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a-Weyl's theorem and perturbations

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Abstract. We study the stability of a-Weyl's theorem under perturbations by operators in some known classes. We establish in particular that if T is a finite a-isoloid operator, then a-Weyl's theorem is transmitted from T to $T+R$ for every Riesz operator R commuting with T.

1. Introduction. Throughout this paper, X will denote an infinite-dimensional complex Banach space, $\mathcal{L}(X)$ the algebra of all linear bounded operators on X, and $\mathcal{K}(X)$ its ideal of compact operators. For an operator $T \in \mathcal{L}(X)$, write T^* for its adjoint; $\mathrm{N}(T)$ for its kernel; $\mathrm{R}(T)$ for its range; $\sigma(T)$ for its spectrum; $\sigma_{\mathrm{ap}}(T)$ for its approximate point spectrum; and $\sigma_{\mathrm{p}}(T)$ for its point spectrum.

For an operator $T \in \mathcal{L}(X)$, the *ascent* $\mathrm{a}(T)$ and *descent* $\mathrm{d}(T)$ are given by $\mathrm{a}(T) = \inf\{n \geq 0 : \mathrm{N}(T^n) = \mathrm{N}(T^{n+1})\}$ and $\mathrm{d}(T) = \inf\{n \geq 0 : \mathrm{R}(T^n) = \mathrm{R}(T^{n+1})\}$, respectively; the infimum over the empty set is taken to be ∞. If the ascent and descent of $T \in \mathcal{L}(X)$ are both finite, then $\mathrm{a}(T) = \mathrm{d}(T) = p$, $X = \mathrm{N}(T^p) \oplus \mathrm{R}(T^p)$ and $\mathrm{R}(T^p)$ is closed (see [16]).

Also, an operator $T \in \mathcal{L}(X)$ is called *semi-Fredholm* if $\mathrm{R}(T)$ is closed and either $\dim \mathrm{N}(T)$ or $\operatorname{codim} \mathrm{R}(T)$ is finite. For such an operator the *index* is defined by $\mathrm{ind}(T) = \dim \mathrm{N}(T) - \operatorname{codim} \mathrm{R}(T)$, and if the index is finite, T is said to be *Fredholm*. For $T \in \mathcal{L}(X)$, the *essential spectrum* $\sigma_{\mathrm{e}}(T)$, the *semi-Fredholm spectrum* $\sigma_{\mathrm{SF}}(T)$, the *Weyl spectrum* $\sigma_{\mathrm{w}}(T)$, the *Browder spectrum* $\sigma_{\mathrm{b}}(T)$, the *essential approximate point spectrum* $\sigma_{\mathrm{ea}}(T)$ and the *Browder essential approximate point spectrum* $\sigma_{\mathrm{ab}}(T)$ are given by

$$\sigma_{\mathrm{e}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$

$$\sigma_{\mathrm{SF}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm}\},$$

$$\sigma_{\mathrm{w}}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm of index } 0\},$$

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$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm of finite ascent and descent}\},$$
$$\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm of non-positive index}\},$$
$$\sigma_{ab}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm of finite ascent}\}.$$

It is well known that

$$\sigma_{ea}(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T)$$

and

$$\sigma_{ea}(T) \subseteq \sigma_{ab}(T) \subseteq \sigma_b(T).$$

For a subset K of $\mathbb{C}$, we write $\operatorname{iso} K$ for its isolated points and $\operatorname{acc} K$ for its accumulation points. A complex number λ is said to be a *Riesz point* of $T \in \mathcal{L}(X)$ if $\lambda \in \operatorname{iso} \sigma(T)$ and the spectral projection corresponding to the set $\{\lambda\}$ has finite-dimensional range. The set of all Riesz points of T is denoted by $\Pi_o(T)$, and we note that $\Pi_o(T) = \operatorname{iso} \sigma(T) \cap \varrho_e(T)$ where $\varrho_e(T) = \mathbb{C} \setminus \sigma_e(T)$ (see [3] or [11]). Also, from [4] we recall that if $T \in \mathcal{L}(X)$ and $\lambda \in \sigma(T)$, then $\lambda \in \Pi_o(T)$ if and only if $T - \lambda$ is Fredholm of finite ascent and descent. Consequently, $\sigma_b(T) = \sigma(T) \setminus \Pi_o(T) = \sigma_e(T) \cup \operatorname{acc} \sigma(T)$.

The set of isolated points λ in the spectrum (resp. approximate spectrum) for which $\mathrm{N}(T - \lambda)$ is non-zero and finite-dimensional is denoted by $\Pi_{oo}(T)$ (resp. $\Pi_{oo}^{a}(T)$).

Definition. Let T be a bounded operator on X. We will say that

(i) *Weyl's theorem holds for* T if $\sigma_w(T) = \sigma(T) \setminus \Pi_{oo}(T)$.
(ii) *a-Weyl's theorem holds for* T if $\sigma_{ea}(T) = \sigma_{ap}(T) \setminus \Pi_{oo}^{a}(T)$.
(iii) *Browder's theorem holds for* T if $\sigma_w(T) = \sigma_b(T)$.
(iv) *a-Browder's theorem holds for* T if $\sigma_{ea}(T) = \sigma_{ab}(T)$.

It is well known that the following implications hold ([3], [13]):

a-Weyl's theorem $\Rightarrow$ Weyl's theorem $\Rightarrow$ Browder's theorem;

a-Weyl's theorem $\Rightarrow$ a-Browder's theorem $\Rightarrow$ Browder's theorem.

In this paper, we examine the stability of a-Weyl's theorem under perturbations by operators in some known classes. We prove that if $T \in \mathcal{L}(X)$ is a finite a-isoloid operator that satisfies a-Weyl's theorem and F is a bounded operator commuting with T and for which there exists a positive integer n such that F^n has finite rank, then Weyl's theorem holds for $T + F$. Further, we establish that if, in addition, T is finite a-isoloid, then $T + R$ obeys Weyl's theorem where R is an arbitrary Riesz operator commuting with T.

2. a-Weyl's theorem under perturbations.

Before stating our results, we need to introduce the following two subspaces that will play a fundamental role in this paper.

Let T be a bounded operator on X. The *quasi-nilpotent part* of T is defined by

$$\mathrm{H_o}(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\},$$

and the *analytic core* of T by

$$\mathrm{K}(T) := \{x \in X : \exists \{x_n\}_{n\geq 0} \subseteq X \text{ and } \exists c > 0 \text{ such that } x = x_0, \\ Tx_{n+1} = x_n \text{ and } \|x_n\| \leq c^n \|x\| \text{ for all } n \geq 0\}.$$

These subspaces are T-hyperinvariant, i.e. if S is a bounded operator on X that commutes with T, then $S\mathrm{H_o}(T) \subseteq \mathrm{H_o}(T)$ and $S\mathrm{K}(T) \subseteq \mathrm{K}(T)$, and generally not closed. However, if $\mathrm{H_o}(T)$ is closed, then $T_{|\mathrm{H_o}(T)}$ is quasi-nilpotent (see [10]). Also, if T semi-Fredholm or semi-regular (i.e. $\mathrm{R}(T)$ is closed and $\mathrm{N}(T^n) \subseteq \mathrm{R}(T)$ for all positive integers n), then $\mathrm{K}(T) = \bigcap_{n=1}^{\infty} \mathrm{R}(T^n)$ is closed (see [10]). The following facts are easy to verify: $T(\mathrm{K}(T)) = \mathrm{K}(T)$ and $\bigcup_{n=1}^{\infty} \mathrm{N}(T^n) \subseteq \mathrm{H_o}(T)$; if T is injective with closed range then $\mathrm{H_o}(T) = \{0\}$.

From Theorem 1.6 of [8], we recall the following useful characterization: $\lambda \notin \operatorname{acc}\sigma(T)$ if and only if $X = \mathrm{H_o}(T-\lambda) \oplus \mathrm{K}(T-\lambda)$ where the direct sum is topological; and in this case, $\mathrm{H_o}(T-\lambda)$ is non-zero precisely when λ is an isolated point of the spectrum.

The equivalences (i)–(v) in the following lemma were first established in [2] (see also [1, Chapter 3, §2]); we give here the proof for completeness.

LEMMA 2.1. *Let T be a semi-Fredholm operator. The following assertions are equivalent*:

(i) *T has finite ascent*;
(ii) $\mathrm{H_o}(T) \cap \mathrm{K}(T) = \{0\}$;
(iii) $\mathrm{H_o}(T)$ *is finite-dimensional*;
(iv) *there exists a positive integer p for which* $\mathrm{H_o}(T) = \mathrm{N}(T^p)$;
(v) $\mathrm{H_o}(T)$ *is closed.*

Moreover, 0 *is an isolated point of* $\sigma_{\mathrm{ap}}(T)$ *if and only if* $\mathrm{H_o}(T)$ *is a non-zero closed subspace.*

Proof. First, since T is semi-Fredholm, the Kato decomposition [7, Theorem 4] provides two closed T-invariant subspaces X_1, X_2 such that $X = X_1 \oplus X_2$, X_1 is finite-dimensional, $T_1 := T_{|X_1}$ is nilpotent and $T_2 := T_{|X_2}$ is semi-regular. Consequently, $X_1 \subseteq \mathrm{H_o}(T)$, $\mathrm{H_o}(T) = X_1 \oplus \mathrm{H_o}(T) \cap X_2$ and $\mathrm{K}(T) = \bigcap \mathrm{R}(T^n) = \mathrm{K}(T_2)$.

(i)⇒(ii). Since T_2 is semi-regular, $\overline{\mathrm{H_o}(T_2)} = \overline{\bigcup_n \mathrm{N}(T_2^n)}$ by [10, Lemma 1.1]. Moreover, T has finite ascent; then so does T_2 and hence $\mathrm{H_o}(T_2) = \mathrm{N}(T_2^p)$ where $p = \mathrm{a}(T_2)$. Consequently, $\mathrm{H_o}(T_2)$ is closed and so $\mathrm{H_o}(T) \cap X_2 = \mathrm{H_o}(T_2) = \{0\}$ (see [8]). Thus $\mathrm{H_o}(T) \cap \mathrm{K}(T) = \mathrm{H_o}(T_2) \cap \mathrm{K}(T_2) = \{0\}$.

(ii)⇒(iii). Since $\mathrm{H_o}(T_2) \subseteq \mathrm{H_o}(T) \cap \mathrm{K}(T) = \{0\}$, we see that $\mathrm{H_o}(T) = X_1$ is finite-dimensional.

(iii)⇒(iv). If $\mathrm{H}_{\mathrm{o}}(T)$ is finite-dimensional then $T_{|\mathrm{H}_{\mathrm{o}}(T)}$ is nilpotent, and therefore there exists $p \geq 1$ such that $\mathrm{H}_{\mathrm{o}}(T) \subseteq \mathrm{N}(T^p)$. Thus, $\mathrm{H}_{\mathrm{o}}(T) = \mathrm{N}(T^p)$.

(iv)⇒(v) is clear.

(v)⇒(i). From the fact that $\mathrm{H}_{\mathrm{o}}(T_2) = \mathrm{H}_{\mathrm{o}}(T) \cap X_2$ is closed and T_2 is semi-regular, we deduce that $\mathrm{H}_{\mathrm{o}}(T_2) = \{0\}$. Thus T_2 is injective, and because X_1 is finite-dimensional, we conclude that T has finite ascent.

For the "moreover" part suppose that $\mathrm{H}_{\mathrm{o}}(T)$ is a non-zero closed subspace. It follows from the proof of (i)⇒(ii) that 0 is an isolated point of $\sigma_{\mathrm{ap}}(T)$. Conversely, if $0 \in \operatorname{iso} \sigma_{\mathrm{ap}}(T)$, and because $\mathrm{R}(T)$ is closed, we find that $\mathrm{N}(T)$, and consequently $\mathrm{H}_{\mathrm{o}}(T)$, is non-zero. Let $\lambda \neq 0$ in a connected neighborhood of 0 be such that $T-\lambda$ is injective with closed range. Then $T_2-\lambda$ is injective with closed range and $\mathrm{H}_{\mathrm{o}}(T_2-\lambda) = \{0\}$, which implies that $\mathrm{H}_{\mathrm{o}}(T_2) = \{0\}$ by Lemma 1.3 of [10]. Finally, $\mathrm{H}_{\mathrm{o}}(T) = X_1$ is closed. ∎

Obviously, it follows from the previous lemma that every semi-Fredholm operator with finite ascent has a non-positive index.

For an operator T, we denote by $\Pi_{\mathrm{o}}^{\mathrm{a}}(T)$ the set of all isolated points λ of $\sigma_{\mathrm{ap}}(T)$ for which $T-\lambda$ is semi-Fredholm.

Remark. Let T be a bounded operator on X. As immediate consequences of Lemma 2.1, we derive the following assertions:

(i) $\Pi_{\mathrm{o}}^{\mathrm{a}}(T) \subseteq \Pi_{\mathrm{oo}}^{\mathrm{a}}(T)$ and $\sigma_{\mathrm{ab}}(T) = \sigma_{\mathrm{ap}}(T) \setminus \Pi_{\mathrm{o}}^{\mathrm{a}}(T) = \operatorname{acc} \sigma_{\mathrm{ap}}(T) \cup \sigma_{\mathrm{SF}}(T)$.
(ii) If T satisfies a-Browder's theorem, then a-Weyl's theorem holds for T if and only if $\Pi_{\mathrm{o}}^{\mathrm{a}}(T) = \Pi_{\mathrm{oo}}^{\mathrm{a}}(T)$.
(iii) If a-Weyl's theorem holds for T then so does a-Browder's theorem. Indeed, a-Weyl's theorem for T implies $\Pi_{\mathrm{oo}}^{\mathrm{a}}(T) \cap \sigma_{\mathrm{SF}}(T) \subseteq \Pi_{\mathrm{oo}}^{\mathrm{a}}(T) \cap \sigma_{\mathrm{ea}}(T) = \emptyset$, and so $\Pi_{\mathrm{oo}}^{\mathrm{a}}(T) \subseteq \Pi_{\mathrm{o}}^{\mathrm{a}}(T) = \operatorname{iso} \sigma_{\mathrm{ap}}(T) \cap \varrho_{\mathrm{SF}}(T)$. Thus, $\Pi_{\mathrm{o}}^{\mathrm{a}}(T) = \Pi_{\mathrm{oo}}^{\mathrm{a}}(T)$ and $\sigma_{\mathrm{ea}}(T) = \sigma_{\mathrm{ab}}(T)$.

An operator $R \in \mathcal{L}(X)$ is called *Riesz* if $R-\lambda$ is Fredholm for every non-zero complex number λ. It is well known that the restriction of R to one of its closed invariant subspace is a Riesz operator (see [4]). In [15], it is shown by M. Schechter and R. Whitley that if T is a semi-Fredholm operator that commutes with R, then $T+R$ is semi-Fredholm and $\operatorname{ind}(T+R) = \operatorname{ind}(T)$.

Lemma 2.1 allows us to derive a shorter proof of the following result due to V. Rakočević [14].

Proposition 2.2. *Let $T \in \mathcal{L}(X)$ be a semi-Fredholm operator and R be a Riesz operator that commutes with T. The following assertions hold*:

(i) *If T has finite ascent then so does $T+R$.*
(ii) *If T has finite descent then so does $T+R$.*

Proof. (i) Suppose first that T is injective. It follows that the operator $S := T + R$ is semi-Fredholm and $\operatorname{ind}(S) = \operatorname{ind}(T) \leq 0$. Therefore $\mathrm{N}(S^p)$ is finite-dimensional and hence $T\mathrm{N}(S^p) = \mathrm{N}(S^p)$ for every $p \geq 1$; consequently,

$$\mathrm{N}(S^p) \subseteq \mathrm{K}(T) \quad \text{for all } p \in \mathbb{N}. \tag{2.1}$$

On the other hand, $\mathrm{K}(T) = \bigcap \mathrm{R}(T^n)$ is closed, and since $T_{|\mathrm{K}(T)}$ is invertible and $R_{|\mathrm{K}(T)}$ is a Riesz operator that commutes with $T_{|\mathrm{K}(T)}$, [6, Theorem 3.5] implies that the restriction of S to $\mathrm{K}(T)$ has finite ascent, that is, by (2.1), S has finite ascent. Now, if T is semi-Fredholm with finite ascent, then, by Lemma 2.1, $\mathrm{H}_{\mathrm{o}}(T) = \mathrm{N}(T^d)$ is finite-dimensional, where d is a positive integer. Consider the maps $\widehat{T}$, $\widehat{S}$, $\widehat{R}$ on $X/H_0(T)$ induced respectively by T, S and R. It is straightforward that $\widehat{T}$ is injective with closed range and $\widehat{R}$ is a Riesz operator commuting with $\widehat{T}$. Therefore $\widehat{S}$ is semi-Fredholm of finite ascent $k = a(\widehat{S})$ and so $\mathrm{N}(S^p) \subseteq (S^k)^{-1}(\mathrm{H}_{\mathrm{o}}(T))$ for all positive integer p. Moreover, because S is semi-Fredholm with $\operatorname{ind}(S) = \operatorname{ind}(T) < \infty$, $\mathrm{N}(S)$ is finite-dimensional and hence so is $(S^k)^{-1}(\mathrm{H}_{\mathrm{o}}(T))$. Thus S has finite ascent, as desired.

(ii) By duality. ∎

The following corollary follows from the previous proposition and the fact that the essential approximate point spectrum is invariant under commuting Riesz perturbation.

COROLLARY 2.3. *If $T \in \mathcal{L}(X)$ satisfies a-Browder's theorem and R is a Riesz operator commuting with T, then $T + R$ satisfies a-Browder's theorem.*

For a bounded operator T on X, we use $\Pi^{\mathrm{a}}_{\mathrm{of}}(T)$ to denote the set of isolated points λ of $\sigma_{\mathrm{ap}}(T)$ such that $\mathrm{N}(T - \lambda)$ is finite-dimensional. Evidently, $\Pi^{\mathrm{a}}_{\mathrm{o}}(T) \subseteq \Pi^{\mathrm{a}}_{\mathrm{oo}}(T) \subseteq \Pi^{\mathrm{a}}_{\mathrm{of}}(T)$.

PROPOSITION 2.4. *Let T be a bounded operator on X. If R is a Riesz operator that commutes with T, then*

$$\Pi^{\mathrm{a}}_{\mathrm{of}}(T + R) \cap \sigma_{\mathrm{ap}}(T) \subseteq \operatorname{iso} \sigma_{\mathrm{ap}}(T).$$

To prove this proposition, we need the following elementary lemma:

LEMMA 2.5. *Let $T \in \mathcal{L}(X)$ be a quasi-nilpotent operator with finite-dimensional kernel. If R is a Riesz operator that commutes with T, then $\sigma(T + R)$ is a finite set.*

Proof. Suppose to the contrary that there exists a sequence $\{\lambda_n\}$ of distinct numbers in $\sigma(T+R)\backslash\{0\}$. It follows that $T-\lambda_n$ is invertible, and since R is a Riesz operator that commutes with T, we find that $T+R-\lambda_n$ is Fredholm with index zero. Therefore $\mathrm{N}(T + R - \lambda_n)$ is a non-zero finite-dimensional subspace because $T + R - \lambda_n$ is non-invertible, and hence the restriction of T to $\mathrm{N}(T + R - \lambda_n)$ is nilpotent. Consequently, $\mathrm{N}(T + R - \lambda_n) \cap \mathrm{N}(T)$ is not trivial and so it contains a non-zero element x_n. Since each x_n is an eigenvector of $T+R$ associated to λ_n, and the numbers λ_n are mutually distinct, we can easily check that $\{x_n\}$ consists of linearly independent vectors of $\mathrm{N}(T)$. Thus $\mathrm{N}(T)$ has infinite dimension, which is the desired contradiction. ■

Proof of Proposition 2.4. Assume that $\lambda \in \Pi^{\mathrm{a}}_{\mathrm{of}}(T+R)$. Then there exists a punctured neighbourhood U of λ such that $T+R-\mu$ is injective with closed range for all $\mu \in U$. Therefore, by Proposition 2.2, $T-\mu$ is a semi-Fredholm operator with finite ascent and hence Lemma 2.1 implies that $\mathrm{H_o}(T-\mu)$ is finite-dimensional and $\mathrm{H_o}(T-\mu) \cap \mathrm{K}(T-\mu) = \{0\}$ for $\mu \in U$. On the other hand, by Theorem 3.5 of [10], the closed subspaces $\mathrm{H_o}(T-\mu) + \mathrm{K}(T-\mu) = \mathrm{H_o}(T-\mu) \oplus \mathrm{K}(T-\mu)$ are constant on U. Let Z denote one of them and T_{o} and R_{o} be respectively the restrictions of T and R to Z.

We claim that λ is not an accumulation point of $\sigma(T_{\mathrm{o}})$. Let $\mu \in U$. Since $(T-\mu)_{|\mathrm{K}(T-\mu)}$ is invertible, $(T+R-\mu)_{|\mathrm{K}(T-\mu)}$ is Fredholm with index zero, and hence so is $T_{\mathrm{o}}+R_{\mathrm{o}}-\mu$ because $\mathrm{H_o}(T-\mu)$ is finite-dimensional. Moreover, $T+R-\mu$ is injective, therefore $T_{\mathrm{o}}+R_{\mathrm{o}}-\mu$ is invertible. This shows that $\lambda \notin \operatorname{acc}\sigma(T_{\mathrm{o}}+R_{\mathrm{o}})$ and consequently

$$Z = \mathrm{H_o}(T_{\mathrm{o}}+R_{\mathrm{o}}-\lambda) \oplus \mathrm{K}(T_{\mathrm{o}}+R_{\mathrm{o}}-\lambda).$$

Write $T_{\mathrm{o}} = T_1 + T_2$ and $R_{\mathrm{o}} = R_1 + R_2$ with respect to this decomposition. Since $T_1+R_1-\lambda$ is a quasi-nilpotent operator with finite-dimensional kernel, Lemma 2.5 ensures that $\sigma(T_1)$ is finite, and hence there exists a punctured neighbourhood V_1 of λ such that $V_1 \cap \sigma(T_1) = \emptyset$. Also, because $T_2+R_2-\lambda$ is invertible, $T_2-\lambda$ has finite ascent and descent. Consequently, there exists a punctured neighbourhood V_2 of λ such that $V_2 \cap \sigma(T_2) = \emptyset$. Now, if we let $V = V_1 \cap V_2 \cap U$, we find that $V \cap \sigma(T_{\mathrm{o}}) = \emptyset$. Finally, we have $\mathrm{N}(T-\mu) \subseteq \mathrm{H_o}(T-\mu) \subseteq Z$ and so $\mathrm{N}(T-\mu) = \mathrm{N}(T_{\mathrm{o}}-\mu) = \{0\}$ for $\mu \in V$. But for such μ, $T-\mu$ is semi-Fredholm, hence $T-\mu$ is injective with closed range. This completes the proof. ■

An operator $T \in \mathcal{L}(X)$ is said to be *a-isoloid* if all isolated points of $\sigma_{\mathrm{ap}}(T)$ are eigenvalues of T.

THEOREM 2.6. *Let T be an a-isoloid operator on X that satisfies a-Weyl's theorem. If F is an operator that commutes with T and for which there exists a positive integer n such that F^n has finite rank, then $T+F$ satisfies a-Weyl's theorem.*

Proof. First observe that F is a Riesz operator. Since a-Browder's theorem holds for $T+F$, by Corollary 2.3, it suffices to establish that $\Pi^{\mathrm{a}}_{\mathrm{oo}}(T+F) = \Pi^{\mathrm{a}}_{\mathrm{o}}(T+F)$. Let $\lambda \in \Pi^{\mathrm{a}}_{\mathrm{oo}}(T+F)$. If $T-\lambda$ is injective with closed range, then $T+F-\lambda$ is semi-Fredholm, and therefore $\lambda \in \Pi^{\mathrm{a}}_{\mathrm{o}}(T+F)$. Suppose that $\lambda \in \sigma_{\mathrm{ap}}(T)$. Then it follows by Proposition 2.4 that $\lambda \in \operatorname{iso}\sigma_{\mathrm{ap}}(T)$. Furthermore, since the restriction of $(T+F-\lambda)^n$ to $\mathrm{N}(T-\lambda)$ has finite-dimensional range and kernel, we infer that also $\mathrm{N}(T-\lambda)$ is finite-dimensional, and so $\lambda \in \Pi^{\mathrm{a}}_{\mathrm{oo}}(T)$ because T is a-isoloid. On the other hand, a-Weyl's theorem for T implies that $\Pi^{\mathrm{a}}_{\mathrm{oo}}(T) \cap \sigma_{\mathrm{ea}}(T) = \emptyset$. Consequently, $T-\lambda$ is semi-Fredholm and hence so is $T+F-\lambda$, which implies that $\lambda \in \Pi^{\mathrm{a}}_{\mathrm{o}}(T+F)$. The other inclusion is trivial, thus $T+F$ satisfies a-Weyl's theorem. ∎

In the following corollary, we recapture the result of D. S. Djordjević [5].

COROLLARY 2.7. *Let $T \in \mathcal{L}(X)$ be an a-isoloid operator. If a-Weyl's theorem holds for T, then it also holds for $T+F$ for every finite rank operator F commuting with T.*

Notice that in the preceding result, it is essential to require that T is a-isoloid. Indeed, if we let $\mathcal{F}(X)$ denote the set of finite rank operators on X, $\mathcal{N}(X)$ the set of nilpotent operators on X and $\{T\}'$ the set of operators commuting with T, then we have:

PROPOSITION 2.8. *Let T be a bounded operator such that $\mathcal{F}(X) \cap \{T\}' \nsubseteq \mathcal{N}(X)$. If a-Weyl's theorem holds for $T+F$ for every finite rank operator F that commutes with T, then T is a-isoloid.*

Proof. Suppose that T is not a-isoloid and let λ be an isolated point of $\sigma_{\mathrm{ap}}(T)$ such that $\mathrm{N}(T-\lambda) = \{0\}$. By hypothesis, there exists a finite rank operator F that is not nilpotent and commutes with T. Observe that F cannot be quasi-nilpotent, because if not, the restriction of F to $\mathrm{R}(F)$ is nilpotent, and hence so is F. Since the spectrum of any finite rank operator on X is finite and contains 0, we have $X = X_1 \oplus X_2$ where $X_1 = \mathrm{H}_{\mathrm{o}}(F)$ and $X_2 = \mathrm{K}(F)$. Furthermore, X_1 and X_2 are T-invariant, and from the fact that F is not quasi-nilpotent and $F_{|X_2}$ is an invertible operator of finite rank, we deduce that X_2 is a non-zero subspace of finite dimension.

Let $T = T_1 \oplus T_2$ be the decomposition of T with respect to $X = X_1 \oplus X_2$, and let α be a complex number for which $\lambda - \alpha \in \sigma_{\mathrm{ap}}(T_2) = \sigma_{\mathrm{p}}(T_2)$. Also, consider the operator $\widetilde{F} = 0 \oplus \alpha I_2$. Clearly $\widetilde{F}$ is a finite-rank operator that commutes with T and $\sigma_{\mathrm{ap}}(T+\widetilde{F}) = \sigma_{\mathrm{ap}}(T_1) \cup \sigma_{\mathrm{ap}}(T_2+\alpha)$. But since $\lambda \in \mathrm{iso}\,\sigma_{\mathrm{ap}}(T)$ and $T-\lambda$ is injective, it follows that $\lambda \notin \sigma_{\mathrm{ap}}(T_2)$ and $\lambda \in \mathrm{iso}\,\sigma_{\mathrm{ap}}(T_1) \subseteq \mathrm{iso}\,\sigma_{\mathrm{ap}}(T+\widetilde{F})$. Moreover, $\mathrm{N}(T+\widetilde{F}-\lambda) = \mathrm{N}(T_2-(\lambda-\alpha))$ is a non-trivial subspace of finite dimension, so $\lambda \in \Pi^{\mathrm{a}}_{\mathrm{oo}}(T+\widetilde{F})$. On the other hand, since $\lambda \notin \Pi^{\mathrm{a}}_{\mathrm{o}}(T) = \mathrm{iso}\,\sigma_{\mathrm{ap}}(T) \cap \varrho_{\mathrm{SF}}(T)$, $T-\lambda$ is not semi-Fredholm, and hence also $T+\widetilde{F}-\lambda$ is not semi-Fredholm, which implies that $\lambda \notin \Pi^{\mathrm{a}}_{\mathrm{o}}(T+\widetilde{F})$. Therefore $T+\widetilde{F}$ does not satisfy a-Weyl's theorem, which is the desired contradiction. ∎

A bounded operator T on X is called *finite a-isoloid* if every isolated point of $\sigma_{\mathrm{ap}}(T)$ is an eigenvalue of T of finite multiplicity.

THEOREM 2.9. *Let T be a finite a-isoloid operator on X that satisfies a-Weyl's theorem. If R is a Riesz operator that commutes with T, then $T+R$ satisfies a-Weyl's theorem.*

Proof. Since $T+R$ obeys a-Browder's theorem, it suffices to show that $\Pi^{\mathrm{a}}_{\mathrm{oo}}(T+R) = \Pi^{\mathrm{a}}_{\mathrm{o}}(T+R)$. Let $\lambda \in \Pi^{\mathrm{a}}_{\mathrm{oo}}(T+R)$. If $T-\lambda$ is injective with closed range, then $T+R-\lambda$ is semi-Fredholm and hence $\lambda \in \Pi^{\mathrm{a}}_{0}(T+R)$. Suppose that $\lambda \in \sigma_{\mathrm{ap}}(T)$. It follows by Proposition 2.4 that λ is an isolated point of $\sigma_{\mathrm{ap}}(T)$, and because T is finite a-isoloid, we see that $\lambda \in \Pi^{\mathrm{a}}_{\mathrm{oo}}(T)$. On the other hand, a-Weyl's theorem for T implies that $\sigma_{\mathrm{ea}}(T) \cap \Pi^{\mathrm{a}}_{\mathrm{oo}}(T) = \emptyset$, therefore $T-\lambda$ is semi-Fredholm and hence so is $T+R-\lambda$. Consequently, $\lambda \in \Pi^{\mathrm{a}}_{\mathrm{o}}(T+R)$. The other inclusion is trivial and so $T+R$ satisfies a-Weyl's theorem. ∎

COROLLARY 2.10. *Let T be an a-finite-isoloid operator on X that satisfies a-Weyl's theorem. If K is a compact operator commuting with T, then a-Weyl's theorem holds for $T+K$.*

For the special case of quasi-nilpotent perturbations, we provide a relatively weak condition that ensures the stability of a-Weyl's theorem.

PROPOSITION 2.11. *Let $T \in \mathcal{L}(X)$ be such that $\sigma_{\mathrm{p}}(T) \cap \operatorname{iso}\sigma_{\mathrm{ap}}(T) \subseteq \Pi^{\mathrm{a}}_{\mathrm{oo}}(T)$. If T satisfies a-Weyl's theorem then so does $T+Q$ for every quasi-nilpotent operator Q commuting with T.*

Proof. We note first that $\sigma_{\mathrm{ap}}(T+Q) = \sigma_{\mathrm{ap}}(T)$ and $\sigma_{\mathrm{ea}}(T+Q) = \sigma_{\mathrm{ea}}(T)$ (see [9] and [15]); in particular we have $\Pi^{\mathrm{a}}_{\mathrm{o}}(T+Q) = \Pi^{\mathrm{a}}_{\mathrm{o}}(T)$. Since, by Corollary 2.3, a-Browder's theorem holds for $T+Q$, we only have to show that $\Pi^{\mathrm{a}}_{\mathrm{oo}}(T+Q) = \Pi^{\mathrm{a}}_{\mathrm{o}}(T+Q)$. Let $\lambda \in \Pi^{\mathrm{a}}_{\mathrm{oo}}(T+Q)$. It follows that the restriction of $T-\lambda$ to the finite-dimensional subspace $\mathrm{N}(T+Q-\lambda)$ is not invertible and so $\mathrm{N}(T-\lambda)$ is non-trivial. Consequently, $\lambda \in \sigma_{\mathrm{p}}(T) \cap \operatorname{iso}\sigma_{\mathrm{ap}}(T) \subseteq \Pi^{\mathrm{a}}_{\mathrm{oo}}(T)$. But a-Weyl's theorem for T implies that $\Pi^{\mathrm{a}}_{\mathrm{oo}}(T) = \Pi^{\mathrm{a}}_{\mathrm{o}}(T)$. Thus $\lambda \in \Pi^{\mathrm{a}}_{\mathrm{o}}(T) = \Pi^{\mathrm{a}}_{\mathrm{o}}(T+Q)$, which completes the proof. ∎

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