum_{i=0}^{n-1}\lambda^i f_i(x) \quad \text{converges for all } \lambda \in \mathbb{K},$$

where $\mathbb{K}$ is the unit circle in the complex plane.

Proof. The proof will be given for ergodic T, but it is also valid when T is an arbitrary MPT by ergodic decomposition. Since the assertion is true for additive processes, by the Wienner–Wintner theorem, we can assume (by passing to F' if necessary) that $f_i \geq 0$ for each $i \geq 1$. Fix $k \in \mathbb{Z}^+$, and define

$$g_i^k(x) = \begin{cases} T^{i-k}f_k(x) & \text{for } i > k, \\ f_i(x) & \text{for } 0 \leq i \leq k. \end{cases}$$

Using the same notation as in Proposition 2.1 (and its proof), it follows that

$$0 \leq T^i(v_i - v_k) \leq f_i - g_i^k \leq T^i b_k \quad \text{if } i > k,$$
$$f_i - g_i^k = 0 \quad \text{if } i \leq k,$$

where $b_k = \delta - v_k$. Therefore $\sum_{i=0}^{n-1}(f_i - g_i^k) \leq \sum_{i=0}^{n-1}T^i b_k$. Because $v_k \uparrow \delta$, we have $\|b_k\|_1 = \|\delta - v_k\|_1 \downarrow 0$ as $k \to \infty$.

Now, for each $n > k$ and $\lambda \in \mathbb{K}$, we have

$$\sum_{i=0}^{n-1} \lambda^i g_i^k = \sum_{i=0}^{k-1} \lambda^i f_i + \sum_{i=k}^{n-1} \lambda^i T^{i-k} f_k = \sum_{i=0}^{k-1} \lambda^i f_i + \lambda^k \sum_{i=0}^{n-k-1} \lambda^i T^i f_k.$$

Hence, by the Wiener–Wintner theorem, there exists a set $N_k \in \Sigma$, depending on f_k only, with $\mu(N_k) = 0$, such that for $x \notin N_k$,

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} \lambda^i g_i^k(x) \quad \text{exists for all } \lambda \in \mathbb{K}.$$

Let $N' = \bigcup_{k \geq 0} N_k$. Then $\mu(N') = 0$, and for $x \notin N'$,

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} \lambda^i g_i^k(x) \quad \text{exists for all } \lambda \in \mathbb{K} \text{ and all } k \geq 0.$$

Next, observe that, by Birkhoff's individual ergodic theorem,

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^i b_k(x) = \int b_k \quad \text{for almost every } x.$$

Let $M \in \Sigma$ be the set with $\mu(M) = 0$ such that for $x \notin M$,

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^i b_k(x) = \int b_k$$

for all $k \geq 0$. If $N = M \cup N'$, then $\mu(N) = 0$. Furthermore, if $\mathbf{a} = \{a_i\}$ and $\mathbf{a}^k = \{a_i^k\}$, where $a_i = f_i(x)$ and $a_i^k = g_i^k(x)$ for a fixed $x \in X \setminus N$, then

$$0 \leq \|\mathbf{a} - \mathbf{a}^k\|_1 = \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} |a_i - a_i^k| = \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} |f_i(x) - g_i^k(x)|$$

$$\leq \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} T^i b_k(x) = \int b_k \, d\mu = \|b_k\|_1 \downarrow 0 \quad \text{as } k \to \infty.$$

Hence $\|\mathbf{a} - \mathbf{a}^k\|_1 \to 0$. Therefore, by Lemma 3.1 of [BO] (or Lemma 2.2(b) of [JO]), $n^{-1} \sum_{i=0}^{n-1} \lambda^i f_i(x)$ converges for all $\lambda \in \mathbb{K}$. ∎

REMARK. The limit in Theorem 2.2 is 0 for all $\lambda \neq 1$ if T is weakly mixing [BeL].

3. Weights induced by admissible processes. An important consequence of Theorem 2.2 is that, for a.e. x, the sequence $\{f_i(x)\}$, where $F = \{f_i\} \subset L_p$ is a strongly p-bounded admissible superadditive process, has Fourier coefficients, i.e., is a Hartman almost periodic sequence [LOT]. Furthermore, by the a.e. convergence of the averages of such processes [Ç], it is easy to see that $\{f_i(x)\} \in W_p$, $1 \leq p < \infty$, where $W_p := \{\mathbf{a} = \{a_i\} :$

$\|\mathbf{a}\|_{W_p}^p = \limsup_n n^{-1} \sum_{i=0}^{n-1} |a_i|^p < \infty\}$, and W_∞ is the class of bounded sequences. It is known that, even in the additive case, a Hartman sequence need not be a good sequence for a.e. convergence [LOT]. On the other hand, if $F \subset L_\infty$ is a 1-bounded admissible process, by the observations above and Corollary 4.3 of [BeL], for a.e. x, $\{f_i(x)\}$ is a sequence good a.e. for any dynamical system $(\Omega, \Sigma', \nu, S)$, where S has Lebesgue spectrum. Thus, it is natural to ask whether the sequences of weights induced by admissible superadditive processes are good a.e. or in the mean in L_p. Now we turn to the study of this question.

THEOREM 3.1. *Let (X, Σ, μ) be a Lebesgue space and let $F = \{f_i\} \subset L_p$, $1 \le p \le \infty$, be a strongly p-bounded T-admissible process. Let q be the dual index. Then for a.e. $x \in X$, the sequence $\mathbf{a} = \{f_i(x)\}$ is good a.e. in L_q.*

Proof. First, observe that if $\{f_i\}$ is not positive, then the process $\{f_i'\} = \{f_i - T^i f_0\}$ is a positive T-admissible strongly p-bounded process. By the return times theorem [Bo, Ru], for a.e. $x \in X$ the sequence $\{T^i f_0(x)\}$ is good a.e. in L_q. Hence, without loss of generality, we can assume that the process F is positive.

Now we will use the same machinery as in Theorem 2.2. Define $\mathbf{a}^k = \{a_i^k\}_{i=0}^\infty$, $k \ge 0$, as $a_i^k = g_i^k(x)$. First, assume that $F \subset L_\infty$. Then there is a null set $N \subset X$ such that for $x \notin N$, the sequence $\mathbf{a}^k$ is good a.e. in L_1 for all $k \ge 0$ [Bo, Ru]. For such an x, if $\mathbf{a} = \{f_i(x)\}$, then

$$0 \le \|\mathbf{a}^k - \mathbf{a}\|_{W_\infty} \le \sup_i |T^i(\delta - v_k)(x)| \le \sup_i \|T^i(\delta - v_k)\|_{L_\infty(X)}$$

$$\le \|(\delta - v_k)\|_{L_\infty(X)} \quad \text{since } T \text{ is a contraction on } L_\infty.$$

Since $v_k \uparrow \delta$ a.e., we have $\lim_k \|\mathbf{a}^k - \mathbf{a}\|_{W_\infty} \le \lim_k \|\delta - v_k\|_{L_\infty} = 0$. Now, let (Ω, Σ, ν, S) be a dynamical system and $g \in L_\infty(\Omega)$. Then

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} a_i S^i g(\omega) - \frac{1}{m} \sum_{i=0}^{m-1} a_i S^i g(\omega) \right| \le \left| \frac{1}{n} \sum_{i=0}^{n-1} (a_i - a_i^k) S^i g(\omega) \right|$$

$$+ \left| \frac{1}{n} \sum_{i=0}^{n-1} a_i^k S^i g(\omega) - \frac{1}{m} \sum_{i=0}^{m-1} a_i^k S^i g(\omega) \right| + \left| \frac{1}{m} \sum_{i=0}^{m-1} (a_i - a_i^k) S^i g(\omega) \right|$$

$$\le \|\mathbf{a} - \mathbf{a}^k\|_{W_\infty} \left[\frac{1}{n} \sum_{i=0}^{n-1} S^i |g(\omega)| + \frac{1}{m} \sum_{i=0}^{m-1} S^i |g(\omega)| \right]$$

$$+ \left| \frac{1}{n} \sum_{i=0}^{n-1} a_i^k S^i g(\omega) - \frac{1}{m} \sum_{i=0}^{m-1} a_i^k S^i g(\omega) \right|.$$

Since $\|\mathbf{a}^k - \mathbf{a}\|_{W_\infty} \to 0$, and since $n^{-1} \sum_{i=0}^{n-1} S^i |g|$ converges a.e. to a function g^* with $\int |g^*| < \infty$, it follows that $\{n^{-1} \sum_{i=0}^{n-1} a_i S^i g(\omega)\}$ is Cauchy, hence

converges a.e. Thus, by [LOT, Proposition 3.1], $n^{-1} \sum_{i=0}^{n-1} a_i S^i g$ converges a.e. for all $g \in L_1(\Omega)$.

For $F \subset L_p, 1 \leq p < \infty$, we adopt the method of proof of [LOT, Proposition 3.2]. By [LOT, Theorem 3.6], for each k, $\mathbf{a}^k$ is good a.e. in L_q, hence in L_∞. Again, if (Ω, Σ, ν, S) is a dynamical system and $f \in L_\infty(\Omega)$, then for a.e. $\omega \in \Omega$,

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} a_i S^i f(\omega) - \frac{1}{m} \sum_{i=0}^{m-1} a_i S^i f(\omega) \right| \leq \left| \frac{1}{n} \sum_{i=0}^{n-1} (a_i - a_i^k) S^i f(\omega) \right|$$

$$+ \left| \frac{1}{n} \sum_{i=0}^{n-1} a_i^k S^i f(\omega) - \frac{1}{m} \sum_{i=0}^{m-1} a_i^k S^i f(\omega) \right| + \left| \frac{1}{m} \sum_{i=0}^{m-1} (a_i - a_i^k) S^i f(\omega) \right|$$

$$\leq \|f\|_\infty \left[\frac{1}{n} \sum_{i=0}^{n-1} |a_i - a_i^k| + \frac{1}{m} \sum_{i=0}^{m-1} |a_i - a_i^k| \right]$$

$$+ \left| \frac{1}{n} \sum_{i=0}^{n-1} a_i^k S^i f(\omega) - \frac{1}{m} \sum_{i=0}^{m-1} a_i^k S^i f(\omega) \right|.$$

Since $\|\mathbf{a}^k - \mathbf{a}\|_{W_1} \leq \|b_k\|_1 \to 0$, we conclude that $\{n^{-1} \sum_{i=0}^{n-1} a_i S^i f(\omega)\}$ is Cauchy, hence converges. Since $\{f_i(x)\} \in W_p$ is good a.e. in L_∞, [LOT, Proposition 3.1] shows that $\{f_i(x)\}$ is good a.e. in L_q. ∎

REMARKS. 1. Theorem 3.1 extends the return times theorem to sequences generated by admissible processes.

2. Theorem 2.2 follows from Theorem 3.1 (use all possible rotations since a 1-bounded admissible process in L_1 is strongly bounded [ÇF]).

The fact that the sequences considered in Theorem 3.1 are good for the norm convergence follows from the method employed in the previous theorem and [ÇLO, Theorem 4.3] when $p = 1$, and [ÇLO, Theorem 4.2] when $1 < p < \infty$:

PROPOSITION 3.2. *Let* $F = \{f_i\} \subset L_p$, $1 \leq p < \infty$, *be a strongly p-bounded admissible process. Then for almost every x the sequence* $\{f_i(x)\}$ *is good in the p-mean.*

Theorem 3.1 can also be interpreted as stating that if (X, Σ, μ) is a Lebesgue space, $T : X \to X$ is an invertible MPT, and $F \subset L_p, 1 \leq p \leq \infty$, is a strongly p-bounded T-admissible process, then there exists a set $N \in \Sigma$ with $\mu(N) = 0$ such that for any $x \in X \setminus N$ the sequence $\{f_i(x)\}$ is good a.e. for L_q, where q is the dual index. This fact, combined with [Ç, Theorem 3.1], leads to a further generalization of the return times theorem:

THEOREM 3.3. *Let (X, Σ, μ) be a Lebesgue space and $T : X \to X$ be an invertible MPT. If $F \subset L_p, 1 \leq p \leq \infty$, is a strongly p-bounded T-admissible process, then there exists a set $N \in \Sigma$ with $\mu(N) = 0$ such that for any $x \in X \setminus N$ the sequence $\{f_i(x)\}$ is good a.e. for any strongly q-bounded admissible process (in L_q) relative to invertible MPTs, where q is the dual index.*

As observed in [LOT, Proposition 1.5], a sequence induced by an additive process need not be a Besicovitch sequence. Hence, a sequence $\{f_i(x)\}$ induced by an admissible superadditive process need not be a Besicovitch sequence either. Indeed, $\{f_i(x)\}$ is a linear combination of a Besicovitch sequence, a shift orthogonal sequence and a sequence with zero mean. However, with an additional condition on T one obtains Besicovitch sequences induced by admissible processes.

THEOREM 3.4. *Let $\{f_i\} \subset L_\infty^+$ be a 1-bounded T-admissible process. Assume that T has discrete spectrum. Then for a.e. x, the sequence of weights $\{f_i(x)\}$ is bounded Besicovitch.*

Proof. Define sequences $\{\mathbf{s}^k\}$ and $\mathbf{s}$ by $s_i^k = g_i^k(x)$ and $s_i = f_i(x)$ for all $i \geq 1$, where g_i^k is as defined in the proof of Theorem 2.2. For each k, there is $N_k \subset X$ with $\mu(N_k) = 0$ such that $\mathbf{s}^k$ is bounded Besicovitch [BeL, Theorem 3.22]. Obviously, $\|\mathbf{s}^k\|_\infty \leq \|\mathbf{s}\|_\infty$. As in the proof of Theorem 2.2,

$$\|\mathbf{s}^k - \mathbf{s}\|_{W_1} = \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} |s_i^k - s_i| \leq \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} T^i b_k = \|b_k\|_1 \downarrow 0.$$

Thus, $\mathbf{s}$ is in the sup-dominated W_1-closure of the space of bounded Besicovitch sequences. Hence by Lemma 3.23 of Baxter–Olsen [BO], for any fixed $x \in \bigcap N_k^c$ the sequence $\mathbf{s}$ is bounded Besicovitch. ∎

Next, we will introduce a class of good sequences of weights induced by admissible processes and investigate some interesting properties of such weights. To do so, in the following definition and in Theorem 3.5 below we will assume that (X, Σ, μ, T) is a *strictly L-stable system* [BrK], where X is a compact connected metric space, and $\{v_i\} \subset L_\infty^+$ is an increasing sequence of Riemann integrable functions on X with $v_i \uparrow v$ uniformly.

DEFINITION. For any $x \in X$ the sequence $\mathbf{a} = \{a_k\}$, where $a_k = v_k(T^k x)$, is called *near-uniform*.

REMARKS. 1. Every uniform sequence is near-uniform. To see this, let $v_i = \chi_Y$, where (X, Σ, m, T, Y) is the apparatus for the uniform sequence [BrK].

2. By Theorem 3.1, for a.e. $x \in X$, near-uniform sequences are good a.e. in L_1.

THEOREM 3.5. *Near-uniform sequences are good a.e. in L_1 for all $x \in X$.*

Proof. By the method of proof of Theorem 3.1, all one needs to show is that

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} f_i(x) \quad \text{exists for all } x \in X,$$

where $f_i = T^i v_i$. By the uniform convergence, v is necessarily Riemann integrable, and $\gamma_F = \|v\|_1$. Given $\varepsilon > 0$, find continuous functions r and s such that

$$0 \le r \le v \le s \quad \text{for every } x \in X, \quad \text{and} \quad \int (s - r) < \varepsilon.$$

Furthermore, we can assume that, for large enough K, if $k \ge K$, we have $r \le v_k$. Defining g_i^k as before, for $k \ge K$, we see that $n^{-1} \sum_{i=0}^{n-1} g_i^k \le n^{-1} \sum_{i=0}^{n-1} f_i(x)$. Since $T^i r \le T^i v_k$ for all $i \ge 0$, we have $T^i r \le T^{i-k} f_k$. Similarly, $v_i \le s$ for all i implies that $f_i \le T^i s$ for all i. Hence, by ignoring $\sum_{i=0}^{k-1} g_i^k$ if necessary, we have

$$0 \le \frac{1}{n} \sum_{i=0}^{n-1} T^i r \le \frac{1}{n} \sum_{i=0}^{n-1} g_i^k \le \frac{1}{n} \sum_{i=0}^{n-1} f_i \le \frac{1}{n} \sum_{i=0}^{n-1} T^i s.$$

Since T is L-stable, both $\lim n^{-1} \sum_{i=0}^{n-1} T^i r(x) = \int r$ and $\lim n^{-1} \sum_{i=0}^{n-1} T^i s(x) = \int s$ exist for every $x \in X$. Hence, for every $x \in X$,

$$0 \le \limsup \frac{1}{n} \sum_{i=0}^{n-1} f_i(x) - \liminf \frac{1}{n} \sum_{i=0}^{n-1} f_i(x) \le \lim \frac{1}{n} \sum_{i=0}^{n-1} T^i (s - r)(x)$$

$$= \int (s - r) < \varepsilon.$$

Hence $\lim n^{-1} \sum_{i=0}^{n-1} f_i(x)$ exists for all $x \in X$. ∎

REMARK. Let $\{A_n\}$ and A be measurable with $A_n \uparrow A$ such that $\sum \mu(A_n \setminus A_{n-1}) < \infty$. Then, letting $v_i = \chi_{A_i}$ and $v = \chi_A$, the observations above imply immediately that the "return-time sequence" $\{n_k\}$ defined by

$$n_k = \inf\{l \in \mathbb{Z} : T^l(x) \in A_k, \, l \ge n_{k-1}\}$$

is good a.e. in L_1. Also, it follows that the frequency of the relation $T^n x \in A_n$ exists a.e. and is equal to $\mu(A)$ iff T is ergodic.

4. Purely subadditive parts of admissible processes. By the "decomposition theorem" of Kingman any bounded T-superadditive process $F = \{F_n\} \subset L_1$ can be decomposed into a difference $G - H$, where $G = \{G_n\}$ is an additive process and $H = \{H_n\}$ is a positive subadditive process [AS]. In fact, H is a *purely subadditive process*, in the sense that it does not dominate any nonzero T-additive process and $\lim_n n^{-1} H_n = 0$ a.e. (and hence

in norm) [AS]. Furthermore, $G_n = \sum_{k=0}^{n-1} T^k \delta$, where δ is an exact dominant of F. If $H = \{H_n\}$ is the purely subadditive part of a superadditive process, then necessarily $H_n = \sum_{i=0}^{n-1} h_i \geq 0$ for each $n \geq 0$, but this does not imply that each $h_i \geq 0$. However, if F is admissible, then by Proposition 2.1 (and with the same notation as in its proof), $h_i = T^i \delta - f_i \geq 0$. Also, $\|h_i\|_1 = \|\delta - v_i\|_1$ for each $i \geq 0$. Hence we have

PROPOSITION 4.1. *If $H = \{H_n\} \subset L_1$ is the purely subadditive part of a T-admissible process, where $H_n = \sum_{i=0}^{n-1} h_i$, then $h_i \geq 0$ for all $i \geq 0$.*

The limit in Theorem 2.2 is identified (as 0 for $\lambda \neq 1$) if T is weakly mixing. For more general cases of weights involving admissible processes, one utilizes the positivity of the purely subadditive part of an admissible process and the following observation.

PROPOSITION 4.2. *If $H = \{h_n\} \subset L_1$ is the purely subadditive part of a T-admissible process and $\mathbf{a} \in l_\infty$ is a sequence which is good a.e., then $\lim_n n^{-1} \sum_{k=0}^{n-1} a_k h_k = 0$ a.e. and in norm.*

Proof. If $\mathbf{a}$ is good a.e., then by [Ç, Theorem 3.1] it is good a.e. for any bounded admissible process $F \subset L_1$. In particular, if $H = \{h_n\}$ is the purely subadditive part of F, then $\lim_n n^{-1} \sum_{k=0}^{n-1} a_k h_k$ exists a.e. Furthermore (see [AS]),

$$0 \leq \lim_n \frac{1}{n} \left| \sum_{k=0}^{n-1} a_k h_k \right| \leq \lim_n \frac{1}{n} \sum_{k=0}^{n-1} h_k = 0 \quad \text{a.e. and in norm.} \quad \blacksquare$$

REMARKS. 1. If $\mathbf{a} \in l_\infty$ is a sequence which is good a.e. or good in the 1-mean (in particular, $a_n = \lambda^n$, $\lambda \in \mathbb{C}$), then for any bounded T-admissible process $F \subset L_1$,

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} a_k f_k = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} a_k T^k \delta,$$

where δ the exact dominant for F.

2. In [LOT] a series representation of the limit of weighted averages of additive processes in L_p, $1 \leq p < \infty$, has been obtained. Proposition 4.2, combined with their results and the remark above, also yields a series representation of the limit of weighted averages for admissible processes.

3. It is known that the spectrum $\sigma(\mathbf{a}) = \{\lambda \in \mathbb{C} : |\lambda| = 1, c(\lambda) \neq 0\}$ of a Hartman sequence $\mathbf{a}$ is countable [Ka]. If T is invertible and F is a 1-bounded admissible process with purely subadditive part $H = \{h_k\}$, then by Proposition 4.1, $\lim_n n^{-1} \sum_{k=0}^{n-1} \lambda^k h_k = 0$ a.e., and hence

$$c(\lambda) = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \lambda^k f_k = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \lambda^k T^k \delta.$$

So, the spectrum of the weights induced by F is the same as the spectrum of the weights induced by the additive process defined by its exact dominant. Consequently, following the arguments in [LOT, Theorem 3.15], the limit in Theorem 3.1 is equal to the limit of the additive part.

We close this section with the following observation on the behavior of the purely subadditive part of an admissible process. Recall that, if a sequence $\{a_i\}$ has mean zero (i.e. $\lim_n n^{-1} \sum_{k=0}^{n-1} a_i = 0$), then there exists a set $K \subset \mathbb{N}$ of zero density such that $\lim_{n \in \mathbb{N} \setminus K, \, n \to \infty} a_n = 0$. So, for a fixed $x \in X$, since $\lim_n n^{-1} \sum_{k=0}^{n-1} h_i = 0$, there exists a set $K \subset \mathbb{N}$ of density 0 such that $\lim_{n \in \mathbb{N} \setminus K, \, n \to \infty} h_n(x) = 0$. On the other hand, since $H_n \leq 0$, we also have

$$0 \leq \int \frac{1}{n} H_n = \frac{1}{n} \int \sum_{k=0}^{n-1} h_i = \frac{1}{n} \sum_{k=0}^{n-1} \int h_i.$$

By subadditivity, $H_n \leq \sum_{k=0}^{n-1} T^i h_0$, which implies that

$$0 \leq \frac{1}{n} H_n \leq \frac{1}{n} \sum_{k=0}^{n-1} T^i h_0$$

and the right hand side converges, so is bounded. Hence the sequence $\{n^{-1} H_n\}$ is bounded. Now, if $g_n = n^{-1} \sum_{k=0}^{n-1} T^i h_0$, then $n^{-1} H_n \leq g_n$, and $g_n \to g^*$ a.e. for some $g^* \in L_1$. Since $n^{-1} H_n \to 0$ a.e., by the generalized Lebesgue convergence theorem, $\lim \int g_n \to \int g^*$ implies that $0 = \lim_n n^{-1} \int H_n = \lim_n n^{-1} \sum_{k=0}^{n-1} \int h_i$. Consequently, there exists a set $K \subset \mathbb{N}$ of density 0 such that $\lim_{n \in \mathbb{N} \setminus K, n \to \infty} \int h_n(x) = 0$. Since, by admissibility, $h_m \leq T^k h_{m-k}$ for any $m \geq k \geq 0$, we must have $\lim_{n \to \infty} \int h_n(x) = 0$ while n ranges through all positive integers.

5. Norm convergence of averages along generalized Besicovitch weights. In what follows we will still consider a probability space (X, Σ, μ) and a measure preserving transformation $T : X \to X$, not necessarily invertible. We will also consider the group U of all unimodular functions on X:

$$U = \{u \in L_\infty(X) : |u(x)| = 1 \text{ a.e. on } X\}.$$

One can deduce from the Wiener–Wintner theorem that for every $f \in L_1$ there exists a null set N such that for $x \notin N$ and any $u \in U$ the averages

$$\frac{1}{n} \sum_{k=0}^{n-1} u^k(x) f(T^k x)$$

converge whenever $|u(x)| = 1$. This observation naturally motivates the question of whether it is possible to obtain the norm and a.e. convergence

of the averages of the form

$$(*) \qquad \frac{1}{n} \sum_{k=0}^{n-1} u^k(T^k x) f(T^k x) = \frac{1}{n} \sum_{k=0}^{n-1} [(u^k f) \circ T^k](x)$$

if $u \in U$ and $f \in L_1(X)$.

In this section we shall examine the norm convergence of the averages $(*)$. Note that, due to the nature of these averages, a standard argument with the telescoping sum effect for a uniformly bounded transformation in a Banach space appears to be of no help. Still, it would be desirable to obtain the L_1-norm convergence of the averages $(*)$ for all $u \in U$ and $f \in L_1$. The argument in this section will allow us to show that the norm convergence takes place when $u \in U$ assumes countably many values; in this case we will write $u \in U_c$. Define

$$U_1 = \{u \in U : \{u^k : k \in \mathbb{Z}\} \text{ is precompact in } L_1\}.$$

As is seen below the class U_1 is not empty. In fact, Corollary 5.3 provides a method of constructing nontrivial representatives of U_1. At the same time, the next example indicates that $U_1 \neq U$. The authors are thankful to the anonymous referee for this example.

EXAMPLE 5.1. If $X = \mathbb{K}$ with Lebesgue's measure, and $u(z) = z$, $z \in \mathbb{K}$, then $u \notin U_1$. In order to see this consider the sequence $u_n(z) = z^n$, $n = 1, 2, \ldots$, and show that, given any $h \in L_\infty(\mathbb{K})$, we have $\int u_n h \, d\mu \to 0$ as $n \to \infty$. Indeed, for every positive integer m,

$$\int z^m \, d\mu = \int_0^1 e^{itm} \, dt \to 0$$

as $m \to \infty$, therefore, $\int u_n p \, d\mu \to 0$ for every polynomial p on $\mathbb{K}$. Since the polynomials are dense in $L_1(\mathbb{K})$, we have $u_n \to 0$ weakly in L_1. Together with the facts that U is closed in L_1 and $0 \notin U$, this implies that $\{u^n : n \in \mathbb{N}\}$ cannot have a subsequence converging in L_1, hence $\{u^k : k \in \mathbb{Z}\}$ is not precompact in L_1.

A sequence $\{E_n\}_{n=1}^\infty \subset \Sigma$ will be called a *disjoint measurable partition* (*d.m.p.*) of X if $\mu(E_n) > 0$ for each n, $X = \bigcup E_n$, and $E_i \cap E_j = \emptyset$ for $i \neq j$.

LEMMA 5.2. *Let $g_n : X \to \mathbb{K}$ be a sequence of measurable functions, and let $\{E_i\}$ be a d.m.p. of X such that $g_n|_{E_i} \equiv \lambda_{ni} = $ const for all $n, i = 1, 2, \ldots$. Then $\{g_n\}$ has a subsequence $\{g_{n_m}\}$ converging in L_1-norm.*

Proof. We define the sets G_m inductively as follows. Let $G_0 = \{g_n\}$ and $I_0 = \mathbb{N}$. Let ν_i be an accumulation point of the set $\{\lambda_{ni}\}_{n \in I_{m-1}}$, and let

$$G_m = \bigcap_{i=1}^{m} \{g_n \in G_{m-1} : |\nu_i - \lambda_{ni}| < [2^i m \mu(E_i)]^{-1}\},$$

$$I_m = \{n : g_n \in G_m\}, \quad m = 1, 2, \ldots.$$

Define g_0 to be such that $g_0|_{E_m} \equiv \nu_m$, $m = 1, 2, \ldots$, and pick $g_{n_m} \in G_m$, $m \geq 1$. Then we have

$$
\begin{aligned}
\|g_0 - g_{n_m}\|_1 &= \sum_{i=1}^{m} \int_{E_i} |g_0 - g_{n_m}| \, d\mu + \sum_{i=m+1}^{\infty} \int_{E_i} |g_0 - g_{n_m}| \, d\mu \\
&\leq \sum_{i=1}^{m} \int_{E_i} |\nu_i - \lambda_{n_m i}| \, d\mu + 2 \sum_{i=m+1}^{\infty} \mu(E_i) \\
&< \frac{1}{m} \sum_{i=1}^{m} \frac{1}{2^i} + 2 \sum_{i=m+1}^{\infty} \mu(E_i) \\
&< \frac{1}{m} + 2 \sum_{i=m+1}^{\infty} \mu(E_i) \to 0 \quad \text{as } m \to \infty. \blacksquare
\end{aligned}
$$

COROLLARY 5.3. $U_c \subset U_1$.

REMARK. The authors do not know whether or not $U_c = U_1$.

Now we shall state the following straightforward fact without proof.

LEMMA 5.4. *Let $(B, \|\cdot\|)$ be a Banach space, and let $\{A_n\} \subset B$ be a sequence such that for every $\varepsilon > 0$ there is a convergent sequence $\{\widehat{A}_n\} \subset B$ and a positive integer N for which*

$$\|A_n - \widehat{A}_n\| < \varepsilon \quad \text{for all } n \geq N.$$

Then the sequence $\{A_n\}$ also converges in B.

Let $\mathbb{L}_\infty(X)$ denote the *-algebra of all measurable functions on X (and not their equivalence classes). Since μ is finite, there exists a *-homomorphism $L_\infty(X) \ni g \mapsto \widetilde{g} \in \mathbb{L}_\infty(X)$ such that $\widetilde{1}(x) = 1$ and $\widetilde{0}(x) = 0$ for all $x \in X$. Such a homomorphism is called a *lifting* (see [Ku], for example).

THEOREM 5.5. *For every $u \in U_c$ and $f \in L_\infty$, the averages*

$$A_n(u; f) = \frac{1}{n} \sum_{k=0}^{n-1} (u^k f) \circ T^k$$

converge in L_p-norm for all $1 \leq p < \infty$.

Proof. We start with an adaptation of the scheme of the proof of Theorem 5 in [Ry]. Let $[u]$ denote the L_1-closure of the group $\{u^k : k \in \mathbb{Z}\}$, which, by Corollary 5.3, is a compact group. Consider the Haar measure

on $[u]$. Define

$$X' = X \otimes [u]$$

and $T_u : X' \to X'$ by

$$T_u(x, v) = (Tx, uv).$$

By the invariance property of Haar measure, T_u is a MPT on X'. Let $L_\infty \ni g \mapsto \widetilde{g} \in \mathbb{L}_\infty$ be a lifting. If we define f' on X' by $f'(x, v) = (\widetilde{v}f)(x) = \widetilde{v}(x)f(x)$, then f' is measurable by Theorem A.5 (see Appendix), so $f' \in L_\infty(X')$. By Birkhoff's individual ergodic theorem, the averages

$$\frac{1}{n}\sum_{k=0}^{n-1} f'(T_u^k(x, v)) = \frac{1}{n}\sum_{k=0}^{n-1} f'(T^k x, u^k v) = \frac{1}{n}\sum_{k=0}^{n-1} (\widetilde{u^k v f})(T^k x)$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} (u^k v f)(T^k x)$$

converge for almost all $(x, v) \in X'$. It follows from Fubini's theorem that, for almost all $v \in [u]$, the averages

$$A_n(u, v; f) = \frac{1}{n}\sum_{k=0}^{n-1} (u^k v f) \circ T^k$$

converge a.e. on X. Since the L_∞-norms of these averages are bounded (by $\|f\|_\infty$), the bounded convergence theorem yields L_p-convergence of $A_n(u, v; f)$ for almost every $v \in [u]$.

Since the Haar measure of an open set is positive, any L_1-neighborhood of the unit $1 \in [u]$ contains a $v \in [u]$ for which $\{A_n(u, v; f)\}$ converges in L_p. Therefore, for $\varepsilon > 0$ one can find $v \in [u]$ such that

$$\{A_n(u, v; f)\} \text{ converges in } L_p \quad \text{and} \quad \|1 - v\|_1 < 2\left(\frac{\varepsilon}{2\|f\|_\infty}\right)^p.$$

Then

$$\|A_n(u; f) - A_n(u, v; f)\|_p$$

$$= \left\|\frac{1}{n}\sum_{k=0}^{n-1}(u^k f - u^k v f) \circ T^k\right\|_p \leq \frac{1}{n}\sum_{k=0}^{n-1}\|u^k f - u^k v f\|_p$$

$$\leq \frac{1}{n}\sum_{k=0}^{n-1}\|u^k\|_\infty\|f - vf\|_p = \|f - vf\|_p \leq \|1 - v\|_p\|f\|_\infty$$

$$\leq \|1 - v\|_\infty^{(p-1)/p} \cdot \|1 - v\|_1^{1/p} \cdot \|f\|_\infty$$

$$< 2^{(p-1)/p} \cdot 2^{1/p} \cdot \frac{\varepsilon}{2\|f\|_\infty} \cdot \|f\|_\infty = \varepsilon.$$

Thus, by Lemma 5.4, $\{A_n(u;f)\}$ converges in L_p-norm for every $1 \leq p < \infty$. ∎

REMARKS. 1. The function F on $X \otimes [u]$ defined by $F(x,v) = \widetilde{v}(x)$ ceases to be well defined if $\widetilde{v}$ is replaced by v. However, it would be possible to avoid the usage of lifting if the group $[u]$ were countable, which, in general, is not true.

2. By Theorem A.5 in the Appendix, the function F is measurable on $X \otimes [u]$ only if u assumes countably many values. That is why in Theorem 5.5 we assume that $u \in U_c$.

COROLLARY 5.6. *Given any $u \in U_c$ and $f \in L_p$, the averages $A_n(u;f)$ converge in L_p-norm for every $1 \leq p < \infty$.*

Proof. Fix $\varepsilon > 0$, and let $h \in L_\infty$ be such that $\|f - h\|_p < \varepsilon$. By Theorem 5.5, the averages $A_n(u;h)$ converge in L_p. We also have

$$\|A_n(u;f) - A_n(u;h)\|_p \leq \frac{1}{n} \sum_{k=0}^{n-1} \|u^k\|_\infty \|f - h\|_p < \varepsilon.$$

Therefore, by Lemma 5.4, we obtain L_p-convergence of $A_n(u;f)$. ∎

COROLLARY 5.7. *For every function $u \in U_c$, the averages*

$$A_n(u) = \frac{1}{n} \sum_{k=0}^{n-1} u^k \circ T^k$$

converge in L_p-norm for every $1 \leq p < \infty$.

Assume now that G is a subset of the group U.

DEFINITION 5.8. $P_s : \mathbb{Z} \to L_\infty$ is called a *trigonometric polynomial over G* if

$$P_s(\cdot) = \sum_{j=1}^{s} r_j u_j^{(\cdot)}$$

for some $\{r_j\}_{j=1}^{s} \subset \mathbb{C}$ and $\{u_j\}_{j=1}^{s} \subset G$.

By linearity, from Corollary 5.6 we obtain the following.

PROPOSITION 5.9. *Let P_s be a trigonometric polynomial over the group U_c. Then, for every $f \in L_p$, the averages*

$$A_n(P_s; f) = \frac{1}{n} \sum_{k=0}^{n-1} (P_s(k)f) \circ T^k$$

converge in L_p for all $1 \leq p < \infty$.

DEFINITION 5.10. Let $1 \leq p \leq \infty$. A sequence $\{b_k\} \subset L_p$ is called a *G_p-Besicovitch (G_p) sequence* if for every $\varepsilon > 0$ there is a trigonometric

polynomial P_s over G such that

$$(**) \qquad \limsup_n \frac{1}{n} \sum_{k=0}^{n-1} \|b_k - P_s(k)\|_p < \varepsilon.$$

A G_p sequence $\{b_k\}$ is called *bounded* if there exists a constant C such that $\|b_k\|_\infty \le C < \infty$ for every k.

THEOREM 5.11. *Let $1 \le q \le p < \infty$ be such that $1/p + 1/q = 1$. If $\{b_k\}$ is a $(U_c)_p$ sequence and $f \in L_\infty$, then the averages*

$$A_n(\{b_k\}; f) = \frac{1}{n} \sum_{k=0}^{n-1} (b_k f) \circ T^k$$

converge in L_p-norm. For $f \in L_p$, the averages $A_n(\{b_k\}; f)$ converge in L_p-norm if $\{b_k\}$ is a bounded $(U_c)_p$ sequence. If $\{b_k\}$ is a $(U_c)_p$ sequence with $\|b_k\|_q \le C < \infty$ for every k, then these averages converge in L_1 for all $f \in L_p$.

Proof. Let $\{b_k\}$ be a $(U_c)_p$ sequence and assume that $f \in L_\infty$. Fix $\varepsilon > 0$ and choose P_s such that $(**)$ is satisfied. Then

$$\|A_n(\{b_k\}; f) - A_n(P_s; f)\|_p \le \frac{1}{n} \sum_{k=0}^{n-1} \|(b_k - P_s(k))f\|_p$$

$$\le \frac{1}{n} \sum_{k=0}^{n-1} \|b_k - P_s(k)\|_p \|f\|_\infty < \varepsilon \|f\|_\infty$$

for all n large enough. Hence, by Lemma 5.4 and Proposition 5.9, the averages $A_n(\{b_k\}; f)$ converge in L_p-norm.

Now, let $\{b_k\}$ be a bounded $(U_c)_p$ sequence, and let $f \in L_p$. Fix $\varepsilon > 0$ and pick $h \in L_\infty$ such that $\|f - h\|_p < \varepsilon$. By the first part of this proof, the sequence $\{A_n(\{b_k\}; h)\}$ converges in L_p. Moreover,

$$\|A_n(\{b_k\}; f) - A_n(\{b_k\}; h)\|_p \le \|b_k\|_\infty \|f - h\|_p < C\varepsilon,$$

which, by Lemma 5.4, yields the desired convergence.

Finally, let $\{b_k\}$ be a $(U_c)_p$ sequence with $\|b_k\|_q \le C < \infty$ for every k. Let $f \in L_p$. Given $\varepsilon > 0$, one can find $h \in L_\infty$ with $\|f - h\|_p < \varepsilon$. Again, we know that $\{A_n(\{b_k\}; h)\}$ converges in L_1. Also,

$$\|A_n(\{b_k\}; f) - A_n(\{b_k\}; h)\|_1 \le \|b_k\|_q \|f - h\|_p < \varepsilon C,$$

and the proof is complete by Lemma 5.4. ∎

6. Almost everywhere convergence along generalized Besicovitch weights. In this section we will study almost everywhere convergence of the averages of the type $(*)$. It turns out that the tools available are far

from leading to a conclusive result if one considers the whole group U. Instead, we will restrict ourselves to the subgroup U_{F} of U_{c} consisting of all unimodular functions that assume a finite number of values.

PROPOSITION 6.1. *For every $u \in U_{\mathrm{F}}$ and any function $f \in L_1$, the averages $(*)$ converge almost everywhere.*

Proof. Pick $u \in U_{\mathrm{F}}$, and let $\{\lambda_1, \ldots, \lambda_m\}$ with $\lambda_i \neq \lambda_j$, $i \neq j$, be the range of u. Then $u = \sum_{i=1}^{m} \lambda_i \chi_{E_i}$, where $E_i = u^{-1}(\lambda_i)$, $i = 1, \ldots, m$, which implies that

$$u^k = \sum_{i=1}^{m} \lambda_i^k \chi_{E_i}$$

for every k. Therefore,

$$A_n(u; f)(x) = \frac{1}{n} \sum_{k=0}^{n-1} \Big(\sum_{i=1}^{m} \lambda_i^k \chi_{E_i} f \Big)(T^k x) = \sum_{i=1}^{m} \frac{1}{n} \sum_{k=0}^{n-1} \lambda_i^k f_i(T^k x),$$

where we set $f_i = f \chi_{E_i}$, so $f_i \in L_1$. Since the averages $n^{-1} \sum_{k=0}^{n-1} \lambda_i^k f_i(T^k x)$ converge a.e. [Ry] for every i, we get the desired convergence. ∎

REMARKS. 1. One can check that if $u \in U$, then the Dunford–Schwartz operator T_u given by $(T_u f)(x) = (uf)(Tx)$, $f \in L_1$, does not generate the averages $(*)$, so one cannot employ the idea of the proof in [LO] to obtain the almost everywhere convergence of $(*)$ for $u \in U$.

2. It may seem that, due to Lemma 6.3 below, applying the argument of Theorem 5.5 one can strengthen the result of Proposition 6.1 if G consists of elements u such that $\{u^k : k \in \mathbb{Z}\}$ is precompact in L_∞. But the L_∞-precompactness of $\{u^k\}$ implies that $u \in U_{\mathrm{F}}$:

PROPOSITION 6.2. *If $u \in U$, then the group $\{u^k : k \in \mathbb{Z}\}$ is precompact in L_∞ if and only if u assumes finitely many values.*

Proof. The "if" part is obvious. Now, let $\{u^k : k \in \mathbb{Z}\}$ be precompact in L_∞. Then, by the spectral theorem, $\{z^k : k \in \mathbb{Z}\}$ is precompact in $C(\Delta)$, where Δ is the closure of the range of u. Assume that Δ is not finite. Then one can find a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \Delta$ converging to some $\lambda_0 \in \Delta$ such that $\lambda_n \neq \lambda_0$ for every n. By Arzelà's theorem, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|z^k - w^k| < \varepsilon$ for all k whenever $|z - w| < \delta$. Let $\varepsilon = 1$; there exists n_0 such that for $n > n_0$ we have $|\lambda_n - \lambda_0| < \delta$, so $|(\lambda_n \lambda_0^{-1})^k - 1| < 1$ for every k. But this is possible only if $\lambda_n \lambda_0^{-1} = 1$ for $n > n_0$, a contradiction. Therefore, Δ is finite, and hence so is the range of u. ∎

To proceed with the almost everywhere convergence along generalized Besicovitch weights we will need the following generalization of Lemma 5.4. Note that in [Ry] a version of this result is used implicitly. However, due to its important role, we found it worthwhile to state and prove it separately.

LEMMA 6.3. *Let $\{A_n\}$ be a sequence of measurable functions such that, given $\varepsilon, \delta > 0$, there exist a set $E \subset X$ with $\mu(E^c) < \varepsilon$, an almost everywhere convergent sequence $\{\widehat{A}_n\}$ and a positive integer N with*

$$\|(A_n - \widehat{A}_n)\chi_E\|_\infty < \delta \quad \text{for all } n \geq N.$$

Then $\{A_n\}$ itself converges almost everywhere. In particular, $\{A_n\}$ converges a.e. if the above condition holds with $E = X$.

Proof. By Egorov's theorem, given $\varepsilon, \delta > 0$, it is enough to find $E \subset X$ with $\mu(E^c) < \varepsilon$ and a number N such that

$$\|(A_m - A_n)\chi_E\|_\infty < \delta \quad \text{whenever } m, n \geq N.$$

By the assumption, we can find an almost everywhere convergent sequence $\{\widehat{A}_n\}$, a set $E_1 \subset X$ with $\mu(E_1^c) < \varepsilon/2$, and a number N_1 such that

$$\|(A_n - \widehat{A}_n)\chi_{E_1}\|_\infty < \delta/3 \quad \text{when } n \geq N_1.$$

Then, by Egorov's theorem, find $E_2 \subset X$ with $\mu(E_2^c) < \varepsilon/2$ such that $\widehat{a}_n$ converges uniformly on E_2. Therefore, it is possible to find a number N_2 for which

$$\|(\widehat{A}_m - \widehat{A}_n)\chi_{E_2}\|_\infty < \delta/3 \quad \text{whenever } m, n \geq N_2.$$

Finally, letting $N = \max\{N_1, N_2\}$ and $E = E_1 \cap E_2$, we get $\mu(E^c) < \varepsilon$ and

$$\|(A_m - A_n)\chi_E\|_\infty$$
$$\leq \|(A_m - \widehat{A}_m)\chi_E\|_\infty + \|(\widehat{A}_m - \widehat{A}_n)\chi_E\|_\infty + \|(A_n - \widehat{A}_n)\chi_E\|_\infty < \delta$$

as soon as $m, n \geq N$. ∎

Now, by linearity, from Proposition 6.1 we obtain the following result.

PROPOSITION 6.4. *Let P_s be a trigonometric polynomial over the group U_F. Then the averages $A_n(P_s; f)$ converge a.e. for every $f \in L_1$.*

THEOREM 6.5. *If $\{b_k\}$ is a $(U_F)_\infty$ sequence (Definition 5.7), then the averages $A_n(\{b_k\}; f)$ converge a.e. for every $f \in L_\infty$. If $\{b_k\}$ is a bounded $(U_F)_\infty$ sequence, then these averages converge for all $f \in L_1$.*

Proof. Fix $\varepsilon > 0$, and let P_s be such that condition $(**)$ is satisfied. By Proposition 6.4, the averages $A_n(P_s; f)$ converge a.e. Further,

$$\|A_n(\{b_k\}; f) - A_n(P_s; f)\|_\infty = \left\|\frac{1}{n}\sum_{k=0}^{n-1}(b_k f)\circ T^k - \frac{1}{n}\sum_{k=0}^{n-1}(P_s(k)f)\circ T^k\right\|_\infty$$

$$\leq \frac{1}{n}\sum_{k=0}^{n-1}\|((b_k - P_s(k))f)\circ T^k\|_\infty$$

$$\leq \frac{1}{n}\sum_{k=0}^{n-1}\|b_k - P_s(k)\|_\infty\|f\|_\infty.$$

Therefore, for large enough n, we obtain

$$\|A_n(\{b_k\}; f) - A_n(P_s; f)\|_\infty \leq \limsup_n \frac{1}{n} \sum_{k=0}^{n-1} \|b_k - P_s(k)\|_\infty \cdot \|f\|_\infty$$
$$< \varepsilon \|f\|_\infty,$$

which, by Lemma 6.3, completes the first part of the proof.

Now, observe that, if $\{b_k\}$ is a bounded $(U_{\mathrm{F}})_\infty$ sequence, then, for every $f \in L_1$,

$$|A_n(\{b_k\}; f)(x)| = \left| \frac{1}{n} \sum_{k=0}^{n-1} b_k(T^k x) \cdot f(T^k x) \right| \leq \|b_k\|_\infty \cdot \frac{1}{n} \sum_{k=0}^{n} |f|(T^k x)$$

almost everywhere, and the averages $n^{-1} \sum_{k=0}^{n-1} |f|(T^k x)$ converge almost everywhere by the pointwise ergodic theorem for $|f| \in L_1$. Hence, $\sup_n |A_n(\{b_k\}; f)(x)| < \infty$ a.e. on X, for all $f \in L_1$. Since we have the a.e. convergence of $A_n(\{b_k\}; f)$ for all $f \in L_\infty$, and L_∞ is dense in L_1, by the Banach principle, the assertion follows. ∎

REMARK. If one replaces U_{F} with $\{u \in U : u(x) \equiv \lambda, |\lambda| = 1\}$, then the classical result [Ry] follows: if $\{\beta_k\} \subset \mathbb{C}$ is a bounded Besicovitch sequence, then the weighted averages

$$A_n(\{\beta_k\}; f)(x) = \frac{1}{n} \sum_{k=0}^{n-1} \beta_k f(T^k x)$$

converge almost everywhere on X for all $f \in L_1$.

Now, using the tools employed to obtain the basic result in Section 2, we generalize Theorem 6.5 to the setting of T-admissible processes.

THEOREM 6.6. *Let $F = \{f_n\} \subset L_1$ be a bounded T-admissible process, where T is an invertible MPT, and $\{b_i\}$ be a bounded $(U_{\mathrm{F}})_\infty$ sequence. Then the averages*

$$A_n(\{b_i\}; F)(x) := \frac{1}{n} \sum_{i=0}^{n-1} b_i(T^i x) f_i(x)$$

converge almost everywhere on X.

Proof. We use the same machinery as in Theorem 2.2. For a fixed $m \in \mathbb{Z}^+$, define

$$g_i^m(x) = \begin{cases} f_m(T^{i-m} x) & \text{for } i > m, \\ f_i(x) & \text{for } 0 \leq i \leq m. \end{cases}$$

Hence $f_i - g_i^m \leq T^i c_m$ and

$$D_n := \frac{1}{n} \sum_{i=0}^{n-1} (f_i - g_i^m) \leq \frac{1}{n} \sum_{i=0}^{n-1} T^i c_m, \quad \text{where} \quad c_m = \delta - v_m.$$

Now, let

$$f^*(x) = \limsup_n A_n(\{b_i\}; F)(x), \qquad f_*(x) = \liminf_n A_n(\{b_i\}; F)(x).$$

Then

$$0 \leq |f^*(x) - f_*(x)| \leq 2 \limsup_n |A_n(\{b_i\}; F)(x) - A_n(\{b_i\}; \{g_i^m\})(x)|$$

$$\leq 2 \limsup_n \left| \frac{1}{n} \sum_{i=0}^{n-1} b_i(T^i x)(f_i - g_i^m)(x) \right|$$

$$\leq 2B \limsup_n D_n(x).$$

Given $\alpha > 0$, define $E = \{\omega : f^*(x) - f_*(x) > \alpha\}$. In order to prove a.e. convergence of $A_n(\{b_i\}; F)$ it is enough to show that $\mu(E) = 0$. First, $\lim_n A_n(\{b_i\}; g_i^m)(x)$ exists a.e. by Theorem 6.5. Thus,

$$E \subset \left\{ x : \sup_n |D_n(x)| > \frac{\alpha}{2B} \right\} \subset \left\{ x : \sup_n \left| \frac{1}{n} \sum_{i=0}^{n-1} T^i c_m(x) \right| > \frac{\alpha}{2B} \right\}.$$

By the a.e. convergence, the additive process $\{\sum_{i=0}^{n-1} T^i c_m\}$ admits a maximal inequality along $\{b_i\}$. Therefore, for some constant C,

$$\mu(E) \leq \mu\left(\left\{ x : \sup_n \left| \frac{1}{n} \sum_{i=0}^{n-1} T^i c_m(x) \right| > \frac{\alpha}{2B} \right\} \right) \leq \frac{2BC}{\alpha} \|c_m\|_1.$$

Since $\|c_m\|_1 \downarrow 0$ as $m \to \infty$, we obtain $\mu(E) = 0$. $\blacksquare$

As mentioned earlier, the Wiener–Wintner theorem implies that, for any function $u \in U$, the averages $n^{-1} \sum_{k=0}^{n-1} u^k(x) f(T^k x)$ converge a.e. for all $f \in L_1$. Consequently, for all $f \in L_1$ and any trigonometric polynomial P_s over U, the averages

$$\frac{1}{n} \sum_{k=0}^{n-1} P_s(k)(x) f(T^k x)$$

converge a.e. Therefore, one can also define weights using linear combinations of u^k's, $u \in U$, and ask whether such weights are good for a.e. convergence. The following theorems are obtained by using the methods used above in this section verbatim. Hence, we will state them only.

THEOREM 6.7. *Let $\{b_k\}$ be a bounded U_∞ sequence (Definition 5.10). For every $f \in L_1$ the weighted averages $n^{-1} \sum_{k=0}^{n-1} b_k(x) f(T^k x)$ converge almost everywhere on X.*

THEOREM 6.8. *Let* $F = \{f_n\} \subset L_1$ *be a bounded T-admissible process, where T is invertible, and $\{b_k\}$ be a bounded U_∞ sequence. Then the averages $n^{-1} \sum_{k=0}^{n-1} b_k(x) f_k(x)$ converge almost everywhere on X.*

7. Identification of the limits in Theorems 5.11, 6.5, and 6.7. As in [Ry], we will identify the limits when T is a weakly mixing transformation. Recall that a measure preserving transformation T is weakly mixing if and only if

(i) T is ergodic, i.e. $f \in L_1(X)$, $f \circ T = f$ imply $f = \text{const}$; and

(ii) $f \in L_1(X), f \circ T = \lambda \cdot f, \lambda \neq 1$ imply $f = 0$.

PROPOSITION 7.1. *Let $\{b_k\}$ be a bounded $(U_c)_p$ sequence (Definition 5.10). Then the averages $A_n(\{b_k\}; f)$ converge in L_p to constants for every $f \in L_p$ and every $\{b_k\}$ if and only if the transformation T is weakly mixing.*

Proof. Repeating a standard argument [Ry], one can check that if the averages $A_n(\{b_k\}; f)$ converge in L_p to constants for all $f \in L_p$, then T is weakly mixing.

Next, assume that T is weakly mixing. Pick $u \in U_c$, and let $f \in L_p$. By Corollary 5.6, we have

$$(1) \qquad A_n(u; f) = \frac{1}{n} \sum_{k=0}^{n-1} (u^k f) \circ T^k \to f_u \quad \text{in } L_p.$$

We shall show that f_u is constant. Applying the spectral theorem to the unitary function u, one has

$$u = \| \cdot \|_\infty\text{-} \lim_{m \to \infty} \sum_{j=1}^{m} \lambda_j(m) \chi_{E_j(m)},$$

where $|\lambda_j(m)| = 1$ for all m and j, and $\mu(E_j(m) \cap E_{j'}(m)) = 0$ for every m whenever $j \neq j'$. Therefore, for a fixed k,

$$\sum_{j=1}^{m} \lambda_j^k(m) \chi_{E_j(m)} = \Big[\sum_{j=1}^{m} \lambda_j(m) \chi_{E_j(m)} \Big]^k \to u^k \quad \text{in } L_\infty$$

as $m \to \infty$, which implies that

$$\Big(\sum_{j=1}^{m} \lambda_j^k(m) \chi_{E_j(m)} f \Big) \circ T^k \to (u^k f) \circ T^k \quad \text{in } L_p$$

for every k as $m \to \infty$. It now follows that

$$(2) \qquad f_{mn} = \frac{1}{n} \sum_{k=0}^{n-1} \Big(\sum_{j=1}^{m} \lambda_j^k(m) \chi_{E_j(m)} f \Big) \circ T^k \to a_n(u; f) \quad \text{in } L_p$$

as $m \to \infty$. On the other hand,

$$f_{mn} = \sum_{j=1}^{m} \frac{1}{n} \sum_{k=0}^{n-1} \lambda_j^k(m) f_j(m) \circ T^k,$$

where $f_j(m) = \chi_{E_j(m)} f \ (\in L_p)$. By Theorem 7 in [Ry], the averages $n^{-1} \sum_{k=0}^{n-1} \lambda_j^k(m) f_j(m) \circ T^k$ converge a.e. to a constant for all j. Therefore, by the bounded convergence theorem, it can be easily seen that these averages converge to a constant in L_p for every j. Thus,

$$(3) \qquad f_{mn} \to c_m = \text{const}, \qquad n \to \infty.$$

Now, due to (1)–(3), it is possible to find a subsequence c_{m_l} converging to f_u in L_p. Then it can be easily verified that f_u is constant.

Let now $\{b_k\}$ be a bounded $(U_c)_p$ sequence. Then for every m one can find a trigonometric polynomial $P_{s(m)}$ over U such that

$$\limsup_{n} \frac{1}{n} \sum_{k=0}^{n-1} \|b_k - P_{s(m)}(k)\|_\infty < \frac{1}{m}.$$

For $h \in L_\infty$, defining

$$A_{mn} = A_n(P_{s(m)}; h) = \frac{1}{n} \sum_{k=0}^{n-1} (P_{s(m)}(k)h) \circ T^k,$$

we get

$$\|A_n(\{b_k\}; h) - A_{mn}\|_p \leq \frac{1}{n} \sum_{k=0}^{n-1} \|b_k - P_{s(m)}(k)\|_p \cdot \|h\|_\infty < \frac{\|h\|_\infty}{m}$$

for all $n \geq n(m)$. Therefore, for some subsequence $\{n_m\}$, we have

$$(4) \qquad \|A_{n_m}(\{b_k\}; h) - A_{mn_m}\|_p \to 0, \qquad m \to \infty.$$

At the same time, by the first part of the proof,

$$(5) \qquad A_{mn_m} \to A_m = \text{const} \qquad \text{in } L_p$$

as $m \to \infty$. Since we also know that the sequence $\{A_{n_m}(\{b_k\}; h)\}$ converges in L_p to some $\bar{h}$, taking (4) and (5) into account, we infer that $\bar{h}$ is constant.

Finally, let $\{h_m\} \subset L_\infty$ be such that $\|f - h_m\|_p < 1/m$. Then, for every m and n, we have

$$\|A_n(\{b_k\}; f) - A_n(\{b_k\}; h_m)\|_p \leq \|b_k\|_\infty \cdot \|f - h_m\|_p < C/m.$$

Remembering that $\{A_n(\{b_k\}; f)\}$ converges in L_p, one can see now that it must converge to a constant. ∎

THEOREM 7.2. *Let $\{b_k\}$ be a bounded $(U_F)_\infty$ sequence. Then the averages $A_n(\{b_k\}; f)$ converge a.e. to constants for every $f \in L_1$ if and only if T is weakly mixing.*

Proof. Again, it is enough to show the "if" part. If T is weakly mixing, since $U_\mathrm{F} \subset U_\mathrm{c}$, by Theorem 7.1, the averages $A_n(\{b_k\}; f)$ converge to constants in L_1 for all $f \in L_1$. Since we also know that these averages converge a.e. to some $\bar f \in L_1$, it follows that $\bar f$ is constant. ∎

In a similar fashion we obtain the following result.

THEOREM 7.3. *If $\{b_k\}$ is a bounded $(U_\mathrm{F})_\infty$ sequence, then the averages $n^{-1} \sum_{k=0}^{n-1} b_k(x) f(T^k x)$ converge a.e. to constants for every $f \in L_1$ if and only if T is weakly mixing.*

Appendix: A measurability criterion. The results of this section are due to A. Ber and V. Chilin (see Acknowledgements).

Let G be a compact group equipped with the Borel σ-algebra. If $\widehat{G}$ is the group of all continuous characters on G, then the following two results are known.

THEOREM A.1 (see [HR, Corollary 22.19]). *For a measurable character $\phi : G \to \mathbb{C}$, we have $\phi \in \widehat{G}$.*

THEOREM A.2 (see [HR, Theorem 24.15]). *If G is Abelian, then it is metrizable if and only if $\widehat{G}$ is countable.*

Let G be a metrizable compact subgroup of $U = \{v \in L_\infty(X) : |v(x)| = 1$ a.e.$\}$. Let $\mathcal{B}$ be the Borel σ-algebra on G. Define a function $F : X \otimes G \to \mathbb{C}$ by $F(x, v) = \widetilde{v}(x)$ (here $L_\infty \ni g \mapsto \widetilde{g} \in \mathbb{L}_\infty$ is a lifting; see Section 5). Also, if $x \in X$, we define $\widehat{x} : G \to \mathbb{C}$ by $\widehat{x}(v) = \widetilde{v}(x)$. Note that $\widehat{x}$ is a character on G for every $x \in X$.

THEOREM A.3. *If $\widehat{x}$ is measurable on G for every $x \in X$, then there exists a d.m.p. $\{X_n\}_{n=1}^\infty$ of X (see Section 5) such that, for any $v \in G$ and n, there is a constant $c_n(v)$ such that $v(x) = c_n(v)$ a.e. on X_n. Moreover, the function $F(x, v) = \widetilde{v}(x)$ is measurable on $X \otimes G$.*

Proof. By Theorem A.1, $\widehat{x} \in \widehat{G}$ for every $x \in X$. Furthermore, since G is Abelian and metrizable, Theorem A.2 shows that $\widehat{G} = \{\phi_n\}_{n=1}^\infty$, where $\phi_i \neq \phi_j$ whenever $i \neq j$. Set $\widehat{x} = \Phi(x)$, and let

$$A_n = \Phi^{-1}(\phi_n), \quad n = 1, 2, \ldots.$$

Then $A_i \cap A_j = \emptyset$ if $i \neq j$ and $X = \bigcup A_n$. Pick $x_n \in A_n$, $n = 1, 2, \ldots,$ arbitrarily. Since

$$x \in A_n \;\Leftrightarrow\; \Phi(x) = \phi_n(= \Phi(x_n)) \;\Leftrightarrow\; \widehat{x} = \widehat{x}_n$$
$$\Leftrightarrow\; \widetilde{v}(x) = \widetilde{v}(x_n) \,\forall v \in G \;\Leftrightarrow\; x \in (\widetilde{v})^{-1}(\widetilde{v}(x_n)) \,\forall v \in G,$$

we have

$$A_n = \bigcap_{v \in G} (\widetilde{v})^{-1}(\widetilde{v}(x_n)).$$

Also, since G is metrizable and compact, it is separable. Let $G = \overline{\{v_m\}_{m=1}^{\infty}}$. If we define

$$X_n = \bigcap_{m=1}^{\infty} (\widetilde{v}_m)^{-1}(\widetilde{v}_m(x_n)), \quad n = 1, 2, \ldots,$$

then each X_n is measurable.

We now show that $X_n = A_n$ for every n. Indeed, because $A_n \subset X_n$ and $X = \bigcup A_n$, it is enough to prove that $X_i \cap X_j = \emptyset$ if $i \neq j$. Assume that $x \in X_i \cap X_j$ and $i \neq j$. Then $\widetilde{v}_m(x) = \widetilde{v}_m(x_i)$ and, at the same time, $\widetilde{v}_m(x) = \widetilde{v}_m(x_j)$, that is, $\widetilde{v}_m(x_i) = \widetilde{v}_m(x_j)$, or $\widehat{x}_i(v_m) = \widehat{x}_j(v_m)$ for all m. Since $\{v_m\}$ is dense in G and $\widehat{x}_i$ and $\widehat{x}_j$ are continuous on G, we get $\widehat{x}_i = \widehat{x}_j$, which implies that $A_i = A_j$, a contradiction. Therefore, $\{X_n\}$ is a d.m.p. of X (we discard the sets X_i with zero measure). If we define $\widetilde{v}(x_n) = c_n(v)$, then for any $x \in X_n$ and $v \in G$ we have $\widetilde{v}(x) = \widetilde{v}(x_n) = c_n(v)$, so $v(x) = c_n(v)$ a.e. on X_n.

We now show that the function F is measurable. Let B be any Borel set in $\mathbb{C}$. Since $x \in X_n$ is equivalent to $\widetilde{v}(x) = \widetilde{v}(x_n)$ for all $v \in G$, we have

$$F^{-1}(B) = F^{-1}(B) \cap \bigcup_{n=1}^{\infty} (X_n \times G) = \bigcup_{n=1}^{\infty} (F^{-1}(B) \cap (X_n \times G))$$

$$= \bigcup_{n=1}^{\infty} \{(x, v) \in X_n \times G : \widetilde{v}(x) \in B\}$$

$$= \bigcup_{n=1}^{\infty} (X_n \times \{v \in G : \widetilde{v}(x_n) \in B\}) \in \Sigma \otimes \mathcal{B},$$

because, by our assumption, the set $\{v \in G : \widetilde{v}(x_n) \in B\} = \{v \in G : \widehat{x}_n(v) \in B\}$ is measurable. ∎

THEOREM A.4. *Let the metric in G be generated by the L_1-norm. If there exists a d.m.p. $\{X_n\}$ of X such that, for every $v \in G$ and n, there is a constant $c_n(v)$ such that $v(x) = c_n(v)$ a.e. on X_n, then the function $F(x, v) = \widetilde{v}(x)$ is measurable on $X \otimes G$.*

Proof. We will write χ_E for the characteristic function of a set $E \in \Sigma$, while $[\chi_E]$ will denote its class in $L_\infty(X)$. Obviously, $[\chi_{X_n}]^\sim = \chi_{E_n}$ for some $E_n \in \Sigma$. Also $\mu(X_n \triangle E_n) = 0$ and $[\chi_{E_n}]^\sim = \chi_{E_n}$, for $n = 1, 2, \ldots$. Define $Y_1 = E_1$ and $Y_n = E_n \setminus \bigcup_{i=i}^{n-1} E_i$, $n \geq 2$. Because $\chi_{Y_n} = \prod_{i=1}^{n-1} \chi_{E_n}(1 - \chi_{E_i})$, and lifting is a homomorphism, it follows that $[\chi_{Y_n}]^\sim = \chi_{Y_n}$ for every n. Since also $\mu(\bigcup_{n=1}^{\infty} Y_n) = 1$, $Y_i \cap Y_j = \emptyset$ if $i \neq j$, and $v(x) = c_n(v)$ a.e. on Y_n for every n, without loss of generality we can assume that $[\chi_{X_n}]^\sim = \chi_{X_n}$, $n = 1, 2, \ldots$. Then $v[\chi_{X_n}] = c_n(v)[\chi_{X_n}]$ implies that $\widetilde{v}(x) = c_n(v)$ for all $x \in X_n$ and $v \in G$.

By Theorem A.3, the proof will be finished if we show that the character $\widehat{x}$ is continuous on G for every $x \in X$. Let $v_k, v \in G$ and $\|v_k - v\|_1 \to 0$ as $k \to \infty$. For each pair k and m we have

$$|v - v_k|[\chi_{X_m}] = |(v - v_k)[\chi_{X_m}]| = |c_m(v) - c_m(v_k)|,$$

which implies that

$$\int\limits_{X_m} |v - v_k| \, d\mu = |c_m(v) - c_m(v_k)| \cdot \mu(X_m).$$

If $x \in X$, then there exists an index n_0 such that $x \in X_{n_0}$. Therefore,

$$|\widehat{x}(v) - \widehat{x}(v_k)| \cdot \mu(X_{n_0})$$
$$= |\widetilde{v}(x) - \widetilde{v}_k(x)| \cdot \mu(X_{n_0}) = |c_{n_0}(v) - c_{n_0}(v_k)| \cdot \mu(X_{n_0})$$
$$= \int\limits_{X_{n_0}} |v - v_k| \, d\mu \leq \int\limits_{X} |v - v_k| \, d\mu = \|v - v_k\|_1 \to 0,$$

that is, $\widehat{x}$ is continuous on G for every $x \in X$. ∎

The following is a direct consequence of Theorems A.3 and A.4.

THEOREM A.5. *Let (X, Σ, μ) be a probability space, and let G be an L_1-compact subgroup of the group U of all unimodular measurable functions on X. Consider the Borel σ-algebra on G. Let $f \mapsto \widetilde{f}$ be a lifting on $L_\infty(X)$. Then the function $F : X \otimes G \to \mathbb{C}$ given by $F(x, v) = \widetilde{v}(x)$ is measurable if and only if there exists a d.m.p. $\{X_n\}$ of X such that for every $v \in G$ and n there is a constant $c_n(v)$ with $v(x) = c_n(v)$ a.e. on X_n.*

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