STUDIA MATHEMATICA 148 (2) (2001)

An exact functional Radon–Nikodym theorem for Daniell integrals

by

E. DE AMO (Almería), I. CHITESCU (Bucureşti)
and M. DÍAZ CARRILLO (Granada)

Abstract. Given two positive Daniell integrals I and J, with J absolutely continuous with respect to I, we find sufficient conditions in order to obtain *an exact Radon–Nikodym derivative* f *of* J *with respect to* I. The procedure of obtaining f is constructive.

1. Introduction. In this paper we consider two positive Daniell integrals I and J on a lattice of functions B which is also a unitary algebra, J being absolutely continuous with respect to I. We give sufficient conditions to obtain, in a constructive manner, an "exact" Radon–Nikodym derivative f of J with respect to I, i.e., to have $J(u) = I(fu)$ for every u in B. Generally, the derivative thus obtained must be in a larger space than B, so the relation $J(u) = I(fu)$ actually holds for the canonical extensions of I and J.

We recognize the strong influence of [6] and [8].

2. Main result. We shall consider a nonempty set X (the total space) and a vector lattice B of functions $f : X \to \mathbb{R}$ (with pointwise operations and order). We shall also assume that B is an algebra with unit $1 \in B$. We denote by $+B$ the positive elements in B.

In what follows, $I : B \to \mathbb{R}$ will be a *positive Daniell integral* (i.e., I is linear, positive and $I(f_n) \searrow 0$ whenever the decreasing sequence (f_n) in B is such that $f_n \searrow 0$ pointwise). For Daniell integrals, see [9] and [10], and for general measure theory, see [5], [7] and [10].

Let also $J : B \to \mathbb{R}$ be a positive linear functional. We shall assume that J is *absolutely continuous with respect to* I, i.e., for all $\varepsilon > 0$ and for all u in $+B$ there exists $\delta > 0$ such that, for all v in $+B$ with $v \le u$ and $I(v) < \delta$ one has $J(v) < \varepsilon$ (see [1] and [3]), which is denoted by $J \ll I$.

2000 *Mathematics Subject Classification*: 28B05, 26D15.

This paper has been jointly written when the second author visited the Universities of Granada and Almería for scientific research, during February 1998.

Notice that J must also be a Daniell integral (due to its absolute continuity with respect to I).

According to the general theory, I generates the vector lattice $L(I)$ of all *I-integrable functions* and J generates the vector lattice $L(J)$ of all *J-integrable functions*. We shall denote by $L_{\mathrm{b}}(I)$ (resp. $L_{\mathrm{b}}(J)$) the set of all bounded I-integrable (resp. J-integrable) functions. Recall that for $f : X \to \mathbb{R}$, to say that f is in $L(I)$ means that for every $\varepsilon > 0$ there exist $g, h : X \to \mathbb{R}$ with the following three properties:

(a) There exists an increasing sequence $(g_n)_n$ in B such that $g_n \nearrow g$ and

$$I^*(g) := \sup_n I(g_n) < \infty.$$

(b) There exists a decreasing sequence $(h_n)_n$ in B such that $h_n \searrow h$ and

$$I_*(h) := \inf_n I(h_n) > -\infty.$$

(c) One has the inequalities $h \le f \le g$ and $0 \le I^*(g) - I_*(h) < \varepsilon$ (the last inequality actually means that $\sup_n [I(g_n - h_n)] < \varepsilon$).

In case $f \ge 0$ one can suppose $h \ge 0$.

Then I can be uniquely extended to a linear positive functional $\bar{I} : L(I) \to \mathbb{R}$ having the property that $I_*(h) \le \bar{I}(f) \le I^*(g)$ for all h and g as above.

Similar considerations apply to the extension of J.

LEMMA 1. *If $J \ll I$, then $L_{\mathrm{b}}(I) \subset L_{\mathrm{b}}(J)$.*

Proof. Let $u \ge 0$ in $L_{\mathrm{b}}(I)$. We shall prove that $u \in L(J)$ (i.e., $u \in L_{\mathrm{b}}(J)$) and this will imply $L_{\mathrm{b}}(I) \subset L_{\mathrm{b}}(J)$, in view of the decomposition $u := u^+ - u^-$ with u^+, u^- in $L_{\mathrm{b}}(I)$ for arbitrary $u \in L_{\mathrm{b}}(I)$.

Consider a number $M > 0$ such that $u \le M$. Take $\varepsilon > 0$. Since $J \ll I$, one can find $\delta > 0$ such that for all $0 \le v \le 2M, v \in B$, the inequality $I(v) < \delta$ implies that $J(v) < \varepsilon$.

We can consider $h \le u \le g$ with $g_n \nearrow g$, $h_n \searrow h \ge 0$, $I^*(g) - I_*(h) < \delta/2$, as above. One can assume $0 \le h_n \le M$, $0 \le g_n \le M$, because $g_n \vee 0 =: g'_n \nearrow g = g \vee 0$ and $g''_n := g'_n \wedge M \nearrow g \wedge M \ge u$; $h_n \vee 0 =: h'_n \searrow h = h \vee 0$; and

$$I^*(g \wedge M) - I_*(h) = \sup_n [I(g''_n - h'_n)] \le I^*(g) - I_*(h) = \sup_n [I(g_n - h_n)] < \delta/2.$$

For all n one has $|g_n - h_n| \le g_n + h_n \le 2M$. On the other hand, $|g_n - h_n| \to |g - h| = g - h$ pointwise. Since all g_n and h_n are in $L(I)$, we can use the measure space generated by I and Lebesgue's Dominated Convergence Theorem to conclude that $\bar{I}(g - h) = \lim_n I(|g_n - h_n|)$.

Since $\bar{I}(g - h) = I^*(g) - I_*(h) < \delta/2$, there exists a natural number n_0 such that $I(|g_n - h_n|) < \delta$ for all $n \ge n_0$. It follows that for all $n \ge n_0$ one has $J(|g_n - h_n|) < \varepsilon$, because $|g_n - h_n| \le 2M$; consequently,

$$J(g_n - h_n) \le J(|g_n - h_n|) < \varepsilon.$$

Since the sequence is increasing, one gets

$$\sup_n J(g_n - h_n) = J^*(g) - J_*(h) \le \varepsilon,$$

which means that $u \in L(I)$, because ε is arbitrary. ∎

LEMMA 2. *One has $\bar{J} \ll \bar{I}$ for bounded functions*; *i.e.*, *for every $\varepsilon > 0$ and every $M > 0$, there exists $\delta > 0$ having the property that if $u \in L(I)$ is such that $0 \le u \le M$ and $\bar{I}(u) < \delta$, then $\bar{J}(u) < \varepsilon$. Consequently*, *if $0 \le u \in L(I)$ is such that $\bar{I}(u) = 0$, one has $u \in L(J)$ and $\bar{J}(u) = 0$.*

Proof. Let $\varepsilon, M > 0$. There exists $\delta_1 > 0$ such that for all $v \in B$ with $0 \le v \le M$ and $I(v) < \delta_1$ one has $J(v) < \varepsilon/2$. Set $\delta := \delta_1/4$.

Now, take $u \in L(I)$ with $0 \le u \le M$ and $\bar{I}(u) < \delta$. Consider h_n, g_n in B with $g_n \nearrow g$, $h_n \searrow h$, $h \le u \le g$ and $I^*(g) - I_*(h) = \sup_n[I(g_n - h_n)] < \delta_1/4$ as above. As we have seen, one can consider that $0 \le h_n \le M$, $0 \le g_n \le M$.

Choose $n_0 \in \mathbb{N}$ such that $\bar{J}(u) \le J^*(g) = \sup_n J(g_n) < J(g_{n_0}) + \varepsilon/2$; therefore

$$\bar{J}(u) < J(g_n) + \varepsilon/2, \quad \forall n \ge n_0. \tag{1}$$

For every $n \in \mathbb{N}$, one has

$$I(g_n) = I(g_n - h_n) + I(h_n) < I(h_n) + \delta_1/4. \tag{2}$$

Since $I_*(h) := \inf_n I(h_n)$, one can find $n_1 \in \mathbb{N}$ such that $I(h_{n_1}) < I_*(h) + \delta_1/4 \le \bar{I}(u) + \delta_1/4$, therefore

$$I(h_n) < \bar{I}(u) + \delta_1/4, \quad \forall n \ge n_1. \tag{3}$$

Now, let $n \ge \max\{n_0, n_1\}$. In view of (2) and (3), one gets

$$I(g_n) < I(h_n) + \delta_1/4 < \bar{I}(u) + \delta_1/2 < 3\delta_1/4 < \delta_1,$$

which implies that $J(g_n) < \varepsilon/2$.

In view of (1), one has

$$\bar{J}(u) < J(g_n) + \varepsilon/2 < \varepsilon \tag{4}$$

and, because ε is arbitrary, (4) shows that $\bar{J} \ll \bar{I}$ for bounded functions in $L(I)$.

For the case $0 \le u \in L(I)$ with $\bar{I}(u) = 0$, one has $u = \sup_n u_n$, where $(u_n)_n$ is the increasing sequence in $L_{\mathrm{b}}(I)$ given by $u_n := u \wedge n$. For every n one has $\bar{I}(u_n) = 0$ and the absolute continuity for bounded functions gives $\bar{J}(u_n) = 0$. Since $\sup_n \bar{J}(u_n) = 0$, Beppo-Levi's theorem implies $u \in L(J)$, and $\bar{J}(u) = 0$. ∎

REMARK. $\bar{J} \ll \bar{I}$ for bounded functions means that for every $\varepsilon > 0$ and $0 \le h \in L_{\mathrm{b}}(I)$, there exists $\delta > 0$ having the property that if $u \in L(I)$ is such that $0 \le u \le h$, the inequality $\bar{I}(u) < \delta$ implies $\bar{J}(u) < \varepsilon$.

In order to continue our investigations, we introduce, for every $u \ge 0$ in $L_{\mathrm{b}}(I)$ and $\varepsilon > 0$:

(a) the *average range of $\bar{J}$ with respect to $\bar{I}$ on u*, which is the set of real numbers

$$A(\bar{I}, \bar{J})(u) := \{\bar{J}(v)/\bar{I}(v);\ 0 \le v \le u,\ v \in L(I),\ \bar{I}(v) > 0\};$$

(b) the *ε-approximate average range of $\bar{J}$ with respect to $\bar{I}$ on u*, which is the set of real numbers (possibly empty)

$$A_\varepsilon(\bar{I}, \bar{J})(u) := \{x \in \mathbb{R};\ |x - a| \le \varepsilon \text{ for all } a \in A(\bar{I}, \bar{J})(u)\}.$$

We make three assumptions which will be discussed and justified at the end of the paper. $\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$ of all natural numbers.

ASSUMPTION 1. This assumption is sequential and inductive, consisting of the following sequence of steps:

$s(1)$: There exists a sequence $(h_{n;1})_{n\in\mathbb{N}}$ or a finite family $(h_{n;1})_{1\le n\le p_1}$ of positive functions in $L(I)$ such that $\bar{I}(h_{n;1}) > 0$ for all n and

$$\sum_n h_{n;1} = 1 \tag{i$_1$}$$

with pointwise covergence.

$s(2)$: For every $n \in \mathbb{N}$ or $1 \le n \le p_1$, there exists a sequence $(h_{(n,i);2})_{i\in\mathbb{N}}$ or a finite family $(h_{(n,i);2})_{1\le i\le p_2}$ of positive functions in $L(I)$ such that $\bar{I}(h_{\alpha;2}) > 0$ for all possible $\alpha := (n, i)$, and for all possible n we have pointwise

$$\sum_i h_{(n,i);2} = h_{n;1}. \tag{i$_2$}$$

This implies $\sum_\alpha h_{\alpha;2} = 1$, where the sum $\sum_\alpha$ is taken pointwise over the set of all possible α.

Assuming that the step $s(n-1)$ for $n \ge 2$ of the assumption has been defined (this pertains to the family $(h_{\alpha;n-1})_\alpha$ where $\alpha \in \mathbb{N}^{n-1}$ ranges over all possible α) we shall write $(\alpha, i_n) \in \mathbb{N}^n$, for every $\alpha = (i_1, \ldots, i_{n-1}) \in \mathbb{N}^{n-1}$ and $i_n \in \mathbb{N}$.

Now we are able to write the next step:

$s(n)$: For every $\alpha \in \mathbb{N}^{n-1}$ in the set of all possible α given by the previous steps, there exists a sequence $(h_{(\alpha,i);n})_{i\in\mathbb{N}}$ or a finite set $(h_{(\alpha,i);n})_{1\le i\le p_n}$ of positive functions in $L(I)$ such that $\bar{I}(h_{\beta;n}) > 0$ for all possible β. Moreover, for all possible α, we have pointwise

$$\text{(i)}_n \qquad \sum_i h_{(\alpha,i);n} = h_{\alpha;n-1}$$

where the sum $\sum_i$ is taken over the set of all possible i. This implies, in view of $\sum_\alpha h_{\alpha;n-1} = 1$ in $s(n-1)$ and in view of (i)_n, that $\sum_\beta h_{\beta;n} = 1$, where β ranges over the set of all possible β.

Final comment upon Assumption 1: For every possible $\alpha \in \mathbb{N}^m$, if $n > m$, one has

$$h_{\alpha;m} = \sum_\beta h_{(\alpha,\beta);n}$$

where the sum runs over all possible $\beta \in \mathbb{N}^{n-m}$, with obvious notations. Note that all the $h_{\alpha;m}$ are in $L_{\mathrm{b}}(I)$.

Assumption 2. For every natural number n and for every $\alpha \in \mathbb{N}^n$ in the set of all possible α, one has

$$A_{2^{-n}}(\bar{I}, \bar{J})(h_{\alpha;n}) \neq \emptyset.$$

Assumption 3. There exists a number $M > 0$ such that for all n in $\mathbb{N}$ and for all $\alpha \in \mathbb{N}^n$ in the set of all possible α, one has

$$A_{2^{-n}}(\bar{I}, \bar{J})(h_{\alpha;n}) \subset [-M, M].$$

The general theory says that if f is a bounded function in $L(I)$ and u is in $L(I)$, then fu is in $L(I)$. We can now state the main result of this paper.

Theorem (An exact Radon–Nikodym theorem for Daniell integrals). *Assume that I, J are as above and Assumptions* 1–3 *are fulfilled. Then there exists a positive bounded function f in $L(I)$ such that*

$$\bar{J}(u) = \bar{I}(fu)$$

for all u in $L(I)$. The function f (called the Radon–Nikodym derivative of $\bar{J}$ with respect to $\bar{I}$) is I-almost unique, which means that if g in $L(I)$ is such that $\bar{J}(u) = \bar{I}(gu)$ for all u in $L(I)$ then $\bar{I}(|f-g|) = 0$.

Proof. We shall construct a sequence $(f_n)_n$ of bounded I-integrable functions.

Let n be in $\mathbb{N}$. In order to construct f_n, we take an element $r_{\alpha;n}$ in each $A_{2^{-n}}(\bar{I}, \bar{J})(h_{\alpha;n})$ for all possible α in $\mathbb{N}^n$, according to Assumption 2. We define $f_n : X \to \mathbb{R}$ pointwise by

$$f_n := \sum_\alpha r_{\alpha;n} h_{\alpha;n}$$

(where $\alpha \in \mathbb{N}^n$ ranges over the set of all possible α).

One has clearly

$$|f_n| \le \sum_\alpha |r_{\alpha;n}| h_{\alpha;n} \le M \sum_\alpha h_{\alpha;n} = M,$$

so f_n is bounded. Here we have used Assumption 3 and again Assumption 1. If μ is the measure induced by the Daniell integral I (according to the general theory), then the functions f_n are clearly μ-measurable and, being bounded, are also μ-integrable, i.e., they are in $L(I)$.

Now we prove that the sequence (f_n) is uniformly Cauchy, which implies that it is uniformly convergent to a function f. Indeed, let $m < n$ in $\mathbb{N}$. We shall prove that for all t in X one has

$$|f_m(t) - f_n(t)| \le 2^{1-m} \tag{5}$$

and this will prove the assertion.

Take $t \in X$. We have (here $\beta \in \mathbb{N}^{n-m}$ is taken to be in the set of all possible such indices)

$$\begin{aligned}
|f_m(t) - f_n(t)| &= \Big| \sum_\alpha r_{\alpha;m} h_{\alpha;m}(t) - \sum_\gamma r_{\gamma;n} h_{\gamma;n}(t) \Big| \\
&\le \sum_\alpha \Big| r_{\alpha;m} h_{\alpha;m}(t) - \sum_\beta r_{(\alpha,\beta);n} h_{(\alpha,\beta);n}(t) \Big| \\
&\le \sum_{(\alpha,\beta)} |r_{\alpha;m} h_{(\alpha,\beta);n}(t) - r_{(\alpha,\beta);n} h_{(\alpha,\beta);n}(t)| \\
&= \sum_{(\alpha,\beta)} h_{(\alpha,\beta);n}(t) |r_{\alpha;m} - r_{(\alpha,\beta);n}|
\end{aligned}$$

(see the final comment upon Assumption 1).

For every (α, β) we take a natural i such that (with obvious notation)

$$0 \le h_{(\alpha,\beta,i);n+1} =: v \le h_{(\alpha,\beta);n} \le h_{\alpha;m}, \qquad \bar{I}(v) > 0.$$

Summing upon all possible (α, β) and finding each time such a $v = v(\alpha, \beta)$, one has

$$\begin{aligned}
|f_m(t) - f_n(t)| &\le \sum_{(\alpha,\beta)} h_{(\alpha,\beta);n}(t) \left(\left| r_{\alpha;m} - \frac{\bar{J}(v)}{\bar{I}(v)} \right| + \left| \frac{\bar{J}(v)}{\bar{I}(v)} - r_{(\alpha,\beta);n} \right| \right) \\
&\le \sum_{(\alpha,\beta)} h_{(\alpha,\beta);n}(t) (2^{-m} + 2^{-n}) \\
&\le 2^{-m+1} \sum_{(\alpha,\beta)} h_{(\alpha,\beta);n}(t) = 2^{-m+1},
\end{aligned}$$

and (5) is proved.

Let $\widetilde{f} : X \to \mathbb{R}$ be the (uniform) limit $\widetilde{f} := \lim_n f_n$.

It is clear (because $|f_n| \leq M$) that $|\widetilde{f}| \leq M$ and so $\widetilde{f}$ is bounded, therefore $\widetilde{f} \in L(I)$ according to the general theory.

We prove that for all u in $L(I)$,

$$\bar{J}(u) = \bar{I}(\widetilde{f}u). \tag{6}$$

We show that (6) holds for every positive bounded $u \leq 1$ in $L(I)$. Indeed, one can write $\lim_n f_n u = \widetilde{f}u$ and $|f_n u| \leq M$, $|\widetilde{f}u| \leq M$ (everything pointwise) and this implies

$$\lim \bar{I}(f_n u) = \bar{I}(\widetilde{f}u). \tag{7}$$

On the other hand, for every n in $\mathbb{N}$,

$$|\bar{J}(u) - \bar{I}(f_n u)| = \Big|\bar{J}(u) - \bar{I}\Big(\Big(\sum_\alpha r_{\alpha;n} h_{\alpha;n}\Big)u\Big)\Big| = \Big|\bar{J}(u) - \sum_\alpha r_{\alpha;n}\bar{I}(uh_{\alpha;n})\Big|$$

(again by dominated convergence).

Because $u = u\sum_\alpha h_{\alpha;n}$, one also has $\bar{J}(u) = \sum_\alpha \bar{J}(uh_{\alpha;n})$ and so

$$|\bar{J}(u) - \bar{I}(f_n u)| = \Big|\sum_\alpha (\bar{J}(uh_{\alpha;n}) - r_{\alpha;n}\bar{I}(uh_{\alpha;n}))\Big|. \tag{8}$$

In case $\bar{I}(uh_{\alpha;n}) = 0$ one has $\bar{J}(uh_{\alpha;n}) = 0$, because $\bar{J} \ll \bar{I}$. In case $\bar{I}(uh_{\alpha;n}) > 0$ one has $0 \leq uh_{\alpha;n} \leq h_{\alpha;n}$ and then

$$\left|\frac{\bar{J}(uh_{\alpha;n})}{\bar{I}(uh_{\alpha;n})} - r_{\alpha;n}\right| \leq 2^{-n},$$

which implies in all situations that

$$|\bar{J}(uh_{\alpha;n}) - r_{\alpha;n}\bar{I}(uh_{\alpha;n})| \leq 2^{-n}\bar{I}(uh_{\alpha;n}) \leq 2^{-n}\bar{I}(h_{\alpha;n}). \tag{9}$$

In view of (8) and (9), one obtains

$$|\bar{J}(u) - \bar{I}(f_n u)| \leq 2^{-n}\sum_\alpha \bar{I}(h_{\alpha;n}) = 2^{-n}\bar{I}(1),$$

which implies

$$\lim \bar{I}(f_n u) = \bar{J}(u). \tag{10}$$

From (7) and (10) we obtain (6), which therefore holds for positive bounded functions u in $L(I)$.

If u is an arbitrary positive function in $L(I)$, we have the pointwise convergence $u_n \nearrow u$, where $u_n := u \wedge n$. Since $\bar{J}(u_n) = \bar{I}(\widetilde{f}u_n)$ for all n, it follows, by passing to suprema, that $\bar{J}(u) = \bar{I}(\widetilde{f}u)$ and (6) is true for all positive functions in $L(I)$. By linearity, (6) holds for all functions in $L(I)$.

If μ is the (complete) measure induced by the Daniell integral I, then (6) implies ($\chi_A =$ the indicator function of A)

$$0 \le \bar{J}(\chi_A) = \int_A \widetilde{f}\, d\mu$$

for all $A \subset X$ with $\chi_A \in L(I)$. General measure theory says that $\widetilde{f}(t) \ge 0$ μ-almost everywhere.

The set $M := \{t \in X;\ \widetilde{f}(t) < 0\}$ has the properties $\chi_M \in L(I)$ and $\mu(M) = \bar{I}(\chi_M) = 0$. Defining $f : X \to \mathbb{R}$ via

$$f(t) := \begin{cases} \widetilde{f}(t), & t \notin M, \\ 0, & t \in M, \end{cases}$$

one has $f \ge 0$ everywhere, $f = \widetilde{f}$ μ-almost everywhere and therefore

$$\bar{J}(u) = \bar{I}(\widetilde{f}u) = \bar{I}(fu)$$

for all u in $L(I)$.

For the unicity, consider another function g in $L(I)$ such that $\bar{J}(u) = \bar{I}(gu)$ for all u in $L(I)$. So, we have $\bar{I}(f\chi_A) = \bar{I}(g\chi_A)$, which means that $\int_A f\, d\mu = \int_A g\, d\mu$ for all $A \subset X$ with $\chi_A \in L(I)$. General measure theory says that $g = f$ μ-almost everywhere, which means

$$0 = \int |f - g|\, d\mu = \bar{I}(|f - g|). \ \blacksquare$$

3. Other results and comments

3.1. We begin with a general result which will furnish material for some comments. Assume therefore that $X \neq \emptyset$ is an abstract set, B a vector lattice of functions $f : X \to \mathbb{R}$ and $I, J : B \to \mathbb{R}$ are linear positive functionals. Using the conventions $\frac{0}{0} := 0$ and $\frac{a}{0} := \infty$ for $a > 0$ we shall modify the previous definitions a little. Namely, for every u in $+B$ and $\varepsilon > 0$, we set

$$A'(I, J)(u) := \{J(v)/I(v);\ 0 \le v \le u,\ v \in B\},$$
$$A'_\varepsilon(I, J)(u) := \{x \in \mathbb{R};\ |x - a| \le \varepsilon,\ a \in A'(I, J)(u)\}.$$

PROPOSITION. (i) *Assume that for all u in $+B$, the set $A'(I, J)(u)$ is bounded* (*e.g. in case there exists a number $M > 0$ such that $J \le MI$*). *Then $J \ll I$.*

(ii) *For every u in $+B$ and every $\varepsilon > 0$, the set $A'_\varepsilon(I, J)(u)$ is closed* (*actually compact*).

(iii) *For every u in $+B$ we have $0 < \varepsilon < \gamma \Rightarrow A'_\varepsilon(I, J)(u) \subset A'_\gamma(I, J)(u)$.*

(iv) *Assuming that u in $+B$ is such that $A'_\varepsilon(I, J)(u) \neq \emptyset$ for all $\varepsilon > 0$, the intersection $\bigcap_{\varepsilon > 0} A'_\varepsilon(I, J)(u)$ contains exactly one point.*

Proof. (i) Assume that for all u in $+B$ the set $A'(I, J)(u)$ is bounded. If $J \ll I$ is false, we can find $\varepsilon_0 > 0$ and u in $+B$ with the property that for all n in $\mathbb{N}$, there exists $0 \le u_n \le u$ in B such that $I(u_n) < 1/n$ and $J(u_n) \ge \varepsilon_0$.

If $I(u_n) = 0$, then $J(u_n)/I(u_n) = \infty$ and $A'(I,J)(u)$ is not bounded.

If $I(u_n) > 0$, then $J(u_n)/I(u_n) \ge n\varepsilon_0$ and, in this case too, $A'(I,J)(u)$ is not bounded. Contradiction, and (i) follows.

In the particular case when $J \le MI$ for some positive M, one has obviously $A'(I,J)(u) \subset [0,M]$.

Assertion (ii) is clear when $A'_\varepsilon(I,J)(u) = \emptyset$. So, assume $A'_\varepsilon(I,J)(u) \neq \emptyset$.

If a number x is such that $x = \lim_n x_n$, where $(x_n)_n$ is a sequence in $A'_\varepsilon(I,J)(u)$, then for an arbitrary fixed $0 \le v \le u$ in B one has

$$\left|\frac{J(v)}{I(v)} - x_n\right| \le \varepsilon$$

for every n. Passing to the limit gives

$$\left|\frac{J(v)}{I(v)} - x\right| \le \varepsilon.$$

The fact that v is arbitrary shows that $x \in A'_\varepsilon(I,J)(u)$.

Point (iii) is trivial. We prove (iv).

For every $\varepsilon > 0$, the nonempty set $A'_\varepsilon(I,J)(u)$ is bounded (for every x and y in $A'_\varepsilon(I,J)(u)$ one has $|x-y| \le |x-a| + |y-a| \le 2\varepsilon$, upon taking some a in $A'_\varepsilon(I,J)(u)$), therefore compact, and the decreasing intersection is nonempty. Put $A := \bigcap_{\varepsilon>0} A'_\varepsilon(I,J)(u)$.

Assume the existence of $x \neq y$ in A; then one has, for a fixed $0 \le v \le u$ in B, the inequalities

$$\left|\frac{J(v)}{I(v)} - x\right| \le \frac{|x-y|}{4} \quad \text{and} \quad \left|\frac{J(v)}{I(v)} - y\right| \le \frac{|x-y|}{4}.$$

Hence

$$|x-y| \le \left|\frac{J(v)}{I(v)} - x\right| + \left|\frac{J(v)}{I(v)} - y\right| \le \frac{|x-y|}{2},$$

which is false. ∎

Commenting on the Proposition, we can say:

(a) Point (i) *motivates* Assumption 3 a little. For example, in the particular case when there exists a positive number M such that $J \le MI$, one quickly sees that $\bar{J} \le M\bar{I}$ and this implies that for every positive u in $L(I)$ one has $A'(I,J)(u) \subset [0,M]$.

(b) Point (iv) can give us some *ideas* in connection with the possible values of the function f. Namely, they should be *close* to the elements in the intersection of the form

$$\bigcap_{\varepsilon>0} A'_\varepsilon(\bar{I},\bar{J})(h_{\alpha;n})$$

for large n. So, f is obtained via a kind of *differentiation* procedure.

(c) In connection with Assumption 2, which says that for large n the average range $A(\bar{I}, \bar{J})(h_{\alpha;n})$ must have very small diameter (see also comment (b)), the following example will be, perhaps, illuminating, putting into evidence a concrete construction of the family $(h_{\alpha;n})$.

3.2. We now give an example to show how the theorem effectively works.

We take $X := [0,1]$ and $B :=$ the algebra of all continuous functions $u : [0,1] \to \mathbb{R}$. The functional $I : B \to \mathbb{R}$ is given by $I(f) := \int_0^1 u(x)\,dx$. In order to construct J we fix a positive function $f \in B$ and then $J : B \to \mathbb{R}$ is given by $J(u) := \int_0^1 f(x)u(x)\,dx$.

If $M := \max\{f(x);\ x \in [0,1]\}$ then $J \le MI$, which shows that $J \ll I$ and Assumption 3 is automatically satisfied.

One knows that $\bar{I}$ is exactly the Lebesgue integral on the space $L(I)$ of all Lebesgue integrable functions, so

$$\bar{I}(u) = \int u\,d\mu, \qquad \forall u \in L(I),$$

where $\mu : \mathcal{M} \to +\mathbb{R}$ is the Lebesgue measure (induced by I over the set $\mathcal{M}$ of all Lebesgue measurable subsets of $[0,1]$). Then $\bar{J}$ acts via

$$\bar{J}(u) = \int fu\,d\mu, \qquad \forall u \in L(I).$$

We now show how Assumptions 1 and 2 can be satisfied. To this end, we use the following

STATEMENT. *Let $U \subset \mathbb{R}$ be a compact interval, μ the Lebesgue measure on U and $f : U \to \mathbb{R}$ a positive continuous function. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every a in U which is not the right end of U, one has the property: for each interval $[a,b] \subset U$ with $b - a < \delta$ and for each Lebesgue integrable function $u : U \to \mathbb{R}$ such that $0 \le u \le \chi_{[a,b]}$ and u is not null μ-almost everywhere on $[a,b]$, the following relation holds:*

$$\frac{\int_{[a,b]} fu\,d\mu}{\int_{[a,b]} u\,d\mu} \in [f(a) - \varepsilon/2, f(a) + \varepsilon/2].$$

Proof. Since f is bounded, there exists a natural number k, not depending on a, such that

$$\frac{1}{k} + \frac{1}{k^2} + \frac{f(a)}{k} < \frac{\varepsilon}{2}. \tag{11}$$

In view of the uniform continuity of f, there exists $\delta > 0$ not depending on a such that

$$M := \sup\{f(x);\ x \in [a,b]\} \le f(a) + 1/k \tag{12}$$

and

$$m := \inf\{f(x);\ x \in [a,b]\} \ge f(a) - 1/k \tag{13}$$

if $[a,b] \subset U$ is such that $b - a < \delta$.

Fix the interval $[a,b]$ as in the Statement. First, we shall prove it for every Riemann integrable function

$$u \in H(a,b) := \{u : U \to \mathbb{R};\ 0 \le u \le \chi_{[a,b]},\ u \ne 0\ \mu\text{-a.e.}\}.$$

There exist positive continuous functions $g,h : [a,b] \to \mathbb{R}$ such that

$$0 \le h \le u \le g \quad \text{and} \quad \int\limits_{[a,b]} h\,d\mu > 0$$

(since $\int_{[a,b]} u\,d\mu > 0$).

In view of the general properties of the Daniell integral, we can find an increasing sequence $(h_n)_n$ of positive continuous functions $h_n : [a,b] \to \mathbb{R}$, $h_n \le u$, and a decreasing sequence $(g_n)_n$ of positive continuous functions $g_n : [a,b] \to \mathbb{R}$, $g_n \ge u$, such that

$$\int\limits_{[a,b]} u\,d\mu = \sup_n \int\limits_{[a,b]} h_n\,d\mu = \inf_n \int\limits_{[a,b]} g_n\,d\mu.$$

One can find a natural n such that

$$(14)\qquad 1 \le \frac{\int_{[a,b]} g_n\,d\mu}{\int_{[a,b]} h_n\,d\mu} \le 1 + \frac{1}{k} \quad \text{and} \quad 1 \ge \frac{\int_{[a,b]} h_n\,d\mu}{\int_{[a,b]} g_n\,d\mu} \ge 1 - \frac{1}{k}.$$

Put $h := h_n$ and $g := g_n$. One has

$$\frac{\int_{[a,b]} fh\,d\mu}{\int_{[a,b]} g\,d\mu} \le \frac{\int_{[a,b]} fu\,d\mu}{\int_{[a,b]} u\,d\mu} \le \frac{\int_{[a,b]} fg\,d\mu}{\int_{[a,b]} h\,d\mu}.$$

But using (12)–(14) gives

$$\int\limits_{[a,b]} fg\,d\mu \le M \int\limits_{[a,b]} g\,d\mu \le \left(f(a) + \frac{1}{k}\right) \int\limits_{[a,b]} g\,d\mu;$$

therefore,

$$\frac{\int_{[a,b]} fg\,d\mu}{\int_{[a,b]} h\,d\mu} \le \left(f(a) + \frac{1}{k}\right) \frac{\int_{[a,b]} g\,d\mu}{\int_{[a,b]} h\,d\mu} \le \left(f(a) + \frac{1}{k}\right)\left(1 + \frac{1}{k}\right);$$

and

$$\int\limits_{[a,b]} fh\,d\mu \ge m \int\limits_{[a,b]} h\,d\mu \ge \left(f(a) - \frac{1}{k}\right) \int\limits_{[a,b]} h\,d\mu;$$

therefore

$$\frac{\int_{[a,b]} fh\,d\mu}{\int_{[a,b]} g\,d\mu} \ge \left(f(a) - \frac{1}{k}\right) \frac{\int_{[a,b]} h\,d\mu}{\int_{[a,b]} g\,d\mu} \ge \left(f(a) - \frac{1}{k}\right)\left(1 - \frac{1}{k}\right).$$

It follows that
$$f(a)-\left(\frac{1}{k}+\frac{f(a)}{k}-\frac{1}{k^2}\right)\le\frac{\int_{[a,b]}fu\,d\mu}{\int_{[a,b]}u\,d\mu}\le f(a)+\left(\frac{1}{k}+\frac{f(a)}{k}+\frac{1}{k^2}\right),$$
which implies, using (11), that
$$f(a)-\frac{\varepsilon}{2}\le\frac{\int_{[a,b]}fu\,d\mu}{\int_{[a,b]}u\,d\mu}\le f(a)+\frac{\varepsilon}{2}.$$

Now, we prove the validity of the result for every Lebesgue integrable function $u\in H(a,b)$. There exists a sequence $(u_n)_n$ of Riemann integrable functions, $u_n:U\to\mathbb{R}$, such that $\|u_n-u\|_1:=\int|u_n-u|\,d\mu\to 0$ and one can suppose that $u_n\to u$ μ-a.e. Then $\|u_n^+-u\|_1\to 0$, where $u_n^+:=u_n\vee 0$ (since $u=u\vee 0$) and hence $\|z_n-u\|_1\to 0$, where $z_n:=u_n^+\wedge\chi_{[a,b]}$ (since $u\wedge\chi_{[a,b]}=u$). We have used the properties of the Banach lattice $L^1(\mu)$. One sees that $0\le z_n\le\chi_{[a,b]}$, z_n are Riemann integrable and, for n greater than some n_0, one must have $\int z_n\,d\mu>0$, because $\int u\,d\mu>0$.

The result already obtained for Riemann integrable functions yields for all $n\ge n_0$,
$$\frac{\int_{[a,b]}fz_n\,d\mu}{\int_{[a,b]}z_n\,d\mu}\in[f(a)-\varepsilon/2,f(a)+\varepsilon/2],$$
and passing to the n-limit, one obtains
$$\frac{\int_{[a,b]}fu\,d\mu}{\int_{[a,b]}u\,d\mu}\in[f(a)-\varepsilon/2,f(a)+\varepsilon/2].\ \blacksquare$$

Using this Statement, one can see that, for given $\varepsilon>0$, there exists $\delta>0$ such that for every $0\le a<b\le 1$ with $b-a<\delta$, one has
$$A_\varepsilon(\bar{I},\bar{J})(\chi_{[a,b]})\ne\emptyset.$$

Indeed, we saw that for given $\varepsilon>0$, one can find $\delta>0$ such that for every a,b as above one has
$$A(\bar{I},\bar{J})(\chi_{[a,b]})\subset[f(a)-\varepsilon/2,f(a)+\varepsilon/2].$$
Then it is immediately seen that
$$[f(a)-\varepsilon/2,f(a)+\varepsilon/2]\subset A_\varepsilon(\bar{I},\bar{J})(\chi_{[a,b]}).$$

We are prepared to satisfy Assumptions 1 and 2. The constructions indicated in the successive steps $s(1),s(2),\ldots$ will give each time a finite number of results. More precisely:

$s(1)$: There exists a natural number p_1 such that for every interval $[a,b]\subset[0,1]$ with $b-a\le 1/p_1$, one has
$$A_{2^{-1}}(\bar{I},\bar{J})(\chi_{[a,b]})\ne\emptyset$$
as we have seen.

Let us divide $[0,1]$ as follows:

$$0 =: x_0 < x_1 < \ldots < x_{p_1} := 1, \qquad x_i - x_{i-1} = 1/p_1.$$

Then we define $(h_{i;1})_{1\le i\le p_1}$ via

$$h_{i;1} := \begin{cases} \chi_{[x_{i-1},x_i[} \in L(I) & \text{for } i < p_1, \\ \chi_{[x_{i-1},x_i]} \in L(I) & \text{for } i = p_1. \end{cases}$$

$s(2)$: There exists a natural number p_2 such that for every interval $[a,b] \subset [0,1]$ with $b-a \le 1/p_2$, one has

$$A_{2^{-2}}(\bar{I},\bar{J})(\chi_{[a,b]}) \neq \emptyset.$$

Then, for every $i = 1,\ldots,p_1$, we divide $[x_{i-1},x_i]$ as follows:

$$x_{i-1} =: x_{i,0} < x_{i,1} < \ldots < x_{i,p_2} := x_i, \qquad x_{i,k} - x_{i,k-1} = \frac{1}{p_1p_2}.$$

Fixing $i = 1,\ldots,p_1-1$, we can define the functions $(h_{(i,k);2})_{1\le k\le p_2}$ by

$$h_{(i,k);2} := \chi_{[x_{i,k-1},x_{i,k}[} \in L(I)$$

and for $i = p_1$ we define $(h_{(i,k);2})_{1\le k\le p_2}$ by

$$h_{(i,k);2} := \begin{cases} \chi_{[x_{i,k-1},x_{i,k}[} \in L(I) & \text{for } k < p_2, \\ \chi_{[x_{i,k-1},x_{i,k}]} \in L(I) & \text{for } k = p_2. \end{cases}$$

We obtained the set of functions $(h_{\alpha;2})_{\alpha\in A}$ with $A := \{1,\ldots,p_1\}\times\{1,\ldots,p_2\}$ and this accomplishes the construction for step $s(2)$.

The procedure continues in the same manner (dividing all intervals into small subintervals of equal length a.s.o.).

The reader can see that, in this way, Assumptions 1 and 2 are satisfied.

One should add that the construction will give as uniform limit of the sequence $(f_n)_n$ so obtained a function which must be μ-almost everywhere equal to the initial f, which is the Radon–Nikodym derivative of J with respect to I.

3.3. We end with some supplementary considerations.

Our theorem gives an *exact* Radon–Nikodym derivative. One knows (see [1]) that for Daniell integrals one cannot generally find *exact* Radon–Nikodym derivatives, only *approximate* Radon–Nikodym derivatives existing always. The price payed in order to obtain this better situation was the following:

(a) We are obliged to work for the Daniell case, more particular than the case of general linear positive functionals. This general extension procedure has been studied in [2], being among the first ones concerned with Loomis systems.

(b) Additional assumptions 1–3 were adopted.

(c) The *exact* Radon–Nikodym derivative one can find generally belongs to the space $L(I)$, which is considerably larger than the initial space B.

The procedure presented is *constructive*, which distinguishes the present work from [1].

References

[1] E. de Amo, I. Chitescu and M. Díaz Carrillo, *An approximate functional Radon–Nikodym theorem*, Rend. Circ. Mat. Palermo 48 (1999), 443–450.
[2] P. Bobillo Guerrero and M. Díaz Carrillo, *Summable and integrable functions with respect to any Loomis system*, Arch. Math. (Basel) 49 (1987), 245–256.
[3] N. Bourbaki, *Intégration* (*Les structures fondamentales de l'analyse, livre VI*), Hermann, Paris, 1956.
[4] R. Cristescu, *Ordered Vector Spaces and Linear Operators*, Ed. Academiei–Abacus Press, 1976.
[5] N. Dinculeanu, *Vector Measures*, Pergamon Press, New York, 1967.
[6] J. W. Hagood, *A Radon–Nikodym theorem and L_p completeness for finitely additive vector measures*, J. Math. Anal. Appl. 113 (1968), 266–279.
[7] P. R. Halmos, *Measure Theory*, Van Nostrand, Princeton, 1950.
[8] H. B. Maynard, *A Radon–Nikodym theorem for finitely additive bounded vector measures*, Pacific J. Math. 83 (1979), 401–413.
[9] W. F. Pfeffer, *Integrals and Measures*, Dekker, New York, 1977.
[10] A. C. Zaanen, *Integration*, North-Holland, Amsterdam, 1967.

Departamento de Álgebra y Análisis Matemático
Universidad de Almería
04120 Almería, Spain
E-mail: edeamo@ualm.es

Faculty of Mathematics
University of Bucharest
Str. Academiei, 14
Bucureşti, Romania

Departamento de Análisis Matemático
Universidad de Granada
18071 Granada, Spain
E-mail: madiaz@ugr.es

Received February 12, 1999
Revised version May 29, 2001 (4262)