Elementary operators on Banach algebras and Fourier transform

by

MILOŠ ARSENOVIĆ and DRAGOLJUB KEČKIĆ (Beograd)

Abstract. We consider elementary operators $x \mapsto \sum_{j=1}^{n} a_jx b_j$, acting on a unital Banach algebra, where $a_j$ and $b_j$ are separately commuting families of generalized scalar elements. We give an ascent estimate and a lower bound estimate for such an operator. Additionally, we give a weak variant of the Fuglede–Putnam theorem for an elementary operator with strongly commuting families $\{a_j\}$ and $\{b_j\}$, i.e. $a_j = a'_j + ia''_j$ ($b_j = b'_j + ib''_j$), where all $a'_j$ and $a''_j$ ($b'_j$ and $b''_j$) commute. The main tool is an $L^1$ estimate of the Fourier transform of a certain class of $C^\infty_{cpt}$ functions on $\mathbb{R}^{2n}$.

0. Introduction. The theory of generalized scalar operators on a Banach space was developed in [6]. Briefly, $a \in \mathcal{A}$ is a generalized scalar element of a unital Banach algebra $\mathcal{A}$ if it has real spectrum, and if for all real $t$, $\|e^{ita}\| \leq C(1 + |t|^s)$, for some constant $C$ depending only on $a$. Also, it is known that these two conditions are equivalent to the existence of a functional calculus for $a$, based on $\mathbb{R}$. If $s = 0$, we say that such an element is pre-hermitian. In that case the condition of having real spectrum is not necessary. Also we can define pre-normal elements as elements of the form $h + ik$ with $h, k$ pre-hermitian. Many properties of pre-hermitian, pre-normal, and generalized scalar elements can be found in [6] and [5]. In Section 1 we review results concerning such elements, necessary for reading this note.

In [13], a functional calculus for several commuting operators on a Banach space, using Fourier transform, was developed. In Section 2, we prove two results about $L^1$ behaviour of the Fourier transforms of a family of $C^\infty_{cpt}$ functions. These results have a central role in further applications to the theory of elementary operators on a unital Banach algebra.

Section 3 contains applications of the results from Section 2 to elementary operators on a unital Banach algebra $\mathcal{A}$, i.e. to mappings $\Lambda : \mathcal{A} \to \mathcal{A}$ of the form

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These operators were introduced by Lumer and Rosenblum [11]. They have been investigated in many papers, first on the algebra $B(H)$ of all bounded operators on a separable Hilbert space $H$. For important results on elementary operators acting on a Banach algebra, or on the algebra of all bounded operators on a Banach space, the reader is referred to [12], [17], [18] and references therein.

We give three independent applications. The first of them is an ascent estimate for an elementary operator (1), with generalized scalar $a_j$ and $b_j$. For a linear mapping $\Lambda : E \to E$ on an arbitrary linear space $E$, the ascent $\text{asc}(\Lambda)$ is defined as the least positive integer $k$ such that $\ker(\Lambda^k) = \ker(\Lambda^{k+1})$. If no such positive integer exists we set $\text{asc}(\Lambda) = +\infty$.

We estimate the ascent of the operator (1) in terms of the orders of $a_j$, $b_j$ and the dimension of the set $\sigma(a_1, \ldots, a_n) \times \sigma(b_1, \ldots, b_n)$.

The second application is a weak variant of the Fuglede–Putnam theorem for the operator (1), where \{a_j\} and \{b_j\} are strongly commuting families. This means that $a_j = a'_j + ia''_j$, $b_j = b'_j + ib''_j$, where \{a'_1, a''_1, \ldots, a'_n, a''_n\} and \{b'_1, b''_1, \ldots, b'_n, b''_n\} are commuting families of generalized scalar elements. This weak Fuglede–Putnam theorem asserts that $\Lambda(x) = 0$ implies $(\Lambda^*)^k(x) = 0$ for some positive integer $k$, where $\Lambda^*(x) = \sum_{j=1}^n a_j^* x b_j^*$, and $a_j^* = a'_j - ia''_j$, $b_j^* = b'_j - ib''_j$. We determine $k$ in terms of the orders of $a'_j$, $a''_j$, $b'_j$, $b''_j$ and, once again, the dimension of the set $\sigma(a'_1, a''_1, \ldots, a'_n, a''_n) \times \sigma(b'_1, b''_1, \ldots, b'_n, b''_n)$.

The third application is a norm estimate for the solution of the equation
\[
\sum_{j=1}^n a_j x b_j = y,
\]
in terms of the right hand side, provided that $0 \notin \{\lambda_1 \mu_1 + \cdots + \lambda_n \mu_n \mid \lambda_j \in \sigma(a_j), \mu_j \in \sigma(b_j)\}$.

Finally, we conclude this note with some questions that we have not been able to answer.

1. Preliminaries

**Definition 1.1.**

(a) We say that an element $a \in \mathcal{A}$ is **hermitian** if $\|e^{ita}\| = 1$ for all real $t$. The set of all hermitian elements of the algebra $\mathcal{A}$ is denoted by $\mathcal{H}(\mathcal{A})$.

(b) We say that an element $a \in \mathcal{A}$ is **pre-hermitian** if there exists $M < \infty$ such that $\|e^{ita}\| \leq M$ for all real $t$. The set of all pre-hermitian elements of $\mathcal{A}$ is denoted by $\mathcal{H}_1(\mathcal{A})$. 
(c) We say that an element $a \in A$ is normal if $a = h + ik$ for some $h, k \in \mathcal{H}(A)$ such that $hk = kh$, and pre-normal if $a = h + ik$ for some $h, k \in \mathcal{H}_1(A)$ such that $hk = kh$.

(d) The numerical range of $a \in A$ is the set

$$W(a) = \{ f(a) \mid f \in A^*, \| f \| = 1, f(e) = 1 \}.$$ 

**Proposition 1.1.**

(a) $W(a)$ is always a closed convex subset of $\mathbb{C}$, and $\sigma(a) \subseteq W(a)$, where $\sigma(a)$ is the spectrum of $a$.

(b) $a \in A$ is hermitian if and only if $W(a) \subseteq \mathbb{R}$, if and only if $\| 1 + ita \| = 1 + o(t)$ as $\mathbb{R} \ni t \to 0$.

(c) A real linear combination of two hermitian elements is always hermitian.

(d) For a finite family of mutually commuting pre-hermitian elements, there exists a norm on $A$ equivalent to the original one, making all of them hermitian.

(e) If $a = h + ik$, where $h, k \in \mathcal{H}(A)$, then $h$ and $k$ are uniquely determined.

**Proof.** Statements (a), (b), (c) and (e) are Theorems 1.3, 1.6 and Lemmas 5.2, 5.4 and 5.7 of [5], whereas statement (d) follows easily from Lemma 1.7 of [5].

**Proposition 1.2.**

(a) Let $a = h + ik$ be a pre-normal element, where $h, k \in \mathcal{H}_1(A)$, and suppose $ax = xa$ for some $x \in A$. Then $(h - ik)x = x(h - ik)$, $hx = xh$ and $kx = xk$.

(b) If $a = h + ik$ is a pre-normal element, $h, k \in \mathcal{H}_1(A)$, then $h$ and $k$ are uniquely determined.

**Proof.** (a) The proof of this part is essentially the same as Rosenblum’s well known proof of the Fuglede–Putnam theorem. Nevertheless we shall give it. Set $a^* = h - ik$. From $ax = xa$, it is easy to obtain by induction $\lambda^n a^n x = x \lambda^n a^n$ for all $\lambda \in \mathbb{C}$, and consequently $e^{\lambda a} x = xe^{\lambda a}$. Since $hk = kh$, it follows that $aa^* = a^*a$, and hence $e^{-\lambda a} xe^{\lambda a} = e^{\lambda a - \lambda a^*} xe^{-\lambda a + \lambda a^*}$. If we take $\lambda = \alpha + i\beta$, then we can easily compute $\lambda a - \lambda a^* = 2i(\alpha k - \beta h)$, and also $e^{\lambda a - \lambda a^*} = e^{2i\alpha k} e^{-i2\beta h}$ since $k$ and $h$ commute with each other. Therefore $\| e^{\lambda a - \lambda a^*} \| \leq \| e^{2i\alpha k} \| \| e^{-i2\beta h} \| \leq M$. Now, the entire function $\lambda \mapsto e^{-\lambda a} xe^{\lambda a^*} = \varphi(\lambda)$ is bounded, and according to Liouville’s theorem it is constant. Thus, $e^{-\lambda a} xe^{\lambda a^*} = \varphi(\lambda) = \varphi(0) = x$, i.e. $xe^{\lambda a^*} = e^{\lambda a^*} x$. Expanding both sides of this equation in a series, and comparing the coefficients, we get

$$a^* x = xa^*.$$
Adding (or subtracting) the initial equality we get the second and third equalities of the statement.

(b) Let \( a = h + ik = h_1 + ik_1 \), where \( h, h_1, k, k_1 \) are pre-hermitian elements such that \( hk = kh \) and \( h_1k_1 = k_1h_1 \). Obviously, \( a \) commutes with \( a \), and by the previous part of this proposition, we conclude that all \( h, k, h_1, k_1 \) mutually commute. Now, by Proposition 1.1(d) there exists a norm, equivalent to the initial one, such that \( h, h_1, k, k_1 \) are all hermitian. Now, we have \( h = h_1, k = k_1 \).

The previous proposition allows us to define, for an arbitrary pre-normal \( a = h + ik \in \mathcal{A} \), its adjoint \( a^* = h - ik \).

Recall that from Vidav Palmer’s well known theorem, \( \mathcal{A} = \mathcal{H}(\mathcal{A}) + i\mathcal{H}(\mathcal{A}) \) if and only if \( \mathcal{A} \) is a \( C^* \)-algebra.

Let \( a \in \mathcal{A} \), and let \( L_a, R_a : \mathcal{A} \to \mathcal{A} \) be given by \( L_a(x) = ax \) and \( R_a(x) = xa \). The following proposition carries over some of the properties of \( a \) to the operators \( L_a, R_a \in B(\mathcal{A}) \).

**Proposition 1.3.**

(a) The mappings \( a \mapsto L_a \) and \( a \mapsto R_a \) are isometries and monomorphisms from the algebra \( \mathcal{A} \) to the algebra \( B(\mathcal{A}) \).

(b) The spectra \( \sigma(L_a) \) and \( \sigma(R_a) \) coincide with \( \sigma(a) \).

(c) \( W(L_a) = W(R_a) = W(a) \).

(d) If \( a \) is (pre-)hermitian, then so are both \( L_a \) and \( R_a \).

(e) If \( a = h + ik \) is (pre-)normal, then so are both \( L_a = L_h + iL_k \) and \( R_a = R_h + iR_k \).

We leave an easy proof to the reader.

**Definition 1.2.** We say that \( a \in \mathcal{A} \) is a generalized scalar element if \( e^{ita} \) has polynomial growth for real \( t \), i.e. there are constants \( C, s \) such that

\[
\|e^{ita}\| \leq C(1 + |t|^s),
\]

and the spectrum of \( a \) is real. In this case we say that \( a \) has order \( s \).

It is clear that every pre-hermitian element \( a \) is a generalized scalar element of order 0, i.e. (2) holds with \( s = 0 \). Also, there exists a norm equivalent to the initial one which makes \( a \) hermitian. Changing norm does not change the spectrum. Thus \( a \) has real spectrum.

In [7], for any \( s > 0 \), an example is given of an element \( S \) such that \( \|e^{its}\| \approx |t|^s \) as \( t \to \infty \).

**2. Fourier transform.** The basic tool we use to derive our results is a functional calculus for commuting families of generalized scalar operators, developed in [13].
**Definition 2.1.** \( \hat{L}_1^s = \hat{L}_1^s(\mathbb{R}^n) \) is the set of all inverse Fourier transforms of functions from \( \{ g : \mathbb{R}^n \to \mathbb{C} \mid (1 + |\xi|)^s g(\xi) \in L^1(\mathbb{R}^n) \} \).

In fact, \( \hat{L}_1^s \) is an algebra with respect to pointwise multiplication.

**Theorem 2.1.** Let \( S_1, \ldots, S_n \) be a commuting family of generalized scalar operators acting on a Banach space \( X \), and let \( s_1, \ldots, s_n \) be their orders. Then there is an algebra homomorphism \( \Phi : \hat{L}_1^s \to L(X) \) (\( s = s_1 + \cdots + s_n \)) given by

\[
\Phi(f)(\xi) = f(S_1, \ldots, S_n) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi_1, \ldots, \xi_n) e^{i(\xi_1 S_1 + \cdots + \xi_n S_n)} d\xi,
\]

where \( \hat{f} \) denotes the Fourier transform of \( f \), i.e.

\[
\hat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(y) e^{-ixy} dy.
\]

The homomorphism \( \Phi \) has the following properties:

(i) The integral in (3) converges since \( (1 + |\xi|)^s \hat{f}(\xi) \in L^1(\mathbb{R}^n) \) and exists as a Bochner integral.

(ii) If \( f \equiv 0 \) on the joint Taylor spectrum \( \sigma_T(S_1, \ldots, S_n) \) then we have \( f(S_1, \ldots, S_n) = 0 \), and consequently, if \( f \equiv g \) on \( \sigma_T(S_1, \ldots, S_n) \) then \( f(S_1, \ldots, S_n) = g(S_1, \ldots, S_n) \).

(iii) For \( f \) analytic in a neighborhood of the joint spectrum, \( f(S_1, \ldots, S_n) \) has its usual meaning, obtained by power series expansion of \( f \).

**Remark 2.1.** Although the integral (3) exists as a Bochner integral, for our applications the following property of the integral of a function \( f : \mathbb{R}^n \to L(X) \) suffices: \( \varphi(\int_{\mathbb{R}^n} f dx) = \int_{\mathbb{R}^n} \varphi \circ f dx \) for all bounded linear functionals \( \varphi \in L(X)^* \).

**Remark 2.2.** In [16], it was proved that the Taylor and Harte spectra of a commuting family of generalized scalar elements coincide.

The elementary operator (1) can be expressed as

\[
\Lambda = Q(L_{a_1}, \ldots, L_{a_n}, R_{b_1}, \ldots, R_{b_n}),
\]

where \( Q(x) = x_1 x_{n+1} + x_2 x_{n+2} + \cdots + x_n x_{2n} \) is a quadratic form on \( \mathbb{R}^{2n} \). Our aim is to estimate \( \| e^{it\Lambda} \| \) by calculating \( e^{it\Lambda} \) as \( e^{itQ(L_{a_1}, \ldots, R_{b_n})} \). Unfortunately, \( e^{itQ} \not\in L^1(\mathbb{R}^{2n}) \), so it is impossible to calculate its Fourier transform, as a function. However, we can multiply \( Q \) by a suitable \( C^\infty_{\text{cpt}} \) function which is equal to 1 on the joint spectrum of the \( 2n \)-tuple \( (L_{a_1}, \ldots, L_{a_n}, R_{b_1}, \ldots, R_{b_n}) \).

This spectrum is a compact subset, and we shall derive our results in terms of its dimension.

Let \( K \subseteq \mathbb{R}^{2n} \) be an arbitrary compact set. Recall that \( K \) is said to have **Hausdorff dimension** \( c \) if there exists a positive constant \( N > 0 \) such that for
all $\delta > 0$ there exists a finite decomposition $K = \bigcup_{j=1}^{m} \beta_j$ with the following properties: (i) $\max_{1 \leq j \leq m} \text{diam}(\beta_j) < \delta$ and (ii) $\sum_{j=1}^{m} (\text{diam}(\beta_j))^c \leq N$. We need a somewhat stronger concept of Hausdorff dimension, described in the following definition.

**Definition 2.2.**

(a) We say that a compact set $K$ has balanced Hausdorff dimension $c$ if there exist positive constants $N, P > 0$ such that for all $\delta > 0$ there exists a finite covering $K \subseteq \bigcup_{j=1}^{m} \beta_j$ ($\beta_i \cap \beta_j = \emptyset$!) with the following properties: (i) $\delta/P < \text{diam}(\beta_j) < \delta$ for all $1 \leq j \leq m$ and (ii) $\sum_{j=1}^{m} (\text{diam}(\beta_j))^c \leq N$.

(b) We say that a function $f$ generates $e^{itQ}$ on $K$ if $f \equiv e^{itQ}$ on $K$, $f$ is analytic in a neighborhood of $K$, and $f \in C_{\text{cpt}}^\infty$. The set of all such functions is denoted by $C_Q(K)$.

**Remark 2.3.** One can verify that any subset of $\mathbb{R}^{2n}$ $C^1$-diffeomorphic to a $c$-dimensional simplex has balanced Hausdorff dimension $c$. In particular, every $c$-dimensional compact manifold, with or without boundary, has balanced Hausdorff dimension $c$.

**Lemma 2.2.** Let $K \subseteq \mathbb{R}^{2n}$ be a set of balanced Hausdorff dimension $c$. Then for all $\delta > 0$ there exists an open set $U_\delta \supset K$ such that $m(U_\delta) \leq C(K,n)\delta^{2n-c}$ and $\text{dist}(K,U_\delta^C) \geq \delta/P$.

**Proof.** Given $\delta > 0$, let $K = \bigcup_{j=1}^{m} \beta_j$ be a decomposition of $K$ with properties (i) and (ii) from Definition 2.2(a). Set

$U_{\delta,j} = \{x \in \mathbb{R}^{2n} \mid \text{dist}(x, \beta_j) < d_j = \text{diam}(\beta_j)\},$

and $U_\delta = \bigcup_{j=1}^{m} U_{\delta,j}$. Clearly, $\text{dist}(K,U_\delta^C) \geq \min d_j \geq \delta/P$. Also

$m(U_\delta) \leq \sum_{j=1}^{m} m(U_{\delta,j}) \leq |B_{2n}| \sum_{j=1}^{m} (2d_j)^{2n},$

since $U_{\delta,j}$ is contained in some ball of radius $2d_j$. (Here $|B_{2n}|$ denotes the measure of the unit ball in $\mathbb{R}^{2n}$.) Now, we have

$m(U_\delta) \leq C \sum_{j=1}^{m} d_j^{2n} = C \sum_{j=1}^{m} d_j^c d_j^{2n-c} \leq C \delta^{2n-c} \sum_{j=1}^{m} d_j^c \leq C N \delta^{2n-c}. \quad \blacksquare$

**Lemma 2.3.** Let $K \subseteq \mathbb{R}^{2n}$ be a compact set of balanced Hausdorff dimension $c$.

(a) For large $t$ there exists an open set $U_t \supset K$ such that $m(U_t) \leq C(K,n)/t^{2n-c}$ and $\text{dist}(K,U_t^C) \geq 1/Pt$. 
There exists a $C^\infty$ function $\psi_t : \mathbb{R}^{2n} \to \mathbb{R}$, analytic in a neighborhood of $K$, such that $0 \leq \psi_t(x) \leq 1$, and

$$
\psi_t(x) = \begin{cases} 
1, & x \in K, \\
0, & x \notin U_t,
\end{cases}
$$

and $|\partial^\alpha \psi_t/\partial x^\alpha| \leq C_\alpha t^{\left|\alpha\right|}$ for any multiindex $\alpha$.

For all positive integers $k$ we have

$$
|\Delta^k(e^{itQ(x})\psi_t(x))| = O(t^{2k}) \quad \text{as } t \to \infty.
$$

**Proof.** (a) Put $\delta = 1/t$ in Lemma 2.2.

(b) This part is in fact Proposition 1.3.5 from [2].

(c) Indeed, $\Delta^k$ is a differential operator of order $2k$, so $\Delta^k(e^{itQ(x})\psi_t(x))$ is a finite sum of terms, each containing a partial derivative of $\psi_t(x)$ of order $i$ and a partial derivative of $e^{itQ(x)}$ of order $j$, with $i + j \leq 2k$, and the result follows by parts (a) and (b).

**Theorem 2.4.** Let $K \subseteq \mathbb{R}^{2n}$ be a compact set of balanced Hausdorff dimension $c$. There exists a family of functions $\varphi_t \in C_Q(K)$ such that for any $\varepsilon > 0$ the following estimate holds:

$$
\left\|(1 + |\xi|^s)\hat{\varphi}_t(\xi)\right\|_1 = o(t^{s+c/2+\varepsilon}) \quad (t \to \infty).
$$

**Proof.** Set $\varphi_t(x) = \psi_t(x)e^{itQ(x)}$, where $\psi_t$ are the functions from Lemma 2.3(b). For large $|x|$, using Lemma 2.3(c), we have

$$
\left|(1 + |x|^s)\hat{\varphi}_t(x)\right| = (1 + |x|^s) \frac{1}{(2\pi)^n|x|^{2k}} \left\| \varphi_t(\xi)\Delta^k e^{-ix\xi} d\xi \right\|
$$

$$
= (1 + |x|^s) \frac{1}{(2\pi)^n|x|^{2k}} \left\| \Delta^k \varphi_t(\xi)e^{-ix\xi} d\xi \right\|
$$

$$
\leq C_1 t^{2k-2n+c} \frac{t^{2k-m(U_t)}}{|x|^{2k-s}} \leq C_1 t^{2k-2n+c} \frac{t^{2k-2n+c}}{|x|^{2k-s}}.
$$

Now, using the Cauchy–Schwarz inequality and Plancherel’s theorem we get

$$
\left\|(1 + |x|^s)\hat{\varphi}_t(x)\right\|_1
$$

$$
= \int_{|x| \leq M} \left|(1 + |x|^s)\hat{\varphi}_t(x)\right| dx + \int_{|x| \geq M} \left|(1 + |x|^s)|\hat{\varphi}_t(x)|\right| dx
$$

$$
\leq \left( \int_{|x| \leq M} (1 + |x|^s)^2 dx \right)^{1/2} \left\| \hat{\varphi}_t \right\|_2 + C_1 t^{2k-2n+c} \int_{|x| \geq M} 1/|x|^{2k-s} dx
$$

$$
= \left( \int_{|x| \leq M} (1 + |x|^s)^2 dx \right)^{1/2} \left\| \hat{\varphi}_t \right\|_2 + C_1 t^{2k-2n+c} \int_{|x| \geq M} 1/|x|^{2k-s} dx
$$

$$
= \left( \int_{|x| \leq M} (1 + |x|^s)^2 dx \right)^{1/2} \left\| \hat{\varphi}_t \right\|_2 + C_1 t^{2k-2n+c} \int_{|x| \geq M} 1/|x|^{2k-s} dx.
$$
\[
\leq K'_1 M^s m \{ |x| \leq M \}^{1/2} \| \hat{\varphi}_t \|_2 + C_1 t^{2k-2n+c} \int_{|x| \geq M} 1/|x|^{2k-s} \, dx \\
\leq K_1 M^{n+s} t^{c/2-n} + K_2 t^{2k-2n+c} M^{2n-2k+s}.
\]

If we put \( M = t^{1+ \frac{c}{2k-n}} \) we get \( \| \hat{\varphi}_t \|_1 \leq K t^{(2s+c)k-s(n-c/2)} \). As \( k \) can be arbitrarily large this proves (4). \( \blacksquare \)

Next, we want to improve estimate (4) under additional assumptions.

**Theorem 2.5.** Let \( K \) be a subset of a \( c \)-dimensional affine subspace \( x_0 + V \subseteq \mathbb{R}^{2n} \), where \( V \) is a vector subspace of \( \mathbb{R}^{2n} \), and \( 0 \leq c \leq 2n \) is an integer. Assume \( Q \) is nondegenerate on \( V \). There exists a family of functions \( \varphi_t \in C_Q(K) \) such that

\[
\| \hat{\varphi}_t(\xi) \|_1 = O(t^{c/2}) \quad (t \to \infty).
\]

**Proof.** First, we can assume \( V \) is a linear subspace of \( \mathbb{R}^{2n} \); next we choose a basis in \( V \) such that in the new coordinates \( Q(x) = \sum_{j=1}^{c} \lambda_j y_j^2 \) \((x \in V)\), where \( \lambda_j \in \{1, -1\} \). This basis can be extended, using Witt’s theorem [10, XIV.5], to a basis in \( \mathbb{R}^{2n} \) such that, in the new coordinates,

\[
Q(x) = \sum_{j=1}^{2n} \lambda_j y_j^2, \quad \lambda_j \in \{1, -1\}.
\]

We have here a linear change of variables \( x = By \). Choose \( R > 0 \) such that \( |y_j| \leq R \) for all points in \( K \) and define \( \varphi_t(x) = \psi_t(x) \exp(itQ(x)) \), where

\[
\psi_t(x) = \prod_{p=1}^{c} g(y_p) \prod_{q=1}^{2n-c} f(\sqrt{t} y_{c+q}) = \chi_t(y),
\]

and \( f, g \in C^\infty_{\text{cpt}}(\mathbb{R}) \) are such that \( f = 1 \) in a neighborhood of 0, \( g(y) = 1 \) if \(-R \leq y \leq R\), and \( g(y) = 0 \) for \( y \geq 2R \). We have

\[
\hat{\varphi}_t(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \psi_t(x) \exp(itQ(x)) e^{-ix\cdot \xi} \, dx \\
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \chi_t(y) e^{itQ(By)} e^{-iy\cdot B^t \xi} \, dy \\
= \frac{|\det B|}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \chi_t(y) \exp \left( it \sum \lambda_j y_j^2 \right) e^{-iy\cdot B^t \xi} \, dy \\
= |\det B| \hat{\theta}_t(B^t \xi),
\]

where

\[
\theta_t(y) = \chi_t(y) \exp \left( it \sum \lambda_j y_j^2 \right) = \prod_{p=1}^{c} u_t(y_p) \cdot \prod_{q=1}^{2n-c} v_t(y_{q+c}),
\]

and \( u_t(y) = e^{\pm ity^2} g(y) \), \( v_t(y) = e^{\pm ity^2} f(\sqrt{t} y) \).
Since \( \| \hat{u}_t \|_{L^1(\mathbb{R})} \) does not depend on \( t \) we use Fubini’s theorem to reduce (5) to the following

**Lemma 2.6.** \( \| \hat{u}_t \|_{L^1(\mathbb{R})} = O(\sqrt{t}) \) as \( t \to \infty \).

This lemma follows easily from the following estimates:

1° \( |\hat{u}_t(\eta)| \leq C/\sqrt{t} \) for \( \eta \in [-6Rt, 6Rt] \), where \( C \) does not depend on \( t \) and \( \eta \).

2° \( |\hat{u}_t(\eta)| \leq C/|\eta|^2 \) for \( \eta \geq 6Rt \), where \( C \) does not depend on \( t \) and \( \eta \).

**Proof of 1°.** We have

\[
|\hat{u}_t(\eta)| = \frac{e^{-i\eta^2/4t}}{(2\pi)^n} \int_{-\infty}^{\infty} e^{i(y-\eta/2t)^2} g(y) \, dy = \frac{e^{-i\eta^2/4t}}{(2\pi)^n} \int_{-\infty}^{\infty} e^{itz^2} g(z) \, dz,
\]

where \( g_{\alpha}(z) = g(z + \alpha) \) and \( \alpha = \eta/2t \). Note that \( |\alpha| \leq 3R \). Therefore \( g_{\alpha}(z) \), \( |\alpha| \leq 3R \) is a bounded family of functions in the \( C^\infty_{\text{cpt}}(\mathbb{R}) \) topology, so the stationary phase method [9] gives the following estimate, uniformly over \( |\alpha| \leq 3R \):

\[
\int_{-\infty}^{\infty} e^{itz^2} g_{\alpha}(z) \, dz = O(1/\sqrt{t}).
\]

We present here a proof for the reader’s convenience. First, note that \( g_{\alpha}(z) = g_{\alpha}(0) + z\gamma_{\alpha}(z) \), where \( \gamma_{\alpha}(z) = \int_0^1 g'(zs + \alpha) \, ds \) and therefore

\[
\int_{-\infty}^{\infty} e^{itz^2} g_{\alpha}(z) \, dz = \lim_{A \to \infty} \int_{-A}^{A} g_{\alpha}(0)e^{itz^2} \, dz + \lim_{A \to \infty} \int_{-A}^{A} e^{itz^2}z\gamma_{\alpha}(z) \, dz.
\]

The first limit is equal to \( g_{\alpha}(0)e^{i\pi/4}\sqrt{\pi/4t} \), which is \( O(1/\sqrt{t}) \) uniformly over \( \alpha \). This also shows that the second limit exists. Next,

\[
\lim_{A \to \infty} \int_{-A}^{A} e^{itz^2}z\gamma_{\alpha}(z) \, dz = \frac{1}{2it} \lim_{A \to \infty} \int_{-A}^{A} \gamma_{\alpha}(z) \frac{dz}{dz} e^{itz^2} \, dz
\]

\[
= \frac{1}{2it} \lim_{A \to \infty} \left[ \gamma_{\alpha}(z)e^{itz^2} \bigg|_{-A}^{A} - \int_{-A}^{A} \gamma'_{\alpha}(z)e^{itz^2} \, dz \right]
\]

\[
= -\frac{1}{2it} \lim_{A \to \infty} \int_{-A}^{A} \gamma'_{\alpha}(z)e^{itz^2} \, dz,
\]

because \( \gamma_{\alpha}(z) = (g_{\alpha}(z) - g_{\alpha}(0))/z = o(1) \) as \( |z| \to \infty \).

Next, note that

\[
\gamma'_{\alpha}(z) = \frac{g'_{\alpha}(z)z - g_{\alpha}(z) + g_{\alpha}(0)}{z^2} = \frac{g_{\alpha}(0)}{z^2} \quad \text{for } |z| > 5R,
\]
hence
\[
\int_{-\infty}^{\infty} |\gamma'_\alpha(z)e^{itz^2}| \, dz \leq 2|g_\alpha(0)| \int_{\frac{5R}{2}}^{\infty} \frac{dz}{z^2} + \int_{-\frac{5R}{2}}^{5R} |\gamma'_\alpha(z)| \, dz
\]
\[
\leq \frac{2|g_\alpha(0)|}{5R} + 10R \cdot \max_{\alpha} \left| \frac{1}{0} g''(zs + \alpha)s \, ds \right| = C.
\]
This shows that \( \lim_{A \to \infty} \int_{-A}^{A} \gamma'_\alpha(z)e^{itz^2} \, dz \) is uniformly bounded over \( t \in \mathbb{R} \) and \( |\alpha| \leq 3R \). Hence (7) gives
\[
\lim_{A \to \infty} \int_{-A}^{A} e^{itz^2} z\gamma_\alpha(z) \, dz = O(1/t),
\]
and 1° is proved.

**Proof of 2°.** We can ignore the factor \( e^{-i\eta^2/4t} \). We have
\[
\int_{-\infty}^{\infty} e^{itz^2} g_\alpha(z) \, dz = \frac{1}{2it} \int_{-\infty}^{\infty} g_\alpha(z) \frac{d}{dz}(e^{itz^2}) \, dz
\]
\[
= -\frac{1}{2it} \int_{-\infty}^{\infty} e^{itz^2} \left( \frac{g_\alpha(z)}{z} \right)' \, dz
\]
\[
= -\frac{1}{(2it)^2} \int_{-\infty}^{\infty} (e^{itz^2})' \frac{1}{z} \left( \frac{g_\alpha(z)}{z} \right)' \, dz
\]
\[
= -\frac{1}{4t^2} \int_{-\infty}^{\infty} e^{itz^2} \frac{d}{dz} \left[ \frac{g_\alpha'(z)}{z^2} - \frac{g_\alpha(z)}{z^3} \right] \, dz,
\]
and this is a sum of four terms of the form
\[
\pm \frac{1}{4t^2} \int_{-\infty}^{\infty} e^{itz^2} \frac{G(z + \alpha)}{z^k} \, dz,
\]
where \( G \) stands for one of the functions \( g, g' \) or \( g'' \) and \( k \in \{2, 3, 4\} \). Consider one such term; changing variables one gets
\[
\frac{1}{4t^2} \int_{-\infty}^{\infty} e^{itz(z-\alpha)^2} \frac{G(z)}{(z-\alpha)^k} \, dz = \frac{e^{ita^2}}{4t^2} \int_{-\infty}^{\infty} e^{itz^2} e^{-i\eta z} \frac{G(z)}{(z-\alpha)^k} \, dz
\]
\[
= -e^{ita^2} \frac{1}{4t^2 \eta^2} \int e^{itz^2} \frac{G(z)}{(z-\alpha)^k} \frac{d^2}{dz^2} e^{-i\eta z} \, dz
\]
\[
= -e^{ita^2} \frac{2R}{4t^2 \eta^2} \int_{-2R}^{2R} e^{-i\eta z} \frac{d^2}{dz^2} \left[ e^{itz^2} \frac{G(z)}{(z-\alpha)^k} \right] \, dz.
\]
Since $|\alpha| \geq 3R$, we have a uniform estimate
\[
\left| \frac{d^2}{dz^2} \left[ e^{iz^2} \frac{G(z)}{(z - \alpha)^k} \right] \right| \leq Ct^2
\]
for $|z| \leq 2R$ and $|\alpha| \geq 3R$, hence each of the four terms is estimated by $C/\eta^2$, as needed.

3. Elementary operators. Our first result is a simple consequence of results from the previous section.

**Theorem 3.1.**

(a) Let $a_1, \ldots, a_n$, and $b_1, \ldots, b_n$ be commuting $n$-tuples of generalized scalar elements of a unital Banach algebra $A$, with orders $s_1, \ldots, s_n$, and $r_1, \ldots, r_n$ respectively. Also, let $s = s_1 + \cdots + s_n$, $r = r_1 + \cdots + r_n$ be their total orders. Then the elementary operator $\Lambda$ given by (1) is also a generalized scalar operator. Its order is $r + s + c/2 + \varepsilon$ for any $\varepsilon > 0$, where $c$ is the balanced Hausdorff dimension of the set $K = \sigma(a_1, \ldots, a_n) \times \sigma(b_1, \ldots, b_n)$, where $\sigma$ denotes the joint spectrum defined in [8].

(b) If, in addition, $s = r = 0$ and $K$ is contained in an affine subspace of $\mathbb{R}^{2n}$ of integer dimension $c$, then $\Lambda$ is a generalized scalar operator with order at most $c/2$.

**Proof.** (a) From a result of Harte and Hernandez [8] it follows that
\[
\sigma(L_{a_1}, \ldots, L_{a_n}, R_{b_1}, \ldots, R_{b_n}) \subseteq \sigma(a_1, \ldots, a_n) \times \sigma(b_1, \ldots, b_n) = K.
\]
Also, using Proposition 1.3, it is easy to verify that the operators $L_{a_1}, \ldots, L_{a_n}, \ R_{b_1}, \ldots, R_{b_n}$ form a commuting family of generalized scalar operators on $A$ considered as a Banach space. Take the functions $\varphi_t$ from Theorem 2.4. Since $\varphi_t = \exp(itQ)$ on $K$, from Theorems 2.1 and 2.4 it follows that
\[
\|\exp(it\Lambda)\| = \|\varphi_t(\Lambda)\| \leq \|\hat{f}(\xi)\alpha(1 + |\xi|^s)\|_1 = o(t^{s+r+c/2+\varepsilon}),
\]
where $c$ is the balanced dimension of $K$.

(b) The proof of the second part is the same. The only difference is that we apply Theorem 2.5 instead of Theorem 2.4.

In the worst case the dimension of $K$ might be $2n$, so we get the following corollary.

**Corollary 3.2.** Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be two $n$-tuples of commuting pre-hermitian elements of a unital Banach algebra $A$. Then the operator $\Lambda : A \rightarrow A$ is a generalized scalar operator, and its order is at most $n$.

It seems, from $1^\circ$ in the proof of Lemma 2.6, that this estimate is the best one can obtain via Fourier transform. However, considering the following
example one can conjecture that if $\mathcal{A}$ is a $C^*$-algebra a better estimate $\|e^{itA}\| = O(t^{n/2})$ holds.

**Example 3.1.** Let $H = L^2(0, 1)$, and let $A : H \to H$ be given by $Af(s) = sf(s)$. Consider the mapping $X \mapsto A(X) = AXA$ on $B(H)$. Note that $A : B(H) \to B(H)$ is the adjoint of the multiplication operator $M : \mathfrak{S}_1 \to \mathfrak{S}_1$ of the same form $M(X) = AXA$, where $\mathfrak{S}_1$ stands for the ideal of all nuclear operators. This can be used to reduce the norm estimate of $e^{itA}$ to a norm estimate of $e^{itM}$.

If $X$ is a nuclear operator, then it can be expressed as an integral operator with kernel $K$, $Xf(s) = \int_0^1 K(s, u)f(u) \, du$. Straightforward calculation gives $M^n(X)f(s) = \int_0^1 s^n K(s, u)u^n f(u) \, du$, and

$$e^{itM}(X)f(s) = \left(\sum_{n=0}^{\infty} \frac{t^n s^n M^n(X)/n!}{n!}\right)f(s) = \int_0^1 e^{itsu}K(s, u)f(u) \, du.$$  

Thus $e^{itM}$ is a Schur multiplier with symbol $e^{itsu}$. From [4] it follows that its norm does not exceed $C \text{ess sup}_{0<s<1} \|u \mapsto e^{itsu}\|_{W^\alpha_2}$ for all $\alpha > 1/2$, where $W^\alpha_2$ stands for the Sobolev space of index $\alpha$. It is easy to verify that the last expression is $O(t^\alpha)$.

**Remark 3.1.** The estimate $\|\exp(itA)\| = O(t^{s+r+2n})$ as $t \to \infty$ for $s, r$ integers follows from a paper by Albrecht [1]. If $s, r$ are half-integers then from [1] one can derive only $\|\exp(itA)\| = O(t^{s+r+3n})$. Our estimate is a refinement of the last one.

Let $E$ be an arbitrary linear space, and let $T : E \to E$ be an arbitrary linear mapping. The **ascent** of $T$ is defined as the least integer $m$ such that $\ker T^{m+1} = \ker T^m$. The ascent of $T$ is usually denoted by $\text{asc}(T)$. Clearly $\text{asc}T = 0$ if and only if $T$ is injective. Also $\text{asc}(T) \leq 1$ if and only if ker $T$ and $T(E)$ have trivial intersection. The finite ascent leads to the property of being semifredholm.

**Theorem 3.3.** Let $X$ be a Banach space, and let $S : X \to X$ be a generalized scalar operator of order $s$. Then the ascent of $S$ is finite, and $\text{asc}(S) \leq \lceil s \rceil + 1$.

**Proof.** Indeed, suppose that $S^{k+1}(x) = 0$ for some $x \in X$, where $k > s$ is a positive integer. Then $e^{itS}(x) = \sum_{j=0}^k \frac{(it)^j S^j(x)}{j!}$, and also

$$S^k(x) = k!\left(e^{itS}(x) - \sum_{j=0}^{k-1} \frac{(it)^j S^j(x)}{j!}\right)/(it)^k.$$
Since $S$ is a generalized scalar operator, we obtain

$$
\|S^k(x)\| \leq \frac{k!|e^{itS}|\|x\| + \sum_{j=0}^{k-1} t^j \|S^j(x)\|/j!}{t^k} \\
\leq \frac{k!(Ct^s\|x\| + \sum_{j=0}^{k-1} t^j \|S^j(x)\|/j!)}{t^k} \to 0 \quad (t \to \infty),
$$

implying $S^k(x) = 0$, as required. ■

**Corollary 3.4.** Let $\mathcal{A}$ be a unital Banach algebra, and let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be two $n$-tuples of commuting generalized scalar elements of $\mathcal{A}$, with orders $s_1, \ldots, s_n$ and $r_1, \ldots, r_n$, respectively. If $\Lambda : \mathcal{A} \to \mathcal{A}$ is an elementary operator given by $\Lambda(x) = \sum_{j=1}^n a_j x b_j$, then asc($\Lambda$) < $\infty$. Moreover, asc($\Lambda$) $\leq [s + r + c/2] + 1$, where $s = s_1 + \cdots + s_n$, $r = r_1 + \cdots + r_n$, and $c$ is the balanced Hausdorff dimension of the set $\sigma(a_1, \ldots, a_n) \times \sigma(b_1, \ldots, b_n)$.

**Proof.** It suffices to combine Theorems 3.1 and 3.3. ■

**Remark 3.2.** In [19] it was proved that asc$\Lambda \leq (2 + 8(s + r))n - 1$. The previous corollary, even in the worst case $c = 2n$, is a refinement of this result. Also, if $a_j, b_j$ are pre-hermitian elements with finite spectra we have asc($\Lambda$) $\leq 1$.

We say that the family $\{U_1, \ldots, U_n\}$ is strongly commuting if $U_j = S_j + iT_j$, where $\{S_1, \ldots, S_n, T_1, \ldots, T_n\}$ is a commuting family of generalized scalar elements.

The following theorem is a variant of the classical Fuglede–Putnam theorem.

**Theorem 3.5.** Let $\mathcal{A}$ be a unital Banach algebra, let $a_j = a_j'' + ia_j'$, $b_j = b_j'' + ib_j'$ $\in \mathcal{A}$ $(1 \leq j \leq n)$ be two strongly commuting families, and let $s = \sum_{j=1}^n (s_j' + s_j'')$ and $r = \sum_{j=1}^n (r_j' + r_j'')$ be the total orders of the families $a_j$ and $b_j$. Define $\Lambda(x) = \sum a_j x b_j$ and $\Lambda^*(x) = \sum a_j^* x b_j^*$ ($\Lambda, \Lambda^* : \mathcal{A} \to \mathcal{A}$), where $a_j = a_j' - ia_j''$, $b_j = b_j' - ib_j''$. If $\Lambda(x) = 0$, then $(\Lambda^*)^k(x) = 0$ for some positive integer $k$. Further $k \leq [s + r + c/2] + 1$, where $c$ denotes the balanced Huadorff dimension of $\sigma(a_1', a_1'', \ldots, a_n', a_n'') \times \sigma(b_1', b_1'', \ldots, b_n', b_n'')$.

**Proof.** (a) It is clear that $\Lambda(x) = \Lambda_1(x) + i\Lambda_2(x)$ and $\Lambda^*(x) = \Lambda_1(x) - i\Lambda_2(x)$, where

$$
\Lambda_1(x) = \sum (a_j' x b_j' - a_j'' x b_j''), \quad \Lambda_2(x) = \sum (a_j'' x b_j' + a_j' x b_j'').
$$

It is also clear that $\Lambda_1$ and $\Lambda_2$ commute. From Theorem 3.1 we know that $\|\exp(it\Lambda_1)\|, \|\exp(it\Lambda_2)\| = O(t^\mu)$, where $\mu = s + r + c/2 + \varepsilon$ and $\varepsilon$ is sufficiently small.
Suppose now that $\Lambda(x) = 0$. We have $\Lambda_1(x) = -i\Lambda_2(x)$, and by induction
\[ \Lambda^n_1(x) = (-i\Lambda_2)^n(x), \]
and therefore $\exp(\Lambda_1)(x) = \exp(-i\Lambda_2)(x)$. Let $\lambda = \alpha + i\beta \in \mathbb{C}$, and let $f$ be an arbitrary functional from $\mathcal{A}^*$, the dual space of $\mathcal{A}$ considered as a Banach space. We get
\[
|f(\exp(\Lambda_1)(x))| = |f(\exp(i\beta\Lambda_1)\exp(\alpha\Lambda_1)(x))| = |f(\exp(i\beta\Lambda_1)\exp(-i\alpha\Lambda_2)(x))| \leq \|f\|C(\alpha\beta)^\mu \|x\| \leq \|f\|C_1|\lambda|^{2\mu} \|x\|.
\]
Since $\lambda \mapsto f(\exp(\Lambda_1)(x))$ is an entire function, from Cauchy’s formulae for the coefficients in the power series expansion it follows that this function is a polynomial of degree at most $2\mu$. Hence $f(\Lambda_1^n(x)) = 0$ for all $f \in \mathcal{A}^*$ and $m > 2\mu$. Invoking the Hahn–Banach theorem we conclude that $\Lambda_1^n(x) = 0$ for all $m > 2\mu$. By Corollary 3.4 the ascent of the operator $A_1$ does not exceed $k = [s + r + c/2] + 1$. Since $2\mu > k$, it follows that $\Lambda_1^k(x) = 0$. Also $A_1^j A_2^{k-j} x = i^{k-j} A_1^j A_2^{k-j} x = 0$, and therefore
\[
(\Lambda^*)_k(x) = (A_1 - iA_2)^k(x) = \sum_{j=0}^k (-i)^{k-j} A_1^j A_2^{k-j}(x) = 0. \tag*{\blacksquare}
\]

**Remark 3.3.** Note that for given $a_j = a'_j + ia''_j$, where $a'_j$ and $a''_j$ are commuting generalized scalar elements we do not claim that this representation is unique, so $a_j^*$ is not uniquely determined.

**Remark 3.4.** The worst case is $c = 4n$ from which we get $k \leq [s + r + 2n] + 1$ in any case. The best case is where all $a_j$ and $b_j$ are pre-normal and $c = 0$, for instance pre-normal elements with finite spectra. Then we can claim $k = 1$, and that is the strong Fuglede–Putnam theorem.

Consider the equation $\sum_{j=1}^n a_j x b_j = y$. The problem of estimating the norm of $\|x\|$ in terms of $\|y\|$ is very well known. It amounts to estimating $\|\Lambda^{-1}\|$. See for instance [14] and [3]. In [14] it was proved that
\[
\|x\| \leq \frac{C}{\delta} \left(\frac{\max\{1, \delta\}}{\delta}\right)^s \|y\|,
\]
where $s$ is the order of $\Lambda$ and where $\delta = \inf\{\sum \lambda_j \mu_j | (\lambda_1, \ldots, \lambda_n) \in \sigma(a_1, \ldots, a_n), (\mu_1, \ldots, \mu_n) \in \sigma(b_1, \ldots, b_n)\}$. However, the existence of $s$ was only proved indirectly, and no exact value was given.

The following theorem gives this estimate with an explicit formula for $s$.

**Theorem 3.6.**

(a) Let $\mathcal{A}$ be a unital Banach algebra, and let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be two $n$-tuples of commuting generalized scalar elements of $\mathcal{A}$, with orders $s_1, \ldots, s_n$ and $r_1, \ldots, r_n$, respectively. Also, let $\Lambda : \mathcal{A} \to \mathcal{A}$ be an elementary operator given by (1). If $0 \notin \{\lambda_1 \mu_1 + \cdots + \lambda_n \mu_n \mid \lambda_j, \mu_j \in \mathbb{C}\}$
\((\lambda_1, \ldots, \lambda_n) \in \sigma(a_1, \ldots, a_n), (\mu_1, \ldots, \mu_n) \in \sigma(b_1, \ldots, b_n)\), then the equation
\[
\sum_{j=1}^{n} a_j x b_j = y
\]
has a unique solution for all \(y \in \mathcal{A}\). Moreover

\[\tag{8}
\|x\| \leq \frac{C}{\delta} \left( \max \{1, \delta\} \right)^p \|y\|,
\]
where \(p = s_1 + \cdots + s_n + r_1 + \cdots + r_n + c/2 + \varepsilon\), \(\delta = \inf \{\lambda_1 \mu_1 + \cdots + \lambda_n \mu_n \mid (\lambda_1, \ldots, \lambda_n) \in \sigma(a_1, \ldots, a_n), (\mu_1, \ldots, \mu_n) \in \sigma(b_1, \ldots, b_n)\}\), and \(c\) is the balanced Hausdorff dimension of the set \(\sigma(a_1, \ldots, a_n) \times \sigma(b_1, \ldots, b_n)\).

(b) If, in addition, \(s_j = r_j = 0\) and \(K\) is contained in an affine subspace of \(\mathbb{R}^{2n}\), then \(\varepsilon\) in (a) can be omitted. In other words, \(p = c/2\).

**Proof.** The existence of the unique solution follows easily from Gel’fand theory. Indeed,
\[
\sigma(A) = \sigma(L_{a_1} R_{b_1} + \cdots + L_{a_n} R_{b_n}) \\
\subseteq \sigma(L_{a_1}) \sigma(R_{b_1}) + \cdots + \sigma(L_{a_n}) \sigma(R_{b_n}) \\
= \sigma(a_1) \sigma(b_1) + \cdots + \sigma(a_n) \sigma(b_n) = D.
\]

The proof of (8) was derived in [14], but for the convenience of the reader we shall outline it.

By Theorem 3.1, \(A\) is a generalized scalar operator on a Banach space \(\mathcal{A}\). Moreover \(\|e^{i\xi A}\| \leq M(1 + |t|^p)\), where \(p = s + r + c/2 + \varepsilon\) in part (a) and \(p = c/2\) in part (b). From Theorem 2.1, it follows that
\[
f(A) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{i\xi A} d\xi,
\]
where \(\widehat{f}\) is the Fourier transform of \(f\). Further, we can choose a function \(f_1 \in \hat{L}^p\) equal to \(1/x\) in a neighborhood of \(\{x \in \mathbb{R} \mid |x| \geq 1\}\). Set \(f_\delta(x) = f_1(x/\delta)/\delta\). Obviously, \(f_\delta(x) = 1/x\) in a neighborhood of \(\{x \in \mathbb{R} \mid |x| \geq \delta\} \supseteq D \supseteq \sigma(A)\) for \(\delta = \inf \{\lambda_1 \mu_1 + \cdots + \lambda_n \mu_n \mid \lambda_j \in \sigma(a_j), \mu_j \in \sigma(b_j)\}\) > 0, since \(D\) does not contain 0. Hence we have
\[
\|A^{-1}\| = \|f_\delta(A)\| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\widehat{f_\delta}(\xi)| \|e^{i\xi A}\| d\xi \\
\leq \frac{M}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\widehat{f_\delta}(\xi)|(1 + |\xi|^p) d\xi.
\]
By a change of variables we see that \(\widehat{f_\delta}(\xi) = \widehat{f_1}(\delta \xi)\), and thus
The estimates (8) and (9) are incomparable. Namely, if 

\[ n^\frac{1}{2.5} \] can be upgraded in order to relax the condition on 

\[ K \] by substituting \( \delta \xi = w \). The observation \( \delta^p + |w|^p \leq (\max\{1, \delta\})^p(1 + |w|^p) \) enables us to end the proof. The constant \( C \) can be calculated as 

\[ C = M \cdot \inf \| (1 + |w|^p) \hat{f}(w) \|_{L^1(\mathbb{R})} \], where the infimum is taken over all functions \( f \in L_1^2 \) which are equal to \( 1/x \) in a neighborhood of \( \{x \mid |x| \geq 1\} \). For the existence of such functions see [14] and references therein.

Remark 3.5. In [16] the authors gave another estimate of \( \|x\| \) in terms of \( \|y\| \), using 

\[ d(a_j, b_j) = \inf \{ (\langle u, v \rangle)/|u||v| \mid u \in \sigma(a_j), v \in \sigma(b_j) \}, \]

\[ d(a) = \inf \{ |u| \mid u \in \sigma(a_j) \} \]

and \( d(b) \). This estimate is

\[ \|x\| \leq C \max\{1, d(a_j)^{-s} d(b_j)^{-r}\} \frac{1 + |\log d(a_j, b_j)|}{d(a_j)d(b_j)(a_j, b_j)^{2n+s+r}}. \]

The estimates (8) and (9) are incomparable. Namely, if \( n = 1 \), then \( d(a_j, b_j) = 1, \delta = d(a_j)d(b_j) \), and (9) is sharper than (8). On the other hand, if \( n = 2, \sigma(a_j) = (t, 0), 1 \leq t \leq 2, \sigma(b_j) = (t \cos \varphi, t \sin \varphi), 1 \leq t \leq 2 \), with \( \varphi \) fixed, then \( d(a_j) = d(b_j) = 1, \delta = d(a_j, b_j) = \cos \varphi \), and (8) is sharper than (9) for \( \varphi \) close to \( \pi/2 \).

4. Questions

1. We believe that the additional condition on the set \( K = \sigma(a_1, \ldots, a_n) \times \sigma(b_1, \ldots, b_n) \) to have balanced Hausdorff dimension \( c \) is superfluous, i.e. the usual notion of Hausdorff dimension is sufficient. However, we have no proof.

2. One can try to avoid the nondegeneracy condition in Theorem 2.5, by considering 

\[ Q_\varepsilon(x) = Q(x) + \varepsilon \sum_{j=1}^{2n} x_j^2. \]

3. We are convinced that the technique applied in the proof of Theorem 2.5 can be upgraded in order to relax the condition on \( K \). Namely we believe that, instead of assuming that \( K \) is contained in a linear subspace of \( \mathbb{R}^{2n} \), it is enough to assume that it lies in a \( c \)-dimensional \( C^m \) manifold for suitable \( m \).

4. A more general frame for these investigations is \( \Lambda(x) = \sum a_j xb_j \), \( a_j \in A, b_j \in B \), and \( x \in \mathcal{X} \); here \( A \) and \( B \) are unital Banach algebras, and \( \mathcal{X} \) is a Banach \( \mathcal{A}-\mathcal{B} \)-bimodule. The only problem with this general situation is to determine the relationship between the joint spectrum \( \sigma(L_{a_j}, R_{b_j}) \) and the joint spectra \( \sigma(a_j) \) and \( \sigma(b_j) \). Note that \( a_j, b_j \) are elements of a unital Banach algebra, and \( L_{a_j} \) and \( R_{b_j} \) are left and right multiplications on the bimodule \( \mathcal{X} \). We believe that again a result analogous to that of Harte and Hernandez holds.
5. In \[20\], Shul’man derived an ascent estimate for an elementary operator \( \Lambda : B(H) \rightarrow B(H), \) \( \Lambda(X) = \sum_{j=1}^{n} A_j X B_j, \) where \( A_j \) and \( B_j \) are commuting \( n \)-tuples of normal operators acting on a Hilbert space \( H. \) He proved that \( \text{asc}(\Lambda) \leq n - 1 \) and \( \text{asc}(\Lambda) \leq (c/2], \) where \( c \) is the Hausdorff dimension of the joint spectrum \( \sigma_T(A_1, \ldots, A_n). \) Here \( (c/2] \) denotes the least integer greater than or equal to \( c/2, \) i.e. \( (c/2] = \lfloor c/2 \rfloor + 1 \) for noninteger \( c/2, \) and \( (c/2] = c/2 \) for integer \( c/2. \) (The number \( c \) does not depend on the \( n \)-tuple \( \{B_n\}! \))

Our last question is whether an analogous result holds for pre-normal elements of a unital Banach algebra.

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**References**


Matematički Fakultet
Univerzitet u Beogradu
Studentski trg 16–18
11000 Beograd, Serbia and Montenegro
E-mail: arsenovic@matf.bg.ac.yu
keckic@matf.bg.ac.yu

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