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Domination by positive Banach–Saks operators

by

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Abstract. Given a positive Banach–Saks operator T between two Banach lattices E and F, we give sufficient conditions on E and F in order to ensure that every positive operator dominated by T is Banach–Saks. A counterexample is also given when these conditions are dropped. Moreover, we deduce a characterization of the Banach–Saks property in Banach lattices in terms of disjointness.

1. Introduction. Banach–Saks operators were introduced by Beauzamy in [4]. These operators form an operator ideal in the sense of Pietsch. Recall that given two Banach spaces X and Y, a bounded operator $T : X \to Y$ is called *Banach–Saks* if for every bounded sequence $(x_n)_n$ in X, the sequence of images $(Tx_n)_n$ has a subsequence which is Cesàro convergent in Y (i.e, there exists a subsequence $(Tx_{n_k})_k$ such that $(r^{-1}\sum_{k=1}^{r} Tx_{n_k})_r$ is norm convergent in Y). The Banach–Saks property was considered for the first time by Banach and Saks for L^p spaces, $1 < p < \infty$, in their seminal paper [3]. Of course, X is said to have the Banach–Saks property (or to be Banach–Saks) if the identity $\mathrm{Id} : X \to X$ is a Banach–Saks operator.

In this note we present a domination result in the class of Banach–Saks operators. Precisely, let $T : E \to F$ be a positive operator from the Banach lattice E taking values in the Banach lattice F (that is, $Tx \geq 0$ for every $x \geq 0$ in E). Assume that T is Banach–Saks. We want to impose conditions on the Banach lattices E and F in order to ensure that every positive operator $S : E \to F$ dominated by T (that is, $(T - S)x \geq 0$ for every $0 \leq x \in E$) is also Banach–Saks. Note that such a result can be regarded as a sufficient condition for an operator to be Banach–Saks.

This *domination problem* has been widely studied by some authors in other classes of operators, such as compact operators by Dodds and Fremlim ([7]), weakly compact operators by Wickstead ([20]) and Dunford–Pettis

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operators by Kalton and Saab ([14]). More recently, the present authors have obtained positive results in other classes such as that of strictly singular operators (see [11], [10], [12]).

Observe that this domination problem is trivially solved if any of the spaces involved is Banach–Saks. Thus, in order to avoid trivialities we want to make sure that there are examples to which our results can apply. Many Banach spaces are known to have the Banach–Saks property. For example, every uniformly convex Banach space belongs to this class ([5]). In fact, the Banach–Saks property of a Banach space implies reflexivity. Remarkably, the converse is not true as shown by Baernstein ([2]).

Before presenting our result, let us recall some definitions. A Banach lattice E with an order continuous norm has the *subsequence splitting property* ([19]) if for every bounded sequence $(f_n)_n$ in E, there are a subsequence $(n_k)_k$ and sequences $(g_k)_k$, $(h_k)_k$ in E with $|g_k| \wedge |h_k| = 0$ and $f_{n_k} = g_k + h_k$ such that

(i) $(g_k)_k$ is L-weakly compact (see below),
(ii) $|h_k| \wedge |h_l| = 0$ if $k \neq l$.

The *lower index* of a Banach lattice E is defined as $s(E) = \sup\{q \geq 1 :$ E satisfies an upper q-estimate$\}$. The *upper index* of E is defined as $\sigma(E) = \inf\{q \geq 1 : E$ satisfies a lower q-estimate$\}$.

The subsequence splitting property will be central to our argument below. As one can see in Proposition 3.2, it is an essential hypothesis. The class of spaces with this property is quite broad. For instance, every Banach lattice which does not uniformly contain copies of ℓ_n^∞, for all natural n, has the subsequence splitting property ([13], [9]). This is the case for every Banach lattice E with finite upper index $\sigma(E)$. Also every rearrangement invariant function space which contains no isomorphic copy of c_0 has the subsequence splitting property ([19]).

The main result of the paper is the following

THEOREM 1.1. *Let* $0 \leq S \leq T : E \to F$ *be two positive operators defined on a Banach lattice* E *and taking values in a Banach lattice* F. *Assume that* T *is Banach–Saks. Then* S *is Banach–Saks if one of the following conditions hold*:

(a) E *has the subsequence splitting property and* E' *is order continuous*,
(b) F *has the subsequence splitting property.*

We observe that the Baernstein space is a Banach lattice with the pointwise order. Also it is an order continuous Banach lattice, as also is its dual, because it is reflexive (cf. [16, Thm. 2.4.15]). Moreover, as a set, it is included in ℓ^2, therefore it is 2-concave (see [2]) and hence $\sigma(E) < \infty$ ([15, p. 100]). According to the above it has the subsequence splitting property.

Thus, spaces like the Baernstein space or $L^1[0,1]$ do not have the Banach–Saks property but they satisfy hypotheses (a) and (b) of the theorem above, respectively.

On the other hand, Dodds, Semenov and Sukochev have recently dealt in [8] with the problem of characterizing the Banach–Saks property in rearrangement invariant spaces. To this end they consider a subsequence splitting-type property. On our way to obtaining the domination result we also obtain a characterization of the Banach–Saks property in the more general setting of Banach lattices, in terms of disjointness (see Proposition 3.2 and corollaries after it).

The concepts and notation employed for Banach lattices and positive operators are standard. For any unexplained terms we refer to [1], [15], [16] or [21].

2. Domination theorem. Before presenting the proof of our theorem we need to recall some facts.

First, it is known that for every order continuous Banach lattice E with a weak unit there exists a probability space (Ω, Σ, μ), an (in general not closed) order ideal I of $L^1(\Omega, \Sigma, \mu)$, a lattice norm $\| \, \|_I$ on I and an order isometry ψ_1 between E and $(I, \| \, \|_I)$ such that the canonical inclusion from I into $L^1(\Omega, \Sigma, \mu)$ is continuous with $\|f\|_1 \le \|f\|_I$ (cf. [15, Prop. 1.b.14]).

Let E be a Banach function space with an order continuous norm defined on a finite measure space (Ω, Σ, μ). Recall that a bounded subset $A \subset E$ is *uniformly integrable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|f\chi_B\|_E < \varepsilon$ for every $B \in \Sigma$ with $\mu(B) < \delta$ and every $f \in A$. The concept of L-weakly compact set generalizes the concept of uniformly integrable in the setting of Banach lattices (see [16] for the definition).

The following result is known (see [11, Lemma 3.3] for a proof). It provides the characterization of L-weakly compact sets to be used later.

LEMMA 2.1. *Let E be a Banach lattice with order continuous norm and a weak unit, and hence representable as an order ideal in $L^1(\Omega, \mu)$ for some probability space (Ω, Σ, μ). Then*:

(a) *a bounded subset of E is uniformly integrable if and only if it is L-weakly compact*,
(b) *a norm bounded sequence $(g_n)_n$ in E is convergent if and only if it is uniformly integrable and $\| \, \|_1$-convergent.*

A Banach space E has the *weak Banach–Saks property* (or it is *weakly Banach–Saks*) if every weakly convergent sequence $(f_n)_n$ in E has a subsequence which is Cesàro convergent.

We will make use of an important result due to Szlenk, initially given for $[0,1]$ with the Lebesgue measure ([18]), but easily extended to arbitrary probability spaces.

THEOREM 2.2 (Szlenk). *Let (Ω,Σ,μ) be a probability space. Then $L^1(\Omega,\mu)$ is weakly Banach–Saks.*

LEMMA 2.3. *Let E be an order continuous Banach lattice. If a bounded sequence $(f_n)_n$ in E is L-weakly compact, then there exists a subsequence $(f_{n_k})_k$ which is Cesàro convergent (i.e. $(r^{-1}\sum_{k=1}^r f_{n_k})_r$ is convergent).*

Proof. Denote by M the closed subspace spanned by $(f_n)_n$. Since M is separable, there is a closed ideal G in E, with a weak unit, that contains M (cf. [15, 1.a.9]). Thus, we can assume that E is an order continuous Banach lattice with a weak unit and hence it is representable as an order ideal in $L^1(\Omega,\mu)$ for some probability space (Ω,Σ,μ). Moreover, there is some $K>0$ such that, for every $B\in\Sigma$,

$$\|f_n\chi_B\|_1 \le K\|f_n\chi_B\|_E \quad \text{for all } n \text{ in } \mathbb{N}.$$

Therefore $(f_n)_n$ is uniformly integrable in $L^1(\Omega)$ (Lemma 2.1(a)) and, by the Dunford–Pettis theorem (see [17, Proposition IV-2-3]), we can choose a subsequence $(f_{n_k})_k$ converging weakly to a function f in $L^1(\Omega)$. By Szlenk's theorem, there exists another subsequence $(f_{n_{k_j}})_j$ such that

$$\Big\|\frac{1}{r}\sum_{j=1}^r f_{n_{k_j}} - f\Big\|_1 \to 0 \quad \text{as } r\to\infty$$

for some function $f\in L^1(\Omega,\mu)$. Observe that the sequence $(r^{-1}\sum_{j=1}^r f_{n_{k_j}})_r$ is also uniformly integrable in E and therefore $(r^{-1}\sum_{j=1}^r f_{n_{k_j}})_r$ converges to f in E by Lemma 2.1(b). ■

REMARK 2.4. Lemma 2.3 extends Theorem 4.10 in [8] to the setting of order continuous Banach lattices, with slightly more restrictive assumptions. Note that L-weak compactness implies both weak compactness (cf. [16, Prop. 3.6.5]) and uniform integrability (Lemma 2.1) used in Theorem 4.10 cited above.

LEMMA 2.5. *Let E be a Banach lattice such that E' is order continuous. If $0\le S\le T: E\to F$ are two positive operators and T is a Banach–Saks operator, then for every bounded sequence $(h_n)_n$ in E of mutually disjoint elements there exists a subsequence $(Sh_{n_k})_k$ which is Cesàro convergent.*

Proof. By considering the decomposition $h_n = h_n^+ - h_n^-$, where h_n^+ and h_n^- are the positive and negative parts of h_n, respectively, we can assume that h_n is positive for every $n\in\mathbb{N}$. Note that the sequence $(h_n)_n$ is weakly null, since E' is order continuous ([16, Thm. 2.4.14]). Since T is a Banach–Saks operator, there exists a subsequence $(h_{n_k})_k$ such that $(r^{-1}\sum_{k=1}^{r} Th_{n_k})_r$ is convergent. In fact, its limit must be zero, since $(Th_{n_k})_k$ is weakly null. Finally, observe that S is dominated by T and that $(h_{n_k})_k$ is a positive sequence. Therefore we have

$$0 \le \frac{1}{r}\sum_{k=1}^{r} Sh_{n_k} \le \frac{1}{r}\sum_{k=1}^{r} Th_{n_k}.$$

The conclusion follows. ∎

THEOREM 2.6. *Let E and F be two Banach lattices such that E has the subsequence splitting property and E' is order continuous*, *and let* $0 \le S \le T : E \to F$ *be two positive operators. If T is Banach–Saks*, *then so is S.*

Proof. Let $(f_n)_n$ be a bounded sequence in E. Since E has the subsequence splitting property we can extract a subsequence $(n_k)_k$ and write, for all k,

$$f_{n_k} = g_k + h_k,$$

where the sequence $(g_k)_k$ is L-weakly compact in E, $(h_k)_k$ is bounded pairwise disjoint, and $(g_k)_k$ and $(h_k)_k$ are mutually disjoint. Use jointly Lemma 2.3 and Lemma 2.5 to conclude. ∎

LEMMA 2.7. *Let E be an order continuous Banach lattice. Let $(h_k)_k$ be a pairwise disjoint bounded sequence in E such that $(r^{-1}\sum_{k=1}^{r} h_k)_r$ is convergent to some $h \in E$. Then $h = 0$.*

Proof. As above, we can think of E as an order ideal of $L^1(\mu)$ over some probability space (Ω, Σ, μ). Since the inclusion of E into $L^1(\mu)$ is continuous, the sequence $(r^{-1}\sum_{k=1}^{r} h_k)_r$ converges to h in the norm of $L^1(\mu)$; hence there exists a subsequence $(r_j)_j$ such that $(r_j^{-1}\sum_{k=1}^{r_j} h_k)_{r_j}$ converges to h everywhere. The disjointness of $(h_k)_k$ implies that h must be 0. ∎

THEOREM 2.8. *Let E and F be two Banach lattices such that F has the subsequence splitting property*, *and let* $0 \le S \le T : E \to F$ *be two positive operators. If T is Banach–Saks*, *then so is S.*

Proof. Choose an arbitrary bounded sequence $(f_n)_n$ in E. We may assume that $f_n \ge 0$ for every $n \in \mathbb{N}$. Since T is a Banach–Saks operator and F has the subsequence splitting property, there exist a subsequence $(n_k)_k$, a uniformly integrable sequence $(g_k)_k$ of positive elements in F, and a pairwise disjoint sequence $(h_k)_k$ of positive elements in F such that $(r^{-1}\sum_{k=1}^{r} Tf_{n_k})_r$ is convergent and $Tf_{n_k} = g_k + h_k$ for all k. Moreover, $(r^{-1}\sum_{k=1}^{r} g_k)_r$ is convergent by Lemma 2.3, while $(r^{-1}\sum_{k=1}^{r} h_k)_r$ converges to zero by Lemma 2.7. Since $0 \le Sf_{n_k} \le Tf_{n_k}$ for every $k \in \mathbb{N}$, by the Riesz decomposition property ([15, p. 2]), we can choose two positive sequences

$(g'_k)_k$ and $(h'_k)_k$ in F such that

$$0 \le g'_k \le g_k, \quad 0 \le h'_k \le h_k, \quad Sf_{n_k} = g'_k + h'_k.$$

Clearly, the sequence $(r^{-1}\sum_{k=1}^{r} h'_k)_r$ converges to zero and the sequence $(g'_k)_k$ has a Cesàro convergent subsequence since it is uniformly integrable. The proof is complete. ∎

Proof of Theorem 1.1. Put together Theorems 2.6 and 2.8. ∎

As previously said, we cannot expect counterexamples to the domination problem if either of the spaces involved is Banach–Saks. Also notice that the space E given by Baernstein ([2]) is not suitable to furnish a counterexample, since it meets the requirements of Theorem 1.1. As pointed out above, E is 2-concave and hence it has the subsequence splitting property.

To obtain a counterexample to the question of domination, we take advantage of the work in [11].

EXAMPLE 2.9. There exist two operators $0 \le S \le T : \ell^1 \to L^\infty[0,1]$ such that T is Banach–Saks and yet S is not.

Indeed, consider the isometry $\widetilde{S} : \ell^1 \to L^\infty[0,1]$ that takes the nth element e_n of the canonical basis of ℓ^1 to the nth Rademacher function r_n on $[0,1]$ (cf. [6, p. 203]). The isometry $\widetilde{S}$ cannot be Banach–Saks since ℓ^1 is not a Banach–Saks space. Now, consider the positive operators $S_1, S_2 : \ell^1 \to L^\infty[0,1]$ defined by $S_1(e_n) = r_n^+$ and $S_2(e_n) = r_n^-$, where r_n^+ and r_n^- denote the positive and negative parts of r_n, respectively. Clearly $\widetilde{S} = S_1 - S_2$ and $0 \le S_1, S_2 \le T$, where T is the rank one operator defined by $T(x) = (\sum_{n=1}^{\infty} x_n)\chi_{[0,1]}$. Note that the operator T is Banach–Saks, being compact, and yet neither S_1 nor S_2 is Banach–Saks, since we have the equalities $T = S_1 + S_2$ and $\widetilde{S} = S_1 - S_2$.

3. Final remarks. The previous work yields a characterization of the Banach–Saks property for Banach lattices in terms of disjointness, under some general assumptions. To this end we introduce the following.

DEFINITION 3.1. A Banach lattice E is *disjointly Banach–Saks* if, for every pairwise disjoint bounded sequence $(x_n)_n$ in E, there is a subsequence $(n_k)_k$ such that the sequence $(x_{n_k})_k$ is Cesàro convergent.

This definition differs from that in [8] since we do not assume the sequence involved to be weakly null. Think of $L^1[0,1]$. This definition also differs from the Banach–Saks property. Indeed, the space c_0 is not Banach–Saks, since it is not reflexive, and yet it is disjointly Banach–Saks. Note that c_0 fails to have the subsequence splitting property. In contrast, we have the following proposition.

PROPOSITION 3.2. *Let E be a Banach lattice with the subsequence splitting property and with E' order continuous. The following conditions are equivalent*:

(a) *E is a Banach–Saks space*,
(b) *E is a weakly Banach–Saks space*,
(c) *E is a disjointly Banach–Saks space.*

Note that the Baernstein space is a Banach lattice which does not have any of the properties above. In fact, the canonical basis of ℓ^1 contains no Cesàro convergent subsequence (cf. [2]).

Proof of the proposition. It is obvious that (a) implies (b). If we assume (b) and $(h_n)_n \subseteq E$ is a bounded pairwise disjoint sequence, then it is weakly null, since E' is order continuous ([16, Thm. 2.4.14]); therefore (c) follows. To see that (c) implies (a), let $(f_n)_n$ be a bounded sequence in E. As we saw in the proof of Theorem 2.6, there exists a subsequence $(n_k)_k$ such that $f_{n_k} = g_k + h_k$, where $(g_k)_k$ is uniformly integrable and $(h_k)_k$ is pairwise disjoint. By Lemma 2.3, there exists a subsequence $(g_{k_j})_j$ which is Cesàro convergent in E. On the other hand, the sequence $(h_k)_k$ has a Cesàro convergent subsequence, by assumption. Hence (a) holds. ∎

COROLLARY 3.3. *Let E be a Banach lattice with the subsequence splitting property. Then E is a Banach–Saks space if and only if E is a disjointly Banach–Saks space.*

Observe that $L^1[0,1]$ is weakly Banach–Saks but not disjointly Banach–Saks.

COROLLARY 3.4. *Let E be a Banach lattice with the subsequence splitting property. Then E is a weakly Banach–Saks space if and only if every pairwise disjoint weakly convergent sequence in E has a subsequence which is Cesàro convergent.*

Notice that the last results extends, to the more general setting of Banach lattices, the result given for rearrangement invariant spaces in [8, Theorem 4.5].

References

[1] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Academic Press, 1985.
[2] A. Baernstein, *On reflexivity and summability*, Studia Math. 42 (1972), 91–94.
[3] S. Banach and S. Saks, *Sur la convergence forte dans les champs L^p*, ibid. 2 (1930), 51–57.
[4] B. Beauzamy, *Propriété de Banach–Saks*, ibid. 66 (1980), 227–235.
[5] J. Diestel, *Geometry of Banach Spaces—Selected Topics*, Lecture Notes in Math. 485, Springer, 1975.

[6] J. Diestel, *Sequences and Series in Banach Spaces*, Springer, 1984.

[7] P. G. Dodds and D. H. Fremlin, *Compact operators in Banach lattices*, Israel J. Math. 34 (1979), 287–320.

[8] P. G. Dodds, E. Semenov, and F. A. Sukochev, *The Banach–Saks property in rearrangement invariant spaces*, Studia Math. 162 (2004), 263–294.

[9] T. Figiel, N. Ghoussoub, and W. B. Johnson, *On the structure of nonweakly compact operators on Banach lattices*, Math. Ann. 257 (1981), 317–334.

[10] J. Flores and F. L. Hernández, *Domination by positive disjointly strictly singular operators*, Proc. Amer. Math. Soc. 129 (2001), 1979–1986.

[11] —, *Domination by positive strictly singular operators*, J. London Math. Soc. 66 (2002), 433–452.

[12] J. Flores and C. Ruiz, *Domination by positive narrow operators*, Positivity 7 (2003), 303–321.

[13] W. B. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri, *Symmetric structures in Banach spaces*, Mem. Amer. Math. Soc. 19 (1979).

[14] N. J. Kalton and P. Saab, *Ideal properties of regular operators between Banach lattices*, Illinois J. Math. 29 (1985), 382–400.

[15] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer, 1979.

[16] P. Meyer-Nieberg, *Banach Lattices*, Springer, 1991.

[17] J. Neveu, *Discrete-Parameter Martingales*, Math. Library 10, North-Holland, Amsterdam, 1975.

[18] W. Szlenk, *Sur les suites faiblement convergentes dans l'espace l*, Studia Math. 25 (1965), 337–341.

[19] L. Weis, *Banach lattices with the subsequence splitting property*, Proc. Amer. Math. Soc. 105 (1989), 87–96.

[20] A. W. Wickstead, *Extremal structure of cones of operators*, Quart. J. Math. Oxford Ser. (2) 32 (1981), 239–253.

[21] A. C. Zaanen, *Riesz Spaces II*, North-Holland, Amsterdam, 1983.

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