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On (A,m)-expansive operators

by

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Abstract. We give several conditions for (A,m)-expansive operators to have the single-valued extension property. We also provide some spectral properties of such operators. Moreover, we prove that the A-covariance of any $(A,2)$-expansive operator $T \in \mathcal{L}(\mathcal{H})$ is positive, showing that there exists a reducing subspace $\mathcal{M}$ on which T is $(A,2)$-isometric. In addition, we verify that Weyl's theorem holds for an operator $T \in \mathcal{L}(\mathcal{H})$ provided that T is $(T^*T,2)$-expansive. We next study (A,m)-isometric operators as a special case of (A,m)-expansive operators. Finally, we prove that every operator $T \in \mathcal{L}(\mathcal{H})$ which is $(T^*T,2)$-isometric has a scalar extension.

1. Introduction. Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, then we shall use the notations $\sigma(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$, $\sigma_{re}(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, and $\sigma_{su}(T)$ for the spectrum, essential spectrum, left essential spectrum, right essential spectrum, point spectrum, approximate point spectrum, and surjective spectrum of T, respectively.

Throughout this paper, fix a positive operator $A \in \mathcal{L}(\mathcal{H})$, and we denote

$$B_A^m(T) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*j} A T^j$$

for an operator $T \in \mathcal{L}(\mathcal{H})$ and a nonnegative integer m. We say that $T \in \mathcal{L}(\mathcal{H})$ is (A,m)-*expansive* if $B_A^m(T) \le 0$ for some positive integer m. In particular, (I,m)-expansive operators are simply called m-*expansive* operators. Moreover, if $B_A^m(T) = 0$, then T is said to be (A,m)-*isometric*. We say that $(A,1)$-isometric operators are A-*isometric*, while (I,m)-isometric operators are m-*isometric*.

An operator $T \in \mathcal{L}(\mathcal{H})$ is called (A,m)-*hyperexpansive* if T is (A,n)-expansive for all positive integer $n \le m$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *completely A-hyperexpansive* if it is (A,n)-expansive for all positive inte-

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gers n. In the special case when T is (I,m)-hyperexpansive (resp. completely I-hyperexpansive), we say that T is *m-hyperexpansive* (resp. *completely hyperexpansive*). When $B_A^m(T) \geq 0$, we say that T is *(A,m)-contractive.* If T is (A,n)-contractive for all positive integers n, then T is said to be *completely A-contractive.*

J. Agler showed in [1] that if $T \in \mathcal{L}(\mathcal{H})$ is subnormal, then $\|T\| \leq 1$ if and only if $B_A^m(T) \geq 0$ for all positive integers m. J. Agler and M. Stankus extended these inequalities to the concept of m-isometric operators. In particular, they provided the structure of 2-isometric operators (see [2] and [3] for more details). Since every 2-isometric operator is completely hyperexpansive, several mathematicians have started investigating completely hyperexpansive operators (see [7] and [28] for more details). For this, it is important to study m-expansive operators. We refer the reader to [14] for more information about m-expansivity. Recently, O. Ahmed and A. Saddi introduced the concept of (A,m)-isometric operators. They gave several generalizations of well known facts on m-isometric operators according to semi-Hilbertian space structures.

If $T \in \mathcal{L}(\mathcal{H})$ is m-expansive, then we have $B_{T^*T}^m(T) = T^* B_I^m(T) T \leq 0$, which means that T is (T^*T,m)-expansive. Hence it is natural to consider (A,m)-expansive operators. In this paper, we give several conditions for (A,m)-expansive operators to have the single-valued extension property. We also provide some spectral properties of such operators. Moreover, we prove that the A-covariance of any $(A,2)$-expansive operator $T \in \mathcal{L}(\mathcal{H})$ is positive, showing that there exists a reducing subspace $\mathcal{M}$ on which T is $(A,2)$-isometric. In addition, we verify that Weyl's theorem holds for an operator $T \in \mathcal{L}(\mathcal{H})$ provided that T is $(T^*T,2)$-expansive. We next study (A,m)-isometric operators as a special case of (A,m)-expansive operators. Finally, we prove that every operator $T \in \mathcal{L}(\mathcal{H})$ which is $(T^*T,2)$-isometric has a scalar extension.

2. Preliminaries. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *upper semi-Fredholm* if it has closed range and $\dim\ker(T) < \infty$, and T is called *lower semi-Fredholm* if it has closed range and $\dim(\mathcal{H}/\mathrm{ran}(T)) < \infty$. When T is either upper semi-Fredholm or lower semi-Fredholm, it is called *semi-Fredholm.* The *index* of a semi-Fredholm operator $T \in \mathcal{L}(\mathcal{H})$ is defined by

$$\mathrm{ind}(T) := \dim\ker(T) - \dim(\mathcal{H}/\mathrm{ran}(T)).$$

Note that $\mathrm{ind}(T)$ is an integer or $\pm\infty$. We say that T is *Fredholm* if it is both upper and lower semi-Fredholm. In particular, a Fredholm operator $T \in \mathcal{L}(\mathcal{H})$ of index zero is called *Weyl.* The *Weyl spectrum* of T is given by $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$. We say that *Weyl's theorem holds* *for* T if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$$

where

$$\pi_{00}(T) := \{\lambda \in \mathrm{iso}(\sigma(T)) : 0 < \dim \ker(T - \lambda) < \infty\}$$

and $\mathrm{iso}(\sigma(T))$ denotes the set of all isolated points of $\sigma(T)$. A *hole* in $\sigma_e(T)$ is a nonempty bounded component of $\mathbb{C} \setminus \sigma_e(T)$, and a *pseudohole* in $\sigma_e(T)$ is a nonempty component of $\sigma_e(T) \setminus \sigma_{le}(T)$ or of $\sigma_e(T) \setminus \sigma_{re}(T)$. The *spectral picture* of T is the structure consisting of $\sigma_e(T)$ and the collection of holes and pseudoholes in $\sigma_e(T)$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single-valued extension property at* z_0 if for every neighborhood G of z_0 and any analytic function $f : G \to \mathcal{H}$, $(T - z)f(z) \equiv 0$ implies $f(z) \equiv 0$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single-valued extension property* (or SVEP) if it has the single-valued extension property at every z in $\mathbb{C}$. For an operator $T \in \mathcal{L}(\mathcal{H})$ and a vector $x \in \mathcal{H}$, the set $\rho_T(x)$ consists of elements z_0 in $\mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in $\mathcal{H}$, which satisfies $(T - z)f(z) \equiv x$. We let $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$ and

$$H_T(F) := \{x \in \mathcal{H} : \sigma_T(x) \subseteq F\},$$

where F is a subset of $\mathbb{C}$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Dunford's property* (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathcal{H}$ of $\mathcal{H}$-valued analytic functions if $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then so does $f_n(z)$.

We say that $T \in \mathcal{L}(\mathcal{H})$ is *scalar* of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous unital homomorphism of topological algebras

$$\Phi : C_0^m(\mathbb{C}) \to \mathcal{L}(\mathcal{H})$$

such that $\Phi(z) = T$, where as usual z stands for the identity function on $\mathbb{C}$ and $C_0^m(\mathbb{C})$ for the space of all m times continuously differentiable functions with compact support. An operator is *subscalar* of order m if it is similar to the restriction of a scalar operator of order m to an invariant subspace. The following implications are well known (see [10] and [20] for more details):

$$\text{scalar} \Rightarrow \text{property } (\beta) \Rightarrow \text{Dunford's property (C)} \Rightarrow \text{SVEP}.$$

Let z be the coordinate in the complex plane $\mathbb{C}$ and let $d\mu(z)$ denote the planar Lebesgue measure. Fix a complex separable Hilbert space $\mathcal{H}$ and a bounded (connected) open subset U of $\mathbb{C}$. We denote by $L^2(U, \mathcal{H})$ the

Hilbert space of measurable functions $f : U \to \mathcal{H}$ such that

$$\|f\|_{2,U} = \Big(\int\limits_U \|f(z)\|^2 \, d\mu(z) \Big)^{1/2} < \infty.$$

The subspace of functions $f \in L^2(U, \mathcal{H})$ which are analytic in U, i.e., $\bar{\partial} f = 0$, is denoted by

$$A^2(U, \mathcal{H}) = L^2(U, \mathcal{H}) \cap \mathcal{O}(U, \mathcal{H}),$$

where $\mathcal{O}(U, \mathcal{H})$ denotes the Fréchet space of $\mathcal{H}$-valued analytic functions on U with the uniform topology. The space $A^2(U, \mathcal{H})$ is a Hilbert space, called the *Bergman space* for U.

For a fixed nonnegative integer m, the vector valued Sobolev space $W^m(U, \mathcal{H})$ with respect to $\bar{\partial}$ and of order m is the space of those functions $f \in L^2(U, \mathcal{H})$ whose derivatives $\bar{\partial} f, \ldots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U, \mathcal{H})$. Endowed with the norm

$$\|f\|^2_{W^m} = \sum_{i=0}^{m} \|\bar{\partial}^i f\|^2_{2,U},$$

$W^m(U, \mathcal{H})$ becomes a Hilbert space contained continuously in $L^2(U, \mathcal{H})$.

We remark that the linear operator M of multiplication by z on $W^m(U, \mathcal{H})$ is continuous and it has a spectral distribution

$$\Phi_M : C_0^m(\mathbb{C}) \to \mathcal{L}(W^m(U, \mathcal{H}))$$

of order m defined by $\Phi_M(\varphi) f = \varphi f$ for $\varphi \in C_0^m(\mathbb{C})$ and $f \in W^m(U, \mathcal{H})$. Therefore, M is a scalar operator of order m (see [26] for more details).

3. (A, m)-expansivity. In this section, we study (A, m)-expansive and (A, m)-contractive operators. We first consider the single-valued extension property for (A, m)-expansive operators.

THEOREM 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ and let $0 \notin \sigma_p(A)$. Then the following statements hold:*

(i) *Suppose that T is (A, m)-expansive for some positive integer m. If m is even, then T has the single-valued extension property. If m is odd, then T has the single-valued extension property at each $\lambda_0 \in \mathbb{C}$ with $|\lambda_0| \le 1$ or $|\lambda_0| \ge \|T\|$.*

(ii) *If T is (A, m)-contractive for some positive odd integer m, then T has the single-valued extension property at each $\lambda_0 \in \mathbb{C}$ with $|\lambda_0| \ge \min\{1, \|T\|\}$.*

Proof. Let $\lambda_0 \in \mathbb{C}$ and let D be any open neighborhood of λ_0 in $\mathbb{C}$. Assume that $f : D \to \mathcal{H}$ is any analytic function on D such that

$$(T - \lambda) f(\lambda) \equiv 0 \quad \text{on } D. \tag{3.1}$$

From (3.1), it follows that $(T^j - \lambda^j)f(\lambda) \equiv 0$ on D for all positive integers j. This implies that

$$0 \geq \langle B_A^m(T)f(\lambda), f(\lambda)\rangle = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\langle A^{1/2}T^j f(\lambda), A^{1/2}T^j f(\lambda)\rangle \tag{3.2}$$
$$= \sum_{j=0}^{m}(-1)^j\binom{m}{j}|\lambda|^{2j}\|A^{1/2}f(\lambda)\|^2 = (1-|\lambda|^2)^m\|A^{1/2}f(\lambda)\|^2$$

for all $\lambda \in D$.

(i) Suppose that T is (A,m)-expansive for some even integer m. Since m is even, we deduce from (3.2) that $A^{1/2}f(\lambda) \equiv 0$ on D. Since $0 \notin \sigma_p(A)$, we have $f(\lambda) \equiv 0$ on D. Thus T has the single-valued extension property at every $\lambda_0 \in \mathbb{C}$, i.e., T has the single-valued extension property.

Suppose that T is (A,m)-expansive for some odd integer m. If $|\lambda_0| \leq 1$, then we can choose an open disk D_0 in D so that $|\lambda| < 1$ for all $\lambda \in D_0$. Then (3.2) ensures that $A^{1/2}f(\lambda) \equiv 0$ on D_0, and so $f(\lambda) \equiv 0$ on D_0 since $0 \notin \sigma_p(A)$. By the identity theorem, $f(\lambda) \equiv 0$ on D. Hence T has the single-valued extension property at λ_0. If $|\lambda_0| \geq \|T\|$, then there is an open disk D_1 in D such that $T - \lambda$ is invertible for all $\lambda \in D_1$, and so it is obvious that $f(\lambda) \equiv 0$ on D by (3.1) and the identity theorem.

(ii) Suppose that T is (A,m)-contractive for some odd integer m. By applying the proof of (i), it is enough to show that T has the single-valued extension property at all λ_0 with $|\lambda_0| \geq 1$. Fix such a λ_0. Note that

$$0 \geq -\langle B_A^m(T)f(\lambda), f(\lambda)\rangle \tag{3.3}$$
$$= \left\langle \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}A^{1/2}T^j f(\lambda), A^{1/2}T^j f(\lambda)\right\rangle$$
$$= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}|\lambda|^{2j}\|A^{1/2}f(\lambda)\|^2 = (|\lambda|^2-1)^m\|A^{1/2}f(\lambda)\|^2$$

for all $\lambda \in D$. Since $|\lambda_0| \geq 1$, we can choose an open disk D_0 in D so that $|\lambda| > 1$ for all $\lambda \in D_0$. Then (3.3) ensures that $A^{1/2}f(\lambda) \equiv 0$ on D_0. Since $0 \notin \sigma_p(A)$, we have $f(\lambda) \equiv 0$ on D_0. By the identity theorem, $f(\lambda) \equiv 0$ on D. Hence T has the single-valued extension property at λ_0. ∎

COROLLARY 3.2. *Let m be a positive integer and let $0 \notin \sigma_p(A)$. Then the following assertions hold:*

(i) *If $m > 1$, then (A,m)-hyperexpansive operators have the single-valued extension property. Moreover, every completely hyperexpansive operator has the single-valued extension property.*
(ii) *Every (A,m)-isometric operator has the single-valued extension property.*

Proof. (i) If $T \in \mathcal{L}(\mathcal{H})$ is (A,m)-hyperexpansive for some $m > 1$, then it is $(A,2)$-expansive, and thus it has the single-valued extension property from Theorem 3.1. The latter assertion holds obviously.

(ii) From Theorem 3.1, it suffices to assume that m is odd. If $T \in \mathcal{L}(\mathcal{H})$ is an (A,m)-isometric operator, then it is also $(A,m+1)$-isometric by the identity $B_A^{m+1}(T) = B_A^m(T) - T^* B_A^m(T)T$. Hence the conclusion follows from Theorem 3.1. ∎

The following corollary gives some immediate consequences of Theorem 3.1 and [20, Theorems 3.3.8, 3.3.9, Propositions 1.3.2, 1.2.16].

COROLLARY 3.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be an (A,m)-expansive operator for some even positive integer m and let $0 \notin \sigma_p(A)$. Then the following statements hold:*

- (i) *$f(T)$ has the single-valued extension property and $f(\sigma_T(x)) = \sigma_{f(T)}(x)$ for any analytic function f on a neighborhood of $\sigma(T)$ and any $x \in \mathcal{H}$.*
- (ii) *$\sigma(T) = \sigma_{su}(T) = \bigcup\{\sigma_T(x) : x \in \mathcal{H}\}$.*
- (iii) *If F_1 and F_2 are disjoint closed sets in $\mathbb{C}$, then $H_T(F_1 \cup F_2) = H_T(F_1) \oplus H_T(F_2)$ as an algebraic direct sum.*

In the following proposition, we give some spectral properties of (A,m)-expansive operators.

PROPOSITION 3.4. *Let $T \in \mathcal{L}(\mathcal{H})$ be (A,m)-expansive for some positive integer m and let $0 \notin \sigma_{ap}(A)$.*

- (i) *If m is even, then $\sigma_{ap}(T) \subseteq \partial\mathbb{D}$. Hence either $\sigma(T) \subseteq \partial\mathbb{D}$ or $\sigma(T) = \overline{\mathbb{D}}$.*
- (ii) *If m is odd, then $\sigma_{ap}(T) \subseteq \mathbb{C} \setminus \mathbb{D}$. In particular, T is injective and* $\operatorname{ran}(T)$ *is closed.*

Proof. If $\lambda \in \sigma_{ap}(T)$, then there is a sequence $\{x_n\}_{n=1}^{\infty}$ of unit vectors in $\mathcal{H}$ such that $\lim_{n\to\infty} \|(T-\lambda)x_n\| = 0$. Since $\lim_{n\to\infty} \|(T^j - \lambda^j)x_n\| = 0$ for $j = 1, \ldots, m$, we have

$$\big| \|A^{1/2}T^j x_n\| - |\lambda|^j \|A^{1/2}x_n\| \big| \le \|A^{1/2}(T^j - \lambda^j)x_n\| \to 0 \quad \text{as } n \to \infty$$

for $j = 1, \ldots, m$. In addition, we note that

$$\begin{aligned}
0 \ge \langle B_A^m(T)x_n, x_n\rangle &= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \|A^{1/2}T^j x_n\|^2 \\
&= \sum_{j=0}^{m} (-1)^j \binom{m}{j} [(\|A^{1/2}T^j x_n\|^2 - |\lambda|^{2j}\|A^{1/2}x_n\|^2) + |\lambda|^{2j}\|A^{1/2}x_n\|^2] \\
&= \sum_{j=0}^{m} (-1)^j \binom{m}{j} (\|A^{1/2}T^j x_n\|^2 - |\lambda|^{2j}\|A^{1/2}x_n\|^2) + (1-|\lambda|^2)^m \|A^{1/2}x_n\|^2
\end{aligned}$$

for all n. Hence

$$0 \geq (1 - |\lambda|^2)^m \limsup_{n\to\infty} \|A^{1/2}x_n\|^2.$$

Since $0 \notin \sigma_{ap}(A)$, it must be the case that $\limsup_{n\to\infty} \|A^{1/2}x_n\| \neq 0$, and so

$$(1 - |\lambda|^2)^m \leq 0. \tag{3.4}$$

(i) If m is even, then $0 \leq (1-|\lambda|^2)^m \leq 0$ from (3.4), and so $|\lambda| = 1$. This means that $\sigma_{ap}(T) \subseteq \partial\mathbb{D}$, and so

$$\partial\sigma(T) \subseteq \sigma_{ap}(T) \subseteq \partial\mathbb{D}. \tag{3.5}$$

Suppose that $\sigma(T) \not\subseteq \partial\mathbb{D}$. In order to show that $\sigma(T) = \overline{\mathbb{D}}$, we first claim that $0 \in \sigma(T)$. Let $\lambda \in \sigma(T) \cap \mathbb{D}$. Since λ is an interior point of $\sigma(T)$ by (3.5), we can choose the largest positive number r such that $\{z \in \mathbb{C} : |z - \lambda| \leq r\}$ $\subseteq \sigma(T)$. Since $r(T) = \max\{|z| : z \in \sigma(T)\} = \max\{|z| : z \in \partial\sigma(T)\} = 1$, it follows that $\sigma(T) \subseteq \overline{\mathbb{D}}$. Hence $r \leq 1 - |\lambda|$. If $r < 1 - |\lambda|$, then there exists $z \in \partial\sigma(T)$ with $|z - \lambda| = r$ by the maximality of r. But this contradicts (3.5). Thus $r = 1 - |\lambda|$. That is,

$$\{z \in \mathbb{C} : |z - \lambda| \leq 1 - |\lambda|\} \subseteq \sigma(T) \quad \text{for any } \lambda \in \sigma(T) \cap \mathbb{D}. \tag{3.6}$$

Since $\sigma(T) \not\subseteq \partial\mathbb{D}$ and $\sigma(T) \subseteq \overline{\mathbb{D}}$, we can select a point $\lambda_0 \in \sigma(T) \cap \mathbb{D}$. It is enough to assume that $\lambda_0 \neq 0$. If $|\lambda_0| < 1/2$, then (3.6) implies that

$$0 \in \{z \in \mathbb{C} : |z - \lambda_0| \leq 1 - |\lambda_0|\} \subseteq \sigma(T).$$

Otherwise, take a positive integer N satisfying that $1/N < 1-|\lambda_0|$. If we set $\lambda_1 := (|\lambda_0| - 1/N)e^{i\mathrm{Arg}\lambda_0}$, then $|\lambda_0| - (1 - |\lambda_0|) < |\lambda_0| - 1/N = |\lambda_1| < |\lambda_0|$ and so $\lambda_1 \in \{z \in \mathbb{C} : |z-\lambda_0| \leq 1-|\lambda_0|\} \subseteq \sigma(T)$ by (3.6). If $|\lambda_1| < 1/2$, then from (3.6),

$$0 \in \{z \in \mathbb{C} : |z - \lambda_1| \leq 1 - |\lambda_1|\} \subseteq \sigma(T).$$

Otherwise, put $\lambda_2 := (|\lambda_0|-2/N)e^{i\mathrm{Arg}\lambda_0}$. Then $|\lambda_1|-(1-|\lambda_1|) < |\lambda_1|-1/N = |\lambda_2| < |\lambda_1|$ and so $\lambda_2 \in \{z \in \mathbb{C} : |z - \lambda_1| \leq 1 - |\lambda_1|\} \subseteq \sigma(T)$ by (3.6). Repeating this procedure, we find a sequence $\{\lambda_n\}$ where

$$|\lambda_n| = |\lambda_0| - n/N \quad \text{and} \quad \{z \in \mathbb{C} : |z - \lambda_n| \leq 1 - |\lambda_n|\} \subseteq \sigma(T)$$

for all $n \geq 1$. Taking a positive integer n_0 such that $|\lambda_0| - n_0/N < 1/2$, we find that $0 \in \sigma(T)$.

Choose the largest positive number s so that $\{z \in \mathbb{C} : |z| \leq s\} \subseteq \sigma(T)$. Since $\sigma(T) \subseteq \overline{\mathbb{D}}$, it follows that $s \leq 1$. But, if $s < 1$, then we obtain a point $z \in \partial\sigma(T)$ with $|z| = s$, which contradicts (3.5), and so $s = 1$. This means that $\sigma(T) = \overline{\mathbb{D}}$.

(ii) Suppose that m is odd. If $|\lambda| < 1$, then $0 < (1 - |\lambda|^2)^m \leq 0$ from (3.4), which is a contradiction. Hence $\sigma_{ap}(T) \subseteq \mathbb{C} \setminus \mathbb{D}$.

In particular, since $0 \notin \sigma_{ap}(T)$ from (i) and (ii), it follows that T is injective and $\mathrm{ran}(T)$ is closed. ∎

REMARK. Let $T \in \mathcal{L}(\mathcal{H})$ be (A, m)-expansive for some even integer m and let $0 \notin \sigma_{ap}(A)$. We observe that if T is not invertible, then $\sigma(T) = \overline{\mathbb{D}}$ from Proposition 3.4. In addition, since $0 \notin \sigma(I)$, Theorem 3.1 and Proposition 3.4 hold for m-expansive operators without any spectral assumptions.

Since every (A, m)-isometric operator is $(A, m+1)$-isometric, one can recapture the result in [4] that if $T \in \mathcal{L}(\mathcal{H})$ is an (A, m)-isometric operator and $0 \notin \sigma_{ap}(A)$, then $\sigma_{ap}(T) \subseteq \partial\mathbb{D}$ and either $\sigma(T) \subseteq \partial\mathbb{D}$ or $\sigma(T) = \overline{\mathbb{D}}$. From this, we get the following corollary.

COROLLARY 3.5. *Let $T \in \mathcal{L}(\mathcal{H})$ and let $0 \notin \sigma_{ap}(A)$. If both T and T^* are (A, m)-isometric for some positive integer m, then $\sigma(T) \subseteq \partial\mathbb{D}$.*

Proof. If $\sigma(T) \nsubseteq \partial\mathbb{D}$, then $0 \in \sigma(T) \setminus \sigma_{ap}(T)$ from Proposition 3.4, and so $\overline{\mathrm{ran}(T)} \neq \mathcal{H}$. Hence $0 \in \sigma_{ap}(T^*)$. But this contradicts Proposition 3.4, since T^* is (A, m)-isometric. ∎

Next we deal with (A, m)-expansive operators which are complex symmetric. Recall that an operator $C : \mathcal{H} \to \mathcal{H}$ is called a *conjugation* if C is antilinear (i.e., $C(\alpha x + \beta y) = \overline{\alpha} Cx + \overline{\beta} Cy$ for all $\alpha, \beta \in \mathbb{C}$ and all $x, y \in \mathcal{H}$), C is isometric (i.e., $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$), and $C^2 = I$. We say that an operator $T \in \mathcal{L}(\mathcal{H})$ is *complex symmetric* if there is a conjugation C on $\mathcal{H}$ such that $CTC = T^*$ (see [16] for more details).

PROPOSITION 3.6. *Let $T \in \mathcal{L}(\mathcal{H})$ be (A, m)-expansive for some positive integer m and let $0 \notin \sigma_p(A)$. If T is complex symmetric, then the following assertions hold:*

(i) *If m is even, then both T and T^* have the single-valued extension property.*
(ii) *If m is odd, then both T and T^* have the single-valued extension property at each $\lambda_0 \in \mathbb{C}$ with $|\lambda_0| \leq 1$ or $|\lambda_0| \geq \|T\|$.*

Proof. Since T is complex symmetric, there exists a conjugation C such that $CTC = T^*$. Since T is (A, m)-expansive, we get

$$\begin{aligned} 0 &\geq \left\langle Cx, \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*j} A T^j Cx \right\rangle \\ &= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \langle Cx, CT^j CACT^{*j} x \rangle \\ &= \left\langle \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^j (CAC) T^{*j} x, x \right\rangle \end{aligned}$$

for all $x \in \mathcal{H}$. This means that $\sum_{j=0}^{m}(-1)^j\binom{m}{j}T^j(CAC)T^{*j} \le 0$, and so T^* is (CAC,m)-expansive. In addition, note that CAC is a positive operator with $0 \notin \sigma_p(CAC)$. Thus we complete the proof by invoking Theorem 3.1. ∎

Corollary 3.7. *Let $T \in \mathcal{L}(\mathcal{H})$ be an (A,m)-expansive operator for some odd positive integer m and $0 \notin \sigma_{ap}(A)$. If T is complex symmetric, then $\sigma(T) \subseteq \mathbb{C} \setminus \mathbb{D}$.*

Proof. Let $\lambda_0 \in \mathbb{D}$. Since T and T^* have the single-valued extension property at λ_0 by Proposition 3.6, we deduce from [5, Corollary 2.50] that $\lambda_0 \notin \sigma(T) \setminus \sigma_{ap}(T)$, that is, $\lambda_0 \notin \sigma(T)$ or $\lambda_0 \in \sigma_{ap}(T)$. But since $\sigma_{ap}(T) \subseteq \mathbb{C} \setminus \mathbb{D}$ by Proposition 3.4, we see that $\lambda_0 \notin \sigma(T)$. Thus $\sigma(T) \subseteq \mathbb{C} \setminus \mathbb{D}$. ∎

We next verify that all powers of an (A,m)-expansive operator are again (A,m)-expansive. As in [14], we define an operation $\diamond$ by

$$(T^{*m}AT^m) \diamond (T^{*k}AT^k) := T^{*m}(T^{*k}AT^k)T^m$$

for all nonnegative integers m,k and extend this by linearity to (finite) linear combinations of $\{T^{*m}AT^m\}_{m=0}^{\infty}$. Then it is easy to check that $\diamond$ is commutative and associative. We denote $B_A^0(T) := 0$.

Lemma 3.8. *For all operators $T \in \mathcal{L}(\mathcal{H})$ and all nonnegative integers m,k, we have*

$$B_A^k(T) \diamond B_A^m(T) = B_A^{m+k}(T).$$

Proof. We fix any nonnegative integer m and then use induction on k. The given identity trivially holds for $k=0$. If $B_A^k(T) \diamond B_A^m(T) = B_A^{m+k}(T)$ for some positive integer k, then it follows that

$$\begin{aligned} B_A^{k+1}(T) \diamond B_A^m(T) &= B_A^k(T) \diamond B_A^1(T) \diamond B_A^m(T) = B_A^{m+k}(T) \diamond B_A^1(T) \\ &= B_A^{m+k+1}(T), \end{aligned}$$

which completes the proof. ∎

Proposition 3.9. *If $T \in \mathcal{L}(\mathcal{H})$ is (A,m)-expansive for some positive integer m, then T^n is also (A,m)-expansive for every positive integer n.*

Proof. Fix any positive integer n. We will use induction to show that

$$B_A^m(T^n) = \sum_{j=0}^{m}\binom{m}{j}(T^{*(n-1)j}B_A^{m-j}(T^{n-1})T^{(n-1)j}) \diamond B_A^j(T) \tag{3.7}$$

for every positive integer m. Since

$$B_A^1(T^n) = B_A^1(T^{n-1}) \diamond A + (T^{*n-1}AT^{n-1}) \diamond B_A^1(T), \tag{3.8}$$

we see that (3.7) holds for $m=1$. Assume that (3.7) is true for some positive integer $m = k$. Then from (3.8) and Lemma 3.8 we obtain

$$
\begin{aligned}
&B_A^{k+1}(T^n) = B_A^k(T^n) \diamond B_A^1(T^n) \\
&= \sum_{j=0}^{k} \binom{k}{j} (T^{*(n-1)j} B_A^{k-j}(T^{n-1}) T^{(n-1)j}) \diamond B_A^j(T) \diamond B_A^1(T^{n-1}) \\
&\quad + \sum_{j=0}^{k} \binom{k}{j} (T^{*(n-1)j} B_A^{k-j}(T^{n-1}) T^{(n-1)j}) \diamond B_A^j(T) \diamond (T^{*n-1} A T^{n-1}) \diamond B_A^1(T) \\
&= \sum_{j=0}^{k} \binom{k}{j} \left(T^{*(n-1)j} B_A^{k+1-j}(T^{n-1}) T^{(n-1)j}\right) \diamond B_A^j(T) \\
&\quad + \sum_{j=0}^{k} \binom{k}{j} \left(T^{*(n-1)(j+1)} B_A^{k-j}(T^{n-1}) T^{(n-1)(j+1)}\right) \diamond B_A^{j+1}(T) \\
&= \sum_{j=0}^{k+1} \binom{k+1}{j} \left(T^{*(n-1)j} B_A^{k+1-j}(T^{n-1}) T^{(n-1)j}\right) \diamond B_A^j(T),
\end{aligned}
$$

which means that (3.7) is true for $m = k + 1$. Therefore, (3.7) holds for all positive integers m and n. Note that So $B_A^m(T^n)$ can be expressed as a linear combination of terms of the form

$$T^{*r}(B_A^{m-j}(T^{n-1}) \diamond B_A^j(T))T^r$$

with nonnegative coefficients. Applying (3.7) to $B_A^{m-j}(T^{n-1})$, we have

$$B_A^{m-j}(T^{n-1}) = \sum_{i=0}^{m-j} \binom{m-j}{i} (T^{*(n-2)i} B_A^{m-j-i}(T^{n-2}) T^{(n-2)i}) \diamond B_A^i(T).$$

Then $B_A^m(T^n)$ becomes a linear combination of terms of the form

$$T^{*r}(B_A^{m-j-i}(T^{n-2}) \diamond B_A^i(T) \diamond B_A^j(T))T^r$$

with nonnegative coefficients. Apply (3.7) to $B_A^{m-j-i}(T^{n-2})$ as well. By repeating this procedure, $B_A^m(T^n)$ is finally expressed as a linear combination of terms of the form

$$T^{*r}(B_A^{i_1}(T) \diamond \cdots \diamond B_A^{i_j}(T))T^r$$

with nonnegative coefficients and $i_1 + \cdots + i_j = m$. From Lemma 3.8, we have

$$T^{*r}(B_A^{i_1}(T) \diamond \cdots \diamond B_A^{i_j}(T))T^r = T^{*r} B_A^{i_1+\cdots+i_j}(T) T^r = T^{*r} B_A^m(T) T^r.$$

Hence, if T is (A, m)-expansive, then $B_A^m(T) \leq 0$ and so T^n is also (A, m)-expansive. ∎

REMARK. From the proof of Proposition 3.9, we observe that every power of an (A, m)-isometric operator is also (A, m)-isometric, where m is any positive integer.

Next we consider $(A, 2)$-expansive operators. We define the *A-covariance operator* for an $(A, 2)$-expansive operator $T \in \mathcal{L}(\mathcal{H})$ by

$$\Delta_A(T) := -B_A^1(T) = T^*AT - A.$$

THEOREM 3.10. *Let $T \in \mathcal{L}(\mathcal{H})$. Then the following assertions are valid:*

(i) *If T is $(A, 2)$-expansive, then $\Delta_A(T) \geq 0$, i.e., T is $(A, 1)$-expansive.*
(ii) *If T is an invertible $(A, 2)$-expansive operator, then T is $(A, 1)$-isometric.*

Proof. (i) We first claim that

$$T^{*k}AT^k \leq k\Delta_A(T) + A \tag{3.9}$$

for all positive integers $k \geq 2$. Since $B_A^2(T) \leq 0$, we obtain

$$T^{*2}AT^2 \leq 2T^*AT - A = 2\Delta_A(T) + A. \tag{3.10}$$

Thus (3.9) is true for $k = 2$. Suppose that (3.9) holds for all integers l with $2 \leq l \leq k$. Since $T^*\Delta_A(T)T \leq \Delta_A(T)$ by the definition of $(A, 2)$-expansive operators, we see from (3.10) that

$$T^{*k+1}AT^{k+1} \leq T^{*2}[(k-1)\Delta_A(T) + A]T^2 \leq (k+1)\Delta_A(T) + A.$$

Hence (3.9) holds for all positive integers $k \geq 2$. So it follows that

$$\langle \Delta_A(T)x, x\rangle \geq \frac{1}{k}\|A^{1/2}T^kx\|^2 - \frac{1}{k}\langle Ax, x\rangle$$

for any $x \in \mathcal{H}$ and any positive integer $k \geq 2$, which yields

$$\langle \Delta_A(T)x, x\rangle \geq \limsup_{k\to\infty} \frac{1}{k}\|A^{1/2}T^kx\|^2 \geq 0$$

for any $x \in \mathcal{H}$, that is, $\Delta_A(T) \geq 0$. This means that T is $(A, 1)$-expansive since $B_A^1(T) = -\Delta_A(T)$.

(ii) If T is an invertible $(A, 2)$-expansive operator, then it is easy to see that T^{-1} is $(A, 2)$-expansive as well. Thus $\Delta_A(T^{-1}) = T^{-1*}AT^{-1} - A \geq 0$ by (i). This implies that

$$T^*AT - A = -T^*(T^{-1*}AT^{-1} - A)T \leq 0.$$

Since $\Delta_A(T) = T^*AT - A \geq 0$ from (i), we conclude that $T^*AT = A$. ■

We say that an operator $T \in \mathcal{L}(\mathcal{H})$ is *purely* $(A, 1)$-*contractive* if T is $(A, 1)$-contractive and there is no nonzero reducing subspace of $\mathcal{H}$ on which T is $(A, 1)$-isometric (see [29] for more details).

COROLLARY 3.11. *If $T \in \mathcal{L}(\mathcal{H})$ is $(A, 2)$-expansive, then there exists a unique closed subspace $\mathcal{M} \subseteq \mathcal{H}$ reducing both T and $\Delta_A(T)$ such that*

$T^{2*}AT^2 - 2T^*AT + A = 0$ *on* $\mathcal{M}$ *and* $T_{\mathcal{M}^\perp}$ *is purely* $(\Delta_A(T)|_{\mathcal{M}^\perp}, 1)$-*contractive.*

Proof. We see from Theorem 3.10 that $\Delta_A(T) \geq 0$. Moreover, we know that $T^*\Delta_A(T)T \leq \Delta_A(T)$, which means that T is $(\Delta_A(T), 1)$-contractive. Hence it follows from [29, Proposition 2.1] that there exists a unique closed subspace $\mathcal{M} \subseteq \mathcal{H}$ reducing both T and $\Delta_A(T)$ such that $T|_{\mathcal{M}}$ is $(\Delta_A(T)|_{\mathcal{M}}, 1)$-isometric, that is, $T^{2*}AT^2 - 2T^*AT + A = 0$ on $\mathcal{M}$, and $T_{\mathcal{M}^\perp}$ is purely $(\Delta_A(T)|_{\mathcal{M}^\perp}, 1)$-contractive. ∎

COROLLARY 3.12. *Let* $T \in \mathcal{L}(\mathcal{H})$ *be* $(A, 2)$-*expansive and suppose that* $0 \notin \sigma(\Delta_T(A))$. *Then the following assertions are valid:*

(i) T *is similar to a contraction.*
(ii) T^* *is a* $(\Delta_A(T)^{-1}, 1)$-*contractive operator.*

Proof. (i) We note that $\Delta_A(T)$ is an invertible positive operator from Theorem 3.10. Then we obtain

$$\begin{aligned}(\Delta_A(T)^{1/2}T\Delta_A(T)^{-1/2})^*&(\Delta_A(T)^{1/2}T\Delta_A(T)^{-1/2})\\ &= \Delta_A(T)^{-1/2}(T^*\Delta_A(T)T)\Delta_A(T)^{-1/2}\\ &\leq \Delta_A(T)^{-1/2}\Delta_A(T)\Delta_A(T)^{-1/2} = I.\end{aligned}$$

This means that $\Delta_A(T)^{1/2}T\Delta_A(T)^{-1/2}$ is a contraction. Hence T is similar to a contraction.

(ii) Since $T^*\Delta_A(T)T \leq \Delta_A(T)$ and $\Delta_A(T) \geq 0$ from Theorem 3.10 one can define an operator $\widehat{T} \in \mathcal{L}(\overline{\operatorname{ran}(\Delta_A(T))})$ by the relation

$$\widehat{T}\Delta_A(T)^{1/2}x = \Delta_A(T)^{1/2}Tx, \quad x \in \mathcal{H}.$$

Then

$$\|\widehat{T}\Delta_A(T)^{1/2}x\|^2 = \|\Delta_A(T)^{1/2}Tx\|^2 \leq \|\Delta_A(T)^{1/2}x\|^2$$

for all $x \in \mathcal{H}$. Thus $\widehat{T}$ is a contraction on $\overline{\operatorname{ran}(\Delta_A(T))}$. This implies that

$$T\Delta_A(T)^{-1}T^* = \Delta_A(T)^{-1/2}\widehat{T}(\widehat{T})^*\Delta_A(T)^{-1/2} \leq \Delta_A(T)^{-1},$$

and so T^* is $(\Delta_A(T)^{-1}, 1)$-contractive. ∎

We now consider Weyl's theorem for an operator $T \in \mathcal{L}(\mathcal{H})$ that is $(T^*T, 2)$-expansive. Recall that $T \in \mathcal{L}(\mathcal{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$.

LEMMA 3.13. *If* $T \in \mathcal{L}(\mathcal{H})$ *is a* $(T^*T, 2)$-*expansive operator with* $0 \notin \sigma(T^*T)$, *then it is isoloid, i.e.,* $\operatorname{iso}(\sigma(T)) \subseteq \sigma_p(T)$.

Proof. Let $\lambda \in \operatorname{iso}(\sigma(T))$. From Proposition 3.4, it suffices to assume that $\sigma(T) \subseteq \partial\mathbb{D}$. In particular, T is invertible. Since T is $(T^*T, 1)$-isometric from Theorem 3.10, it is similar to an isometry by [25, Theorem 3.7], which is hyponormal. Since every hyponormal operator is isoloid, the proof is complete. ∎

THEOREM 3.14. *If $T \in \mathcal{L}(\mathcal{H})$ is a $(T^*T, 2)$-expansive operator with $0 \notin \sigma(T^*T)$, then Weyl's theorem holds for T.*

Proof. We write

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } \mathcal{H} = \overline{\operatorname{ran}(T)} \oplus \ker(T^*).$$

If P denotes the orthogonal projection of $\mathcal{H}$ onto $\overline{\operatorname{ran}(T)}$, then

$$\langle T_3(I-P)x, (I-P)x\rangle = \langle (I-P)x, T^*(I-P)x\rangle = 0$$

for all $x \in \mathcal{H}$, and so $T_3 = 0$. Moreover, since $T \in \mathcal{L}(\mathcal{H})$ is $(T^*T, 2)$-expansive, $T^{3^*}T^3 - 2T^{2^*}T^2 + T^*T \leq 0$, which implies that

$$\langle (T_1^{2^*}T_1^2 - 2T_1^*T_1 + I)Tx, Tx\rangle = \langle (T^{3^*}T^3 - 2T^{2^*}T^2 + T^*T)x, x\rangle \leq 0$$

for all $x \in \mathcal{H}$, i.e., T_1 is 2-expansive.

CLAIM. *Weyl's theorem holds for T_1.*

If $\sigma(T_1) \subseteq \partial\mathbb{D}$, then Theorem 3.10 shows that T_1 is unitary and so satisfies Weyl's theorem by [9]. We now assume that $\sigma(T_1) \not\subseteq \partial\mathbb{D}$. Since $\sigma(T_1) = \overline{\mathbb{D}}$ from Proposition 3.4, it is evident that $\operatorname{iso}(\sigma(T_1)) = \emptyset$, which ensures that

$$\sigma(T_1) \setminus \sigma_w(T_1) \supseteq \emptyset = \pi_{00}(T_1).$$

Conversely, let $\lambda \in \sigma(T_1) \setminus \sigma_w(T_1)$. Since $T_1 - \lambda$ is Weyl but not invertible, it is easy to see that $0 < \dim\ker(T_1 - \lambda) = \dim\ker(T_1^* - \overline{\lambda}) < \infty$. If λ is an interior point of $\sigma(T_1)$, we can choose $\varepsilon > 0$ such that $T_1 - \gamma$ is Weyl but not invertible for all $\gamma \in \mathbb{C}$ with $|\gamma - \lambda| < \varepsilon$ (indeed, take $A = T_1 - \lambda$ and $Y = (T_1 - \gamma) - (T_1 - \lambda)$ in [11, Theorem XI.3.12]). Thus we get

$$0 < \dim\ker(T_1 - \gamma) = \dim\ker(T_1^* - \overline{\gamma}) < \infty \text{ for all } \gamma \in \mathbb{C} \text{ with } |\gamma - \lambda| < \varepsilon.$$

Since $\operatorname{ran}(T_1 - \lambda)$ has finite codimension and $\sigma_p(T_1 - \lambda)$ contains a neighborhood of 0, T_1 does not have the single-valued extension property from [15, Theorem 10]. However, this contradicts Theorem 3.1, and so $\lambda \in \partial\sigma(T_1) \setminus \sigma_w(T_1)$. Hence it follows from [11, Theorem XI.6.8] that $\lambda \in \operatorname{iso}(\sigma(T_1))$. Therefore $\lambda \in \pi_{00}(T_1)$.

From the above claim, Weyl's theorem holds for T_1. Furthermore, since T_1 is $(T_1^*T_1, 2)$-expansive, it is isoloid by Lemma 3.13. Since the spectral picture of a zero operator has no pseudoholes, from [21, Theorem 2.4] it suffices to prove that Weyl's theorem holds for $T_1 \oplus 0$. Every zero operator is clearly isoloid, and so we conclude from [22, Corollary 11] that Weyl's theorem holds for $T_1 \oplus 0$. Thus Weyl's theorem holds for T. ∎

COROLLARY 3.15. *If $T \in \mathcal{L}(\mathcal{H})$ is a $(T^*T, 2)$-expansive operator with $0 \notin \sigma(T^*T)$, then the following statements hold:*

(i) *$\sigma_w(f(T)) = f(\sigma_w(T))$ for any analytic function $f(z)$ on a neighborhood of $\sigma(T)$.*
(ii) *Weyl's theorem holds for $f(T)$ where $f(z)$ is any analytic function on $\sigma(T)$.*

Proof. (i) Since $0 \notin \sigma(T^*T)$, the operator T has the single-valued extension property from Theorem 3.1. Hence the conclusion follows from [5, Corollary 3.72].

(ii) Since T is isoloid and satisfies Weyl's theorem by Lemma 3.13 and Theorem 3.14, we obtain

$$f(\sigma_w(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T))$$

from [23]. Since $f(\sigma_w(T)) = \sigma_w(f(T))$ by (i), it follows that

$$\sigma_w(f(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T)).$$

Hence Weyl's theorem holds for $f(T)$. ∎

4. (A, m)-isometries. In this section, we study (A, m)-isometries as a special case of (A, m)-expansive operators. First, we give some spectral properties of (A, m)-isometric operators.

PROPOSITION 4.1. *If $T \in \mathcal{L}(\mathcal{H})$ is an (A, m)-isometric operator for some positive integer m and $0 \notin \sigma_{ap}(A)$, then $\sigma_p(T)^* \subseteq \sigma_p(T^*)$ and $\sigma_{ap}(T)^* \subseteq \sigma_{ap}(T^*)$.*

Proof. Let $z \in \sigma_{ap}(T)$ and $0 \notin \sigma_{ap}(A)$. Then there exists a sequence $\{x_n\}$ of unit vectors in $\mathcal{H}$ such that $\lim_{n\to\infty} \|(T - z)x_n\| = 0$, and we can choose $\delta > 0$ such that $\|Ax_n\| \geq \delta$ for all n. Since $\sigma_{ap}(T) \subseteq \partial\mathbb{D}$ from Proposition 3.4, we have

$$\begin{aligned}
0 &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} A T^j x_n \\
&= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} A (T^j - z^j) x_n + \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} z^j T^{*j} A x_n \\
&= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} A (T^j - z^j) x_n + z^m \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \overline{z}^{m-j} T^{*j} A x_n \\
&= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} A (T^j - z^j) x_n + z^m (T^* - \overline{z})^m A x_n
\end{aligned}$$

for all n. Since $\lim_{n\to\infty}\|T^{*j}A(T^j-z^j)x_n\|=0$ for $j=0,1,\dots,m$, we obtain

$$\|(T^*-\overline{z})^m Ax_n\|=\frac{1}{|z|^m}\left\|\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}A(T^j-z^j)x_n\right\|\to 0 \quad \text{as } n\to\infty.$$

Hence either $\overline{z}\in\sigma_{ap}(T^*)$ or $\lim_{n\to\infty}\|(T^*-\overline{z})^{m-1}Ax_n\|=0$. Since $\|Ax_n\|\geq\delta$, we can show that $\overline{z}\in\sigma_{ap}(T^*)$ inductively. Similarly, $\sigma_p(T)^*\subseteq\sigma_p(T^*)$. ■

Remark. Let $T\in\mathcal{L}(\mathcal{H})$ be (A,m)-isometric for some positive integer m and let $0\notin\sigma_{ap}(A)$. Fix $\lambda\in\mathbb{D}$. Since $\sigma_{le}(T)\subseteq\sigma_{ap}(T)\subseteq\partial\mathbb{D}$ from Proposition 3.4, we know that $T-\lambda$ is semi-Fredholm. Since $\sigma_p(T)\subseteq\partial\mathbb{D}$, we see that $\operatorname{ind}(T-\lambda)\leq 0$.

Next we examine the behavior of the A-covariance

$$\Delta_A(T):=\frac{(-1)^{m-1}}{(m-1)!}B_A^{m-1}(T)$$

when $T\in\mathcal{L}(\mathcal{H})$ is (A,m)-isometric. As explained in [4], for any (A,m)-isometric operator T, we have

$$T^{*k}AT^k=\sum_{n=0}^{m-1}\frac{(-1)^n}{n!}\binom{k}{n}B_A^n(T). \tag{4.1}$$

The identity (4.1) yields the following lemma.

Lemma 4.2 ([4]). *If $T\in\mathcal{L}(\mathcal{H})$ is (A,m)-isometric for some positive integer m, then $\Delta_A(T)\geq 0$.*

We apply Lemma 4.2 to generalize some results of [2].

Proposition 4.3. *If $T\in\mathcal{L}(\mathcal{H})$ is an invertible (A,m)-isometric operator for some positive even integer m, then it is $(A,m-1)$-isometric.*

Proof. Since T^{-1} is (A,m)-isometric, Lemma 4.2 implies that $\Delta_A(T^{-1})\geq 0$. Since m is even, we obtain

$$\Delta_A(T)=-T^{*m-1}\Delta_A(T^{-1})T^{m-1}\leq 0.$$

Hence $\Delta_A(T)=0$ again by Lemma 4.2. ■

Recall that an operator $T\in\mathcal{L}(\mathcal{H})$ is called *finitely cyclic* if there exist a finite number of vectors $x_1,\dots,x_n\in\mathcal{H}$ such that

$$\bigvee\{T^k x_j : k=0,1,\dots,\ j=1,\dots,n\}=\mathcal{H}.$$

For the case $n=1$, we say that T is *cyclic.*

Proposition 4.4. *If $T\in\mathcal{L}(\mathcal{H})$ is a finitely cyclic $(A,2)$-isometric operator, then $\Delta_A(T)=T^*AT-I$ is compact.*

Proof. Let k be any positive integer. Since T is finitely cyclic, so is T^k. Hence there exist $x_1, \ldots, x_n \in \mathcal{H}$ such that

$$\overline{\operatorname{ran}(T^k)} \cup \operatorname{span}\{x_1, \ldots, x_n\} = \mathcal{H},$$

and so $\operatorname{ran}(T^k)^\perp \subseteq \operatorname{span}\{x_1, \ldots, x_n\}$, which means that $\operatorname{ran}(T^k)^\perp$ is finite-dimensional. Let P_k denote the orthogonal projection of $\mathcal{H}$ onto $\overline{\operatorname{ran}(T^k)}$, and put $\varTheta_k := \varDelta_A(T) - P_k \varDelta_A(T) P_k$ for any positive integer k. If $x \in \overline{\operatorname{ran}(T^k)}$, then

$$\varTheta_k x = (I - P_k)\varDelta_A(T)x \in \operatorname{ran}(T^k)^\perp \subseteq \operatorname{span}\{x_1, \ldots, x_n\}.$$

Moreover, for any $x \in \operatorname{ran}(T^k)^\perp$, we can write $x = \sum_{j=1}^n a_j x_j$ for some complex numbers $a_1, \ldots, a_n$, and so

$$\varTheta_k x = \sum_{j=1}^n a_j \varTheta_k x_j \in \operatorname{span}\{\varTheta_k x_1, \ldots, \varTheta_k x_n\}.$$

Therefore, each $\varTheta_k$ has finite rank. Now let $y \in \operatorname{ran}(T^k)$ be given with $y = T^k x$ for some $x \in \mathcal{H}$. Since T is $(A, 2)$-isometric, $\varDelta_A(T) = T^* \varDelta_A(T) T$. Thus we have

$$\begin{aligned} \langle P_k \varDelta_A(T) P_k y, y\rangle &= \langle \varDelta_A(T) y, y\rangle = \langle {T^*}^k \varDelta_A(T) T^k x, x\rangle \\ &= \langle \varDelta_A(T) x, x\rangle. \end{aligned} \tag{4.2}$$

In addition, since it follows from (4.1) that ${T^*}^k A T^k = k\varDelta_A(T) + A$, we get

$$\langle \varDelta_A(T)x, x\rangle = \frac{1}{k}\|A^{1/2}T^k x\|^2 - \frac{1}{k}\|A^{1/2}x\|^2. \tag{4.3}$$

Since $\varDelta_A(T) \geq 0$ by Lemma 4.2, we know that $P_k \varDelta_A(T) P_k \geq 0$, and so (4.2) and (4.3) yield

$$\begin{aligned} \|(P_k \varDelta_A(T) P_k)^{1/2} y\|^2 &= \langle P_k \varDelta_A(T) P_k y, y\rangle = \langle \varDelta_A(T)x, x\rangle \\ &= \frac{1}{k}\|A^{1/2}T^k x\|^2 - \frac{1}{k}\|A^{1/2}x\|^2 \leq \frac{1}{k}\|A^{1/2}\|^2\|y\|^2. \end{aligned}$$

This gives $\lim_{k\to\infty} \|P_k \varDelta_A(T) P_k\| = 0$. Hence $\varDelta_A(T)$ is the uniform limit of the sequence $\{\varTheta_k\}$ of operators of finite rank, and so $\varDelta_A(T)$ is compact. ∎

Next we show that every operator $T \in \mathcal{L}(\mathcal{H})$ which is $(T^*T, 2)$-isometric has a scalar extension.

LEMMA 4.5. *Every* 2*-isometric operator is subscalar of order* 4.

Proof. Let $T \in \mathcal{L}(\mathcal{H})$ be 2-isometric and choose a positive number σ with $\|T^*T - I\| \leq \sigma$. By [3, Proposition 5.12 and Theorem 5.80], T has a Brownian unitary extension B of the form

$$B = \begin{pmatrix} V & \sigma E \\ 0 & U \end{pmatrix}$$

where V is an isometry, U is unitary, and E is a Hilbert space isomorphism onto $\ker(V^*)$. Since V and U are hyponormal, B is subscalar of order 4 by [19]. Since T is the restriction of B to an invariant subspace, it is subscalar of order 4. ∎

THEOREM 4.6. *If $T \in \mathcal{L}(\mathcal{H})$ is $(T^*T, 2)$-isometric, then it is subscalar of order* 8.

Proof. Since T is $(T^*T, 2)$-isometric, $T^{*3}T^3 - 2T^{*2}T^2 + T^*T = 0$. Setting $\mathcal{M} = \overline{\operatorname{ran}(T)}$, we can write

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \quad \text{on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$$

where $T_1 = T|_{\mathcal{M}}$ is 2-isometric and T_2 is a bounded linear operator (see the proof of Theorem 3.14). For any bounded open disk D in $\mathbb{C}$ containing $\sigma(T)$, define the map $V : \mathcal{M} \oplus \mathcal{M}^\perp \to H(D)$ by

$$Vh = 1 \mathbin{\widetilde{\otimes}} h \ \left(\equiv 1 \otimes h + \overline{(T - z)W^8(D, \mathcal{M}) \oplus W^8(D, \mathcal{M}^\perp)}\right)$$

where

$$H(D) := W^8(D, \mathcal{M}) \oplus W^8(D, \mathcal{M}^\perp)/\overline{(T - z)W^8(D, \mathcal{M}) \oplus W^8(D, \mathcal{M}^\perp)}$$

and $1 \otimes h$ denotes the constant function sending any $z \in D$ to $h \in \mathcal{M} \oplus \mathcal{M}^\perp$.

CLAIM. *V is one-to-one and has closed range.*

Let $f_n = f_{n,1} \oplus f_{n,2} \in W^8(D, \mathcal{M}) \oplus W^8(D, \mathcal{M}^\perp)$ and $h_n = h_{n,1} \oplus h_{n,2} \in \mathcal{M} \oplus \mathcal{M}^\perp$ be sequences such that

$$\lim_{n\to\infty} \|(T - z)f_n + 1 \otimes h_n\|_{\oplus_1^2 W^8} = 0. \tag{4.4}$$

This implies that

$$\begin{aligned} &\lim_{n\to\infty} \|(T_1 - z)f_{n,1} + T_2 f_{n,2} + 1 \otimes h_{n,1}\|_{W^8} = 0, \\ &\lim_{n\to\infty} \|z f_{n,2} - 1 \otimes h_{n,2}\|_{W^8} = 0. \end{aligned} \tag{4.5}$$

By the definition of the norm for the Sobolev space, (4.5) implies that

$$\begin{aligned} &\lim_{n\to\infty} \|(T_1 - z)\bar{\partial}^i f_{n,1} + T_2 \bar{\partial}^i f_{n,2}\|_{2,D} = 0, \\ &\lim_{n\to\infty} \|z\bar{\partial}^i f_{n,2}\|_{2,D} = 0, \end{aligned} \tag{4.6}$$

for $i = 1, \ldots, 8$. Since the zero operator is hyponormal, it follows from [26] that

$$\lim_{n\to\infty} \|(I - P)\bar{\partial}^i f_{n,2}\|_{2,D} = 0 \quad \text{for } i = 1, \ldots, 6, \tag{4.7}$$

where P denotes the orthogonal projection of $L^2(D, \mathcal{H})$ onto $A^2(D, \mathcal{H})$. Hence we deduce from (4.6) that

$$\lim_{n\to\infty} \|zP\bar{\partial}^i f_{n,2}\|_{2,D} = 0 \quad \text{for } i = 1, \ldots, 6.$$

Since the zero operator has the property (β), we have

$$\lim_{n\to\infty} \|P\bar{\partial}^i f_{n,2}\|_{2,D_0} = 0 \quad \text{for } i = 1, \dots, 6, \tag{4.8}$$

where $\sigma(T) \subsetneq D_0 \subsetneq D$. Combining (4.7) and (4.8), we have

$$\lim_{n\to\infty} \|\bar{\partial}^i f_{n,2}\|_{2,D_0} = 0 \quad \text{for } i = 1, \dots, 6.$$

Thus (4.6) ensures that

$$\lim_{n\to\infty} \|(T_1 - z)\bar{\partial}^i f_{n,1}\|_{2,D_0} = 0 \quad \text{for } i = 1, \dots, 6.$$

Since T_1 is 2-isometric, it is subscalar of order 4 by Lemma 4.5. Then an application of some results of [13] yields

$$\lim_{n\to\infty} \|\bar{\partial}^i f_{n,1}\|_{2,D_0} = 0 \quad \text{for } i = 1, 2,$$

which gives

$$\lim_{n\to\infty} \|z\bar{\partial}^i f_{n,1}\|_{2,D_0} = 0 \quad \text{for } i = 1, 2.$$

By [26], we get

$$\lim_{n\to\infty} \|(I - P) f_{n,1}\|_{2,D_0} = 0. \tag{4.9}$$

Therefore (4.5), (4.7), and (4.9) imply that

$$\lim_{n\to\infty} \|(T - z)Pf_n + 1\otimes h_n\|_{2,D_0} = 0$$

where $Pf_n := Pf_{n,1} \oplus Pf_{n,2}$. Let Γ be a closed curve in D_0 surrounding $\sigma(T)$. Then $\lim_{n\to\infty} \|Pf_n(z) + (T - z)^{-1} h_n\| = 0$ uniformly on Γ. Applying the Riesz–Dunford functional calculus, we obtain

$$\lim_{n\to\infty} \left\| \frac{1}{2\pi i} \int_{\Gamma} Pf_n(z)\, dz + h_n \right\| = 0.$$

But $\frac{1}{2\pi i} \int_{\Gamma} Pf_n(z)\, dz = 0$ by Cauchy's theorem and hence $\lim_{n\to\infty} \|h_n\| = 0$, which completes the proof of our claim.

Now the class of a vector f or an operator S on $H(D)$ will be denoted by $\widetilde{f}$, respectively $\widetilde{S}$. Let M be the operator of multiplication by z on $W^8(D, \mathcal{H})$. As noted in Section 2, M is a scalar operator of order 8 and has a spectral distribution Φ_M. Since the range of $T - z$ is invariant under M, $\widetilde{M}$ can be well-defined. Moreover, consider the spectral distribution $\Phi_M : C_0^8(\mathbb{C}) \to W^8(D, \mathcal{M}) \oplus W^8(D, \mathcal{M}^{\perp})$ given by $\Phi_M(\varphi) f = \varphi f$ for $\varphi \in C_0^8(\mathbb{C})$ and $f \in W^8(D, \mathcal{M}) \oplus W^8(D, \mathcal{M}^{\perp})$. Then the spectral distribution Φ_M of M commutes with $T - z$, and so $\widetilde{M}$ is still a scalar operator of order 8 with $\widetilde{\Phi_M}$ as a spectral distribution. Since

$$VTh = \widetilde{1 \otimes Th} = \widetilde{z \otimes h} = \widetilde{M}(\widetilde{1 \otimes h}) = \widetilde{M}Vh$$

for all $h \in \mathcal{M} \oplus \mathcal{M}^{\perp}$, we have $VT = \widetilde{M}V$. In particular, $\operatorname{ran}(V)$ is invariant for $\widetilde{M}$. Furthermore, $\operatorname{ran}(V)$ is closed by the above claim. So $\operatorname{ran}(V)$ is an invariant subspace of the scalar operator $\widetilde{M}$. Since T is similar to the restriction $\widetilde{M}|_{\text{ran}(V)}$, we conclude that T is subscalar of order 8. ∎

Recall that an operator $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is called a *quasiaffinity* if it has trivial kernel and dense range. An operator $S \in \mathcal{L}(\mathcal{H})$ is said to be a *quasiaffine transform* of an operator $T \in \mathcal{L}(\mathcal{K})$ if there is a quasiaffinity $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that $XS = TX$. Furthermore, operators $S \in \mathcal{L}(\mathcal{H})$ and $T \in \mathcal{L}(\mathcal{K})$ are *quasisimilar* if there are quasiaffinities $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $Y \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ such that $XS = TX$ and $SY = YT$.

For an operator $T \in \mathcal{L}(\mathcal{H})$, we define a *spectral maximal space* of T to be a T-invariant subspace $\mathcal{M}$ of $\mathcal{H}$ with the property that M contains any T-invariant subspace $\mathcal{N}$ of $\mathcal{H}$ such that $\sigma(T|_{\mathcal{N}}) \subseteq \sigma(T|_{\mathcal{M}})$.

COROLLARY 4.7. *If $T \in \mathcal{L}(\mathcal{H})$ is $(T^*T, 2)$-isometric, then the following statements hold:*

(i) *If $T \in \mathcal{L}(\mathcal{H})$ is $(T^*T, 2)$-isometric, then it has a nontrivial invariant subspace.*
(ii) *T has the property (β), Dunford's property* (C), *and the single-valued extension property.*
(iii) *$H_T(F)$ is a spectral maximal subspace of T and $\sigma(T|_{H_T(F)}) \subseteq \sigma(T) \cap F$ for any closed set F in $\mathbb{C}$.*
(iv) *If $S \in \mathcal{L}(\mathcal{H})$ is an $(S^*S, 2)$-isometric operator that is quasisimilar to T, then $\sigma(S) = \sigma(T)$ and $\sigma_e(S) = \sigma_e(T)$.*

Proof. (i) By the proof of Theorem 4.6, we put

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \quad \text{on } \mathcal{H} = \overline{\text{ran}(T)} \oplus \ker(T^*)$$

where $T_1 = T|_{\overline{\text{ran}(T)}}$ is 2-isometric and T_2 is a bounded linear operator. From [2], either $\sigma(T_1) \subseteq \partial\mathbb{D}$ or $\sigma(T_1) = \overline{\mathbb{D}}$. If $\sigma(T_1) \subseteq \partial\mathbb{D}$, then T_1 is unitary by [2]. Thus T_1 has a nontrivial invariant subspace, and so is T clearly. If $\sigma(T_1) = \overline{\mathbb{D}}$, then we get from [17] that $\sigma(T) = \sigma(T_1) \cup \{0\} = \overline{\mathbb{D}}$. Then $\sigma(T)$ has nonempty interior. Since T is subscalar by Theorem 4.6, it has a nontrivial invariant subspace from [12].

(ii) From section one, it suffices to prove that T has the property (β). Since the property (β) is transmitted from an operator to its restrictions to closed invariant subspaces, we are reduced by Theorem 4.5 to the case of a scalar operator. Since every scalar operator has the property (β) (see [26]), T has the property (β).

(iii) Since T has Dunford's property (C) by (ii), the assertion follows from [10].

(iv) The proof follows from (ii) and [27]. ∎

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