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On the multiplicity of critical points for parameterized functionals on reflexive Banach spaces

by

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Abstract. Some general multiplicity results for critical points of parameterized functionals on reflexive Banach spaces are established. In particular, one of them improves some aspects of a recent result by B. Ricceri. Applications to boundary value problems are also given.

1. Introduction. Let X be a real reflexive Banach space and let $f : X \to \mathbb{R}$ be a real C^1 functional. Generally speaking, in this paper we will consider the problem of finding multiple critical points of f represented by a linear combination of real functionals with positive parameters. In this direction, many results have been established in recent years. In particular, this topic has been widely studied by B. Ricceri by making use of the theory introduced by that author in [R1]–[R3]. The generality of these results makes them applicable to several questions, including the existence and the multiplicity of solutions of boundary value problems for partial differential equations (see for instance [A2], [AC1]–[AC3], [BC], [BMR], [CCD], [K1]–[K3], [R6], [R7] and the references therein).

Our aim is to give new contributions to the critical points theory for functionals of the above type which are inspired by a very recent result established by B. Ricceri (Theorem 1 of [R4]) as a consequence of the results of [R5] and [R8] (see also [R9]). In doing that, we will make use of variational methods and follow some ideas introduced in [A1].

We stress that one of our main results (Theorem 1 below) improves some aspects of Theorem 1 of [R4].

2. Existence of multiple critical points. Our first result on multiple critical points is as follows:

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Theorem 1. *Let X be a reflexive real Banach space. Let $I : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous and coercive C^1 functional whose derivative admits a continuous inverse on X^*. Finally, let $J, \Psi, \Phi : X \to \mathbb{R}$ be three C^1 functionals with compact derivative satisfying the following conditions:*

(a) $\liminf_{\|x\|\to\infty} \frac{J(x)}{I(x)} \geq 0$;

(b) $\limsup_{\|x\|\to\infty} \frac{J(x)}{I(x)} < \infty$;

(c) $\liminf_{\|x\|\to\infty} \frac{\Psi(x)}{I(x)} = -\infty$;

(d) $\inf_{x\in X} (\Psi(x) + \lambda\Phi(x)) > -\infty$ *for all* $\lambda > 0$;

(e) *there exists a strict local minimum point* $x_0 \in X$ *of* I *such that*

(e$_1$) $I(x_0) = J(x_0) = \Psi(x_0) = \Phi(x_0) = 0$;

(e$_2$) $\liminf_{x\to x_0} \frac{J(x)}{I(x)} \geq 0$;

(e$_3$) $\liminf_{x\to x_0} \frac{\Psi(x)}{I(x)} > -\infty$ *and* $\liminf_{x\to x_0} \frac{\Phi(x)}{I(x)} > -\infty$;

(f) *there exists* $y_0 \in X$ *such that* $J(y_0) < 0$.

Then, for each $\nu \in\,]0, \infty[$ *with* $\nu > -I(y_0)/J(y_0)$, *there exists* $\lambda_0 > 0$ *with the following property: for all* $\lambda \in\,]0, \lambda_0]$ *there exists* $\sigma_\lambda > 0$ *such that, for all* $\sigma \in\,]0, \sigma_\lambda[$,

(α) x_0 *is a critical point of* $I + \nu J + \lambda\Psi + \sigma\Phi$;

(β) *there exist three pairwise distinct critical points* x_1, x_2, x_3 *of* $I + \nu J + \lambda\Psi + \sigma\Phi$, *distinct from* x_0 *and satisfying*

$$(I + \nu J)(x_1) < 0 < (I + \nu J)(x_2), \tag{1}$$

$$(I + \nu J + \lambda\Psi + \sigma\Phi)(x_3) \geq 0. \tag{2}$$

Moreover, x_0, x_1, x_2 *are local minimum points of* $I + \nu J + \lambda\Psi + \sigma\Phi$, *two of which, including* x_0, *are not global and one is global.*

Proof. First, note that the functionals J, Ψ, Φ are sequentially weakly continuous on X (see, for instance, Corollary 41.9 of [Z]). Next, fix $\nu \in\,]0, \infty[$ such that $\nu > -I(y_0)/J(y_0)$. Since I is coercive, in view of condition (a) it follows that

$$\lim_{\|x\|\to\infty} (I(x) + \nu J(x)) = \infty. \tag{3}$$

Therefore, since $I(\cdot)+\nu J(\cdot)$ is sequentially weakly lower semicontinuous, it admits a global minimum point $x_\nu \in X$. Moreover,

$$\inf_{x\in X}(I(x)+\nu J(x)) = I(x_\nu)+\nu J(x_\nu) \le I(y_0)+\nu J(y_0) < 0.$$

Consequently,

$$\emptyset \ne (I+\nu J)^{-1}(]-\infty,0[) \subset (I+\nu J)^{-1}(]-\infty,0])$$

and the set $(I+\nu J)^{-1}(]-\infty,0])$ is sequentially weakly compact in X (and so weakly compact by the Eberlein–Šmulian Theorem). From this fact and since Ψ is sequentially weakly continuous in X, one infers that the real functions

$$g_1(\lambda) = \inf_{(I+\nu J)^{-1}(]-\infty,0])}(I+\nu J+\lambda\Psi),$$
$$g_2(\lambda) = \inf_{(I+\nu J)^{-1}(0)}(I+\nu J+\lambda\Psi)$$

are well defined and continuous (being concave) in $\mathbb{R}$. Since $g_1(0) < g_2(0) = 0$, there exists $\tilde\lambda > 0$ such that

$$g_1(\lambda) < g_2(\lambda) \quad \text{for all } \lambda\in[0,\tilde\lambda]. \tag{4}$$

Now, put

$$\begin{cases} \lambda_0 = \tilde\lambda & \text{if } \liminf_{x\to x_0} \dfrac{\Psi(x)}{I(x)} \ge 0, \\ \lambda_0 = \min\left\{\tilde\lambda, \left(-\liminf_{x\to x_0}\dfrac{\Psi(x)}{I(x)}\right)^{-1}\right\} & \text{if } \liminf_{x\to x_0} \dfrac{\Psi(x)}{I(x)} < 0. \end{cases}$$

Note that from assumption (e_3) we have $\lambda_0 > 0$.

Let $\lambda \in]0,\lambda_0[$ and fix

$$\alpha_\lambda \in]0,\lambda^{-1}[\quad \text{with} \quad \alpha_\lambda > -\liminf_{x\to x_0}\frac{\Psi(x)}{I(x)} \text{ if } \liminf_{x\to x_0}\frac{\Psi(x)}{I(x)} < 0. \tag{5}$$

From assumptions (b) and (c) we can find $x_\lambda \in (I+\nu J)^{-1}(]0,\infty[)$ such that

$$(I+\nu J+\lambda\Psi)(x_\lambda) < g_1(\lambda). \tag{6}$$

In view of (3) and assumption (d), it follows that the functional $I+\nu J+\lambda\Psi+\sigma\Phi$ is coercive for all $\sigma>0$. Since the same functional is sequentially weakly lower semicontinuous as well, the real functions

$$h_1(\sigma) = \inf_{(I+\nu J)^{-1}(]-\infty,0])}(I+\nu J+\lambda\Psi+\sigma\Phi),$$
$$h_2(\sigma) = \inf_{(I+\nu J)^{-1}(0)}(I+\nu J+\lambda\Psi+\sigma\Phi)$$

are well defined and continuous (being concave) in $\mathbb{R}$ (recall that the set $(I+\nu J)^{-1}(]-\infty,0])$ is sequentially weakly compact).

Now, observe that since $\lambda < \tilde{\lambda}$, from (4) and (6) one has

$$(I + \nu J + \lambda\Psi)(x_\lambda) < h_1(0) = g_1(\lambda) < g_2(\lambda) = h_2(0).$$

Hence, there exists $\tilde{\sigma} > 0$ such that

$$(I + \nu J + \lambda\Psi + \sigma\Phi)(x_\lambda) < h_1(\sigma) < h_2(\sigma) \text{ for all } \sigma \in [0, \tilde{\sigma}]. \tag{7}$$

At this point, put

$$\begin{cases} \sigma^* = \tilde{\sigma} & \text{if } \liminf\limits_{x\to x_0} \dfrac{\Phi(x)}{I(x)} \geq 0, \\ \sigma^* = \min\left\{\tilde{\sigma}, \dfrac{1}{2}\left(-\liminf\limits_{x\to x_0} \dfrac{\Phi(x)}{I(x)}\right)^{-1}\right\} & \text{if } \liminf\limits_{x\to x_0} \dfrac{\Phi(x)}{I(x)} < 0. \end{cases}$$

Note that from assumption (e_3) we have $\sigma^* > 0$.

Moreover, put $\sigma_\lambda = (1 - \lambda\alpha_\lambda)\sigma^*/2$, where α_λ is as in (5), and let $\sigma \in]0, \sigma_\lambda[$.

Taking into account the choices of α_λ and σ^*, assumption (e_2) and the fact that x_0 is a strict local minimum point of I, we can find $r > 0$ such that

- $I(x) > I(x_0)$;
- $J(x) \geq -\dfrac{1 - \lambda\alpha_\lambda}{2\nu} I(x)$;
- $\Psi(x) \geq -\alpha_\lambda I(x)$;
- $\Phi(x) \geq -\dfrac{1}{\sigma^*} I(x)$,

for all $x \in B(x_0, r) \setminus \{x_0\}$. Consequently,

$$\begin{aligned}(I + \nu J + \lambda\Psi + \sigma\Phi)(x) &\geq \left(1 - \nu\frac{1 - \lambda\alpha_\lambda}{2\nu} - \alpha_\lambda\lambda - \frac{1}{\sigma^*}\sigma\right) I(x) \\ &> \left(\frac{1 - \lambda\alpha_\lambda}{2} - \frac{1 - \lambda\alpha_\lambda}{2}\right) I(x) = 0 = (I + \nu J + \lambda\Psi + \sigma\Phi)(x_0)\end{aligned}$$

for all $x \in B(x_0, r) \setminus \{x_0\}$. Hence, x_0 is a strict local minimum point for the functional $I + \nu J + \lambda\Psi + \sigma\Phi$. Moreover, since the boundary of the set $(I + \nu J)^{-1}(]-\infty, 0])$ is a subset of $(I + \nu J)^{-1}(0)$ and since, by (7), one has $h_1(\sigma) < h_2(\sigma)$, it is easy to see that $I + \nu J + \lambda\Psi + \sigma\Phi$ has a local minimum at some x_1 belonging to $(I + \nu J)^{-1}(]-\infty, 0[)$.

The coerciveness and the sequential weak lower semicontinuity of the functional $I + \nu J + \lambda\Psi + \sigma\Phi$ ensure the existence of a global minimum point $x_2 \in X$ for this functional. Moreover, the left inequality of (7) implies that $x_2 \in (I + \nu J)^{-1}(]0, \infty[)$. Of course, x_0, x_1 and x_2 are three pairwise distinct critical points of $I + \nu J + \lambda\Psi + \sigma\Phi$ with x_1, x_2 satisfying (1). Since this functional satisfies the Palais–Smale condition (see, for instance, Example

38.25 of [Z]), the existence of a fourth critical point x_3, distinct from the previous ones and satisfying (2), is ensured by Theorem (1. ter) of [GP]. ∎

REMARK 1. As said in the introduction, Theorem 1 improves some aspects of Theorem 1 of [R4]. Indeed, in the latter the following stronger assumptions are required:

- $\liminf_{x \to x_0} \frac{\Phi(x)}{I(x)} \geq 0$;
- there exists $y_0 \in X$ such that $\max\{J(y_0), \Psi(y_0), \Phi(y_0)\} < 0$.

Moreover, Theorem 1 of [R4] guarantees (in a slightly formally different statement) the existence of four critical points for $I + \nu J + \lambda\Psi + \sigma^*\Phi$, for each ν as in Theorem 1, each $\lambda \in]0, \lambda_0[$, where λ_0 is a positive constant explicitly determined, and for some $\sigma^* > 0$ depending on ν, λ. However, we stress that, from a variational point of view, our Theorem 1 and Theorem 1 of [R4] are different results. Indeed, the latter states the existence of three local minima of which only x_0 is not global.

In view of Remark 1, we can revisit Theorem 2 of [R4] which is an application of Theorem 1 of [R4] to a Dirichlet problem associated to quasilinear elliptic equations. Before, let us recall some definitions.

Consider a nonempty open bounded set Ω in $\mathbb{R}^N$ with smooth boundary and let $p > 1$. Let $\mathcal{A}$ be the class of all Carathéodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

- if $p \leq N$, there exist $M \geq 0$ and $q > 0$, with $q + 1 < Np/(N - p)$ if $p < N$, such that
$$|f(x,t)| \leq M(1 + |t|^q) \quad \text{for all } t \in \mathbb{R} \text{ and almost all } x \in \Omega;$$
- if $p > N$, one has $\sup_{|t| \leq r} |f(\cdot, t)| \in L^1(\Omega)$ for all $r > 0$.

Given $f \in \mathcal{A}$, a *weak solution* of the problem
$$\begin{cases} -\Delta_p u = f(x,u) \text{ in } \Omega, \\ u_{|\partial\Omega} = 0, \end{cases}$$
where $-\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, is any $u \in W_0^{1,p}(\Omega)$ satisfying the equation
$$\int_\Omega \big(|\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x) - f(x,u(x))v(x)\big)\,dx = 0$$
for all $v \in W_0^{1,p}(\Omega)$.

Following the same arguments of [R4] but applying our Theorem 1 instead of Theorem 1 of [R4], we obtain the following result that improves Theorem 2 of [R4].

THEOREM 2. *Let $q \in]p,\infty[$, with $q < Np/(N-p)$ if $N > p$, and let $f, g, h \in \mathcal{A}$. Put*

$$F(x,\xi) = \int_0^\xi f(x,t)\,dt, \quad G(x,\xi) = \int_0^\xi f(x,t)\,dt, \quad H(x,\xi) = \int_0^\xi h(x,t)\,dt$$

for all $(x,t) \in \Omega \times \mathbb{R}$ and assume the following conditions hold:

(a) $\displaystyle\lim_{\xi\to\infty} \frac{\operatorname{ess\,inf}_{x\in\Omega} F(x,\xi)}{\xi^p} = \infty, \quad \limsup_{|\xi|\to\infty} \frac{\operatorname{ess\,sup}_{x\in\Omega} F(x,\xi)}{|\xi|^q} < \infty,$

(b) $\displaystyle\lim_{|\xi|\to\infty} \frac{\operatorname{ess\,inf}_{x\in\Omega} G(x,\xi)}{|\xi|^q} = \infty,$

(c) $\displaystyle\limsup_{|\xi|\to\infty} \frac{\operatorname{ess\,sup}_{x\in\Omega} H(x,\xi)}{|\xi|^p} \le 0, \quad \liminf_{|\xi|\to\infty} \frac{\operatorname{ess\,sup}_{x\in\Omega} H(x,\xi)}{|\xi|^p} > -\infty,$

(d) $\displaystyle\limsup_{\xi\to 0} \frac{\operatorname{ess\,sup}_{x\in\Omega} F(x,\xi)}{|\xi|^p} < \infty,$

(e) $\displaystyle\liminf_{\xi\to 0} \frac{\operatorname{ess\,inf}_{x\in\Omega} G(x,\xi)}{|\xi|^p} > -\infty,$

(f) $\displaystyle\limsup_{\xi\to 0} \frac{\operatorname{ess\,sup}_{x\in\Omega} H(x,\xi)}{|\xi|^p} \le 0.$

Moreover, assume that there exist a measurable set $B \subset \Omega$ and $\xi_1 \in \mathbb{R}$ such that

(g) *$H(x,\xi_1) > 0$ for all $x \in B$.*

Then, for each $\nu > 0$ large enough, there exists $\lambda_\nu > 0$ with the following property: for each $\lambda \in]0,\lambda_\nu]$ there exists $\sigma_\lambda > 0$ such that, for all $\sigma \in]0,\sigma_\lambda]$, the problem

$$\begin{cases} -\Delta_p u = \lambda f(x,u) - \sigma g(x,u) + \nu h(x,u) & \text{in } \Omega, \\ u_{|\partial\Omega} = 0, \end{cases}$$

has at least three nonzero weak solutions.

Proof. Apply Theorem 1 with $X = W_0^{1,p}(\Omega)$, $I(u) = (1/p)\int_\Omega |\nabla u|^p dx$, $J(u) = -\int_\Omega H(x,u(x))dx$, $\Psi(u) = -\int_\Omega F(x,u(x))dx$, and $\Phi(u) = \int_\Omega G(x,u(x))dx$ for all $u \in W_0^{1,p}(\Omega)$, $x_0 = 0$, and $y_0 = u_0$, where $u_0 \in W_0^{1,p}(\Omega)$ is a function satisfying $J(u_0) < 0$. Such a function exists thanks to condition (g) (see [R4]). ∎

Theorem 1 guarantees the existence of three distinct local minimum points x_0, x_1 and x_2. Moreover, the local minimum at x_0 is strict. Then the existence of a fourth critical point follows by applying a Mountain Pass Theorem, that is, Theorem (1. ter) of [GP]. Note that the previous result can be applied in two ways: either considering the paths joining x_0 to x_1, or considering the paths joining x_0 to x_2. Doing so, we obtain two critical points distinct from x_0, x_1 and x_2. Nevertheless, we do not know if they are distinct or not in general. However, we are going to show that, by imposing some further condition on the functionals J and Ψ, it is possible to distinguish the critical points which result from mountain pass theorems. This yields the following five critical points result.

THEOREM 3. *Let X be a reflexive real Banach space. Let $I, J_1, J_2, \Psi, \Phi : X \to \mathbb{R}$ be five C^1 functionals such that J_1 is nonnegative, Ψ is nonpositive, and I is coercive, sequentially weakly lower semicontinuous and its derivative has continuous inverse on X^*. Moreover, assume that J_1, J_2, Ψ, Φ have compact derivatives and I, J, Ψ, Φ, with $J := \tau J_1 + J_2$, satisfy conditions* (b)–(e) *of Theorem* 1 *for all $\tau > 0$ and condition* (a) *for $\tau = 0$. Finally, let x_0 be as in condition* (e) *and suppose also that:*

(i) *there exists $y_0 \in X$ satisfying*
- (i_1) *$J_2(y_0) < 0$ and $(I + J_2)(y_0) < 0$;*
- (i_2) $\sup_{t\in[0,1]}(I + J_2)(x_0 + t(y_0 - x_0)) \leq 0$.

Then there exist $\lambda_0, \tau_0 > 0$ with the following property: for all $\lambda \in \,]0, \lambda_0[$ there exists $\sigma_\lambda > 0$ such that, for all $\tau \in \,]0, \tau_0[$ and $\sigma \in \,]0, \sigma_\lambda[$,

(α_1) *x_0 is a critical point of $I + \tau J_1 + J_2 + \lambda\Psi + \sigma\Phi$;*

(β_1) *there exist four pairwise distinct critical points x_1, x_2, x_3, x_4 of $I + \tau J_1 + J_2 + \lambda\Psi + \sigma\Phi$, distinct from x_0 and satisfying*

$$(I + J_2)(x_1) < 0 < (I + J_2)(x_2),$$
$$(I + \tau J_1 + J_2 + \lambda\Psi + \sigma\Phi)(x_4) > (I + \tau J_1 + J_2 + \lambda\Psi + \sigma\Phi)(x_3) \geq 0.$$

Moreover, x_0, x_1, x_2 are local minimum points of $I + \tau J_1 + J_2 + \lambda\Psi + \sigma\Phi$, two of which, including x_0, are not global and one is global.

Proof. Fix $\gamma > 0$. From condition (a) there exists $r_0 > 0$ such that

$$\inf_{\|x-x_0\|=r_0} (I + J_2)(x) > \gamma > 0.$$

Since Ψ and Φ are bounded on bounded sets, the real function

$$(\lambda, \sigma) \in \mathbb{R}^2 \mapsto \inf_{\|x-x_0\|=r_0} (I + J_2 + \lambda\Psi + \sigma\Phi)(x)$$

is continuous. Hence from the previous inequality we infer that there exist $\lambda' > 0$ and $\sigma' > 0$ such that, for all $\lambda \in [0, \lambda']$ and $\sigma \in [0, \sigma']$, one has

$$\inf_{\|x-x_0\|=r_0} (I + J_2 + \lambda\Psi + \sigma\Phi)(x) > \gamma > 0.$$

Consequently, in view of the nonnegativity of J_1, we also have

$$\inf_{\|x-x_0\|=r_0}(I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi)(x)>\gamma>0 \tag{8}$$

for all $\lambda\in[0,\lambda']$, $\sigma\in[0,\sigma']$ and $\tau>0$.

Now, let $y_0\in X$ satisfy (i). Then, from the continuity of the real function

$$(\tau,\sigma)\in\mathbb{R}^2\mapsto\sup_{t\in[0,1]}(I+\tau J_1+J_2+\sigma\Phi)(x_0+t(y_0-x_0)),$$

there exist $\tau_0>0$ and $\sigma''>0$ such that, for all $\tau\in[0,\tau_0]$ and $\sigma\in[0,\sigma'']$, one has

$$\begin{aligned}&(I+\tau J_1+J_2+\sigma\Phi)(y_0)<0,\\ &\sup_{t\in[0,1]}(I+\tau J_1+J_2+\sigma\Phi)(x_0+t(y_0-x_0))\le\gamma.\end{aligned} \tag{9}$$

Consequently, in view of the nonpositivity of Ψ, we also have

$$(I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi)(y_0)<0, \tag{10}$$

$$\sup_{t\in[0,1]}(I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi)(x_0+t(y_0-x_0))\le\gamma, \tag{11}$$

for all $\tau\in[0,\tau_0]$, $\sigma\in[0,\sigma'']$ and $\lambda>0$.

At this point, observe that by condition (i_1) and (9) there exists $\tilde\lambda\in\,]0,\lambda']$ such that

$$(I+\tau_0J_1+J_2)(y_0)<\lambda\Big(\inf_{(I+J_2)^{-1}(]-\infty,0])}\Psi-\Psi(y_0)\Big) \tag{12}$$

for all $\lambda\in[0,\tilde\lambda]$. Since J_1 is nonnegative, it is easy to check that inequality (12) implies

$$(I+\tau J_1+J_2+\lambda\Psi)(y_0)<\inf_{(I+J_2)^{-1}(0)}(I++\tau J_1+J_2+\lambda\Psi) \tag{13}$$

for all $\tau\in[0,\tau_0]$ and $\lambda\in[0,\tilde\lambda]$.

Now, fix $\lambda\in[0,\tilde\lambda]$ and $x_\lambda\in(I+J_2)^{-1}(]0,\infty[)$ with $\|x_\lambda-x_0\|>r_0$ and satisfying

$$(I+\tau_0J_1+J_2+\lambda\Psi)(x_\lambda)<\inf_{(I+\nu J_2)^{-1}(]-\infty,0])}(I+J_2+\lambda\Psi). \tag{14}$$

Observe that the existence of x_λ is guaranteed by conditions (a) and (c).

Exploiting the continuity of the real function

$$\sigma\in\mathbb{R}\mapsto\inf_{(I+J_2)^{-1}(0)}(I+J_2+\lambda\Psi+\sigma\Phi),$$

from (13) and (14) we can find $\tilde\sigma_\lambda\in\,]0,\sigma'']$ such that

$$(I+\tau_0J_1+J_2+\lambda\Psi+\sigma\Phi)(y_0)<\inf_{(I+J_2)^{-1}(0)}(I+J_2+\lambda\Psi+\sigma\Phi), \tag{15}$$

$$(I+\tau_0J_1+J_2+\lambda\Psi+\sigma\Phi)(x_\lambda)<\inf_{(I+J_2)^{-1}(]-\infty,0])}(I+J_2+\lambda\Psi+\sigma\Phi), \tag{16}$$

for all $\sigma\in[0,\tilde\sigma_\lambda]$. Since J_1 is nonnegative, from (10), (15) and (16) we obtain

$$\begin{aligned}(17)\qquad &(I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi)(x_\lambda)\\ &\quad< \inf_{(I+J_2)^{-1}(]-\infty,0])}(I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi)\\ &\quad\le (I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi)(y_0)\\ &\quad< \inf_{(I+J_2)^{-1}(0)}(I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi)\end{aligned}$$

for all $\tau\in[0,\tau_0]$ and all $\sigma\in[0,\sigma_\lambda]$.

Now, fix τ and σ as above. Define λ_0 and σ_λ, where $\lambda\in]0,\lambda_0]$, as in the proof of Theorem 1. Then, arguing as in Theorem 1, we infer that x_0 is a strict local minimum point of $I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi$ and that, in view of (17), the latter functional admits two further distinct local minimum points x_1 and x_2. More precisely, x_2 is a global minimum point for $I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi$ and $x_1\in(I+J_2)^{-1}(]-\infty,0[)$ is a global minimum point for the restriction of this functional to $(I+J_2)^{-1}(]-\infty,0])$. Moreover, observing that $J_1(x_0)=J_2(x_0)=0$ (since, by hypothesis, $(\tau J_1+J_2)(x_0)=0$ for all $\tau\in\,]0,\tau_0]$) and $I(x_0)=\Psi(x_0)=\Phi(x_0)=0$, from (17) we obtain

$$\begin{aligned}(18)\qquad (I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi)(x_2)&\le (I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi)(x_\lambda)\\ &< (I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi)(x_1)<0,\end{aligned}$$

which, in turn, implies

$$x_2\in(I+J_2)^{-1}(]0,\infty[).$$

Now, having in mind (10) and (11) and applying Theorem (1. ter) of [GP], we obtain a fourth critical point x_3 distinct from x_0 which satisfies

$$(19)\qquad 0\le(I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi)(x_3)\le\gamma.$$

Finally, note that, since x_λ satisfies (17), we can apply again Theorem (1. ter) of [GP] that ensures the existence of a fifth critical point x_4 distinct from x_0 and satisfying

$$(I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi)(x_4)=\inf_{\varphi\in\Sigma}\sup_{t\in[0,1]}(I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi)(\varphi(t)),$$

where Σ is the set of all continuous paths $\varphi:[0,1]\to X$ joining x_0 to x_λ, that is, $\varphi(0)=x_0$ and $\varphi(1)=x_\lambda$.

Then, since $\|x_\lambda-x_0\|>r_0$, thanks to (8) we have

$$(20)\qquad (I+\tau J_1+J_2+\lambda\Psi+\sigma\Phi)(x_4)>\gamma.$$

To conclude, observe that inequalities (18)–(20) imply, in particular, that x_1,x_2,x_3,x_4 (and x_0) are pairwise distinct. ∎

Our last result is an application of Theorem 3 to the Dirichlet problem associated to a quasilinear elliptic equation involving a combination of power laws.

THEOREM 4. *Let $r, s, p, q, m \in \mathbb{R}$ be such that $1 < r < s < p < q < m$, with $m < Np/(N-p)$ if $N > p$. Then there exist $\lambda_0, \tau_0 > 0$ with the following property: for all $\lambda \in]0, \lambda_0[$ there exists $\sigma_\lambda > 0$ such that, for all $\tau \in]0, \tau_0[$ and $\sigma \in]0, \sigma_\lambda[$, the problem*

$$(P) \qquad \begin{cases} -\Delta_p u = -\tau u^{r-1} + u^{s-1} + \lambda u^{q-1} - \sigma u^{m-1} & \text{in } \Omega, \\ u_{|\partial\Omega} = 0, \end{cases}$$

has at least four nonzero and nonnegative weak solutions.

Proof. Denote by $\|\cdot\| = (\int_\Omega |\nabla(\cdot)|^p\,dx)^{1/p}$ the standard norm of $W_0^{1,p}(\Omega)$. For each $\lambda, \sigma, \tau \in \mathbb{R}$, the nonnegative weak solutions of problem (P) are exactly the critical points of the functional $I + \tau J_1 + J_2 + \lambda\Psi + \sigma\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$, where

$$I(u) = \frac{1}{p}\|u\|^p, \qquad J_1(u) = \frac{1}{r}\int_\Omega (u_+)^r\,dx, \qquad J_2(u) = -\frac{1}{s}\int_\Omega (u_+)^s\,dx,$$

$$\Psi(u) = -\frac{1}{q}\int_\Omega (u_+)^q\,dx, \qquad \Phi(u) = +\frac{1}{m}\int_\Omega (u_+)^m\,dx,$$

and $u_+(x) = \max\{u(x), 0\}$ (see Lemma 2 of [A1]). By standard results, it is known that I is a sequentially weakly lower semicontinuous and coercive C^1 functional whose derivative admits a continuous inverse on $(W_0^{1,p}(\Omega))^*$ and that J_1, J_2, Ψ, Φ are sequentially weakly continuous C^1 functionals with compact derivative. By embedding theorems, one has

$$\lim_{\|x\|\to\infty} \frac{\tau J_1(x) + J_2(x)}{I(x)} = 0 \quad \text{for all } \tau \in \mathbb{R}$$

and

$$\lim_{\|x\|\to 0} \frac{\Psi(x)}{I(x)} = \lim_{\|x\|\to 0} \frac{\Phi(x)}{I(x)} = 0.$$

Moreover, if we fix a positive function $\varphi \in W_0^{1,p}(\Omega)$, one has

$$\lim_{t\to\infty} \frac{\Psi(t\varphi)}{I(t\varphi)} = -\infty.$$

Observe also that, from Lemma 4 of [A1], one has

$$\liminf_{\|x\|\to 0} \frac{\tau J_1(x) + J_2(x)}{I(x)} \ge 0 \quad \text{for all } \tau > 0.$$

Finally, since

$$\liminf_{|t|\to\infty} (\lambda(t_+)^m - (t_+)^q) \ge 0 \quad \text{for all } \lambda > 0,$$

it is easy to infer that

$$\inf_{u \in W_0^{1,p}(\Omega)} (\Psi + \lambda\Phi)(u) > -\infty \quad \text{for all } \lambda > 0.$$

Now, let $v_0 \in W_0^{1,p}(\Omega) \setminus \{0\}$ be the unique global minimum point of the functional $I + J_2$ (see Lemma 5 of [A1]). Then

$$0 = (I + J_2)'(v_0)(v_0) = pI(v_0) + sJ_2(v_0).$$

Consequently,

$$J_2(v_0) = -\frac{p}{s}I(v_0) < 0 \quad \text{and} \quad (I + J_2)(v_0) = \left(1 - \frac{p}{s}\right)I(v_0) < 0,$$

$$(I + J_2)(tv_0) = t^p I(v_0) + t^s J(v_0) = \left(t^p - \frac{p}{s}t^s\right)I(v_0) \le 0,$$

for all $t \in [0, 1]$. At this point, it is easy to check that all the assumptions of Theorem 3 are satisfied with $x_0 = 0$ and $y_0 = v_0$. Hence, the conclusion follows. ■

REMARK 2. In view of Theorem 2 of [A1], it would be of interest to study problem (P) with no upper bound on the exponents m, q when $p < N$. Another interesting question is whether the four solutions of problem (P) are actually positive in Ω.

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