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Stability of commuting maps and Lie maps

by

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Abstract. Let A be an ultraprime Banach algebra. We prove that each approximately commuting continuous linear (or quadratic) map on A is near an actual commuting continuous linear (resp. quadratic) map on A. Furthermore, we use this analysis to study how close are approximate Lie isomorphisms and approximate Lie derivations to actual Lie isomorphisms and Lie derivations, respectively.

1. Introduction. Commuting maps give rise to the most basic examples of functional identities. The initial results on such maps were obtained at the beginning of the 90's by M. Brešar in a series of papers. These results have been extremely influential and they have initiated a lot of activity on this subject by numerous authors. For an excellent account of this topic, which also includes a comprehensive list of references, we refer the reader to the survey paper by Brešar [6]. The main reason for analysing commuting maps is that they are applicable to many areas. This fact was noticed in [5] for the first time. In that paper Brešar made the first breakthrough in settling Herstein's conjectures on Lie maps for prime rings by using commuting quadratic maps. Incidentally, let us mention that the final step in settling Herstein's conjectures on Lie maps was made in a trilogy of papers by Beidar, Brešar, Chebotar, and Martindale. For the general theory of functional identities and their applications we refer the reader to [7].

The goal of the present paper is to obtain metric versions of the classical algebraic results on commuting maps and their applications to Lie theory.

Recall that a map T from a ring $\mathcal{R}$ into itself is said to be *commuting* if $T(a)a = aT(a)$ for each $a \in \mathcal{R}$. A direct calculation shows that if T is a continuous linear (or quadratic) operator on a Banach algebra A, then

$$\sup\{\|T(a)a - aT(a)\| : a \in A,\ \|a\| = 1\} \leq 2\|T - S\|$$

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for each commuting continuous linear (resp. quadratic) operator S on A. In Section 2 we show that the condition that $\sup_{a\in A,\,\|a\|=1}\|T(a)a-aT(a)\|$ is small implies that T is close to some commuting map. A natural framework for this question is the class of ultraprime Banach algebras, in which algebraic descriptions of commuting and related maps get particularly nice forms.

Section 3 is devoted to approximate Lie isomorphisms and approximate Lie derivations. Given Banach algebras A and B and continuous linear maps $\Phi\colon B\to A$ and $\Delta\colon A\to A$, we measure the Lie multiplicativity of Φ and the Lie derivativity of Δ through the constants

$$\begin{aligned}\operatorname{lmult}(\Phi)&=\sup\{\|\Phi([a,b])-[\Phi(a),\Phi(b)]\|:a,b\in B,\ \|a\|=\|b\|=1\},\\ \operatorname{lder}(\Delta)&=\sup\{\|\Delta([a,b])-[\Delta(a),b]-[a,\Delta(b)]\|:a,b\in A,\ \|a\|=\|b\|=1\},\end{aligned}$$

respectively. We are concerned with whether the conditions of $\operatorname{lmult}(\Phi)$ and $\operatorname{lder}(\Delta)$ being small imply that Φ and Δ are near actual Lie homomorphisms and Lie derivations, respectively. A related problem on approximate Jordan isomorphisms naturally appeared in the recent article [2] on approximately spectrum-preserving maps. In [2] we showed that the classical Herstein theorem on Jordan epimorphisms is stable in the sense that approximate Jordan epimorphisms are either approximate epimorphisms or approximate anti-epimorphisms. Here we use similar techniques, but it should be mentioned that commuting and Lie maps are more demanding in the technical aspect because of the presence of central maps. Let us also mention that in [3] we were concerned with the stability of Herstein's theorems on Jordan epimorphisms and Jordan derivations and the question of giving quantitative estimates of this phenomenon.

In the final section, combining the results obtained with those from [1] and [11] on the stability of homomorphisms and derivations we will get ultimate results for $\mathcal{L}(H)$, where H is a Hilbert space.

1.1. Preliminaries. All Banach algebras and Banach spaces which we consider throughout this paper are assumed to be complex.

Let X be a Banach space. By X^* we denote the topological dual space of X. For a Banach space Y, let $\mathcal{L}(X,Y)$ denote the Banach space of all continuous linear operators from X into Y. We write $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$. We write $\mathcal{L}^2(X)$ for the Banach space of all continuous bilinear maps from $X\times X$ into X. By a *continuous quadratic map* on X we mean a map $Q\colon X\to X$ of the form $Q(x)=F(x,x)$ $(x\in X)$ with $F\in\mathcal{L}^2(X)$. We write $\mathcal{Q}(X)$ for the set of all continuous quadratic maps on X.

We write $A+\mathbb{C}\mathbf{1}$ for the "conditional unitization" of a given Banach algebra A, i.e., $A+\mathbb{C}\mathbf{1}=A$ if A has an identity and $A+\mathbb{C}\mathbf{1}$ is formed by adjoining an identity to A otherwise.

Ultraprimeness is a metric version of primeness which was introduced by M. Mathieu in [12]. Let A be a Banach algebra. For each $a, b \in A$, we write $M_{a,b}$ for the two-sided multiplication operator on A defined by

$$M_{a,b}(x) = axb \quad (x \in A).$$

Recall that A is *prime* if $M_{a,b} = 0$ implies $a = 0$ or $b = 0$. We define

$$\kappa(A) = \inf\{\|M_{a,b}\| : a, b \in A, \|a\| = \|b\| = 1\}.$$

The Banach algebra A is said to be *ultraprime* if $\kappa(A) > 0$. It is clear that each finite-dimensional prime Banach algebra is ultraprime. For each Banach space X, the Banach algebra $\mathcal{L}(X)$ is ultraprime and, more generally, every closed subalgebra A of $\mathcal{L}(X)$ containing the finite rank operators is ultraprime with $\kappa(A) = 1$ (cf. [12]). Every prime C^*-algebra is ultraprime [13].

Throughout the paper, we will use ultraproducts of sequences of Banach algebras as an important tool. From now on, $\mathcal{U}$ is a fixed free ultrafilter on $\mathbb{N}$. For a sequence of Banach spaces (X_n), we write $(X_n)^{\mathcal{U}}$ for the ultraproduct of (X_n) with respect to $\mathcal{U}$. This is the quotient Banach space $\ell^\infty(\mathbb{N}, X_n)/\mathcal{N}_{\mathcal{U}}$, where $\ell^\infty(\mathbb{N}, X_n)$ stands for the space of all bounded sequences (x_n) with $x_n \in X_n$ $(n \in \mathbb{N})$ and $\mathcal{N}_{\mathcal{U}} = \{x \in \ell^\infty(\mathbb{N}, X_n) : \lim_{\mathcal{U}} \|x_n\| = 0\}$. With a slight abuse of notation we continue to write (x_n) for the equivalence class it represents; of course, it can be checked that any definition we make is independent of the choice of representative of the equivalence class. The norm on $(X_n)^{\mathcal{U}}$ is given by $\|\mathbf{x}\| = \lim_{\mathcal{U}} \|x_n\|$ for each $\mathbf{x} = (x_n) \in (X_n)^{\mathcal{U}}$. If (A_n) is a sequence of Banach algebras, then the ultraproduct $(A_n)^{\mathcal{U}}$ turns into a Banach algebra with respect to the obvious product $\mathbf{ab} = (a_n b_n)$ for all $\mathbf{a} = (a_n), \mathbf{b} = (b_n) \in (A_n)^{\mathcal{U}}$. For each $n \in \mathbb{N}$, let $T_n \in \mathcal{L}(X_n, Y_n)$ be given for some Banach space Y_n and assume that $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$. Then we can define $(T_n)^{\mathcal{U}} \in \mathcal{L}((X_n)^{\mathcal{U}}, (Y_n)^{\mathcal{U}})$ according to the rule $(x_n) \mapsto (T_n(x_n))$. Moreover,

$$\|(T_n)^{\mathcal{U}}\| = \lim_{\mathcal{U}} \|T_n\|.$$

In particular, if $f_n \in X_n^*$ $(n \in \mathbb{N})$ are given such that $\sup_{n\in\mathbb{N}} \|f_n\| < \infty$, then $(f_n)^{\mathcal{U}} \in ((X_n)^{\mathcal{U}})^*$ is defined through $(x_n) \mapsto \lim_{\mathcal{U}} f_n(x_n)$. In the same manner, if $F_n \in \mathcal{L}^2(X_n)$ and $Q_n \in \mathcal{Q}(X_n)$ $(n \in \mathbb{N})$ are given with the property that $\sup_{n\in\mathbb{N}} \|F_n\| < \infty$ and $\sup_{n\in\mathbb{N}} \|Q_n\| < \infty$, then $(F_n)^{\mathcal{U}} \in \mathcal{L}^2((X_n)^{\mathcal{U}})$ and $(Q_n)^{\mathcal{U}} \in \mathcal{Q}((X_n)^{\mathcal{U}})$ are defined according to the rules $((x_n), (y_n)) \mapsto (F_n(x_n, y_n))$ and $(x_n) \mapsto (Q_n(x_n))$, respectively. Moreover,

$$\|(F_n)^{\mathcal{U}}\| = \lim_{\mathcal{U}} \|F_n\|, \tag{1.1}$$

$$\|(Q_n)^{\mathcal{U}}\| = \lim_{\mathcal{U}} \|Q_n\|. \tag{1.2}$$

We refer the reader to [10] for the basics of ultraproducts.

2. Stability of commuting maps. Let $\mathcal{R}$ be a ring. In what follows, we write $[a,b]=ab-ba$ for all $a,b\in\mathcal{R}$ and we denote by $\mathcal{Z}(\mathcal{R})$ the centre of $\mathcal{R}$. A map $T\colon \mathcal{R}\to\mathcal{R}$ is said to be *commuting* if

$$[T(a),a]=0 \quad (a\in\mathcal{R}). \tag{2.1}$$

2.1. Commuting linear maps. Typical examples of commuting additive maps are provided by the maps of the form

$$T(a)=\lambda a+\mu(a) \quad (a\in\mathcal{R}) \tag{2.2}$$

with $\lambda\in\mathcal{Z}(\mathcal{R})$ and $\mu\colon\mathcal{R}\to\mathcal{Z}(\mathcal{R})$ being an additive map. The usual issue when treating commuting additive maps is to determine which assumptions on the ring $\mathcal{R}$ should be required in order to conclude that all of them are of the standard form (2.2). M. Brešar found out that actually every commuting additive map of a prime ring of characteristic different from 2 is of the form (2.2) if we allow λ and $\mu(a)$ ($a\in\mathcal{R}$) to belong to the so-called *extended centroid* of $\mathcal{R}$ rather than to the centre of $\mathcal{R}$ [7, Corollary 5.28]. This is an enlargement of $\mathcal{Z}(\mathcal{R})$ and we refer the reader to [7, Appendix A] for the details about this concept. We will neglect this ring-theoretical device because of the following property.

Remark 2.1. The centre of any ultraprime Banach algebra A is trivial (either zero or one-dimensional) [12, Proposition 3.4]. Further, its extended centroid is nothing but the complex field [12, Corollary 4.7]. On account of [7, Corollary 5.28], any commuting linear map $T\colon A\to A$ is of the form (2.2) with $\lambda\in\mathbb{C}$ and $\mu\colon A\to\mathcal{Z}(A)$ linear (and clearly μ is continuous whenever T is continuous).

We now turn our attention to Banach algebras. We measure to what extent a continuous linear operator T from a Banach algebra A into itself satisfies condition (2.1) by considering the constant

$$\operatorname{com}(T)=\sup\{\|[T(a),a]\| : a\in A,\ \|a\|=1\}.$$

The subset of $\mathcal{L}(A)$ consisting of commuting maps is denoted by $\mathrm{LCom}(A)$. This is a closed linear subspace of $\mathcal{L}(A)$. Note that the maps $T\mapsto\operatorname{com}(T)$ and $T\mapsto\operatorname{dist}(T,\mathrm{LCom}(A))$ are seminorms on $\mathcal{L}(A)$ which vanish precisely on $\mathrm{LCom}(A)$. Moreover, if $T\in\mathcal{L}(A)$ and $S\in\mathrm{LCom}(A)$, then

$$\|[T(a),a]\|=\|[(T-S)(a),a]\|\le 2\|T-S\|\,\|a\|^2$$

for each $a\in A$ and therefore

$$\operatorname{com}(T)\le 2\operatorname{dist}(T,\mathrm{LCom}(A)) \quad (T\in\mathcal{L}(A)).$$

We are interested in whether there is a constant $M>0$ such that

$$\operatorname{dist}(T,\mathrm{LCom}(A))\le M\operatorname{com}(T) \quad (T\in\mathcal{L}(A)).$$

It is worth pointing out that, for each $T \in \mathcal{L}(A)$, $\mathrm{com}(T)$ is nothing but the norm of the quadratic map $a \mapsto [T(a), a]$ on A. Equality (1.2) then gives the following useful property.

LEMMA 2.2. *Let (A_n) be a sequence of Banach algebras and assume that $T_n \in \mathcal{L}(A_n)$ $(n \in \mathbb{N})$ are given with $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$. Then*

$$\mathrm{com}((T_n)^{\mathcal{U}}) = \lim_{\mathcal{U}} \mathrm{com}(T_n).$$

LEMMA 2.3. *Let (A_n) be a sequence of Banach algebras. Then*

$$\kappa((A_n)^{\mathcal{U}}) = \lim_{\mathcal{U}} \kappa(A_n).$$

Proof. Write $\mathbf{A} = (A_n)^{\mathcal{U}}$. Let $\mathbf{a} = (a_n), \mathbf{b} = (b_n) \in \mathbf{A}$. Then $M_{\mathbf{a},\mathbf{b}} = (M_{a_n,b_n})$ and therefore

$$\|M_{\mathbf{a},\mathbf{b}}\| = \lim_{\mathcal{U}} \|M_{a_n,b_n}\| \geq \lim_{\mathcal{U}}(\kappa(A_n)\|a_n\|\,\|b_n\|) = \lim_{\mathcal{U}} \kappa(A_n)\|\mathbf{a}\|\,\|\mathbf{b}\|.$$

This clearly implies that $\kappa(\mathbf{A}) \geq \lim_{\mathcal{U}} \kappa(A_n)$.

In order to prove the reverse inequality, for each $n\in\mathbb{N}$, we pick $a_n, b_n \in A_n$ with $\|a_n\| = \|b_n\| = 1$ and $\|M_{a_n,b_n}\| \leq \kappa(A_n) + 1/n$. We then consider $\mathbf{a}, \mathbf{b} \in \mathbf{A}$ given by $\mathbf{a} = (a_n)$ and $\mathbf{b} = (b_n)$. We have

$$\|M_{\mathbf{a},\mathbf{b}}\| = \lim_{\mathcal{U}} \|M_{a_n,b_n}\| \leq \lim_{\mathcal{U}}(\kappa(A_n) + 1/n) = \lim_{\mathcal{U}} \kappa(A_n). \blacksquare$$

LEMMA 2.4. *Let (A_n) be a sequence of Banach algebras such that $(A_n)^{\mathcal{U}}$ has an identity. Then $\{n \in \mathbb{N} : A_n$ has an identity$\} \in \mathcal{U}$.*

Proof. Let $\mathbf{1} = (u_n)$ be the identity of $\mathbf{A} = (A_n)^{\mathcal{U}}$. For every $n \in \mathbb{N}$, let L_{u_n} and R_{u_n} denote the operators of left and right multiplication by u_n on A_n, respectively. Then (L_{u_n}) and (R_{u_n}) are the identity operator on $\mathbf{A}$ and therefore $\lim_{\mathcal{U}} \|I_{A_n} - L_{u_n}\| = \lim_{\mathcal{U}} \|I_{A_n} - R_{u_n}\| = 0$. Accordingly, $\{n \in \mathbb{N} : \|I_{A_n} - L_{u_n}\|, \|I_{A_n} - R_{u_n}\| < 1\} \in \mathcal{U}$. On the other hand, the property $\|I_{A_n} - L_{u_n}\|, \|I_{A_n} - R_{u_n}\| < 1$ implies that both L_{u_n} and R_{u_n} are bijective linear maps from A_n onto itself, which, according to [8, Proposition 2], implies that A_n has an identity. $\blacksquare$

THEOREM 2.5. *For each $K > 0$ there exists $M > 0$ such that*

$$\mathrm{dist}(T, \mathrm{LCom}(A)) \leq M\,\mathrm{com}(T)$$

for each Banach algebra A with $\kappa(A) \geq K$ and $T \in \mathcal{L}(A)$.

Proof. Assume towards a contradiction that the assertion in the theorem is false. Then there exist a constant $K > 0$, a sequence of Banach algebras (A_n) with $\kappa(A_n) \geq K$ $(n \in \mathbb{N})$ and a sequence (F_n) with $F_n \in \mathcal{L}(A_n)$ $(n \in \mathbb{N})$ such that $\mathrm{dist}(F_n, \mathrm{LCom}(A_n)) > n\,\mathrm{com}(F_n)$ $(n \in \mathbb{N})$. Set $G_n = \mathrm{dist}(F_n, \mathrm{LCom}(A_n))^{-1} F_n$ $(n \in \mathbb{N})$. Then

$$\mathrm{dist}(G_n, \mathrm{LCom}(A_n)) = 1 \quad (n \in \mathbb{N}) \tag{2.3}$$

and

$$\operatorname{com}(G_n) < 1/n \quad (n \in \mathbb{N}). \tag{2.4}$$

Since the sequence (G_n) is not necessarily bounded, we replace it with a bounded one that still satisfies both (2.3) and (2.4). To this end, on account of (2.3), we can choose a sequence (H_n) with $H_n \in \operatorname{LCom}(A_n)$ $(n \in \mathbb{N})$ and $\lim_{n\to\infty} \|G_n - H_n\| = 1$. We then define $T_n = G_n - H_n$ for each $n \in \mathbb{N}$. It is clear that $\operatorname{dist}(T_n, \operatorname{LCom}(A_n)) = \operatorname{dist}(G_n, \operatorname{LCom}(A_n))$ $(n \in \mathbb{N})$ and then (2.3) gives

$$\operatorname{dist}(T_n, \operatorname{LCom}(A_n)) = 1 \quad (n \in \mathbb{N}). \tag{2.5}$$

Furthermore, $\operatorname{com}(T_n) = \operatorname{com}(G_n)$ $(n \in \mathbb{N})$ and therefore (2.4) yields

$$\lim_{n\to\infty} \operatorname{com}(T_n) = 0. \tag{2.6}$$

We now consider $\mathbf{A} = (A_n)^{\mathcal{U}}$ and $\mathbf{T} = (T_n)^{\mathcal{U}} \in \mathcal{L}(\mathbf{A})$.

On account of Lemma 2.2 and (2.6), $\operatorname{com}(\mathbf{T}) = 0$ and therefore $\mathbf{T}$ is commuting. According to Lemma 2.3, $\kappa(\mathbf{A}) = \lim_{\mathcal{U}} \kappa(A_n) \geq K$ and hence the Banach algebra $\mathbf{A}$ is ultraprime. Remark 2.1 then gives $\lambda \in \mathbb{C}$ and a continuous linear map $\Phi\colon \mathbf{A} \to \mathcal{Z}(\mathbf{A})$ such that

$$\mathbf{T}(\mathbf{a}) = \lambda\mathbf{a} + \Phi(\mathbf{a}) \quad (\mathbf{a} \in \mathbf{A}). \tag{2.7}$$

Our next goal is to show that $\Phi = (\mu_n)^{\mathcal{U}}$ for some bounded sequence (μ_n) of continuous linear maps $\mu_n\colon A_n \to \mathcal{Z}(A_n)$ $(n \in \mathbb{N})$. We first assume that $\mathbf{A}$ does not have an identity. Then $\mathcal{Z}(\mathbf{A}) = \{0\}$ so that $\Phi = 0$ and hence we can take $\mu_n = 0$ for each $n \in \mathbb{N}$. We now assume that $\mathbf{A}$ has an identity. Let $\mathbf{1}$ be the identity of $\mathbf{A}$ and let $\varphi\colon \mathbf{A} \to \mathbb{C}$ be a continuous linear functional such that $\Phi(\mathbf{a}) = \varphi(\mathbf{a})\mathbf{1}$ $(\mathbf{a} \in \mathbf{A})$. On account of Lemma 2.4, we have $U = \{n \in \mathbb{N} : A_n \text{ has an identity}\} \in \mathcal{U}$. For each $n \in U$, let $\mathbf{1}_n$ be the identity of A_n. Then $\kappa(A_n)\|\mathbf{1}_n\|^2 \leq \|M_{\mathbf{1}_n,\mathbf{1}_n}\| = 1$ and so $\|\mathbf{1}_n\| \leq K^{-1/2}$ for each $n \in U$. Define $(u_n) \in \mathbf{A}$ by $u_n = \mathbf{1}_n$ for each $n \in U$ and $u_n = 0$ elsewhere. Then $\mathbf{1} = (u_n)$ and (2.7) can be written as

$$\lim_{\mathcal{U}} \|T_n(a_n) - \lambda a_n - \varphi(\mathbf{a})u_n\| = 0 \quad (\mathbf{a} = (a_n) \in \mathbf{A}). \tag{2.8}$$

For each $n \in U$, let $f_n\colon A_n \to \mathbb{C}$ be a continuous linear functional such that $f_n(u_n) = 1$ and $\|f_n\| = \|u_n\|^{-1} \leq 1$. For each $n \in \mathbb{N} \setminus U$ we consider f_n to be the zero functional on A_n. Further, we define a bounded sequence (φ_n) of continuous linear functionals $\varphi_n\colon A_n \to \mathbb{C}$ by

$$\varphi_n(a) = f_n(T_n(a)) - \lambda f_n(a) \quad (a \in A_n,\ n \in \mathbb{N}).$$

Our next objective is to prove that $\varphi = (\varphi_n)^{\mathcal{U}}$, which clearly implies that $\Phi = (\mu_n)^{\mathcal{U}}$, where $\mu_n\colon A_n \to \mathcal{Z}(A_n)$ is defined by $\mu_n(a) = \varphi_n(a)u_n$ for all

$a \in A_n$ and $n \in \mathbb{N}$. Let $\mathbf{a} = (a_n) \in \mathbf{A}$. Then for each $n \in U$,

$$\begin{aligned}|\varphi_n(a_n) - \varphi(\mathbf{a})| &= \left|f_n(T_n(a_n) - \lambda a_n - \varphi(\mathbf{a})u_n)\right| \\ &\le \|T_n(a_n) - \lambda a_n - \varphi(\mathbf{a})u_n\|\end{aligned}$$

and hence (2.8) yields $\varphi(\mathbf{a}) = \lim_{\mathcal{U}} \varphi_n(a_n)$, as claimed.

Finally, (2.7) now reads

$$\lim_{\mathcal{U}} \|T_n - \lambda I_{A_n} - \mu_n\| = 0.$$

Since the map $\lambda I_{A_n} + \mu_n$ lies in $\mathrm{LCom}(A_n)$ for each $n \in \mathbb{N}$, it follows that

$$\lim_{\mathcal{U}} \mathrm{dist}(T_n, \mathrm{LCom}(A_n)) \le \lim_{\mathcal{U}} \|T_n - \lambda I_{A_n} - \mu_n\| = 0,$$

which contradicts (2.5). ■

2.2. Commuting quadratic maps. M. Brešar characterized commuting traces of biadditive maps on a ring $\mathcal{R}$ and applied the result obtained to describe commutativity preserving maps, Lie isomorphisms, and Lie derivations [5]. Typical examples of commuting traces of biadditive maps on a ring $\mathcal{R}$ are provided by the maps of the form

$$Q(a) = \lambda a^2 + \mu(a)a + \nu(a) \quad (a \in \mathcal{R}) \tag{2.9}$$

with $\lambda \in \mathcal{Z}(\mathcal{R})$, $\mu\colon \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ an additive map, and $\nu\colon \mathcal{R} \to \mathcal{Z}(\mathcal{R})$ the trace of a biadditive map. The standard problem in this context is to find whether every commuting trace of a biadditive map on $\mathcal{R}$ is given by the formula (2.9). It turns out that this is indeed the case if $\mathcal{R}$ is a prime ring of characteristic different from 2 and we allow λ and $\mu(a), \nu(a)$ $(a \in \mathcal{R})$ to belong to the extended centroid of $\mathcal{R}$ rather than to the centre of $\mathcal{R}$ [7, Theorem 5.32]. On account of the central closability, for ultraprime Banach algebras the above mentioned result reads as follows.

Proposition 2.6. *Let A be an ultraprime Banach algebra and let $Q \in \mathcal{Q}(A)$ be a commuting map. Then there exist $\lambda \in \mathbb{C}$, $\mu \in A^*$, and $\nu \in \mathcal{Q}(A)$ with $\nu(A) \subset \mathcal{Z}(A)$ such that $Q(a) = \lambda a^2 + \mu(a)a + \nu(a)$ for each $a \in A$.*

Proof. If A is commutative, then A is isomorphic to $\mathbb{C}$ and the decomposition of Q obviously holds true.

We now turn to the case where A is not commutative. On account of Remark 2.1 and [7, Theorem 5.32], Q is of the form (2.9), where $\lambda \in \mathbb{C}$, $\mu\colon A \to \mathbb{C}$ is a linear functional, and $\nu\colon A \to \mathcal{Z}(A)$ is a quadratic map. In order to show that both μ and ν are continuous we pick $u, v \in A$ with $[u, v] \neq 0$ and we now observe that

$$\mu(a)[u, v] = \frac{1}{2}[Q(a + u) - Q(a - u) - 2\lambda(au + ua) - 2\mu(u)a, v] \quad (a \in A),$$

which shows that μ is continuous. Since $\nu(a) = Q(a) - \lambda a^2 - \mu(a)a$ $(a \in A)$, it follows that ν is also continuous. ■

Let A be a Banach algebra and $Q \in \mathcal{Q}(A)$. The constant

$$\operatorname{com}(Q) = \sup\{\|[Q(a), a]\| : a \in A,\ \|a\| = 1\}$$

still makes sense and it measures to what extent Q satisfies condition (2.1). The subset of $\mathcal{Q}(A)$ consisting of commuting maps is denoted by $\mathrm{QCom}(A)$. This is a closed linear subspace of $\mathcal{Q}(A)$. Note that the maps $Q \mapsto \operatorname{com}(Q)$ and $Q \mapsto \operatorname{dist}(Q, \mathrm{QCom}(A))$ are seminorms on $\mathcal{Q}(A)$ which vanish precisely on $\mathrm{QCom}(A)$. Moreover, if $Q \in \mathcal{Q}(A)$ and $Q' \in \mathrm{QCom}(A)$, then $\|[Q(a), a]\| = \|[(Q - Q')(a), a]\| \le 2\|Q - Q'\|\,\|a\|^2$ for each $a \in A$ and therefore

$$\operatorname{com}(Q) \le 2 \operatorname{dist}(Q, \mathrm{QCom}(A)) \quad (Q \in \mathcal{Q}(A)).$$

We are interested in whether there is a constant $M > 0$ such that

$$\operatorname{dist}(Q, \mathrm{QCom}(A)) \le M \operatorname{com}(Q) \quad (Q \in \mathcal{Q}(A)).$$

In the next theorem we need to measure the commutativity of a given Banach algebra A. To this end, we introduce the constant

$$\chi(A) = \sup\{\|ab - ba\| : a, b \in A,\ \|a\| = \|b\| = 1\}.$$

This is the norm of the bilinear map $(a, b) \mapsto [a, b]$ on A and so (1.1) yields the following property.

Lemma 2.7. *Let (A_n) be a sequence of Banach algebras. Then*

$$\chi((A_n)^{\mathcal{U}}) = \lim_{\mathcal{U}} \chi(A_n).$$

Theorem 2.8. *For each $K > 0$ there exists $M > 0$ such that*

$$\operatorname{dist}(Q, \mathrm{QCom}(A)) \le M \operatorname{com}(Q)$$

for each Banach algebra A with $\kappa(A) \ge K$ and $Q \in \mathcal{Q}(A)$.

Proof. This follows by the same method as in Theorem 2.5. Suppose the assertion of the theorem is false. Then there exist a constant $K > 0$, a sequence of Banach algebras (A_n) with $\kappa(A_n) \ge K$ $(n \in \mathbb{N})$, and a sequence (Q_n) with $Q_n \in \mathcal{Q}(A_n)$ $(n \in \mathbb{N})$ such that

$$\lim_{n\to\infty} \|Q_n\| = 1, \quad \operatorname{dist}(Q_n, \mathrm{QCom}(A_n)) = 1 \quad (n \in \mathbb{N}), \tag{2.10}$$

and

$$\lim_{n\to\infty} \operatorname{com}(Q_n) = 0.$$

For each $n \in \mathbb{N}$, let $F_n : A_n \times A_n \to A_n$ be the symmetric continuous bilinear map such that $Q_n(a) = F_n(a, a)$ $(a \in A_n)$.

Take $\mathbf{A} = (A_n)^{\mathcal{U}}$, $\mathbf{F} = (F_n)^{\mathcal{U}} \in \mathcal{L}^2(\mathbf{A})$, and $\mathbf{Q} = (Q_n)^{\mathcal{U}} \in \mathcal{Q}(\mathbf{A})$. Of course, $\mathbf{F}$ is symmetric and $\mathbf{Q}(\mathbf{a}) = \mathbf{F}(\mathbf{a}, \mathbf{a})$ $(\mathbf{a} \in \mathbf{A})$.

It is a simple matter to check that $\operatorname{com}(\mathbf{Q}) = \lim_{\mathcal{U}} \operatorname{com}(Q_n) = 0$, which implies that $\mathbf{Q}$ is commuting. In fact, Lemma 2.2 still holds true because the map com is now given by the norm of the trace of a trilinear map and

(1.2) works for this case. Next, according to Lemma 2.3, $\mathbf{A}$ is ultraprime. By Proposition 2.6, there exist $\lambda\in\mathbb{C}$, a continuous linear functional $\mathbf{M}\colon \mathbf{A}\to\mathbb{C}$, and a continuous quadratic map $\mathbf{N}\colon \mathbf{A}\to\mathcal{Z}(\mathbf{A})$ such that

$$\mathbf{Q}(\mathbf{a}) = \lambda\mathbf{a}^2 + \mathbf{M}(\mathbf{a})\mathbf{a} + \mathbf{N}(\mathbf{a}) \quad (\mathbf{a}\in\mathbf{A}). \tag{2.11}$$

Further, let $\mathbf{G}\colon \mathbf{A}\times\mathbf{A}\to\mathcal{Z}(\mathbf{A})$ be the symmetric bilinear map associated to $\mathbf{N}$. Linearization of (2.11) gives

$$2\mathbf{F}(\mathbf{a},\mathbf{b}) = \lambda(\mathbf{ab}+\mathbf{ba})+(\mathbf{M}(\mathbf{b})\mathbf{a}+\mathbf{M}(\mathbf{a})\mathbf{b})+2\mathbf{G}(\mathbf{a},\mathbf{b}) \quad (\mathbf{a},\mathbf{b}\in\mathbf{A}). \tag{2.12}$$

Our objective is to prove that $\mathbf{M}=(\mu_n)^{\mathcal{U}}$ and $\mathbf{N}=(\nu_n)^{\mathcal{U}}$ for some bounded sequences (μ_n) of continuous linear functionals $\mu_n\colon A_n\to\mathbb{C}$ and (ν_n) of continuous quadratic maps $\nu_n\colon A_n\to\mathcal{Z}(A_n)$.

We first assume that $\mathbf{A}$ is commutative. Then $\mathbf{A}$ is trivial so that we can take $\mathbf{M}=\mathbf{N}=0$ in (2.11). Accordingly, the maps $\mu_n=\nu_n=0$ $(n\in\mathbb{N})$ satisfy our requirement.

We now assume that $\mathbf{A}$ is not commutative. Then $\chi(\mathbf{A})>0$ and we can pick $0<\varrho<\chi(\mathbf{A})$. From Lemma 2.7, it follows that $V=\{n\in\mathbb{N} : \varrho<\chi(A_n)\}\in\mathcal{U}$. For each $n\in V$, we choose $b_n,c_n\in A_n$ with $\|b_n\|=\|c_n\|=1$ and $\varrho<\|[b_n,c_n]\|$. Furthermore, for each $n\in V$, we take a continuous linear functional $g_n\colon A_n\to\mathbb{C}$ such that $g_n([b_n,c_n])=1$ and $\|g_n\|=\|[b_n,c_n]\|^{-1}$ $<\varrho^{-1}$. For each $n\in\mathbb{N}\setminus V$, we consider g_n to be the zero functional on A_n. Let $\mathbf{b},\mathbf{c}\in\mathbf{A}$ be given by $\mathbf{b}=(b_n)$ and $\mathbf{c}=(c_n)$. On account of (2.12), we have

$$\mathbf{M}(\mathbf{a})[\mathbf{b},\mathbf{c}] = [2\mathbf{F}(\mathbf{a},\mathbf{b})-\lambda(\mathbf{ab}+\mathbf{ba})-\mathbf{M}(\mathbf{b})\mathbf{a},\mathbf{c}] \quad (\mathbf{a}\in\mathbf{A}).$$

By applying the continuous linear functional $(g_n)^{\mathcal{U}}$ on $(A_n)^{\mathcal{U}}$ we arrive at

$$\mathbf{M}(\mathbf{a}) = \lim_{\mathcal{U}} g_n([2F_n(a_n,b_n)-\lambda(a_nb_n+b_na_n)-\mathbf{M}(\mathbf{b})a_n,c_n]) \qquad (\mathbf{a}=(a_n)\in\mathbf{A}),$$

which implies that $\mathbf{M}=(\mu_n)^{\mathcal{U}}$, where (μ_n) is the bounded sequence of continuous linear functionals $\mu_n\colon A_n\to\mathbb{C}$ given by

$$\mu_n(a) = g_n([2F_n(a,b_n)-\lambda(ab_n+b_na)-\mathbf{M}(\mathbf{b})a,c_n]) \quad (a\in A_n,\ n\in\mathbb{N}).$$

In order to show that $\mathbf{N}=(\nu_n)^{\mathcal{U}}$ for some bounded sequence (ν_n) of continuous quadratic maps $\nu_n\colon A_n\to\mathcal{Z}(A_n)$ $(n\in\mathbb{N})$, we consider two different cases. First, we assume that $\mathbf{A}$ does not have an identity. Then $\mathcal{Z}(\mathbf{A})=\{0\}$, so that $\mathbf{N}=0$ and then we take $\nu_n=0$ for each $n\in\mathbb{N}$. We now assume that $\mathbf{A}$ has an identity. Let $\psi\colon \mathbf{A}\to\mathbb{C}$ be a quadratic functional such that $\mathbf{N}(\mathbf{a})=\psi(\mathbf{a})\mathbf{1}$ $(\mathbf{a}\in\mathbf{A})$. On account of Proposition 2.4 we find that $U=\{n\in\mathbb{N} : A_n \text{ has an identity}\}\in\mathcal{U}$. Define $u_n\in A_n$ and $f_n\in A_n^*$ as in the proof of Theorem 2.5. Then (2.11) can be written as

$$\lim_{\mathcal{U}} \|Q_n(a_n)-\lambda a_n^2-\mu_n(a_n)a_n-\psi(\mathbf{a})u_n\| = 0 \quad (\mathbf{a}=(a_n)\in\mathbf{A}),$$

which implies that

$$\psi(\mathbf{a}) = \lim_{\mathcal{U}} f_n(Q_n(a_n) - \lambda a_n^2 - \mu_n(a_n)a_n) \quad (\mathbf{a} = (a_n) \in \mathbf{A}).$$

We thus get $\mathbf{N} = (\nu_n)^{\mathcal{U}}$, where $\nu_n \colon A_n \to \mathcal{Z}(A_n)$ is defined by

$$\nu_n(a) = f_n(Q_n(a) - \lambda a^2 - \mu_n(a)a)u_n \quad (a \in A_n,\, n \in \mathbb{N}).$$

Finally, (2.11) reads as $\mathbf{Q} = (P_n)^{\mathcal{U}}$, where $P_n \in \mathrm{QCom}(A_n)$ is defined by

$$P_n(a) = \lambda a^2 + \mu_n(a)a + \nu_n(a) \quad (a \in A_n,\, n \in \mathbb{N}).$$

Hence

$$\lim_{\mathcal{U}} \mathrm{dist}(Q_n, \mathrm{QCom}(A_n)) \le \lim_{\mathcal{U}} \|Q_n - P_n\| = 0,$$

which contradicts (2.10). ∎

3. Stability of Lie maps

3.1. Lie isomorphisms. Recall that an additive map Φ from a ring $\mathcal{R}$ into a ring $\mathcal{S}$ is a *homomorphism* if

$$\Phi(ab) = \Phi(a)\Phi(b) \quad (a, b \in \mathcal{R}),$$

and an *antihomomorphism* if

$$\Phi(ab) = \Phi(b)\Phi(a) \quad (a, b \in \mathcal{R}).$$

Finally, Φ is said to be a *Lie homomorphism* if

$$\Phi([a, b]) = [\Phi(a), \Phi(b)] \quad (a, b \in \mathcal{R}).$$

The obvious example of a Lie homomorphism $\Phi \colon \mathcal{R} \to \mathcal{S}$ is a map of the form

$$\Phi = \Psi + \tau, \tag{3.1}$$

where $\Psi \colon \mathcal{R} \to \mathcal{S}$ is either a homomorphism or the negative of an antihomomorphism and $\tau \colon \mathcal{R} \to \mathcal{Z}(\mathcal{S})$ is an additive map sending commutators to zero. A basic question is whether every Lie homomorphism arises as described in (3.1). As an application of the analysis of the commuting traces of biadditive maps, M. Brešar showed that if $\mathcal{R}$ is any ring and $\mathcal{S}$ is a noncommutative prime ring with characteristic different from 2, then every Lie isomorphism from $\mathcal{R}$ onto $\mathcal{S}$ is in the standard form (3.1), where we allow Ψ and τ to map into appropriate enlargements, namely $\mathcal{S} + \mathcal{C}$ and $\mathcal{C}$, respectively, where $\mathcal{C}$ denotes the extended centroid of $\mathcal{S}$ [7, Corollary 6.5]. The translation of this result to our framework is the following.

Proposition 3.1. *Let B be a Banach algebra, let A be an ultraprime Banach algebra, and let $\Phi \in \mathcal{L}(B, A)$ be a Lie isomorphism. Then $\Phi = \Psi + \tau\mathbf{1}$, where $\Psi \in \mathcal{L}(B, A + \mathbb{C}\mathbf{1})$ is either a homomorphism or the negative of an antihomomorphism and $\tau \in B^*$ sends commutators to zero.*

Proof. If A is commutative, then A is isomorphic to $\mathbb{C}$ and the claimed decomposition of Φ obviously holds true.

We now assume that A is not commutative. From [7, Corollary 6.5] and Remark 2.1 it follows that $\Phi = \Psi + \tau\mathbf{1}$, where $\Psi\colon B \to A + \mathbb{C}\mathbf{1}$ is either a homomorphism or the negative of an antihomomorphism and $\tau\colon B \to \mathbb{C}$ is a linear functional sending commutators to zero. It remains to prove that both Ψ and τ are continuous. Let us consider only the homomorphism case. We can take $v \in B$ and $a \in A$ with $[\Phi(v), a] \neq 0$. It is easily seen that

$$\tau(u)[\Phi(v), a] = [-\Phi(uv) + \Phi(u)\Phi(v) - \tau(v)\Phi(u), a] \qquad (u \in B).$$

This implies that τ is continuous. Since $\Psi = \Phi - \tau\mathbf{1}$ we see that Ψ is continuous. ∎

Our next concern will be the stability of the above mentioned result. To treat this issue we introduce the following measures of multiplicativity, antimultiplicativity, and Lie multiplicativity of a given map $\Phi \in \mathcal{L}(B, A)$, where A and B are Banach algebras:

$$\begin{aligned} \operatorname{mult}(\Phi) &= \sup\{\|\Phi(ab) - \Phi(a)\Phi(b)\| : a, b \in B,\ \|a\| = \|b\| = 1\},\\ \operatorname{amult}(\Phi) &= \sup\{\|\Phi(ab) - \Phi(b)\Phi(a)\| : a, b \in B,\ \|a\| = \|b\| = 1\},\\ \operatorname{lmult}(\Phi) &= \sup\{\|\Phi([a,b]) - [\Phi(a), \Phi(b)]\| : a, b \in B,\ \|a\| = \|b\| = 1\}. \end{aligned}$$

Further, for $f \in B^*$ we put

$$\|f\|_t = \sup\{|f([a,b])| : a, b \in B,\ \|a\| = \|b\| = 1\}.$$

It is worth pointing out that the constants introduced above are nothing but the norms of the bilinear maps

$$\begin{aligned} &(a,b) \mapsto \Phi(ab) - \Phi(a)\Phi(b),\\ &(a,b) \mapsto \Phi(ab) - \Phi(b)\Phi(a),\\ &(a,b) \mapsto \Phi([a,b]) - [\Phi(a), \Phi(b)],\\ &(a,b) \mapsto f([a,b]), \end{aligned}$$

respectively. Consequently, (1.1) yields the following.

LEMMA 3.2. *Let (A_n) and (B_n) be sequences of Banach algebras and assume that $f_n \in B_n^*$ and $\Phi_n \in \mathcal{L}(B_n, A_n)$ $(n \in \mathbb{N})$ are given with $\sup_{n\in\mathbb{N}} \|f_n\| < \infty$ and $\sup_{n\in\mathbb{N}} \|\Phi_n\| < \infty$. Then*

$$\begin{aligned} \operatorname{mult}((\Phi_n)^{\mathcal{U}}) &= \lim_{\mathcal{U}} \operatorname{mult}(\Phi_n),\\ \operatorname{amult}((\Phi_n)^{\mathcal{U}}) &= \lim_{\mathcal{U}} \operatorname{amult}(\Phi_n),\\ \operatorname{lmult}((\Phi_n)^{\mathcal{U}}) &= \lim_{\mathcal{U}} \operatorname{lmult}(\Phi_n),\\ \|(f_n)^{\mathcal{U}}\|_t &= \lim_{\mathcal{U}} \|f_n\|_t. \end{aligned}$$

Note that the maps mult, amult, and lmult vanish exactly on the sets $\mathrm{Hom}(B, A)$ of all homomorphisms, $\mathrm{AHom}(B, A)$ of all antihomomorphisms, and $\mathrm{LHom}(B, A)$ of all Lie homomorphisms from B into A, respectively. If $A = B$ then we write briefly $\mathrm{Hom}(A)$, $\mathrm{AHom}(A)$, and $\mathrm{LHom}(A)$ instead of $\mathrm{Hom}(A, A)$, $\mathrm{AHom}(A, A)$, and $\mathrm{LHom}(A, A)$, respectively. The map $\|\cdot\|_t$ vanishes on the linear space of all continuous linear functionals on B sending commutators to zero.

To our knowledge, there are two basic choices of assumptions on a map $\Phi \in \mathcal{L}(B, A)$ that could be required to conclude that Φ is an approximate Lie homomorphism (i.e. $\mathrm{lmult}(\Phi)$ is small). The first choice is to consider $\Phi = \Psi + \tau\mathbf{1}$, where $\Psi \in \mathcal{L}(B, A + \mathbb{C}\mathbf{1})$ and $\tau \in B^*$ are such that $\min\{\mathrm{mult}(\Psi), \mathrm{amult}(-\Psi)\}$ and $\|\tau\|_t$ are small (here we are restricting ourselves to the case when $\mathcal{Z}(A)$ is trivial). This pattern of thinking leads to introduce the following constants:

$$\begin{aligned}
\mathrm{smult}_+(\Phi) &= \inf\{\mathrm{mult}(\Phi - \tau\mathbf{1}) + \|\tau\|_t : \tau \in B^*\},\\
\mathrm{smult}_-(\Phi) &= \inf\{\mathrm{amult}(\tau\mathbf{1} - \Phi) + \|\tau\|_t : \tau \in B^*\},\\
\mathrm{smult}(\Phi) &= \min\{\mathrm{smult}_+(\Phi), \mathrm{smult}_-(\Phi)\}.
\end{aligned}$$

The second choice is to assume that $\mathrm{dist}(\Phi, \mathrm{LHom}(B, A))$ is small.

Proposition 3.3. *Let A be an ultraprime Banach algebra. Then there exists a constant $K > 0$ such that*

$$\mathrm{lmult}(\Phi) \le K\, \mathrm{smult}(\Phi)$$

and

$$\mathrm{lmult}(\Phi) \le \big(2 + 2\|\Phi\| + 4\,\mathrm{dist}(\Phi, \mathrm{LHom}(B, A))\big)\, \mathrm{dist}(\Phi, \mathrm{LHom}(B, A))$$

for each Banach algebra B and $\Phi \in \mathcal{L}(B, A)$.

Proof. Let $\Phi \in \mathcal{L}(B, A)$. Pick $\tau \in B^*$ and write $\Psi = \Phi - \tau\mathbf{1}$. For all $a, b \in B$, we have

$$\Phi([a, b]) - [\Phi(a), \Phi(b)] = \Psi([a, b]) - [\Psi(a), \Psi(b)] + \tau([a, b])\mathbf{1}$$

and so

$$\|\Phi([a, b]) - [\Phi(a), \Phi(b)]\| \le \|\Psi([a, b]) - [\Psi(a), \Psi(b)]\| + \|\tau\|_t \|\mathbf{1}\|\, \|a\|\, \|b\|.$$

We can write $\Psi([a, b]) - [\Psi(a), \Psi(b)]$ in two ways. On the one hand,

$$\Psi([a, b]) - [\Psi(a), \Psi(b)] = (\Psi(ab) - \Psi(a)\Psi(b)) - (\Psi(ba) - \Psi(b)\Psi(a)),$$

which gives $\|\Psi([a, b]) - [\Psi(a), \Psi(b)]\| \le 2\,\mathrm{mult}(\Psi)\|a\|\|b\|$. On the other hand, we have

$$\begin{aligned}
\Psi([a, b]) - [\Psi(a), \Psi(b)] = {}&((-\Psi)(ba) - (-\Psi)(a)(-\Psi)(b))\\
&- ((-\Psi)(ab) - (-\Psi)(b)(-\Psi)(a)),
\end{aligned}$$

which yields $\|\Psi([a, b]) - [\Psi(a), \Psi(b)]\| \le 2\,\mathrm{amult}(-\Psi)\|a\|\, \|b\|$. This proves the first inequality in the proposition.

Set $\Psi \in \mathrm{LHom}(B, A)$ and write $\Theta = \Phi - \Psi$. For all $a, b \in A$ we have

$$\begin{aligned}\Phi([a,b]) - [\Phi(a), \Phi(b)] &= \Theta([a,b]) + [\Psi(a), \Psi(b)] - [\Phi(a), \Phi(b)] \\ &= \Theta([a,b]) - [\Psi(a), \Theta(b)] - [\Theta(a), \Phi(b)]\end{aligned}$$

and so

$$\begin{aligned}\|\Phi([a,b]) - [\Phi(a), \Phi(b)]\| &\le (2\|\Theta\| + 2\|\Theta\|(\|\Psi\| + \|\Phi\|))\|a\|\,\|b\| \\ &\le (2\|\Theta\| + 2\|\Theta\|(\|\Theta\| + 2\|\Phi\|))\|a\|\,\|b\|.\end{aligned}$$

This gives $\mathrm{lmult}(\Phi) \le 2\|\Theta\| + 2\|\Theta\|(\|\Theta\| + 2\|\Phi\|)$, which establishes the second inequality in the proposition. ∎

We are now interested in whether $\mathrm{lmult}(\Phi)$ being small implies that $\mathrm{smult}(\Phi)$ is small. In Section 4 we address the question of whether $\mathrm{lmult}(\Phi)$ being small implies that $\mathrm{dist}(\Phi, \mathrm{LHom}(B, A))$ is also small.

Theorem 3.4. *For each $K, M, \varepsilon > 0$ there exists $\delta > 0$ such that if A and B are Banach algebras with $\kappa(A) \ge K$ and $\Phi \in \mathcal{L}(B, A)$ is bijective with $\|\Phi\|, \|\Phi^{-1}\| \le M$, and* $\mathrm{lmult}(\Phi) < \delta$, *then* $\mathrm{smult}(\Phi) < \varepsilon$.

Proof. Suppose the assertion is false. Then there exist $K, M, \varepsilon > 0$, sequences (A_n) and (B_n) of Banach algebras, and a sequence (Φ_n) of bijective continuous linear maps $\Phi_n \colon B_n \to A_n$ ($n \in \mathbb{N}$) with

$$\kappa(A_n) \ge K, \tag{3.2}$$

$$\|\Phi_n\|, \|\Phi_n^{-1}\| \le M, \tag{3.3}$$

$$\mathrm{lmult}(\Phi_n) < 1/n, \tag{3.4}$$

$$\mathrm{smult}(\Phi_n) \ge \varepsilon, \tag{3.5}$$

for each $n \in \mathbb{N}$.

Set $\mathbf{A} = (A_n)^{\mathcal{U}}$, $\mathbf{B} = (B_n)^{\mathcal{U}}$, and $\Phi = (\Phi_n)^{\mathcal{U}}$. From (3.3) it follows that Φ is bijective with inverse given by $(\Phi_n^{-1})^{\mathcal{U}}$. We claim that Φ is a Lie isomorphism. Indeed, if $\mathbf{u} = (u_n), \mathbf{v} = (v_n) \in \mathbf{B}$, then (3.4) yields

$$\begin{aligned}\|\Phi([\mathbf{u}, \mathbf{v}]) - [\Phi(\mathbf{u}), \Phi(\mathbf{v})]\| &= \lim_{\mathcal{U}} \|\Phi_n([u_n, v_n]) - [\Phi_n(u_n), \Phi_n(v_n)]\| \\ &\le \lim_{\mathcal{U}} (\mathrm{lmult}(\Phi_n)\|u_n\|\|v_n\|) = 0.\end{aligned}$$

Furthermore, according to Lemma 2.3 and (3.2), $\kappa(\mathbf{A}) \ge K$ and hence $\mathbf{A}$ is ultraprime.

On account of Proposition 3.1, $\Phi = \Psi + \tau\mathbf{1}$, where $\Psi \in \mathcal{L}(\mathbf{B}, \mathbf{A} + \mathbb{C}\mathbf{1})$ is either a homomorphism or the negative of an antihomomorphism and $\tau \in \mathbf{B}^*$ vanishes on commutators. Our purpose is to show that $\Psi = (\Psi_n)^{\mathcal{U}}$ and $\tau = (\tau_n)^{\mathcal{U}}$, where $\Psi_n \in \mathcal{L}(B_n, A_n + \mathbb{C}\mathbf{1})$ and $\tau_n \in B_n^*$ for each $n \in \mathbb{N}$. We will consider two cases according to the degree of algebraicity of $\mathbf{A}$.

We first assume that the degree of algebraicity of $\mathbf{A}$ is greater than 2. For each $n \in \mathbb{N}$ we define $Q_n \in \mathcal{Q}(A_n)$ by

$$Q_n(a) = \Phi_n((\Phi_n^{-1}(a))^2) \quad (a \in A_n).$$

According to (3.3), $\|Q_n\| \le M^3$ $(n \in \mathbb{N})$ and therefore we can consider $\mathbf{Q} \in \mathcal{Q}(\mathbf{A})$ given by $\mathbf{Q} = (Q_n)^{\mathcal{U}}$. It is clear that $\mathbf{Q}(\mathbf{a}) = \Phi(\Phi^{-1}(\mathbf{a})^2)$ for each $\mathbf{a} \in \mathbf{A}$. Since Φ is a Lie isomorphism, it follows that

$$[\mathbf{Q}(\mathbf{a}), \mathbf{a}] = \mathbf{Q}([\Phi^{-1}(\mathbf{a})^2, \Phi^{-1}(\mathbf{a})]) = 0$$

for each $\mathbf{a} \in \mathbf{A}$. Hence $\mathbf{Q}$ is commuting. From the proof of Theorem 2.8 it may be concluded that

$$\mathbf{Q}(\mathbf{a}) = \lambda \mathbf{a}^2 + \mathbf{M}(\mathbf{a})\mathbf{a} + \mathbf{N}(\mathbf{a}) \quad (\mathbf{a} \in \mathbf{A}) \tag{3.6}$$

for some $\lambda \in \mathbb{C}$, $\mathbf{M} = (\mu_n)^{\mathcal{U}}$ with $\mu_n \in A_n^*$ $(n \in \mathbb{N})$, and $\mathbf{N} = (\nu_n)^{\mathcal{U}}$ with $\nu_n \in \mathcal{Q}(A_n)$ and $\nu_n(A_n) \subset \mathcal{Z}(A_n)$ $(n \in \mathbb{N})$. Taking $\mathbf{a} = \Phi(\mathbf{u})$ with $\mathbf{u} \in \mathbf{B}$ in (3.6) we arrive at

$$\Phi(\mathbf{u}^2) = \lambda \Phi(\mathbf{u})^2 + \mathbf{M}(\Phi(\mathbf{u}))\Phi(\mathbf{u}) + \mathbf{N}(\Phi(\mathbf{u})). \tag{3.7}$$

On the other hand, we have

$$\begin{aligned} \Phi(\mathbf{u}^2) &= \Psi(\mathbf{u}^2) + \tau(\mathbf{u}^2)\mathbf{1} = \sigma\Psi(\mathbf{u})^2 + \tau(\mathbf{u}^2)\mathbf{1} \\ &= \sigma\Phi(\mathbf{u})^2 - 2\sigma\tau(\mathbf{u})\Phi(\mathbf{u}) + (\sigma\tau(\mathbf{u})^2 + \tau(\mathbf{u}^2))\mathbf{1} \quad (\mathbf{u} \in \mathbf{B}), \end{aligned} \tag{3.8}$$

where $\sigma = 1$ in the case where Ψ is a homomorphism and $\sigma = -1$ in the case where Ψ is the negative of an antihomomorphism. Comparing (3.7) and (3.8) we get

$$(\lambda - \sigma)\Phi(\mathbf{u})^2 + (\mathbf{M}(\Phi(\mathbf{u})) + 2\sigma\tau(\mathbf{u}))\Phi(\mathbf{u}) + \mathbf{N}(\Phi(\mathbf{u})) - (\sigma\tau(\mathbf{u})^2 + \tau(\mathbf{u}^2))\mathbf{1} = 0$$

for each $\mathbf{u} \in \mathbf{B}$. This can be written as follows:

$$\begin{aligned} (\lambda - \sigma)\mathbf{a}^2 &+ (\mathbf{M}(\mathbf{a}) + 2\sigma\tau(\Phi^{-1}(\mathbf{a})))\mathbf{a} \\ &+ \mathbf{N}(\mathbf{a}) - (\sigma\tau(\Phi^{-1}(\mathbf{a}))^2 + \tau(\Phi^{-1}(\mathbf{a})^2))\mathbf{1} = 0 \end{aligned}$$

for each $\mathbf{a} \in \mathbf{A}$.

We claim that

$$\tau(\Phi^{-1}(\mathbf{a})) = -\frac{1}{2\sigma}\mathbf{M}(\mathbf{a}) \quad (\mathbf{a} \in \mathbf{A}). \tag{3.9}$$

Since $\deg(\mathbf{A}) > 2$, it follows that $\lambda = \sigma$ and

$$(\mathbf{M}(\mathbf{a}) + 2\sigma\tau(\Phi^{-1}(\mathbf{a})))\mathbf{a} \in \mathcal{Z}(\mathbf{A}) \tag{3.10}$$

for each $\mathbf{a} \in \mathbf{A}$. If $\mathbf{a} \in \mathbf{A} \setminus \mathcal{Z}(\mathbf{A})$, then (3.10) yields (3.9). Let $\mathbf{a} \in \mathcal{Z}(\mathbf{A})$ and pick $\mathbf{b} \in \mathbf{A} \setminus \mathcal{Z}(\mathbf{A})$ (such an element exists because $\dim(\mathcal{Z}(\mathbf{A})) \le 1$). Then $a + b \in \mathbf{A} \setminus \mathcal{Z}(\mathbf{A})$ and therefore

$$\begin{aligned} \tau(\Phi^{-1}(\mathbf{a})) &= \tau(\Phi^{-1}(\mathbf{a} + \mathbf{b})) - \tau(\Phi^{-1}(\mathbf{b})) \\ &= -\frac{1}{2\sigma}\mathbf{M}(\mathbf{a} + \mathbf{b}) + \frac{1}{2\sigma}\mathbf{M}(\mathbf{b}) = -\frac{1}{2\sigma}\mathbf{M}(\mathbf{a}), \end{aligned}$$

which gives (3.9), as claimed. From (3.9) it may be concluded that $\tau = (\tau_n)^{\mathcal{U}}$ and $\Psi = (\Psi_n)^{\mathcal{U}}$, where

$$\tau_n = -\frac{1}{2\sigma}\mu_n \circ \Phi_n \colon B_n \to \mathbb{C}, \qquad \Psi_n = \Phi_n + \frac{1}{2\sigma}\mu_n \circ \Phi_n \colon B_n \to A_n + \mathbb{C}\mathbf{1}$$

for each $n \in \mathbb{N}$.

Having established the case where $\deg(\mathbf{A}) > 2$, we now turn to the case where $\deg(\mathbf{A}) \le 2$. Then $\mathbf{A}$ is finite-dimensional (in fact, it is isomorphic either to $\mathbb{C}$ or to $M_2(\mathbb{C})$). By [4, Theorem 3.1], $W = \{n \in \mathbb{N} : \dim(A_n) = \dim(\mathbf{A})\} \in \mathcal{U}$. Since Φ_n is a bijective linear map, it follows that $\dim(B_n) = \dim(A_n) = \dim(\mathbf{A})$ for each $n \in W$, and [4, Theorem 3.1] then gives $\dim(\mathbf{B}) = \dim(\mathbf{A})$. From [10, Theorem 7.1] it follows that $\tau = (\tau_n)^{\mathcal{U}}$ for some sequence (τ_n) with $\tau_n \in B_n^*$ ($n \in \mathbb{N}$). This clearly implies that $\Psi = (\Psi_n)^{\mathcal{U}}$ where $\Psi_n = \Phi_n - \tau_n \in \mathcal{L}(B_n, A_n)$ ($n \in \mathbb{N}$).

Finally, we are in a position to get a contradiction. Having shown that $\Psi = (\Psi_n)^{\mathcal{U}}$ and $\tau = (\tau_n)^{\mathcal{U}}$ where $\Psi_n \in \mathcal{L}(B_n, A_n + \mathbb{C}\mathbf{1})$ and $\tau_n \in B_n^*$ for each $n \in \mathbb{N}$, we can now apply Lemma 3.2 to get

$$\operatorname{mult}(\Psi) = \lim_{\mathcal{U}} \operatorname{mult}(\Phi_n - \tau_n\mathbf{1}), \qquad \operatorname{amult}(-\Psi) = \lim_{\mathcal{U}} \operatorname{amult}(\tau_n\mathbf{1} - \Phi_n).$$

From the definition we see that

$$\operatorname{mult}(\Phi_n - \tau_n\mathbf{1}) \ge \operatorname{smult}_+(\Phi_n) - \|\tau_n\|_t$$

and

$$\operatorname{amult}(\tau_n\mathbf{1} - \Phi_n) \ge \operatorname{smult}_-(\Phi_n) - \|\tau_n\|_t$$

for each $n \in \mathbb{N}$. Since either Ψ is a homomorphism or $-\Psi$ is an antihomomorphism, it may be concluded that

$$\begin{aligned} 0 &= \min\{\operatorname{mult}(\Psi), \operatorname{amult}(-\Psi)\} \\ &= \lim_{\mathcal{U}} \min\{\operatorname{mult}(\Phi_n - \tau_n\mathbf{1}), \operatorname{amult}(\tau_n\mathbf{1} - \Phi_n)\} \\ &\ge \lim_{\mathcal{U}} (\operatorname{smult}(\Phi_n) - \|\tau_n\|_t) = \lim_{\mathcal{U}} \operatorname{smult}(\Phi_n) - \lim_{\mathcal{U}} \|\tau_n\|_t. \end{aligned}$$

Since $\lim_{\mathcal{U}} \|\tau_n\|_t = \|\tau\|_t = 0$, it follows that $\lim_{\mathcal{U}} \operatorname{smult}(\Phi_n) = 0$, contrary to (3.5), and the proof is complete. ∎

3.2. Lie derivations. Let $\mathcal{R}$ be a ring and $\mathcal{X}$ be an $\mathcal{R}$-bimodule. Recall that an additive map $\Delta \colon \mathcal{R} \to \mathcal{X}$ is a *derivation* if

$$\Delta(ab) = \Delta(a)b + a\Delta(b) \qquad (a, b \in \mathcal{R})$$

and it is a *Lie derivation* if

$$\Delta([a, b]) = [\Delta(a), b] + [a, \Delta(b)] \qquad (a, b \in \mathcal{R}).$$

A typical example of a Lie derivation $\Delta \colon \mathcal{R} \to \mathcal{X}$ is provided by the map

$$\Phi = D + \tau, \tag{3.11}$$

where $D\colon \mathcal{R} \to \mathcal{X}$ is a derivation and τ is an additive map from $\mathcal{R}$ into the centre of $\mathcal{X}$ sending commutators to zero. A basic problem is to determine whether every Lie derivation is in the standard form (3.11). M. Brešar showed that if $\mathcal{R}$ is a prime ring with characteristic different from 2, then every Lie derivation $\Delta\colon \mathcal{R} \to \mathcal{R}$ is as in (3.11) provided that we allow D and τ to map into the appropriate enlargements $\mathcal{RC}+\mathcal{C}$ and $\mathcal{C}$, respectively, where $\mathcal{C}$ is the extended centroid of $\mathcal{R}$ [7, Corollary 6.9]. In the context of ultraprime Banach algebras this result reads as follows.

Proposition 3.5. *Let A be an ultraprime Banach algebra and let $\Delta \in \mathcal{L}(A)$ be a Lie derivation. Then $\Delta = D + \tau\mathbf{1}$, where $D \in \mathcal{L}(A, A + \mathbb{C}\mathbf{1})$ is a derivation and $\tau \in A^*$ sends commutators to zero.*

Proof. If A is commutative, then A is isomorphic to $\mathbb{C}$ and the proposition holds true.

We now proceed with the case where A is not commutative. From [7, Corollary 6.9] and Remark 2.1 it follows that $\Delta = D + \tau\mathbf{1}$, where $D\colon A \to A + \mathbb{C}\mathbf{1}$ is a derivation and $\tau\colon A \to \mathbb{C}$ is a linear functional sending commutators to zero. The only point remaining is the continuity. Since A is not commutative, we can pick $b, c \in A$ with $[b, c] \neq 0$. It is immediate to check that

$$\tau(a)[b,c] = [-\Delta(ab) + \Delta(a)b + a\Delta(b) - \tau(b)a, c] \quad (a \in A),$$

which shows that τ is continuous and finally D is continuous because $D = \Delta - \tau\mathbf{1}$. ∎

Our next objective is to analyse the stability of the preceding result. To this end, for a given continuous linear map Δ from a Banach algebra A into a Banach A-bimodule X, we define the constants

$$\begin{aligned}\operatorname{der}(\Delta) &= \sup\{\|\Delta(ab) - \Delta(a)b - a\Delta(b)\| : \|a\| = \|b\| = 1\},\\ \operatorname{lder}(\Delta) &= \sup\{\|\Delta([a,b]) - [\Delta(a),b] - [a,\Delta(b)]\| : \|a\| = \|b\| = 1\}.\end{aligned}$$

It should be mentioned that $\operatorname{der}(\Delta)$ and $\operatorname{lder}(\Delta)$ are nothing but the norms of the bilinear maps

$$\begin{aligned}(a,b) &\mapsto \Delta(ab) - \Delta(a)b - a\Delta(b),\\ (a,b) &\mapsto \Delta([a,b]) - [\Delta(a),b] - [a,\Delta(b)],\end{aligned}$$

respectively. Consequently, (1.1) yields the following.

Lemma 3.6. *Let (A_n) be a sequence of Banach algebras and assume that, for each $n \in \mathbb{N}$, a Banach A_n-bimodule X_n and $\Delta_n \in \mathcal{L}(A_n, X_n)$ are given with $\sup_{n\in\mathbb{N}} \|\Delta_n\| < \infty$. Then*

$$\operatorname{der}((\Delta_n)^{\mathcal{U}}) = \lim_{\mathcal{U}} \operatorname{der}(\Delta_n), \qquad \operatorname{lder}((\Delta_n)^{\mathcal{U}}) = \lim_{\mathcal{U}} \operatorname{lder}(\Delta_n).$$

The maps $\Delta \mapsto \operatorname{der}(\Delta)$ and $\Delta \mapsto \operatorname{lder}(\Delta)$ define seminorms on $\mathcal{L}(A, X)$ vanishing on the linear subspaces $\operatorname{Der}(A, X)$ consisting of all derivations and $\operatorname{LDer}(A, X)$ consisting of all Lie derivations, respectively. We abbreviate $\operatorname{Der}(A, A)$ to $\operatorname{Der}(A)$ and $\operatorname{LDer}(A, A)$ to $\operatorname{LDer}(A)$. Following the pattern of the preceding subsection, we associate the following constant to $\Delta \in \mathcal{L}(A)$:

$$\operatorname{sder}(\Delta) = \inf\{\operatorname{der}(\Delta - \tau\mathbf{1}) + \|\tau\|_t : \tau \in A^*\}.$$

The map $\Delta \mapsto \operatorname{sder}(\Delta)$ is a seminorm on $\mathcal{L}(A)$.

Proposition 3.7. *Let A be an ultraprime Banach algebra. Then there exists a constant $K > 0$ with*

$$\operatorname{lder}(\Delta) \le K \operatorname{sder}(\Delta) \le 3K \operatorname{dist}(\Delta, \operatorname{LDer}(A)) \qquad (\Delta \in \mathcal{L}(A)).$$

Proof. Let $\Delta \in \mathcal{L}(A)$. Pick $\tau \in A^*$ and write $D = \Delta - \tau\mathbf{1}$. For all $a, b \in A$, we have

$$\begin{aligned}\Delta([a,b]) - [\Delta(a), b] - [a, \Delta(b)] &= D([a,b]) - [D(a), b] - [a, D(b)] + \tau([a,b])\mathbf{1}\\ &= (D(ab) - D(a)b - aD(b))\\ &\quad - (D(ba) - D(b)a - bD(a)) + \tau([a,b])\mathbf{1},\end{aligned}$$

which implies that

$$\|\Delta([a,b]) - [\Delta(a), b] - [a, \Delta(b)]\| \le (2\operatorname{der}(D) + \|\tau\|_t\|\mathbf{1}\|)\|a\|\,\|b\|,$$

and therefore $\operatorname{lder}(\Delta) \le 2\operatorname{der}(D) + \|\tau\|_t\|\mathbf{1}\|$. This establishes the first inequality in the proposition.

Pick $D \in \operatorname{LDer}(A)$. Then there exists $\tau \in A^*$ sending commutators to zero such that $D - \tau\mathbf{1}$ is a derivation. Accordingly,

$$\begin{aligned}\operatorname{sder}(\Delta) &\le \operatorname{der}(\Delta - \tau\mathbf{1}) + \|\tau\|_t = \operatorname{der}(\Delta - \tau\mathbf{1})\\ &\le \operatorname{der}(\Delta - D) + \operatorname{der}(D - \tau\mathbf{1}) = \operatorname{der}(\Delta - D) \le 3\|\Delta - D\|.\end{aligned}$$

This gives the second inequality in the proposition. ∎

We are interested in whether the three seminorms $\operatorname{lder}(\cdot)$, $\operatorname{sder}(\cdot)$, and $\operatorname{dist}(\cdot, \operatorname{LDer}(A))$ are actually pairwise equivalent. Our next concern is the analysis of $\operatorname{lder}(\cdot)$ and $\operatorname{sder}(\cdot)$. The next section will be concerned with $\operatorname{dist}(\cdot, \operatorname{LDer}(A))$.

Theorem 3.8. *For each $K, M, \varepsilon > 0$ there exists $\delta > 0$ such that if A is a Banach algebra with $\kappa(A) \ge K$ and $\Delta \in \mathcal{L}(A)$ is such that $\|\Delta\| \le M$ and $\operatorname{lder}(\Delta) < \delta$, then $\operatorname{sder}(\Delta) < \varepsilon$.*

Proof. This follows by the same method as in Theorem 3.4. To obtain a contradiction, suppose the assertion is false. Then there exist $K, M, \varepsilon > 0$, a sequence of Banach algebras (A_n) with $\kappa(A_n) \ge K$ $(n \in \mathbb{N})$, and a sequence (Δ_n) with $\Delta_n \in \mathcal{L}(A_n)$, $\|\Delta_n\| \le M$, $\operatorname{lder}(\Delta_n) < 1/n$, and $\operatorname{sder}(\Delta_n) \ge \varepsilon$ $(n \in \mathbb{N})$.

We then consider $\mathbf{A} = (A_n)^{\mathcal{U}}$ and $\Delta = (\Delta_n)^{\mathcal{U}} \in \mathcal{L}(\mathbf{A})$. We claim that Δ is a Lie derivation on $\mathbf{A}$. Indeed, if $\mathbf{a} = (a_n), \mathbf{b} = (b_n) \in \mathbf{A}$, then

$$\begin{aligned}\|\Delta([\mathbf{a},\mathbf{b}]) - [\Delta(\mathbf{a}),\mathbf{b}] - [\mathbf{a},\Delta(\mathbf{b})]\| \\ &= \lim_{\mathcal{U}} \|\Delta_n([a_n,b_n]) - [\Delta_n(a_n),b_n] - [a_n,\Delta_n(b_n)]\| \\ &\le \lim_{\mathcal{U}} (\mathrm{lder}(\Delta_n)\|a_n\|\,\|b_n\|) = 0.\end{aligned}$$

From Lemma 2.3 it follows that $\mathbf{A}$ is ultraprime and Proposition 3.5 then gives $\Delta = \mathbf{D} + \tau\mathbf{1}$, where $\mathbf{D} \in \mathcal{L}(\mathbf{A}, \mathbf{A} + \mathbb{C}\mathbf{1})$ is a derivation and $\tau \in \mathbf{A}^*$ vanishes on commutators. Our next concern will be to show that $\mathbf{D} = (D_n)^{\mathcal{U}}$ and $\tau = (\tau_n)^{\mathcal{U}}$, where $D_n \in \mathcal{L}(A_n, A_n + \mathbb{C}\mathbf{1})$ and $\tau_n \in A_n^*$ for each $n \in \mathbb{N}$.

Let us first consider the case when $\deg(\mathbf{A}) > 2$. For each $n \in \mathbb{N}$ we define $Q_n \in \mathcal{Q}(A_n)$ by

$$Q_n(a) = \Delta_n(a^2) - \Delta_n(a)a - a\Delta_n(a) \quad (a \in A_n).$$

Then $\|Q_n\| \le 3M$ $(n \in \mathbb{N})$ so that we can consider the map $\mathbf{Q} = (Q_n)^{\mathcal{U}} \in \mathcal{Q}(\mathbf{A})$. Of course, $\mathbf{Q}(\mathbf{a}) = \Delta(\mathbf{a}^2) - \Delta(\mathbf{a})\mathbf{a} - \mathbf{a}\Delta(\mathbf{a})$ for each $\mathbf{a} \in \mathbf{A}$. Since Δ is a Lie derivation, we see that

$$0 = \Delta([\mathbf{a}^2,\mathbf{a}]) = [\Delta(\mathbf{a}^2),\mathbf{a}] + [\mathbf{a}^2,\Delta(\mathbf{a})] = [\mathbf{Q}(\mathbf{a}),\mathbf{a}]$$

for each $\mathbf{a} \in \mathbf{A}$. Therefore $\mathbf{Q}$ is commuting, and from the proof of Theorem 2.8 we deduce that

$$\mathbf{Q}(\mathbf{a}) = \lambda\mathbf{a}^2 + \mathbf{M}(\mathbf{a})\mathbf{a} + \mathbf{N}(\mathbf{a}) \quad (\mathbf{a} \in \mathbf{A})$$

for some $\lambda \in \mathbb{C}$, $\mathbf{M} = (\mu_n)^{\mathcal{U}}$ with $\mu_n \in A_n^*$ $(n \in \mathbb{N})$, and $\mathbf{N} = (\nu_n)^{\mathcal{U}}$ with $\nu_n \in \mathcal{Q}(A_n)$ and $\nu_n(A_n) \subset \mathcal{Z}(A_n)$ $(n \in \mathbb{N})$.

We claim that

$$\tau(\mathbf{a}) = -\frac{1}{2}\mathbf{M}(\mathbf{a}) \quad (\mathbf{a} \in \mathbf{A}). \tag{3.12}$$

To see this, we now compute $\Delta(\mathbf{a}^2)$ in two ways. On the one hand, we have

$$\Delta(\mathbf{a}^2) = \Delta(\mathbf{a})\mathbf{a} + \mathbf{a}\Delta(\mathbf{a}) + \lambda\mathbf{a}^2 + \mathbf{M}(\mathbf{a})\mathbf{a} + \mathbf{N}(\mathbf{a}).$$

On the other hand, we have

$$\begin{aligned}\Delta(\mathbf{a}^2) &= \mathbf{D}(\mathbf{a}^2) + \tau(\mathbf{a}^2)\mathbf{1} = \mathbf{D}(\mathbf{a})\mathbf{a} + \mathbf{a}\mathbf{D}(\mathbf{a}) + \tau(\mathbf{a}^2)\mathbf{1} \\ &= \Delta(\mathbf{a})\mathbf{a} + \mathbf{a}\Delta(\mathbf{a}) - 2\tau(\mathbf{a})\mathbf{a} + \tau(\mathbf{a}^2)\mathbf{1}.\end{aligned}$$

We thus get

$$\lambda\mathbf{a}^2 + (\mathbf{M}(\mathbf{a}) + 2\tau(\mathbf{a}))\mathbf{a} + (\mathbf{N}(\mathbf{a}) - \tau(\mathbf{a}^2)\mathbf{1}) = 0$$

for each $a \in \mathbf{A}$. Since $\deg(\mathbf{A}) > 2$, it follows that $\lambda = 0$ and

$$(\mathbf{M}(\mathbf{a}) + 2\tau(\mathbf{a}))\mathbf{a} \in \mathcal{Z}(\mathbf{A})$$

for each $\mathbf{a} \in \mathbf{A}$. If $\mathbf{a} \in \mathbf{A} \setminus \mathcal{Z}(\mathbf{A})$, then the preceding property obviously gives (3.12). We now fix $\mathbf{b} \in \mathbf{A} \setminus \mathcal{Z}(\mathbf{A})$. If $\mathbf{a} \in \mathcal{Z}(\mathbf{A})$, then $\mathbf{a}+\mathbf{b} \in \mathbf{A} \setminus \mathcal{Z}(\mathbf{A})$

and therefore

$$\tau(\mathbf{a}) = \tau(\mathbf{a}+\mathbf{b}) - \tau(\mathbf{b}) = -\frac{1}{2}\mathbf{M}(\mathbf{a}+\mathbf{b}) + \frac{1}{2}\mathbf{M}(\mathbf{b}) = -\frac{1}{2}\mathbf{M}(a).$$

From (3.12) it follows that $\mathbf{D} = (D_n)^{\mathcal{U}}$ and $\tau = (\tau_n)^{\mathcal{U}}$ where

$$\tau_n = -\frac{1}{2}\mu_n \in A_n^* \quad (n \in \mathbb{N})$$

and

$$D_n = \varDelta_n - \frac{1}{2}\mu_n \mathbf{1} \in \mathcal{L}(A_n, A_n + \mathbb{C}\mathbf{1}) \quad (n \in \mathbb{N}).$$

We now proceed with the case $\deg(\mathbf{A}) \le 2$. Then $\mathbf{A}$ is finite-dimensional and [10, Theorem 7.1] shows that $\tau = (\tau_n)^{\mathcal{U}}$ for some sequence (τ_n) with $\tau_n \in A_n^*$ $(n \in \mathbb{N})$. This implies that $\mathbf{D} = (D_n)^{\mathcal{U}}$ where $D_n = \varDelta_n - \tau_n \in \mathcal{L}(A_n, A_n + \mathbb{C}\mathbf{1})$ $(n \in \mathbb{N})$.

From the definition we see that

$$\operatorname{sder}(\varDelta_n) \le \operatorname{der}(D_n) + \|\tau_n\|_t \quad (n \in \mathbb{N})$$

and therefore

$$\begin{aligned}\lim_{\mathcal{U}} \operatorname{sder}(\varDelta_n) &\le \lim_{\mathcal{U}} (\operatorname{der}(D_n) + \|\tau_n\|_t) = \lim_{\mathcal{U}} \operatorname{der}(D_n) + \lim_{\mathcal{U}} \|\tau_n\|_t \\ &= \lim_{\mathcal{U}} \operatorname{der}(D_n) + \|\tau\|_t = \lim_{\mathcal{U}} \operatorname{der}(D_n),\end{aligned}$$

which finally gives $\varepsilon \le \lim_{\mathcal{U}} \operatorname{der}(D_n)$. However, since $\mathbf{D}$ is a derivation, Lemma 3.6 gives $\lim_{\mathcal{U}} \operatorname{der}(D_n) = \operatorname{der}(\mathbf{D}) = 0$, which is impossible. ∎

4. Lie maps on operator algebras. Throughout this section we restrict our attention to the Banach algebra $\mathcal{L}(H)$ for a Hilbert space H. Our purpose is to relate lmult and lder to $\operatorname{dist}(\cdot, \operatorname{LHom}(\mathcal{L}(H)))$ and $\operatorname{dist}(\cdot, \operatorname{LDer}(\mathcal{L}(H)))$. First of all, it should be pointed out that in the case when H is infinite-dimensional the only linear functional on $\mathcal{L}(H)$ sending commutators to zero is the zero functional. This implies that every Lie automorphism of $\mathcal{L}(H)$ is either an automorphism or the negative of an antiautomorphism of $\mathcal{L}(H)$ and every Lie derivation on $\mathcal{L}(H)$ is, in fact, a derivation. We now turn our attention to the stability problem.

LEMMA 4.1. *Let H be an infinite-dimensional Hilbert space. Then*

$$\|f\| \le 2\|f\|_t \quad (f \in \mathcal{L}(H)^*).$$

Proof. Let $T \in \mathcal{L}(H)$. On account of [9, Corollary of Theorem 8], we have $T = [P,Q] + [R,S]$ with $P,Q,R,S \in \mathcal{L}(H)$. Further, it is straightforward to check that the operators constructed in [9, Lemma 2] satisfy the preceding factorization and are such that $\|P\|, \|R\| \le \|T\|$ and $\|Q\|, \|S\| \le 1$. If

$f \in \mathcal{L}(H)$, then

$$\begin{aligned}\|f(T)\| &\le \|f([P,Q])\| + \|f([R,S])\|\\ &\le \|f\|_t\|P\|\,\|Q\| + \|f\|_t\|R\|\,\|S\| \le 2\|f\|_t\|T\|,\end{aligned}$$

which proves the lemma. ∎

THEOREM 4.2. *Let H be an infinite-dimensional separable Hilbert space. For any $M,\varepsilon > 0$ there exists $\delta > 0$ such that if $\Phi\colon \mathcal{L}(H) \to \mathcal{L}(H)$ is a bijective continuous linear map with $\|\Phi\|, \|\Phi^{-1}\| \le M$ and $\operatorname{lmult}(\Phi) < \delta$ then*

$$\min\{\operatorname{dist}(\Phi, \operatorname{Hom}(\mathcal{L}(H))), \operatorname{dist}(\Phi, -\operatorname{AHom}(\mathcal{L}(H)))\} < \varepsilon.$$

Proof. Suppose, contrary to our claim, that there exist $M,\varepsilon > 0$ and a sequence (Φ_n) of bijective continuous linear maps $\Phi_n\colon \mathcal{L}(H) \to \mathcal{L}(H)$ with $\|\Phi_n\|, \|\Phi_n^{-1}\| \le M$, $\lim_{n\to\infty} \operatorname{lmult}(\Phi_n) = 0$, and

$$\min\{\operatorname{dist}(\Phi_n, \operatorname{Hom}(\mathcal{L}(H))), \operatorname{dist}(\Phi_n, -\operatorname{AHom}(\mathcal{L}(H)))\} \ge \varepsilon \qquad (n \in \mathbb{N}). \tag{4.1}$$

From Theorem 3.4 we deduce that $\lim_{n\to\infty} \operatorname{smult}(\Phi_n) = 0$. Consequently, we can assume that either

$$\lim_{n\to\infty} \operatorname{smult}_+(\Phi_n) = 0 \tag{4.2}$$

or

$$\lim_{n\to\infty} \operatorname{smult}_-(\Phi_n) = 0. \tag{4.3}$$

We begin by considering the case when (4.2) holds. Then we get a sequence (τ_n) in $\mathcal{L}(H)^*$ such that $\lim_{n\to\infty} \operatorname{mult}(\Phi_n - \tau_n\mathbf{1}) = 0$ and $\lim_{n\to\infty} \|\tau_n\|_t = 0$. From [11, Proposition 6.3] we deduce that there is a sequence (Ψ_n) in $\operatorname{Hom}(\mathcal{L}(H))$ with $\lim_{n\to\infty} \|\Phi_n - \tau_n\mathbf{1} - \Psi_n\| = 0$. Moreover, from Lemma 4.1 we see that $\lim_{n\to\infty} \|\tau_n\| = 0$, and consequently

$$\lim_{n\to\infty} \operatorname{dist}(\Phi_n, \operatorname{Hom}(\mathcal{L}(H))) \le \lim_{n\to\infty} \|\Phi_n - \Psi_n\| = 0,$$

which contradicts (4.1).

We now turn to the case when (4.3) holds. Then there exists a sequence (σ_n) in $\mathcal{L}(H)^*$ with $\lim_{n\to\infty} \operatorname{amult}(\sigma_n\mathbf{1} - \Phi_n) = 0$ and $\lim_{n\to\infty} \|\sigma_n\|_t = 0$. From [2, Proposition 3.8] it may be concluded that there is a sequence (Θ_n) in $\operatorname{AHom}(\mathcal{L}(H))$ with $\lim_{n\to\infty} \|\sigma_n\mathbf{1} - \Phi_n - \Theta_n\| = 0$. By Lemma 4.1, $\lim_{n\to\infty} \|\sigma_n\| = 0$, and hence

$$\lim_{n\to\infty} \operatorname{dist}(\Phi_n, -A\operatorname{Hom}(\mathcal{L}(H))) \le \lim_{n\to\infty} \|\Phi_n - (-\Theta_n)\| = 0,$$

contrary to (4.1). ∎

LEMMA 4.3. *Let A be an ultraprime Banach algebra, X be a Banach A-bimodule, and $T \in \mathcal{L}(A,X)$. Let $T_I\colon I \to X$ be the restriction of T to a nonzero two-sided ideal I of A. Then $\|T\| \le \kappa(A)^{-1}(\operatorname{der}(T) + 2\|T_I\|)$.*

Proof. Let $a, b \in A$ and $c \in I$ with $\|a\| = \|b\| = \|c\| = 1$. Then

$$\begin{aligned}\|T(a)bc\| &\leq \|T(a)bc + aT(bc) - T(abc)\| + \|T(abc)\| + \|aT(bc)\| \\ &\leq \operatorname{der}(T) + 2\|T_I\|.\end{aligned}$$

This implies that

$$\|M_{T(a),c}\| \leq \operatorname{der}(T) + 2\|T_I\|.$$

Since $\kappa(A)\|T(a)\| \leq \|M_{T(a),c}\|$, it follows that

$$\|T(a)\| \leq \kappa(A)^{-1}(\operatorname{der}(T) + 2\|T_I\|)$$

and this establishes the result. ∎

LEMMA 4.4. *Let H be a Hilbert space. Then there exists $M > 0$ such that*

$$\operatorname{dist}(\varDelta, \operatorname{Der}(\mathcal{L}(H))) \leq M \operatorname{der}(\varDelta)$$

for each continuous linear operator $\varDelta\colon \mathcal{L}(H) \to \mathcal{L}(H)$.

Proof. Let $\mathcal{K}(H)$ and $\mathcal{N}(H)$ denote the two-sided ideals of $\mathcal{L}(H)$ consisting of the compact linear operators on H and the nuclear operators on H, respectively. Then $\mathcal{K}(H)$ is an amenable Banach algebra, $\mathcal{N}(H)$ is a Banach $\mathcal{K}(H)$-bimodule, and we can identify $\mathcal{L}(H)$ as the dual of that $\mathcal{K}(H)$-bimodule. Consequently, we can apply [1, Theorem 3.1] to get a constant $C > 0$ with the property that for every continuous linear map $\varGamma\colon \mathcal{K}(H) \to \mathcal{L}(H)$ there exists $T \in \mathcal{L}(H)$ such that

$$\|\varGamma - \operatorname{ad}_{\mathcal{K}(H)}(T)\| \leq C \operatorname{der}(\varGamma).$$

Here, for each $T \in \mathcal{B}(H)$ we write $\operatorname{ad}(T)$ for the inner derivation on $\mathcal{L}(H)$ implemented by T, i.e. $\operatorname{ad}(T)(S) = [T, S]$ $(S \in \mathcal{L}(H))$, and we write $\operatorname{ad}_{\mathcal{K}(H)}(T)$ for the restriction of $\operatorname{ad}(T)$ to $\mathcal{K}(H)$.

Let $\varDelta \in \mathcal{L}(\mathcal{L}(H))$. Then we apply the preceding property to the restriction $\varDelta_{\mathcal{K}(H)}$ of $\varDelta$ to $\mathcal{K}(H)$ to get $T \in \mathcal{L}(H)$ such that

$$\|\varDelta_{\mathcal{K}(H)} - \operatorname{ad}_{\mathcal{K}(H)}(T)\| \leq C \operatorname{der}(\varDelta_{\mathcal{K}(H)}) \leq C \operatorname{der}(\varDelta).$$

Lemma 4.3 then yields

$$\|\varDelta - \operatorname{ad}(T)\| \leq \operatorname{der}(\varDelta - \operatorname{ad}(T)) + 2C \operatorname{der}(\varDelta) = (2C+1) \operatorname{der}(\varDelta),$$

which implies that $\operatorname{dist}(\varDelta, \operatorname{Der}(\mathcal{L}(H))) \leq (2C+1)\operatorname{der}(\varDelta)$, as required. ∎

THEOREM 4.5. *Let H be a Hilbert space. Then there exists $M > 0$ such that*

$$\operatorname{dist}(\varDelta, \operatorname{Der}(\mathcal{L}(H))) \leq M \operatorname{lder}(\varDelta)$$

for each continuous linear operator $\varDelta\colon \mathcal{L}(H) \to \mathcal{L}(H)$.

Proof. Suppose the assertion is false. Then there exists a sequence $(\varGamma_n)$ in $\mathcal{L}(\mathcal{L}(H))$ such that $\operatorname{dist}(\varGamma_n, \operatorname{Der}(\mathcal{L}(H))) = 1$ for each $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \operatorname{lder}(\varGamma_n) = 0.$$

For each $n \in \mathbb{N}$ we define $\Delta_n = \Gamma_n - \delta_n$ where $\delta_n \in \mathrm{Der}(\mathcal{L}(H))$ is chosen so that $\lim_{n\to\infty} \|\Delta_n\| = 1$. Moreover, we have

$$\mathrm{dist}(\Delta_n, \mathrm{Der}(\mathcal{L}(H))) = \mathrm{dist}(\Gamma_n, \mathrm{Der}(\mathcal{L}(H))) = 1 \quad (n \in \mathbb{N}) \tag{4.4}$$

and $\mathrm{lder}(\Delta_n) = \mathrm{lder}(\Gamma_n)$ $(n \in \mathbb{N})$, which yields $\lim_{n\to\infty} \mathrm{lder}(\Delta_n) = 0$. From Theorem 3.8 it follows that $\lim_{n\to\infty} \mathrm{sder}(\Delta_n) = 0$. This implies that there exists a sequence (τ_n) in $\mathcal{L}(H)^*$ such that

$$\lim_{n\to\infty} \mathrm{der}(\Delta_n - \tau_n \mathbf{1}) = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\tau_n\|_t = 0.$$

From Lemma 4.4 we deduce that there exists a sequence (D_n) in $\mathrm{Der}(\mathcal{L}(H))$ with

$$\lim_{n\to\infty} \|\Delta_n - \tau_n \mathbf{1} - D_n\| = 0.$$

By Lemma 4.1, we have $\lim_{n\to\infty} \|\tau_n\| = 0$. Consequently, $\lim_{n\to\infty} \|\Delta_n - D_n\| = 0$, which contradicts (4.4). ∎

In order to give a full picture about the behaviour of the Lie maps on $\mathcal{B}(H)$ for a Hilbert space H, we now complete the information given in Theorems 4.2 and 4.5 by considering the finite-dimensional case. It is well-known that Lie automorphisms and Lie derivations of the matrix algebra $\mathbb{M}_n$ are of the standard form. Actually, every Lie automorphism of $\mathbb{M}_n$ is given by $a \mapsto uau^{-1} + \alpha\, \mathrm{tr}_n(a)\mathbf{1}$ or $a \mapsto -ua^tu^{-1} + \alpha\, \mathrm{tr}_n(a)\mathbf{1}$ for some invertible $u \in \mathbb{M}_n$ and $\alpha \in \mathbb{C}$, where tr_n and $(\cdot)^t$ stand for the trace and transposition on $\mathbb{M}_n$, respectively. Every Lie derivation on $\mathbb{M}_n$ is given by $\mathrm{ad}(v) + \alpha\, \mathrm{tr}_n \mathbf{1}$ for some $v \in \mathbb{M}_n$ and $\alpha \in \mathbb{C}$. The stability of both $\mathrm{LHom}(\mathbb{M}_n)$ and $\mathrm{LDer}(\mathbb{M}_n)$ is provided by the following results.

Proposition 4.6. *Let A and B be finite-dimensional Banach algebras. For any $M, \varepsilon > 0$ there exists $\delta > 0$ such that if $\Phi \in \mathcal{L}(B, A)$ with $\|\Phi\| \le M$ and* $\mathrm{lmult}(\Phi) < \delta$ *then* $\mathrm{dist}(\Phi, \mathrm{LHom}(B, A)) < \varepsilon$.

Proof. The proof of [11, Proposition 1.3] carries over almost verbatim. Let $M, \varepsilon > 0$. Then the set

$$\mathcal{C}_{M,\varepsilon} = \{T \in \mathcal{L}(B, A) : \|T\| \le M,\ \mathrm{dist}(T, \mathrm{LHom}(B, A)) \ge \varepsilon\}$$

is compact. Further, we consider the decreasing net $(\mathcal{G}_\delta)_{\delta>0}$ of open sets given by

$$\mathcal{G}_\delta = \{T \in \mathcal{L}(B, A) : \mathrm{lmult}(T) > \delta\} \quad (\delta > 0).$$

Then

$$\mathcal{C}_{M,\varepsilon} \subset \mathcal{L}(B, A) \setminus \mathrm{LHom}(B, A) = \bigcup_{\delta>0} \mathcal{G}_\delta.$$

Consequently, there exists $\delta > 0$ such that $\mathcal{C}_{M,\varepsilon} \subset \mathcal{G}_\delta$, which is the desired conclusion. ∎

PROPOSITION 4.7. *Let A be a finite-dimensional Banach algebra. Then there exists $M > 0$ such that*

$$\operatorname{dist}(\Delta, \operatorname{LDer}(A)) \leq M \operatorname{lder}(\Delta) \quad \textit{for each } \Delta \in \mathcal{L}(A).$$

Proof. The seminorms $\operatorname{dist}(\cdot, \operatorname{LDer}(A))$ and $\operatorname{lder}(\cdot)$ on $\mathcal{L}(A)$ vanish on $\operatorname{LDer}(A)$ so that both give rise to norms on the quotient $\mathcal{L}(A)/\operatorname{LDer}(A)$. Since this linear space is finite-dimensional, it follows that both norms are equivalent, which proves the proposition. ■

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